Measure Theory

Volume in $\mathbb{R}^d$

A point $x \in \mathbb{R}^d$ is a d-tuple of real numbers $(x_1, x_2, \ldots, x_d)$ for $x_i \in \mathbb{R}$. The standard norm (Euclidian norm) $|x|$ on $\mathbb{R}^d$ is defined as

$$|x| = |(x_1, x_2, \ldots, x_d)| = \sqrt{x_1^2 + x_2^2 + \cdots + x_d^2}.$$

Given two points $x, y \in \mathbb{R}^d$, the distance between them is $|x - y|$.

The complement of $E \subseteq \mathbb{R}^d$, written $E^\complement$, is defined as

$$E^\complement = \left\{x \in \mathbb{R}^d : x \notin E\right\}.$$

If $E, F \in \mathbb{R}^d$, then the complement of F in E, written $E - F$ or $E \backslash F$, is defined as

$$E - F = \{e \in E : e \notin F\}.$$

The distance between two sets $E, F \in \mathbb{R}^d$ is the function $d : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ defined as

$$d(E, F) = \inf_{(x,y) \in E \times F} |x - y|.$$

This can be interpreted as the smallest distance between any two points in the closures of E and F.

Open, Closed, and Compact Sets

The **Open Ball** of radius $r \in \mathbb{R}$ centered at $x \in \mathbb{R}^d$, written $B_r(x)$, is defined as

$$B_r(x) = \left\{y \in \mathbb{R}^d : |y - x| < r\right\}.$$

A subset $E \subseteq \mathbb{R}^d$ is **open** if for all $x \in E$ there exists real $r > 0$ where $B_r(x) \subset E$. E is closed if $E^\complement$ is open. Any union of open sets is open, the union need not be countable. The intersection of finitely many open sets is open. Conversely, arbitrary intersections of closed sets are closed, while only finite unions of closed sets are guaranteed to be closed.

The **closure** of a set $E \subseteq \mathbb{R}^d$, written $\overline{E}$, is defined as

$$\overline{E} = \bigcap_{\alpha \in \mathcal{A}} F_\alpha$$

where $\mathcal{A}$ is the family of all closed set $F \in \mathbb{R}^d$ that contain E.

A set $E \subseteq \mathbb{R}^d$ is **bounded** if there exists real $r > 0$ and $x \in \mathbb{R}^d$ where $E \subset B_r(x)$. A bounded set is **compact** if it is also closed. Compact sets have the Heine-Borel covering property.

Assume $E \subseteq \mathbb{R}^d$ is compact, and

$$E \subset \bigcup_{\alpha \in \mathcal{A}} O_\alpha$$

where O_α is open for all α. Then there exists a finite subset $\mathcal{B} \subseteq \mathcal{A}$ where

$$E \subset \bigcup_{\beta \in \mathcal{B}} O_\beta$$

In other words, every open covering of E contains a finite subcovering.

A point $x \in \mathbb{R}^d$ is a **limit point** of E if for all real $r > 0$, $B_r(x) \cap E \neq \varnothing$.

An **isolated point** in $E \subseteq \mathbb{R}^d$ is a point $x \in E$ where for some real $r > 0$, we have that $B_r(x) \cap E = \{x\}$. If for some real $r > 0$, we have that $B_r(x) \subset E$, then x is an **interior point**. The set of all interior points in E, written E° is called its **interior**. Equivallently, the **closure** of E is the union of E and its limit points. The **boundary** of E, written ∂E, is defined as $\overline{E} \backslash E^\circ$.

A closed set $E \subseteq \mathbb{R}^d$ is **perfect** if it has no isolated points.

Rectangle and Cubes

Definition: a rectangle $R \subseteq R^d$ is given by the product of d one-dimensional closed and bounded intervals, given by $R = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_d . b_d]$ for $a_i, b_i \in \mathbb{R}$. The volume of R, written $|R|$, is defined as

$$|R| = \prod_{i=1}^{d} |a_i - b_i|$$

Definition: a cube $Q \subseteq \mathbb{R}^d$ is a rectangle whose d intervals are of the same length. $|Q| = \ell^d$ where ℓ is the length of the intervals.

A union of rectangles is **almost disjoint** if their interiors are disjoint.

Lemma 1.1: If a rectangle is the almost disjoint union of finitely many other rectangles, i.e.

$$R = \bigcup_{k=1}^{N} R_k,$$

then

$$|R| = \sum_{k=1}^{N} |R_k| \, .$$

Lemma 1.2: If $R_1, R_2, \ldots, R_n \subseteq \mathbb{R}^d$ are rectangles and R is contained in their union, then we have

$$|R| \leq \sum_{k=1}^{N} |R_k| \, .$$

Theorem: Every open subset O of $\mathbb{R}$ can be written uniquely as a countable union of disjoint open intervals.

Proof: For all $x \in O$, let I_x denote the largest open interval such that $x \in I_x \subseteq O$. Thus, if we define $a_x = \inf \{a : a < x, (a, x) \subseteq O\}$ and $b_x = \sup \{b : b > x, (x, b) \subseteq O\}$, we must have that $a_x < x < b_x$. Define $I_x = (a_x, b_x)$. We know that $x \in I_x$ and $I_x \subseteq O$, thus

$$O = \bigcup_{x \in O} I_x.$$

If $I_x \cap I_y \neq \varnothing$, then $x \in I_x \cup I_y \subseteq O$. Since I_x is maximal, we have $I_x \cup I_y \subseteq I_x$. Similarly, $I_x \cup I_y \subseteq I_y$, and thus $I_x = I_y$. Consequently, if $\mathcal{I} = \{I_x\}_{x \in O}$, then any two distinct intervals are disjoint. To show $\mathcal{I}$ is countable, we show that each interval contains a rational number. Since the intervals are disjoint, these numbers are distinct, and thus we can form a bijection from the intervals to a countable set of rational numbers. ■

If O is open and

$$O = \bigcup_{k=1}^{\infty} I_j$$

for disjoint intervals I_j, then the measure of O is

$$m(O) = \sum_{j=1}^{\infty} |I_j| \,.$$

Thus, given a union of two open sets $O_1 \cap O_2 \neq \varnothing$, we have that

$$m(O_1 \cup O_2) = m(O_1) + m(O_2).$$

Theorem: For $d \geq 1$, we have that every open set $O \subseteq \mathbb{R}^d$ can be written as a countable union of almost disjoint closed cubes.

Proof: Form a grid in $\mathbb{R}^d$ of closed cubes with side length 1 and integer vertex coordinates. For each cube Q in this grid, we keep it if $Q \subset O$, remove it if $Q \subset O^{\complement}$, and consider it seperately if $Q \cap O \neq \varnothing$. We then bisect these cubes to obtain side lengths $1/2$ and repeat the above process. Gather all accepted cubes in $\mathcal{Q}$. Since this countable process acts on countable objects, we have that $\mathcal{Q}$ is countable. If $x \in O$, then there exists a cube of side length 2^{-N} that contains x is is contained in O. This cube is either in $\mathcal{Q}$ or is contained in a cube in $\mathcal{Q}$, thus

$$O = \bigcup_{Q \in \mathcal{Q}} Q.$$

■

We can define a measure $m : \mathbb{R}^d \to \mathbb{R}$ where if

$$O = \bigcup_{j=1}^{\infty} O_j$$

then

$$m(O) = \sum_{j=1}^{\infty} |O_j|$$

Definition: The **Cantor** set is defined as $C = \bigcap_{n=1}^{\infty} C_n$ where $C_0 = [0,1], C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1], \ldots$ Thus we have a sequence of compacts sets where

$$C_0 \supset C_1 \supset C_2 \supset \cdots$$

C is closed and compact, and C is totally disconnected, which means that for $x, y \in C$, we have that there exists $z \notin C$ where $x < z < y$. C is perfect, which means it has no isolated points. C has the cardinality of the continuum.

Exterior (Outer) Measure

The exterior measure m_* assigns to any subset of $\mathbb{R}^d$ a notion of size. m_* is generally not additive for unions of disjoint sets. To achieve additivity, we must sacrifice the measurability of some sets. m_* approximates $E \subseteq \mathbb{R}^d$ "from the outside". We define m_* as

$$m_*(E) = \inf \sum_{j=1}^{\infty} |Q_j|$$

where the infimum is taken over countable coverings of E with closed cubes.