Lemma 8.1

Let $\Omega \subseteq \mathbb{C}$ be a domain such that $\partial\Omega$ is a positively oriented simple closed contour. Then for $z_0 \notin \partial\Omega$, we have that

$$\int_{\partial\Omega} \frac{dz}{z - z_0} = \begin{cases} 0 & \text{if } z_0 \notin \overline{\Omega} = \partial\Omega \\ 2\pi i & \text{if } z_0 \in \Omega \end{cases}$$

Proof: Suppose $z_0 \notin \overline{\Omega}$, then for each $z \in \partial\Omega$ define $r(z)$ such that $z_0 \notin D_{r(z)}(z)$. Now define $\Lambda \subseteq \mathbb{C}$ and $f : \mathbb{C} \to \mathbb{C}$ as

$$\Lambda = \left(\bigcup_{z \in \partial\Omega} D_{r(z)}(z) \right) \cup \Omega \qquad \text{and} \qquad f(z) = \frac{1}{z - z_0}.$$

We have that Λ is a domain since it is a union of open balls in $\mathbb{C}$, and thus since $z_0 \notin \Lambda$, we have that f is holomorphic in Λ. Since $\partial\Omega \subseteq \Lambda$, we have that f is holomorphic on $\partial\Omega$, and thus by Cauchy's integral theorem, we obtain

$$\int_{\partial\Omega} \frac{dz}{z - z_0} = \int_{\partial\Omega} f(z)\, dz = 0.$$

Now suppose $z_0 \in \Omega$, then choose real $r > 0$ such that $C = C_r(z_0) \subseteq \Omega$. We can parameterize C by $z = \omega(\theta) = z_0 + re^{i\theta}$ where $0 \leq \theta \leq 2\pi$. Consequently, we have that $dz = ire^{i\theta}\, d\theta$, and thus

$$\int_C \frac{dz}{z - z_0} = \int_0^{2\pi} \frac{ire^{i\theta}}{z_0 + re^{i\theta} - z_0}\, d\theta = \int_0^{2\pi} \frac{ire^{i\theta}}{re^{i\theta}}\, d\theta = i \int_0^{2\pi} d\theta = 2\pi i.$$

Since C is contained in the interior of $\partial\Omega$, we have that

$$\int_C \frac{dz}{z - z_0} = \int_{\partial\Omega} \frac{dz}{z - z_0} = 2\pi i.$$

■

Lemma 8.2

Let f be holomorphic in some domain Ω and let $C \subseteq \Omega$ be a positively oriented simple closed contour. Then, for all z_0 in the interior of C we have that

$$\int_C \frac{f(z) - f(z_0)}{z - z_0}\, dz = 0$$

Proof: Define g as

$$g(z) = \frac{f(z) - f(z_0)}{z - z_0},$$

then because f is holomorphic in Ω, we have that g is holomorphic in $\Omega \setminus \{z_0\}$. Define $g(z_0) = f'(z_0)$, we will show that g is continuous at z_0. Choose real $r > 0$ such that $D_r(z_0) \subseteq \Omega$, and for $n \in \mathbb{N}$, define $r_n = r/n$. We can define a sequence $\{z_n\}$ such that $z_n \in D_{r_n}(z_0)$ for all n. It is easy to show that $\{z_n\} \to z_0$. Since f is holomorphic at z_0, we have that

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0},$$

and thus

$$\lim_{n \to \infty} g(z_n) = \lim_{z_n \to z_0} \frac{f(z_n) - f(z_0)}{z_n - z_0} = f'(z_0),$$

which gives us $\lim_{z \to z_0} g(z) = g(z_0) = f'(z_0)$, showing that g is continuous at z_0. Since we also have that g is holomorphic on $\Omega \setminus \{z_0\}$, we can apply Cauchy's integral theorem on g to obtain

$$\int_C g(z)\, dz = \int_C \frac{f(z) - f(z_0)}{z - z_0}\, dz = 0.$$

∎

Alternate Proof: Since f is holomorphic in Ω, it is continuous at z_0, thus for all $\varepsilon > 0$, there exists $\delta > 0$ such that $z \in D_\delta(z_0) \implies f(z) \in D_\varepsilon(f(z_0))$. Fix ε and choose δ such that $\overline{D}_\delta(z_0) \subseteq \Omega$. Additionally, let ∂D be the boundary of $\overline{D}_\delta(z_0)$. By the ML inequality, we have that

$$|I| = \left| \int_{\partial D} \frac{f(z) - f(z_0)}{z - z_0}\, dz \right| \leq \int_{\partial D} \left| \frac{f(z) - f(z_0)}{z - z_0} \right| dz < \frac{\varepsilon}{\delta} L(\partial D) = \frac{\varepsilon}{\delta} 2\pi\delta = 2\pi\varepsilon,$$

and since $2\pi\varepsilon$ can be arbitrarily small, we have that $I = 0$. Additionally, since z_0 is on the interior of C, we have that ∂D is contained in the interior of C for small enough δ, and thus

$$\int_C \frac{f(z) - f(z_0)}{z - z_0}\, dz = \int_{\partial D} \frac{f(z) - f(z_0)}{z - z_0}\, dz = 0.$$

∎

Theorem 8.1: Cauchy's Integral Formula

Let f be holomorphic in some domain Ω and $C \subseteq \Omega$ be a positively oriented simple closed contour. Then, for all z_0 in the interior of C, we have that

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0}\, dz$$

Proof: We have that

$$\frac{1}{2\pi i} \int_C \frac{f(z_0)}{z - z_0}\, dz = \frac{f(z_0)}{2\pi i} \int_C \frac{dz}{z - z_0} = \frac{f(z_0)}{2\pi i} 2\pi i = f(z_0)$$

and

$$\frac{1}{2\pi i} \int_C \frac{f(z) - f(z_0)}{z - z_0}\, dz = 0,$$

thus

$$\frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0}\, dz = \frac{1}{2\pi i} \int_C \frac{f(z) - f(z_0) + f(z_0)}{z - z_0}\, dz$$

$$= \frac{1}{2\pi i} \int_C \frac{f(z) - f(z_0)}{z - z_0}\, dz + \frac{1}{2\pi i} \int_C \frac{f(z_0)}{z - z_0}\, dz = 0 + f(z_0) = f(z_0).$$

■

Corollary 8.1: Gauss' Mean Value Theorem

Let f be holomorphic in some domain Ω and for $z_0 \in \Omega$ and real $r > 0$, let $C = C_r(z_0) \subseteq \Omega$ be a positively oriented circle. Then we have that

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta})\, d\theta$$

Proof: We have that

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0}\, dz,$$

and parameterizing C with $z = \omega(\theta) = z_0 + re^{i\theta}$ for $0 \leq \theta \leq 2\pi$, we obtain $dz = ire^{i\theta}\, d\theta$, thus

$$f(z_0) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + re^{i\theta})}{re^{i\theta}} ire^{i\theta}\, d\theta = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta})\, d\theta.$$

■

Lemma 8.3

Let C be a contour and g continuous on C. Also define f as

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{g(z)}{z - z_0}\,dz.$$

If R is a region where $C \cap R = \varnothing$, then we have that f is holomorphic in R and that for all $z_0 \in R$,

$$f'(z_0) = \frac{1}{2\pi i}\int_C \frac{g(z)}{(z - z_0)^2}\,dz.$$

Proof: Fix real $r > 0$ such that $D_r(z_0) \subseteq R$, and let $\{z_n\} \subseteq D_r(z_0)$ be a sequence that converges to z_0. We have to show that $\lim_{n\to\infty} A_n = 0$, where

$$A_n = \frac{f(z_n) - f(z_0)}{z_n - z_0} - \frac{1}{2\pi i}\int_C \frac{g(z)}{(z - z_0)^2}\,dz$$

Given our definition of f, we have that

$$\begin{aligned}
\frac{f(z_n) - f(z_0)}{z_n - z_0} &= \frac{1}{2\pi i}\int_C \frac{g(z)}{z_n - z_0}\left[\frac{1}{z - z_n} - \frac{1}{z - z_0}\right] dz \\
&= \frac{1}{2\pi i}\int_C \frac{g(z)}{z_n - z_0}\cdot\frac{z_n - z_0}{(z - z_n)(z - z_0)}\,dz \\
&= \frac{1}{2\pi i}\int_C \frac{g(z)}{(z - z_n)(z - z_0)}\,dz,
\end{aligned}$$

and thus

$$\begin{aligned}
A_n &= \frac{1}{2\pi i}\int_C \frac{g(z)}{(z - z_n)(z - z_0)}\,dz - \frac{1}{2\pi i}\int_C \frac{g(z)}{(z - z_0)^2}\,dz \\
&= \frac{1}{2\pi i}\int_C g(z)\left[\frac{1}{(z - z_0)(z - z_n)} - \frac{1}{(z - z_0)^2}\right] dz \\
&= \frac{1}{2\pi i}\int_C g(z)\frac{z_n - z_0}{(z - z_0)^2(z - z_n)}\,dz \\
&= \frac{z_n - z_0}{2\pi i}\int_C \frac{g(z)}{(z - z_0)^2(z - z_n)}\,dz.
\end{aligned}$$

Define M and D as

$$M = \max_{z\in C}|g(z)| \quad \text{and} \quad D = \min_{w\in C}|z - z_0|\,.$$

Choose $n \in \mathbb{N}$ such that $|z_0 - z_n| < D/2$, then we have that

$$|A_n| = \left|\frac{z_n - z_0}{2\pi i}\int_C \frac{g(z)}{(z - z_0)^2(z - z_n)}\,dz\right| \le \left|\frac{z_n - z_0}{2\pi i}\right|\cdot\frac{ML(C)}{D^2\cdot D/2} = |z_n - z_0|\,\frac{ML}{\pi D^3},$$

but we know that $\lim_{n\to\infty}|z_n - z_0| = 0$, thus $\lim_{n\to\infty}|A_n| = 0$. ■