

Section 1.1

1) Represent the following values in the form $a + bi$, where $a, b \in \mathbb{R}$.

$$(1 + 2i)^3, \quad \frac{5}{-3 + 4i}, \quad \left(\frac{2 + i}{3 - 2i} \right)^2, \quad (1 + i)^n + (1 - i)^n$$

Solution:

$$\begin{aligned} (1 + 2i)^3 &= (-3 + 4i)(1 + 2i) = \boxed{-11 - 2i} \\ \frac{5}{-3 + 4i} &= \frac{5}{-3 + 4i} \cdot \frac{-3 - 4i}{-3 - 4i} = \frac{-15 - 20i}{25} = \boxed{-\frac{3}{5} - \frac{4}{5}i} \\ \left(\frac{2 + i}{3 - 2i} \right)^2 &= \frac{2 + i}{3 - 2i} \cdot \frac{2 + i}{3 - 2i} = \frac{3 + 4i}{5 - 12i} = \frac{3 + 4i}{5 - 12i} \cdot \frac{5 + 12i}{5 + 12i} = \frac{-33 + 56i}{169} \\ &= \boxed{-\frac{33}{169} + \frac{56}{169}i} \end{aligned}$$

Using the binomial theorem, we see that

$$(1 + i)^n + (1 - i)^n = \sum_{k=0}^n \binom{n}{k} i^k + \sum_{k=0}^n \binom{n}{k} (-i)^k = \sum_{k=0}^n \binom{n}{k} [i^k + (-i)^k].$$

Since $i = (-i)^3$ and $-i = i^3$, we have that $i^k + (-i)^k = 0$ for odd k , thus it suffices to consider the even terms of the sum. Additionally, since $i^2 = (-i)^2 = -1$ and $i^4 = (-i)^4 = 1$, we have $i^k + (-i)^k = 2(-1)^k$ for even k . This gives us

$$(1 + i)^n + (1 - i)^n = \boxed{\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} 2(-1)^{2k}}$$

and thus we obtain a real number for all values of n .

2) Letting $z = a + bi$ for $a, b \in \mathbb{R}$, find the real and imaginary parts for the following complex numbers:

$$z^4, \quad \frac{1}{z}, \quad \frac{z - 1}{z + 1}, \quad \frac{1}{z^2}$$

Solution:

$$\begin{aligned} z^4 &= (a + bi)^4 = (a + bi)^2(a + bi)^2 = (a^2 + 2abi - b^2)^2 \\ &= \boxed{a^4 - 6a^2b^2 + b^4 + (4a^3b - 4ab^3)i} \end{aligned}$$

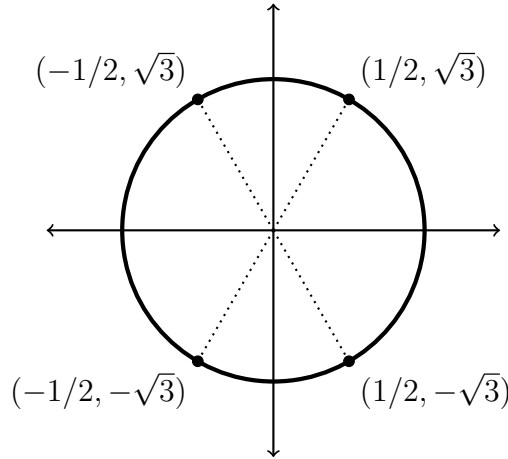
$$\begin{aligned}
\frac{1}{z} &= \frac{1}{a+bi} \cdot \frac{a-bi}{a-bi} = \frac{a-bi}{a^2+b^2} = \boxed{\frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}i} \\
\frac{z-1}{z+1} &= \frac{a-1+bi}{a+1+bi} \cdot \frac{a+1-bi}{a+1-bi} = \frac{(a-1+bi)(a+1-bi)}{(a+1)^2+b^2} = \frac{a^2+b^2-1+2bi}{(a+1)^2+b^2} \\
&= \boxed{\frac{a^2+b^2-1}{(a+1)^2+b^2} + \frac{2b}{(a+1)^2+b^2}i} \\
\frac{1}{z^2} &= \frac{1}{(a+bi)^2} = \frac{1}{a^2-b^2+2abi} = \frac{1}{a^2-b^2+2abi} \cdot \frac{a^2-b^2-2abi}{a^2-b^2-2abi} \\
&= \frac{a^2-b^2-2abi}{a^4+2a^2b^2+b^4} = \boxed{\frac{a^2-b^2}{(a^2+b^2)^2} - \frac{2ab}{(a^2+b^2)^2}i}
\end{aligned}$$

3) We have that

$$\left(\frac{-1 \pm i\sqrt{3}}{2}\right)^3 = \left(\frac{\pm 1 \pm i\sqrt{3}}{2}\right)^6 = 1$$

for all combinations of + and - with the \pm signs.

Proof: Interpreting the complex numbers inside the parentheses as points in \mathbb{R}^2 , we see that they lie on the unit circle for all possible combinations of + and -. The following picture demonstrates this:



Since multiplication of two complex numbers adds their angles and multiplies their magnitudes, it is easy to see that the equation holds. ■

Section 1.2