1 Valuations of Rank One

1.1 p-Adic Valuations of $\mathbb{Q}$

The absolute value is an example of a rank one valuation. For example, the absolute value of a rational number satisfies the following properties:

1.) Non-degeneracy:

$$x \in \mathbb{Q} \implies |x| \geq 0 \text{ and } |x| = 0 \iff x = 0$$

2.) Multiplicative:

$$x, y \in \mathbb{Q} \implies |x|\,|y| = |xy|$$

3.) Triangle Inequality:

$$x, y \in \mathbb{Q} \implies |x + y| \leq |x| + |y|$$

We will now motivate the construction of a stronger valuation on $\mathbb{Q}$.

Theorem: Consider a rational number x. Fixing some prime number p, we can represent x as follows:

$$x = p^{\alpha}\frac{a}{b},$$

where $a, b, \alpha \in \mathbb{Z}$, $p \nmid a$, and $p \nmid b$.

Proof: Fix a prime number p and $x \in \mathbb{Q}$ where $x \neq 0$ and $x = c/d$ for $c, d \in \mathbb{Z}$. Let β_1 and β_2 be the highest powers of p that divide c and d respectively, then

$$x = \frac{c}{d} = \frac{p^{\beta_1}a}{p^{\beta_2}b} = p^{\beta_1 - \beta_2}\frac{a}{b}.$$

Clearly $\beta_1 - \beta_2 \in \mathbb{Z}$, and $p \nmid a$ and $p \nmid b$, so we have obtained our desired form for x. $\blacksquare$