1.) For each equation below, find all solutions for $n \in \mathbb{N}$.

(a) $\phi(n) = n/2$

(b) $\phi(n) = \phi(2n)$

(c) $\phi(n) = 12$

Solution: We first show that (a) is true if and only if n is a power of two. Fix $n = 2^m$ where $m \in \mathbb{N}$, then

$$\phi(n) = \phi(2^m) = 2^m - 2^{m-1} = 2^{m-1} = n/2.$$

Next, suppose n is not a power of two. If n is odd, then $n/2$ is not an integer, and thus cannot be the result of $\phi(n)$. This forces n to be even. Let $\prod_{i\in\mathbb{N}} p_i^{a_i}$ be the prime factorization of n. Since ϕ is multiplicative, and since powers of distinct primes are relatively prime, we have that

$$\phi(n) = \phi\left(\prod_{i\in\mathbb{N}} p_i^{a_i}\right) = \prod_{i\in\mathbb{N}} \phi(p_i^{a_i}).$$

Additionally, because n is even, we know that $a_1 \geq 1$, and since it is not a power of two, we know there exists i where $p_i > 2$ and $a_i \geq 1$. We also know that $\phi(p_i^{a_i}) < p_i^{a_i}$ for all i, thus we have

$$\phi(n) = \prod_{i\in\mathbb{N}} \phi(p_i^{a_i}) = \phi(2^{a_1}) \prod_{i\geq 2} \phi(p_i^{a_i}) = 2^{a_1-1} \prod_{i\geq 2} \phi(p_i^{a_i}) < \frac{2^{a_1} \prod_{i\geq 2} p_i^{a_i}}{2}$$
$$= \frac{\prod_{i\in\mathbb{N}} p_i^{a_i}}{2} = \frac{n}{2}.$$

The inequality is strict, thus showing that $\phi(n) \neq n/2$. This proves the case of (a). ▲

Next, we show that (b) holds only for n that are relatively prime to 2. Let $n \in \mathbb{N}$ and assign $d = (2, n)$. If $d = 1$, the multiplicity of ϕ gives us

$$\phi(2n) = \phi(2)\phi(n) = \phi(n).$$

Otherwise, assume $d \neq 1$. Since d must divide 2, we know that $d = 2$, thus

$$\phi(2n) = \phi(2)\phi(n)\frac{d}{\phi(d)} = \phi(n)\frac{2}{\phi(2)} = 2\phi(n) \neq \phi(n).$$

This holds because $\phi(n) > 0$ for all $n \in \mathbb{N}$, thus proving (b). ■