

Section 1.1

1) Represent the following values in the form $a + bi$, where $a, b \in \mathbb{R}$.

$$(1 + 2i)^3, \quad \frac{5}{-3 + 4i}, \quad \left(\frac{2 + i}{3 - 2i} \right)^2, \quad (1 + i)^n + (1 - i)^n$$

In this case, we assume $n \in \mathbb{N}_0$.

Solution:

$$\begin{aligned} (1 + 2i)^3 &= (-3 + 4i)(1 + 2i) = \boxed{-11 - 2i} \\ \frac{5}{-3 + 4i} &= \frac{5}{-3 + 4i} \cdot \frac{-3 - 4i}{-3 - 4i} = \frac{-15 - 20i}{25} = \boxed{-\frac{3}{5} - \frac{4}{5}i} \\ \left(\frac{2 + i}{3 - 2i} \right)^2 &= \frac{2 + i}{3 - 2i} \cdot \frac{2 + i}{3 - 2i} = \frac{3 + 4i}{5 - 12i} = \frac{3 + 4i}{5 - 12i} \cdot \frac{5 + 12i}{5 + 12i} = \frac{-33 + 56i}{169} \\ &= \boxed{-\frac{33}{169} + \frac{56}{169}i} \end{aligned}$$

Using the binomial theorem, we see that

$$(1 + i)^n + (1 - i)^n = \sum_{k=0}^n \binom{n}{k} i^k + \sum_{k=0}^n \binom{n}{k} (-i)^k = \sum_{k=0}^n \binom{n}{k} [i^k + (-i)^k].$$

Since $i = (-i)^3$ and $-i = i^3$, we have that $i^k + (-i)^k = 0$ for odd k , thus it suffices to consider the even terms of the sum. Additionally, since $i^2 = (-i)^2 = -1$ and $i^4 = (-i)^4 = 1$, we have $i^k + (-i)^k = 2(-1)^k$ for even k . This gives us

$$(1 + i)^n + (1 - i)^n = \boxed{\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} 2(-1)^{2k}}$$

and thus we obtain a real number for all values of n .

2) Letting $z = a + bi$ for $a, b \in \mathbb{R}$, find the real and imaginary parts for the following complex numbers:

$$z^4, \quad \frac{1}{z}, \quad \frac{z - 1}{z + 1}, \quad \frac{1}{z^2}$$

Solution:

$$\begin{aligned} z^4 &= (a + bi)^4 = (a + bi)^2(a + bi)^2 = (a^2 + 2abi - b^2)^2 \\ &= \boxed{a^4 - 6a^2b^2 + b^4 + (4a^3b - 4ab^3)i} \end{aligned}$$

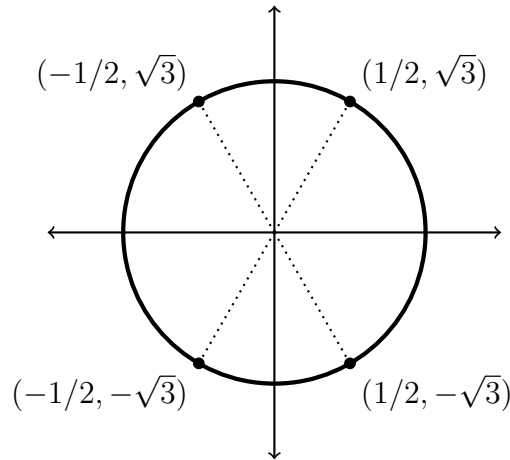
$$\begin{aligned}
\frac{1}{z} &= \frac{1}{a+bi} \cdot \frac{a-bi}{a-bi} = \frac{a-bi}{a^2+b^2} = \boxed{\frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}i} \\
\frac{z-1}{z+1} &= \frac{a-1+bi}{a+1+bi} \cdot \frac{a+1-bi}{a+1-bi} = \frac{(a-1+bi)(a+1-bi)}{(a+1)^2+b^2} = \frac{a^2+b^2-1+2bi}{(a+1)^2+b^2} \\
&= \boxed{\frac{a^2+b^2-1}{(a+1)^2+b^2} + \frac{2b}{(a+1)^2+b^2}i} \\
\frac{1}{z^2} &= \frac{1}{(a+bi)^2} = \frac{1}{a^2-b^2+2abi} = \frac{1}{a^2-b^2+2abi} \cdot \frac{a^2-b^2-2abi}{a^2-b^2-2abi} \\
&= \frac{a^2-b^2-2abi}{a^4+2a^2b^2+b^4} = \boxed{\frac{a^2-b^2}{(a^2+b^2)^2} - \frac{2ab}{(a^2+b^2)^2}i}
\end{aligned}$$

3) We have that

$$\left(\frac{-1 \pm i\sqrt{3}}{2}\right)^3 = \left(\frac{\pm 1 \pm i\sqrt{3}}{2}\right)^6 = 1$$

for all combinations of + and - with the \pm signs.

Proof: Interpreting the complex numbers inside the parentheses as points in \mathbb{R}^2 , we see that they lie on the unit circle for all possible combinations of + and -. The following picture demonstrates this:



Since multiplication of two complex numbers adds their angles and multiplies their magnitudes, it is easy to see that the equation holds. ■

Section 1.2

1.) Compute the values of the following complex square roots:

$$\sqrt{i}, \quad \sqrt{-i}, \quad \sqrt{1+i}, \quad \sqrt{\frac{1-i\sqrt{3}}{2}}$$

Solution: We use the following formula to compute $\sqrt{a+bi}$:

$$\sqrt{a+bi} = \pm \left(\sqrt{\frac{a + \sqrt{a^2 + b^2}}{2}} + i \frac{b}{|b|} \sqrt{\frac{-a + \sqrt{a^2 + b^2}}{2}} \right)$$

This results in the following values:

$$\sqrt{i} = \pm \left(\sqrt{\frac{1}{2}} + i \sqrt{\frac{1}{2}} \right), \quad \sqrt{-i} = \pm \left(\sqrt{\frac{1}{2}} - i \sqrt{\frac{1}{2}} \right),$$

$$\sqrt{1+i} = \pm \left(\sqrt{\frac{1+\sqrt{2}}{2}} + i \sqrt{\frac{-1+\sqrt{2}}{2}} \right),$$

$$\sqrt{\frac{1-i\sqrt{3}}{2}} = \sqrt{\frac{1/2 + \sqrt{1/4 + 3/4}}{2}} - i \sqrt{\frac{-1/2 + \sqrt{1/4 + 3/4}}{2}} = \pm \left(\frac{\sqrt{3}}{2} - \frac{1}{2}i \right)$$

2.) Find all four values of $\sqrt[4]{-1}$.

Solution: We first consider the values of $\sqrt{-1}$, which happen to be $\pm i$. Since $\sqrt[4]{-1} = \sqrt{\sqrt{-1}} = \sqrt{\pm i}$, we have that the four values of $\sqrt[4]{-1}$ are given as follows:

$$\pm \left(\sqrt{\frac{1}{2}} + i \sqrt{\frac{1}{2}} \right), \quad \pm \left(\sqrt{\frac{1}{2}} - i \sqrt{\frac{1}{2}} \right)$$

3.) Find all four values of $\sqrt[4]{i}$ and $\sqrt[4]{-i}$.

Solution: Recalling the previously determined values for \sqrt{i} and $\sqrt{-i}$, we find $\sqrt[4]{i}$ and $\sqrt[4]{-i}$ as follows:

$$\sqrt[4]{i} = \sqrt{\sqrt{i}} = \sqrt{\pm \left(\sqrt{\frac{1}{2}} + i \sqrt{\frac{1}{2}} \right)}$$

$$= \left[\pm \left(\sqrt{\frac{2+\sqrt{2}}{4}} + i\sqrt{\frac{2-\sqrt{2}}{4}} \right), \pm \left(\sqrt{\frac{2-\sqrt{2}}{4}} + i\sqrt{\frac{2+\sqrt{2}}{4}} \right) \right],$$

$$\sqrt[4]{-i} = \sqrt{\sqrt{-i}} = \sqrt{\pm \left(\sqrt{\frac{1}{2}} - i\sqrt{\frac{1}{2}} \right)}$$

$$= \left[\pm \left(\sqrt{\frac{2+\sqrt{2}}{4}} + i\sqrt{\frac{2-\sqrt{2}}{4}} \right), \pm \left(\sqrt{\frac{2-\sqrt{2}}{4}} + i\sqrt{\frac{2+\sqrt{2}}{4}} \right) \right]$$

4.) Given $a, b \in \mathbb{C}$, solve the following quadratic equation for z :

$$z^2 + az + b = 0$$

Solution: We can complete the square as follows:

$$z^2 + az = (z + a/2)^2 - a^2/4 = -b,$$

and since square roots of complex numbers are defined, we can solve for z as follows:

$$z = \pm \sqrt{\frac{a^2}{4} - b} - \frac{a}{2} = \frac{-a \pm 2\sqrt{a^2/4 - b}}{2} = \frac{-a \pm \sqrt{a^2 - 4b}}{2},$$

which is equivalent to the standard real valued quadratic equation for monic quadratic polynomials.

Section 1.3

1.) The set of all matrices of the form

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix},$$

where $a, b \in \mathbb{R}$, combined with the operations of matrix addition and multiplication, is isomorphic to \mathbb{C} .

Proof: Denoting the above set as F , we define a function $\phi : F \rightarrow \mathbb{C}$ where

$$\phi \left(\begin{bmatrix} a & b \\ -b & a \end{bmatrix} \right) = a + bi.$$

Fixing two matrices in F , we see that the following are true:

$$\begin{aligned}\phi\left(\begin{bmatrix} a & b \\ -b & a \end{bmatrix} + \begin{bmatrix} c & d \\ -d & c \end{bmatrix}\right) &= \phi\left(\begin{bmatrix} a+c & b+d \\ -b-d & a+c \end{bmatrix}\right) = a+c + (b+d)i \\ &= (a+bi) + (c+di) = \phi\left(\begin{bmatrix} a & b \\ -b & a \end{bmatrix}\right) + \phi\left(\begin{bmatrix} c & d \\ -d & c \end{bmatrix}\right),\end{aligned}$$

and

$$\begin{aligned}\phi\left(\begin{bmatrix} a & b \\ -b & a \end{bmatrix} \begin{bmatrix} c & d \\ -d & c \end{bmatrix}\right) &= \phi\left(\begin{bmatrix} ac-bd & ad+bc \\ -ad-bc & ac-bd \end{bmatrix}\right) = ac-bd + (ad+bc)i \\ &= (a+bi)(c+di) = \phi\left(\begin{bmatrix} a & b \\ -b & a \end{bmatrix}\right) \phi\left(\begin{bmatrix} c & d \\ -d & c \end{bmatrix}\right).\end{aligned}$$

Since ϕ is obviously a bijection, we have shown that $F \cong \mathbb{C}$. ■

2.) Let I be the ideal in $\mathbb{R}[X]$ generated by $X^2 + 1$, then $\mathbb{R}[X]/I \cong \mathbb{C}$.

Proof: For convenience, denote $\bar{p} = p + I$ for all $p \in \mathbb{R}[X]$. We know that this quotient ring is a field since $X^2 + 1$ is irreducible in $\mathbb{R}[X]$. Additionally, we know that this field is equal to the set $\{\overline{a + bX} : a, b \in \mathbb{R}\}$ of remainders of polynomial division with $X^2 + 1$, and by the division algorithm, any element $\bar{p} \in \mathbb{R}[X]/I$ can be uniquely represented in the form $\overline{a + bX}$. This shows that $\mathbb{R}[X]/I$ is in bijection with \mathbb{C} , hence we are led to consider the function $\phi : \mathbb{R}[X]/I \rightarrow \mathbb{C}$ defined as follows:

$$\phi(\bar{p}) = \phi(\overline{a + bX}) = a + bi.$$

Let $\overline{a + bX}$ and $\overline{c + dX}$ be elements in $\mathbb{R}[X]/I$, then

$$\begin{aligned}\phi(\overline{a + bX} + \overline{c + dX}) &= \phi(\overline{a + c + (b + d)X}) = (a + c) + (b + d)i \\ &= (a + bi) + (c + di) = \phi(\overline{a + bX}) + \phi(\overline{c + dX}).\end{aligned}$$

It can also be seen that

$$\overline{a + bX} \cdot \overline{c + dX} = \overline{ac + (ad + bc)X + bdX^2},$$

and by calculating the remainder of this polynomial divided by $X^2 + 1$, we obtain.

$$\overline{ac + (ad + bc)X + bdX^2} = \overline{ac - bd + (ad + bc)X}.$$

This allows us to see that

$$\phi(\overline{a + bX} \cdot \overline{c + dX}) = \phi(\overline{ac - bd + (ad + bc)X}) = ac - bd + (ad + bc)i$$

$$= (a + bi)(c + di) = \phi(\overline{a + bX}) \phi(\overline{c + dX}),$$

which proves that ϕ is an isomorphism and $R[X]/I \cong \mathbb{C}$. ■