The Symmetric Group

The **symmetric group**, written S_n, is the set of all bijections on a set of order n under composition, where n is typically finite. Elements in S_n are typically written as permutations. Take, for example, $\sigma \in S_5$ group, defined as

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 5 & 1 & 4 \end{pmatrix},$$

which means $\sigma(1) = 3$, $\sigma(2) = 2$, $\sigma(3) = 5$, etc. We have that $|S_n| = n!$.

Cycle Decomposition

A **cycle** is an element of S_n, written $(a_1, a_2, \ldots, a_m)$, where $a_i \in S_n$ and $a_i \mapsto a_{i+1}$ with i mod m. All cycles are a product of disjoint cycles. The **length** of a cycle is the number of non-identity cycles in its disjoint cycle representation. Disjoint cycles commute.

Theorem: Any element $\sigma \in S_n$ has a unique factorization into disjoint cycles up to rearrangement.

Theorem: The order of $\sigma \in S_n$ is the least common multiple of the length of the cycles in its disjoint cycle representation.

Matrix Groups

Given a field F, $GL_n(F)$ is defined as

$$GL_n(F) = \left\{ A : A \in F^{n \times n}, \det(A) \neq 0 \right\}.$$

$GL_n(F)$ is a group under multiplication but not addition.

Extras

Definition: A **group homomorphism** is a function between two groups $(G, *_G)$ and $(H, *_H)$, defined $f : G \to H$ where

$$f(a *_G b) = f(a) *_H f(b)$$

for all $a, b \in G$.