Section 1.1

1) Represent the following values in the form $a + bi$, where $a, b \in \mathbb{R}$.

$$(1+2i)^3, \quad \frac{5}{-3+4i}, \quad \left(\frac{2+i}{3-2i}\right)^2, \quad (1+i)^n + (1-i)^n$$

In this case, we assume $n \in \mathbb{N}_0$.

Solution:

$$(1+2i)^3 = (-3+4i)(1+2i) = \boxed{-11-2i}$$

$$\frac{5}{-3+4i} = \frac{5}{-3+4i} \cdot \frac{-3-4i}{-3-4i} = \frac{-15-20i}{25} = \boxed{-\frac{3}{5} - \frac{4}{5}i}$$

$$\left(\frac{2+i}{3-2i}\right)^2 = \frac{2+i}{3-2i} \cdot \frac{2+i}{3-2i} = \frac{3+4i}{5-12i} = \frac{3+4i}{5-12i} \cdot \frac{5+12i}{5+12i} = \frac{-33+56i}{169}$$

$$= \boxed{-\frac{33}{169} + \frac{56}{169}i}$$

Using the binomial theorem, we see that

$$(1+i)^n + (1-i)^n = \sum_{k=0}^{n} \binom{n}{k} i^k + \sum_{k=0}^{n} \binom{n}{k} (-i)^k = \sum_{k=0}^{n} \binom{n}{k} \left[i^k + (-i)^k\right].$$

Since $i = (-i)^3$ and $-i = i^3$, we have that $i^k + (-i)^k = 0$ for odd k, thus it suffices to consider the even terms of the sum. Additionally, since $i^2 = (-i)^2 = -1$ and $i^4 = (-i)^4 = 1$, we have $i^k + (-i)^k = 2(-1)^k$ for even k. This gives us

$$(1+i)^n + (1-i)^n = \boxed{\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} 2(-1)^{2k}}$$

and thus we obtain a real number for all values of n.

2) Letting $z = a + bi$ for $a, b \in \mathbb{R}$, find the real and imaginary parts for the following complex numbers:

$$z^4, \quad \frac{1}{z}, \quad \frac{z-1}{z+1}, \quad \frac{1}{z^2}$$

Solution:

$$z^4 = (a+bi)^4 = (a+bi)^2(a+bi)^2 = (a^2 + 2abi - b^2)^2$$

$$= \boxed{a^4 - 6a^2b^2 + b^4 + (4a^3b - 4ab^3)i}$$

$$\frac{1}{z} = \frac{1}{a+bi} \cdot \frac{a-bi}{a-bi} = \frac{a-bi}{a^2+b^2} = \boxed{\frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}i}$$

$$\frac{z-1}{z+1} = \frac{a-1+bi}{a+1+bi} \cdot \frac{a+1-bi}{a+1-bi} = \frac{(a-1+bi)(a+1-bi)}{(a+1)^2+b^2} = \frac{a^2+b^2-1+2bi}{(a+1)^2+b^2}$$

$$= \boxed{\frac{a^2+b^2-1}{(a+1)^2+b^2} + \frac{2b}{(a+1)^2+b^2}i}$$

$$\frac{1}{z^2} = \frac{1}{(a+bi)^2} = \frac{1}{a^2-b^2+2abi} = \frac{1}{a^2-b^2+2abi} \cdot \frac{a^2-b^2-2abi}{a^2-b^2-2abi}$$

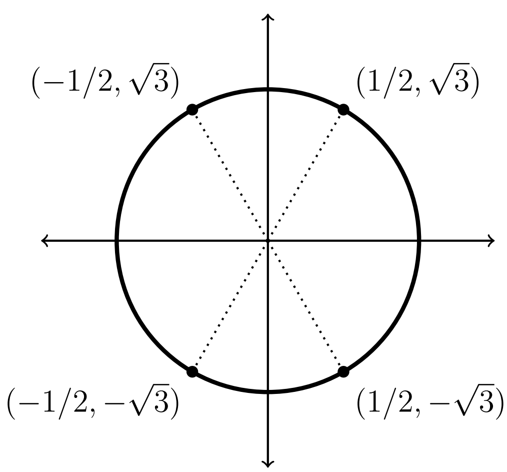
$$= \frac{a^2-b^2-2abi}{a^4+2a^2b^2+b^4} = \boxed{\frac{a^2-b^2}{(a^2+b^2)^2} - \frac{2ab}{(a^2+b^2)^2}i}$$

3) We have that

$$\left(\frac{-1 \pm i\sqrt{3}}{2}\right)^3 = \left(\frac{\pm 1 \pm i\sqrt{3}}{2}\right)^6 = 1$$

for all combinations of $+$ and $-$ with the $\pm$ signs.

Proof: Interpreting the complex numbers inside the parentheses as points in $\mathbb{R}^2$, we see that they lie on the unit circle for all possible combinations of $+$ and $-$. The following picture demonstrates this:



Since multiplication of two complex numbers adds their angles and multiplies their magnitudes, it is easy to see that the equation holds. ■

Section 1.2

1) Compute the values of the following complex square roots:

$$\sqrt{i}, \quad \sqrt{-i}, \quad \sqrt{1+i}, \quad \sqrt{\frac{1-i\sqrt{3}}{2}}$$

Solution: We use the following formula to compute $\sqrt{a+bi}$:

$$\sqrt{a+bi} = \pm\left(\sqrt{\frac{a+\sqrt{a^2+b^2}}{2}} + i\frac{b}{|b|}\sqrt{\frac{-a+\sqrt{a^2+b^2}}{2}}\right)$$

This results in the following values:

$$\sqrt{i} = \boxed{\pm\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)}, \quad \sqrt{-i} = \boxed{\pm\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right)},$$

$$\sqrt{1+i} = \boxed{\pm\left(\sqrt{\frac{1+\sqrt{2}}{2}} + i\sqrt{\frac{-1+\sqrt{2}}{2}}\right)},$$

$$\sqrt{\frac{1-i\sqrt{3}}{2}} = \sqrt{\frac{1/2+\sqrt{1/4+3/4}}{2}} - i\sqrt{\frac{-1/2+\sqrt{1/4+3/4}}{2}} = \boxed{\pm\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)}$$

2) Find all four values of $\sqrt[4]{-1}$.

Solution: We first consider the values of $\sqrt{-1}$, which happen to be $\pm i$. Since $\sqrt[4]{-1} = \sqrt{\sqrt{-1}} = \sqrt{\pm i}$, we have that the four values of $\sqrt[4]{-1}$ are given as follows:

$$\boxed{\pm\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right), \ \pm\left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right)}$$

3) Find all four values of $\sqrt[4]{i}$ and $\sqrt[4]{-i}$.

Solution: Recalling the previously determined values for $\sqrt{i}$ and $\sqrt{-i}$, we find $\sqrt[4]{i}$ and $\sqrt[4]{-i}$ as follows:

$$\sqrt[4]{i} = \sqrt{\sqrt{i}} = \sqrt{\pm\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)}$$

$$= \boxed{\pm\left(\sqrt{\frac{2+\sqrt{2}}{4}} + i\sqrt{\frac{2-\sqrt{2}}{4}}\right),\ \pm\left(\sqrt{\frac{2-\sqrt{2}}{4}} + i\sqrt{\frac{2+\sqrt{2}}{4}}\right)},$$

$$\sqrt[4]{-i} = \sqrt{\sqrt{-i}} = \sqrt{\pm\left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right)}$$

$$= \boxed{\pm\left(\sqrt{\frac{2+\sqrt{2}}{4}} + i\sqrt{\frac{2-\sqrt{2}}{4}}\right),\ \pm\left(\sqrt{\frac{2-\sqrt{2}}{4}} + i\sqrt{\frac{2+\sqrt{2}}{4}}\right)}$$

4) Given $a, b \in \mathbb{C}$, solve the following quadratic equation for z:

$$z^2 + az + b = 0$$

Solution: We can complete the square as follows:

$$z^2 + az = (z + a/2)^2 - a^2/4 = -b,$$

and since square roots of complex numbers are defined, we can solve for z as follows:

$$z = \pm\sqrt{\frac{a^2}{4} - b} - \frac{a}{2} = \frac{-a \pm 2\sqrt{a^2/4 - b}}{2} = \frac{-a \pm \sqrt{a^2 - 4b}}{2},$$

which is equivalent to the standard real valued quadratic equation for monic quadratic polynomials.

Section 1.3

1) Let F be set of all matrices in $\mathbb{R}^{2\times 2}$ that take the form

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix},$$

then F is a field under the operations of matrix addition and multiplication, and furthermore is isomorphic to $\mathbb{C}$.

Proof: The zero matrix and the identity matrix act as the 0 and 1 of F, respectively. It is also easy to verify that F is closed under addition and multiplication. We additionally have that

$$\begin{bmatrix} -a & -b \\ b & -a \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a/(a^2-b^2) & b/(a^2-b^2) \\ -b/(a^2-b^2) & a/(a^2-b^2) \end{bmatrix}$$

serve as the additive and multiplicative inverses respectively. Finally,

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix} \begin{bmatrix} c & d \\ -d & c \end{bmatrix} = \begin{bmatrix} ac-bd & ad+bc \\ -ad-bc & ac-bd \end{bmatrix} = \begin{bmatrix} c & d \\ -d & c \end{bmatrix} \begin{bmatrix} a & b \\ -b & a \end{bmatrix},$$

which verifies the commutativity of F.

Having shown that F is a field, we turn our attention to the function $\phi : F \to \mathbb{C}$ defined as follows:

$$\phi\left(\begin{bmatrix} a & b \\ -b & a \end{bmatrix}\right) = a + bi$$

Fixing two matrices in F, we see that the following are true:

$$\phi\left(\begin{bmatrix} a & b \\ -b & a \end{bmatrix} + \begin{bmatrix} c & d \\ -d & c \end{bmatrix}\right) = \phi\left(\begin{bmatrix} a+c & b+d \\ -b-d & a+c \end{bmatrix}\right) = a + c + (b+d)i$$

$$= (a+bi) + (c+di) = \phi\left(\begin{bmatrix} a & b \\ -b & a \end{bmatrix}\right) + \phi\left(\begin{bmatrix} c & d \\ -d & c \end{bmatrix}\right),$$

and

$$\phi\left(\begin{bmatrix} a & b \\ -b & a \end{bmatrix} \begin{bmatrix} c & d \\ -d & c \end{bmatrix}\right) = \phi\left(\begin{bmatrix} ac-bd & ad+bc \\ -ad-bc & ac-bd \end{bmatrix}\right) = ac - bd + (ad+bc)i$$

$$= (a+bi)(c+di) = \phi\left(\begin{bmatrix} a & b \\ -b & a \end{bmatrix}\right) \phi\left(\begin{bmatrix} c & d \\ -d & c \end{bmatrix}\right).$$

Since ϕ is obviously a bijection, we have shown that $F \cong \mathbb{C}$. ■

2) Let I be the ideal in $\mathbb{R}[X]$ generated by $X^2 + 1$, then $\mathbb{R}[X]/I \cong \mathbb{C}$.

Proof: For convenience, denote $\overline{p} = p + I$ for all $p \in \mathbb{R}[X]$. We know that this quotient ring is a field since $X^2 + 1$ is irreducible in $\mathbb{R}[X]$, and thus $(X^2 + 1)$ is a maximal ideal. Additionally, we know that this field is equal to the set $\left\{\overline{a + bX} : a, b \in \mathbb{R}\right\}$ of remainders of polynomial division with $X^2 + 1$, and by the division algorithm, any element $\overline{p} \in \mathbb{R}[X]/I$ can be uniquely represented in the form $\overline{a + bX}$. This shows that $\mathbb{R}[X]/I$ is in bijection with $\mathbb{C}$, hence we are led to consider the function $\phi : \mathbb{R}[X]/I \to \mathbb{C}$ defined as follows:

$$\phi(\overline{p}) = \phi\left(\overline{a + bX}\right) = a + bi.$$

Let $\overline{a + bX}$ and $\overline{c + dX}$ be elements in $\mathbb{R}[X]/I$, then

$$\phi\left(\overline{a+bX}+\overline{c+dX}\right)=\phi\left(\overline{a+c+(b+d)X}\right)=(a+c)+(b+d)i$$
$$=(a+bi)+(c+di)=\phi\left(\overline{a+bX}\right)+\phi\left(\overline{c+dX}\right).$$

It can also be seen that

$$\overline{a+bX}\cdot\overline{c+dX}=\overline{ac+(ad+bc)X+bdX^2},$$

and by calculating the remainder of this polynomial divided by X^2+1, we obtain.

$$\overline{ac+(ad+bc)X+bdX^2}=\overline{ac-bd+(ad+bc)X}.$$

This allows us to see that

$$\phi\left(\overline{a+bX}\cdot\overline{c+dX}\right)=\phi\left(\overline{ac-bd+(ad+bc)X}\right)=ac-bd+(ad+bc)i$$
$$=(a+bi)(c+di)=\phi\left(\overline{a+bX}\right)\phi\left(\overline{c+dX}\right),$$

which proves that ϕ is an isomorphism and $\mathbb{R}[X]/I\cong\mathbb{C}$. ■

Section 1.4

1) Evaluate

$$\frac{z}{z^2+1}$$

at $z=a+bi$ and $z=a-bi$ and conclude that the two values are conjugate.

Solution: We have that

$$\frac{a+bi}{(a+bi)^2+1}=\frac{a+bi}{a^2-b^2+1+2abi}=\frac{a+bi}{a^2-b^2+1+2abi}\cdot\frac{a^2-b^2+1-2abi}{a^2-b^2+1-2abi}$$
$$=\frac{(a+bi)(a^2-b^2+1-2abi)}{(a^2-b^2+1)^2+4a^2b^2}=\frac{a(a^2-b^2+1)+2ab^2+[b(a^2-b^2+1)-2a^2b]i}{(a^2-b^2+1)^2+4a^2b^2}$$

and

$$\frac{a-bi}{(a-bi)^2+1}=\frac{a-bi}{a^2-b^2+1-2abi}=\frac{a-bi}{a^2-b^2+1-2abi}\cdot\frac{a^2-b^2+1+2abi}{a^2-b^2+1+2abi}$$
$$=\frac{(a-bi)(a^2-b^2+1+2abi)}{(a^2-b^2+1)^2+4a^2b^2}=\frac{a(a^2-b^2+1)+2ab^2-[b(a^2-b^2+1)-2a^2b]i}{(a^2-b^2+1)^2+4a^2b^2},$$

thus their conjugacy has been established.

2) Find the absolute values of the following complex numbers:

$$-2i(3+i)(2+4i)(1+i), \quad \text{and} \quad \frac{(3+4i)(-1+2i)}{(-1-i)(3-i)}$$

Solution:

$$|-2i(3+i)(2+4i)(1+i)| = |-2i| \cdot |3+i| \cdot |2+4i| \cdot |1+i| = 2\sqrt{10}\sqrt{20}\sqrt{2}$$

$$= 2\sqrt{400} = \boxed{40}$$

$$\left|\frac{(3+4i)(-1+2i)}{(-1-i)(3-i)}\right| = \frac{|3+4i| \cdot |-1+2i|}{|-1-i| \cdot |3-i|} = \frac{5\sqrt{5}}{\sqrt{2}\sqrt{10}} = \frac{5\sqrt{5}}{\sqrt{20}} = \frac{50}{20} = \boxed{\frac{5}{2}}$$

3) Prove that

$$\left|\frac{a-b}{1-\overline{a}b}\right| = 1$$

if either $|a| = 1$ or $|b| = 1$.

Proof: First, assume that $|a| = 1$. In this case we have that

$$\left|\frac{a-b}{1-\overline{a}b}\right| = \frac{|a|^2 + |b|^2 - 2\text{Re}(a\bar{b})}{|1|^2 + |\overline{a}b|^2 - 2\text{Re}(a\bar{b})} = \frac{1 + |b|^2 - 2\text{Re}(a\bar{b})}{1 + |b|^2 - 2\text{Re}(a\bar{b})} = 1.$$

Otherwise, we assume $|b| = 1$. Similarly, we find

$$\left|\frac{a-b}{1-\overline{a}b}\right| = \frac{|a|^2 + |b|^2 - 2\text{Re}(a\bar{b})}{|1|^2 + |\overline{a}b|^2 - 2\text{Re}(a\bar{b})} = \frac{|a|^2 + 1 - 2\text{Re}(a\bar{b})}{1 + |\overline{a}|^2 - 2\text{Re}(a\bar{b})}$$

$$= \frac{|a|^2 + 1 - 2\text{Re}(a\bar{b})}{|a|^2 + 1 - 2\text{Re}(a\bar{b})} = 1,$$

thus proving the equality. ■

4) Consider the equation

$$az + b\overline{z} + c = 0,$$

with complex coefficients a, b, and c. Find the conditions on the coefficients under which the equation has a unique solution.

Solution: Consider a solution z to the equation.

5) Fix complex numbers a_i and b_i for $1 \leq i \leq n$, then we have that

$$\left|\sum_{i=1}^{n} a_i b_i\right|^2 = \sum_{i=1}^{n} |a_i|^2 \sum_{i=1}^{n} |b_i|^2 - \sum_{1\leq i<j}^{n} \left|a_i \bar{b}_j - a_j \bar{b}_i\right|^2.$$

Proof: