The Exterior Measure

Definition: Let $E \subseteq \mathbb{R}^d$, then the **exterior measure** m_* of E is defined as

$$m_*(E) = \inf \sum_{j=1}^{\infty} |Q_j| \, ,$$

where $\{Q_j\}$ is a countable covering of E using closed cubes.

We have that $m_*(E) \geq 0$ for any $E \subseteq \mathbb{R}^d$. For any point $x \in \mathbb{R}^d$, we have that $m_*(\{x\}) = 0$; since any point is a closed cube, it covers itself and thus contains a countable cover. Since $|\{x\}| = 0$, we have that $m_*(\{x\}) = 0$. We also have that $m_*(\varnothing) = 0$, as any covering covers $\varnothing$, and since some coverings have zero volume, we have that $m_*(\varnothing) \leq 0$, and thus $= 0$.

Let $Q \subset \mathbb{R}^d$ be a closed cube, then $m_*(Q) = |Q|$. We obviously have that $m_*(Q) \leq |Q|$. Now consider an arbitrary covering $\mathcal{Q}$ where

$$Q \subseteq \bigcup_{j=1}^{\infty} Q_j.$$

Fix $\varepsilon > 0$, and for each $Q_j \in \mathcal{Q}$, choose an open cube S_j such that $Q_j \subset S_j$ and $|S_j| \leq (1+\varepsilon)\,|Q_j|$. Since Q is compact and $\{S_j\}$ is an open cover of Q, we have a finite subcover $\mathcal{S} \subseteq \{S_j\}$. We have that $m_*()$...

Let $Q \subseteq \mathbb{R}^d$ be any open cube, then $m_*(Q) = |Q|$.

Given a rectangle $R \subset \mathbb{R}^d$, we have $m_*(R) = |R|$.

$m_*(\mathbb{R}^d) = \infty$.