Section 1.1

1) Represent the following values in the form $a + bi$, where $a, b \in \mathbb{R}$.

$$(1+2i)^3, \quad \frac{5}{-3+4i}, \quad \left(\frac{2+i}{3-2i}\right)^2, \quad (1+i)^n + (1-i)^n$$

Solution:

$$(1+2i)^3 = (-3+4i)(1+2i) = \boxed{-11-2i}$$

$$\frac{5}{-3+4i} = \frac{5}{-3+4i} \cdot \frac{-3-4i}{-3-4i} = \frac{-15-20i}{25} = \boxed{-\frac{3}{5} - \frac{4}{5}i}$$

$$\left(\frac{2+i}{3-2i}\right)^2 = \frac{2+i}{3-2i} \cdot \frac{2+i}{3-2i} = \frac{3+4i}{5-12i} = \frac{3+4i}{5-12i} \cdot \frac{5+12i}{5+12i} = \frac{-33+56i}{169}$$

$$= \boxed{-\frac{33}{169} + \frac{56}{169}i}$$

Using the binomial theorem, we see that

$$(1+i)^n + (1-i)^n = \sum_{k=0}^{n} \binom{n}{k} i^k + \sum_{k=0}^{n} \binom{n}{k} (-i)^k = \sum_{k=0}^{n} \binom{n}{k} \left[i^k + (-i)^k\right].$$

Since $i = (-i)^3$ and $-i = i^3$, we have that $i^k + (-i)^k = 0$ for odd k, thus it suffices to consider the even terms of the sum. Additionally, since $i^2 = (-i)^2 = -1$ and $i^4 = (-i)^4 = 1$, we have $i^k + (-i)^k = 2(-1)^k$ for even k. This gives us

$$(1+i)^n + (1-i)^n = \boxed{\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} 2(-1)^{2k}}$$

and thus we obtain a real number for all values of n.

2) Letting $z = a + bi$ for $a, b \in \mathbb{R}$, find the real and imaginary parts for the following complex numbers:

$$z^4, \quad \frac{1}{z}, \quad \frac{z-1}{z+1}, \quad \frac{1}{z^2}$$

Solution:

$$z^4 = (a+bi)^4 = (a+bi)^2(a+bi)^2 = (a^2 + 2abi - b^2)^2$$

$$= \boxed{a^4 - 6a^2b^2 + b^4 + (4a^3b - 4ab^3)i}$$

$$\frac{1}{z} = \frac{1}{a+bi} \cdot \frac{a-bi}{a-bi} = \frac{a-bi}{a^2+b^2} = \boxed{\frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}i}$$

$$\frac{z-1}{z+1} = \frac{a-1+bi}{a+1+bi} \cdot \frac{a+1-bi}{a+1-bi} = \frac{(a-1+bi)(a+1-bi)}{(a+1)^2+b^2} = \frac{a^2+b^2-1+2bi}{(a+1)^2+b^2}$$

$$= \boxed{\frac{a^2+b^2-1}{(a+1)^2+b^2} + \frac{2b}{(a+1)^2+b^2}i}$$

$$\frac{1}{z^2} = \frac{1}{(a+bi)^2} = \frac{1}{a^2-b^2+2abi} = \frac{1}{a^2-b^2+2abi} \cdot \frac{a^2-b^2-2abi}{a^2-b^2-2abi}$$

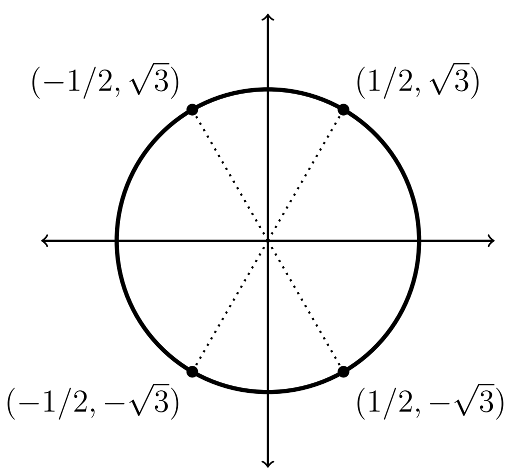
$$= \frac{a^2-b^2-2abi}{a^4+2a^2b^2+b^4} = \boxed{\frac{a^2-b^2}{(a^2+b^2)^2} - \frac{2ab}{(a^2+b^2)^2}i}$$

3) We have that

$$\left(\frac{-1 \pm i\sqrt{3}}{2}\right)^3 = \left(\frac{\pm 1 \pm i\sqrt{3}}{2}\right)^6 = 1$$

for all combinations of $+$ and $-$ with the $\pm$ signs.

Proof: Interpreting the complex numbers inside the parentheses as points in $\mathbb{R}^2$, we see that they lie on the unit circle for all possible combinations of $+$ and $-$. The following picture demonstrates this:



Since multiplication of two complex numbers adds their angles and multiplies their magnitudes, it is easy to see that the equation holds. ∎

Section 1.2

1.)

2.)

3.)

4.)

Section 1.3

1.) The set of all matrices of the form

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix},$$

where $a, b \in \mathbb{R}$, combined with the operations of matrix addition and multiplication, is isomorphic to $\mathbb{C}$.

Proof: Denoting the above set as F, we define a function $\phi : F \to \mathbb{C}$ where

$$\phi\left(\begin{bmatrix} a & b \\ -b & a \end{bmatrix}\right) = a + bi.$$

Fixing two matrices in F, we see that the following are true:

$$\phi\left(\begin{bmatrix} a & b \\ -b & a \end{bmatrix} + \begin{bmatrix} c & d \\ -d & c \end{bmatrix}\right) = \phi\left(\begin{bmatrix} a+c & b+d \\ -b-d & a+c \end{bmatrix}\right) = a + c + (b+d)i$$

$$= (a+bi) + (c+di) = \phi\left(\begin{bmatrix} a & b \\ -b & a \end{bmatrix}\right) + \phi\left(\begin{bmatrix} c & d \\ -d & c \end{bmatrix}\right),$$

and

$$\phi\left(\begin{bmatrix} a & b \\ -b & a \end{bmatrix}\begin{bmatrix} c & d \\ -d & c \end{bmatrix}\right) = \phi\left(\begin{bmatrix} ac-bd & ad+bc \\ -ad-bc & ac-bd \end{bmatrix}\right) = ac - bd + (ad+bc)i$$

$$= (a+bi)(c+di) = \phi\left(\begin{bmatrix} a & b \\ -b & a \end{bmatrix}\right)\phi\left(\begin{bmatrix} c & d \\ -d & c \end{bmatrix}\right).$$

Since ϕ is obviously a bijection, we have shown that $F \cong \mathbb{C}$. ■

2.) We have that $\mathbb{R}[X]/(X^2+1) \cong \mathbb{C}$.

Proof: We know that this is a field since $X^2 + 1$ is clearly irreducible in $\mathbb{R}[X]$.

Additionally, we know that this field is equal to the set $\{a + bX : a, b \in \mathbb{R}\}$,