1.) Find all integers $n \in \mathbb{N}$ that satisfy the following:

a.) $\phi(n) = n/2$

b.) $\phi(n) = \phi(2n)$

c.) $\phi(n) = 12$

Solution:

a.) $n = 2^m$ where $m \in \mathbb{N}$

b.) $(2, n) = 1$

Proof:

a.) Let $n = 2^m$ where $m \in \mathbb{N}$, thus

$$\phi(2^m) = 2^m - 2^{m-1} = 2^{m-1} = n/2.$$

Next, suppose n is not a power of two. If n is odd, then $\phi(n) \neq n/2$ as the codomain of ϕ is $\mathbb{N}$, thus n must be even. Take $\prod p_i^{a_i}$ to be the prime factorization of n extended over all primes. Since ϕ is multiplicative, we find that

$$\phi(n) = \phi\left(\prod p_i^{a_i}\right) = \prod \phi(p_i^{a_i}).$$

Since n is even, we know that $a_1 \geq 1$, and since it is not a power of two, we know there exists i where $p_i > 2$ and $a_i \geq 1$. Let $b_i = \phi(p_i^{a_i})$. If $a_i = 0$, thus $b_i = 1$, otherwise $b_i = p_i^{a_i} - p_i^{a_i - 1} < p_i^{a_i}$, thus $b_1 b_2 b_3 < n$. Using this, we find that

$$\phi(n) = b_1 b_2 b_3 \cdots = 2^{a_i - 1} b_2 b_3 \cdots = \frac{2^{a_1} b_2 b_3 \cdots}{2} < \frac{2^{a_1} 3^{a_2} 5^{a_3} \cdots}{2} = \frac{n}{2},$$

thus $\phi(n) \neq n/2$. ∎

b.) Let $n \in \mathbb{N}$ where $(2, n) = 1$. Since $\phi(2) = 1$, we have that

$$\phi(2n) = \phi(2)\phi(n) = \phi(n).$$

Otherwise, if $(2, n) = d \neq 1$, then $d = 2$, thus

$$\phi(2n) = \phi(2)\phi(n)\frac{d}{\phi(d)} = \phi(n)\frac{2}{\phi(2)} = 2\phi(n) \neq \phi(n),$$

which is true because $\phi(n) > 0$ for all $n \in \mathbb{N}$. ∎

2.) Prove or disprove the following statements:

a.) If $(m, n) = 1$, then $(\phi(m), \phi(n)) = 1$

b.) If n is composite, then $(n, \phi(n)) > 1$

c.) If m and n have the same prime divisors, then $n\phi(m) = m\phi(n)$

3.) Prove that

$$\frac{n}{\phi(n)} = \sum_{d|n} \frac{\mu^2(d)}{\phi(d)}$$