FYS3150 - PROJECT 3

A MODEL FOR THE SOLAR SYSTEM

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ABSTRACT. This project will show the usefulness of writing an object oriented code for a model of our solar system. This will be done using Euler's method and the Velocity Verlet method for solving ordinary differential equations. It is found that the Verlet method is superior in several ways. By experimenting with the laws of physics and the mass of Jupiter, we have seen that Earth could very well end up drifting away from the solar system all together or end up as a moon of Jupiter. Mercury's perihelion precession around the Sun is calculated using a relativistic model for the gravitational force, which is found experimentally to be 42.83 arcseconds, whereas the observed value is 43 arcseconds.

CONTENTS

1. Introduction

Computer simulations of our solar system play a more important role in our daily lives than one might expect. Launching space probes to study distant planets or communication satellites to orbit our own planet requires insights into how the gravitational forces act on objects in space. These insights are gained by testing the theories we have of Newtonian gravity and general relativity, and to reliably do so, we must have numerically stable algorithms.

In the *Theory* and *Methods* sections it will be described the forces that govern the movement of our planets, as well as the ODE solvers we have used in the simulations.

The *Results* section will cover the practical differences of our solvers. It will also include several plots and simulations of our solar system under different circumstances, such as the effects that a massive Jupiter would have on the Earth's orbit, and experimenting with the gravitational laws. We will also test the predictions of Einstein's theory of general relativity.

2. Theory

2.1. **Newton's Gravitational Law.** According to the law of gravity, the force that acts on two bodies of mass M and m is

$$\vec{F}_G = G\frac{Mm}{r^2}\hat{r} \tag{1}$$

where G is the gravitational constant and r is the relative distance between the two bodies. Identifying that $x = r\cos\theta$ and $y = r\sin\theta$, the component-wise forces can be written as

$$F_{G,x} = ma_x = -G\frac{Mm}{r^2}\cos\theta = -G\frac{Mm}{r^3}x \tag{2}$$

$$F_{G,y} = ma_y = -G\frac{Mm}{r^2}\sin\theta = -G\frac{Mm}{r^3}y \tag{3}$$

A similar transformation can be done to obtain an expression for the z-component of the force. For the purposes of this project, it is useful to scale these equations such that we have units that are more accommodating to the scale of the solar system. Assuming a circular motion, we can utilize the centripetal acceleration

$a = \frac{v^2}{r}$. Inserting this into our original equation 1 we have that

$$G\frac{Mm}{r^2} = \frac{mv^2}{r} \tag{4}$$

$$GM = rv^2 \tag{5}$$

$$= r(\omega r)^2 \tag{6}$$

$$= \big(\frac{2\pi}{T}\big)^2 r^3 \tag{7}$$

Setting the radius of this circle equal to $r = 1$ AU, where one AU is defined to be the average distance between the Earth and the Sun, and the orbital period to $T = 1$ yr, we have that

$$GM = 4\pi^2 \mathrm{AU}^3/\mathrm{yr}^2 \tag{8}$$

The acceleration of a body due to the gravitational pull of another can then be written as

$$\vec{a} = \frac{4\pi^2}{r^2}\hat{r} \tag{9}$$

2.2. **Escaping from a gravitational well.** The velocity needed for an object to escape a gravitational well can be derived by looking at the total mechanical energy of the system. Assuming this system is isolated, this energy must always be conserved. That is, $U_i + K_i = U_f + K_f$, where $U = -G\frac{Mm}{r}$ is the gravitational potential energy and $K = \frac{1}{2}mv^2$ is the kinetic energy. Requiring that the final energy is equal to zero, or in other words, the final distance $r_f = \infty$ and final velocity $v_f = 0$, we have that

$$K_i + U_i = 0 + 0 \tag{10}$$

$$\frac{1}{2}mv_i^2 = G\frac{Mm}{r_i} \tag{11}$$

$$v_i = \sqrt{\frac{2GM}{r_i}} \tag{12}$$

In our case, our two objects are the Sun and the Earth, so inserting our known quantity for GM from equation 8, the escape velocity for the earth which is a distance $r_i = 1AU$ from the Sun equates to

$$v_{esc} = 2\sqrt{2}\pi \mathrm{AU}/\mathrm{yr} \tag{13}$$

By changing the gravitational force to

$$F_G = G\frac{Mm}{r^\beta} \tag{14}$$

where $\beta \longrightarrow 3$, we will later see, experimentally, that the escape velocity for earth from the Sun is a factor $\sqrt{2}$ lower than the realistic case in equation 13.

2.3. **Perihelion precession of Mercury.** An important test of the general theory of relativity was to compare its prediction for the perihelion precession of Mercury to the observed value. The reason being that it is the closest planet to the Sun, and the theory's predictions could more easily be observed due to the Sun's massive gravitational pull. According to the theory, Mercury's perihelion should, when all classical effects are subtracted, precess at a rate of 43 arcseconds per century. Modifying the gravitational law to accommodate this relativistic effect, it reads

$$F_G = G\frac{Mm}{r^2}\left[1 + \frac{3l^2}{r^2c^2}\right] \tag{15}$$

where $l = |r \times v|$ is the angular momentum per mass, and c is the speed of light.

3. Methods

3.1. **Euler's method.** Determining the position of a planet orbiting our Sun is a classical second-order ordinary differential equation problem

$$\frac{d^2x(t)}{dx(t)^2} = \frac{F_G}{m} \tag{16}$$

where $x(t)$ is the position, F_G is the gravitational force and m the mass of the object. This can be written as a coupled system of two first-order ODEs, by defining

$$x'(t) = v(t) \tag{17}$$

$$v'(t) = a(t) \tag{18}$$

By Taylor expanding $v'(t)$, truncating after the first derivative and discretizing, we have our new velocity and position given as

$$v_{i+1} = v_i + ha_i + \mathcal{O}(h^2) \tag{19}$$

$$x_{i+1} = x_i + hv_i + \mathcal{O}(h^2) \tag{20}$$

where h is the step size and $\mathcal{O}(h^2)$ is the local error. This is Euler's method, and it can be improved upon slightly by using v_{i+1} in the calculation of x_{i+1}, such that it reads

$$v_{i+1} = v_i + ha_i + \mathcal{O}(h^2) \tag{21}$$

$$x_{i+1} = x_i + hv_{i+1} + \mathcal{O}(h^2) \tag{22}$$

which is the so-called Euler-Chromer method. At only 4 flops (assuming a_i is known) per integration cycle (per dimension), this is a highly efficient algorithm, but it comes at the expense of high numerical stability, as the global error goes as $\mathcal{O}(h)$.

3.2. **The Velocity Verlet method.** Another algorithm which is easily implemented, is the Velocity Verlet method. This algorithm is also based on Taylor expansions[1], but to a higher order, and thus it provides us with a more numerically stable method for solving second-order ODEs;

$$x_{i+1} = x_i + hv_i + \frac{h^2}{2}a_i + \mathcal{O}(h^3) \tag{23}$$

$$v_{i+1} = v_i + \frac{h}{2}(a_i + a_{i+1}) + \mathcal{O}(h^3) \tag{24}$$

Pre-calculating constant factors such as $\frac{h}{2}$ outside the integration loop, this algorithm requires a total of 7 flops per cycle per dimension. The global error of this method goes as $\mathcal{O}(h^2)$.

3.3. **Object orientation.** In a many-body system, such as our solar system, every planet exerts a force on all the other planets. Fortunately, they all follow the same physical laws, so calculating the sum of these forces can be generalized in a simple way. For the purposes of simulating such a system, we have created an object in code which describes each of the planets' properties, such as position, velocity, angular momentum and so forth. This allows us to add an arbitrary number of planets and other celestial bodies to our system, without having to rewrite a single line of code, other than giving each planet the initial conditions, which we have taken from the JPL-website[2].

4. Results

4.1. **Performance of Euler's method and Velocity Verlet.** A 2D simulation of the Earth-Sun system over one year with the Sun fixed in the origin is shown in figure 4.1. Here we have chosen a rather small number of grid points, $n = 1000$, for the purposes of demonstrating the numerical stability of the two methods. Figure 4.2 shows the conservation of mechanical energy and angular momentum for the two methods. The plots are of the absolute difference of these values $\epsilon = |\frac{f_0 - f_1}{f_0}|$ of the initial time step vs the final time step. Figure 4.3 shows the difference in computing time for this simulation.

4.2. **Escape velocities.** Figure 4.4 shows two different simulations on the Earth-sun system. In the first plot, the Earth has been given an initial velocity of $v = 2\sqrt{2}\pi$ AU/yr, which according to equation 13 is the analytically correct velocity for the earth to escape the Sun's gravitational pull entirely. As shown in the plot, the Earth does not escape. In the second plot, a factor of 1.001 has been added to the initial velocity, and after running the simulation for 10 000 years, it's final velocity is $v_f = 0.4$ AU/yr, and it looks as though the Earth will escape. In figure 4.5 we have had a look at how the Earth-Sun system would act if the gravitational force was weaker than it is in reality, as described by equation 14.

[1]The derivation of this algorithm can be found in the lecture notes[1].

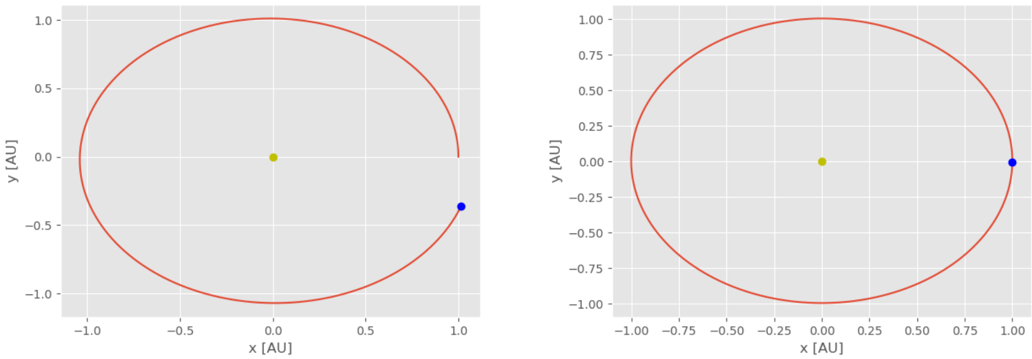

FIGURE 4.1. Earth's trajectory around the Sun for $T = 1$ year, solved with the Euler (left) and Verlet (right) methods with $n = 1000$ gridpoints.

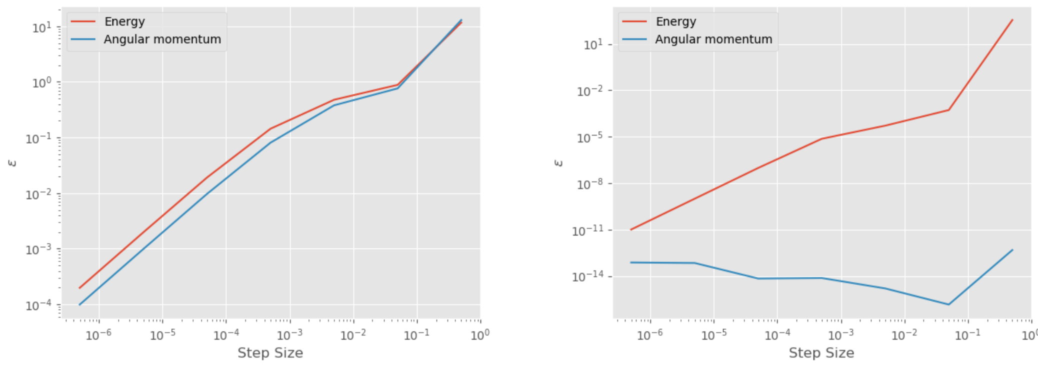

FIGURE 4.2. Absolute error of conservation of energy and angular momentum for Euler's method (left) and the Velocity Verlet method (right) for different step sizes.

The plots show Earth's trajectory around the Sun for ten years with $\beta = 2.9$ and $\beta = 3$ respectively. With $\beta = 2.9$ we can see that Earths orbit is less stable than it normally is, but is still relatively well behaved. In the latter case, the Sun's gravitational pull is not strong enough to keep the Earth in orbit, and it slowly drifts outwards away from the Sun.

4.3. **The Three Body Problem.** We have now added Jupiter to the system, while still holding the Sun fixed in the origin. Experimenting with it's mass and increasing it one thousand fold, shows drastic effects on the Earth's trajectory. Figure 4.6 shows that the Earth eventually would act as a moon of Jupiter. Repeating the simulation, now without the Sun held fixed in the origin, yields similar

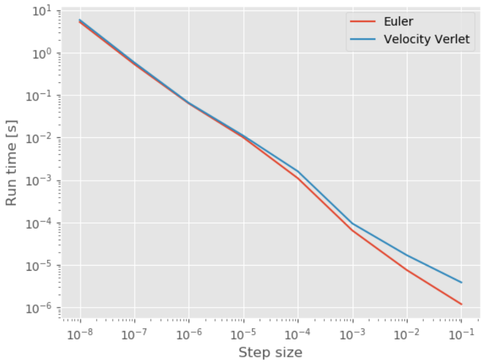

FIGURE 4.3. Program run times for the Euler and Velocity Verlet methods for different time steps.

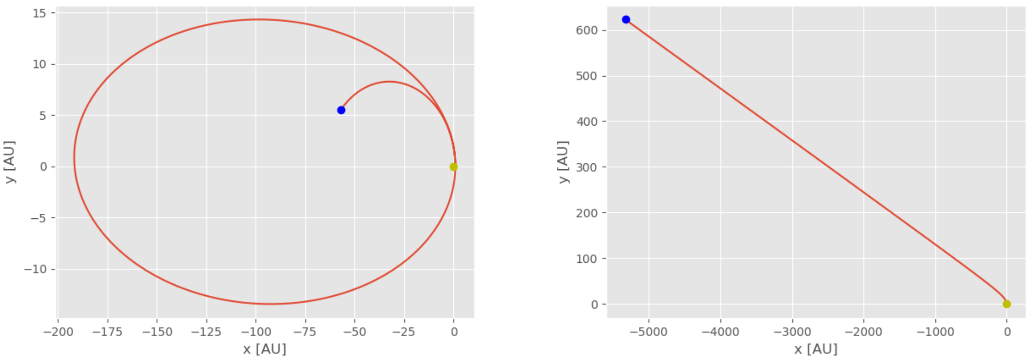

FIGURE 4.4. Simulations of the Earth-Sun system with initial velocities $v_{i,1} = 2\sqrt{2}\pi$ (left) and $v_{i,2} = 1.420 \times 2\pi$ (right). Simulation time for the left case is 1 000 years, and 10 000 years for the right.

results in the case where Jupiter's mass is normal. Figure 4.7 shows simulations where Jupiter's mass is increased ten and one thousand times respectively. In the former case, Jupiter's gravitational pull is massive enough to visibly effect the Sun's orbit around their center of mass. When the mass is increased by one thousand, the Sun and Jupiter act as a binary solar system, with Earth caught in the middle, eventually colliding with Jupiter.

4.4. **Simulating the entire solar system.** Figure 4.8 shows a 3D-plot of the trajectories of all the planets in our solar system. Due to the massive distances between the planets, the inner solar system is not easy to spot on this scale.

4.5. **Perihelion Precession of Mercury.** Figure 4.9 shows the change in Mercury's perihelion precession over the course of 100 years. Einstein's general theory

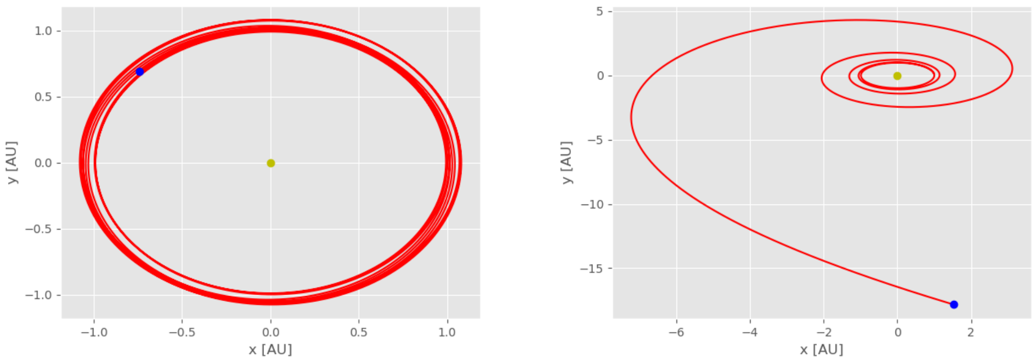

FIGURE 4.5. Earths trajectory around the Sun with $\beta = 2.9$ (left), and $\beta = 3$ (right). Simulation time is ten years.

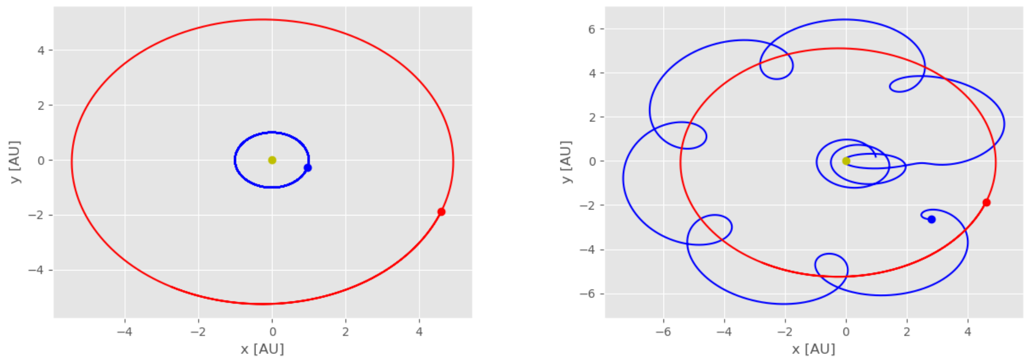

FIGURE 4.6. Simulations of the Earth-Jupiter-Sun system, with the Sun's position fixed. To the right, Jupiter's mass has been increased one thousand times.

of relativity predicts a 43 arcsecond change, our simulation, calculating the gravitational forces with equation 15, run with 10^8 grid points, shows a change in the perihelion angle of $\Delta\theta = 42.83''$.

5. Discussion

5.1. **Performance of Euler's method and Velocity Verlet.** Euler's method is not recommended when one relies on high precission results. It is very easy to implement and requires very few CPU cycles, but the results show that it does a poor job of conserving both energy and angular momentum, even at very small step sizes, as demonstrated in figure 4.2. With $n = 1000$ grid points, it even struggles to reproduce a single orbit of the Earth around the Sun. This algorithm is better suited for 'back of the envelope calculations', and is handy to get a rough view of

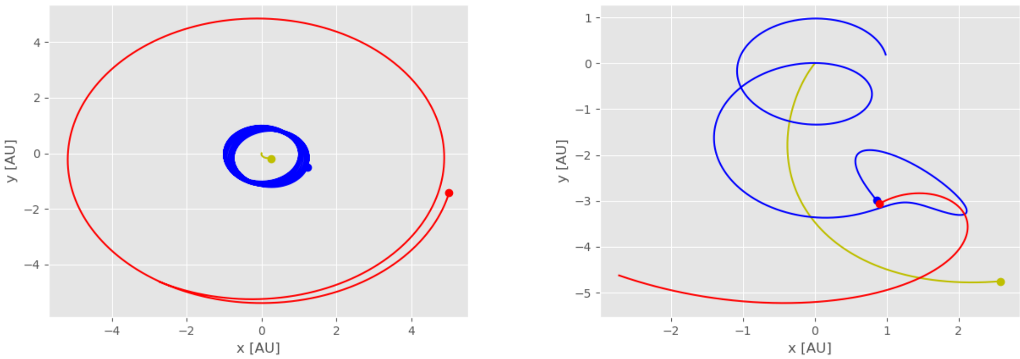

FIGURE 4.7. Simulations of the Earth-Jupiter-Sun system. Jupiter's mass is (to the left) increased ten times, and (to the right) one thousand times.

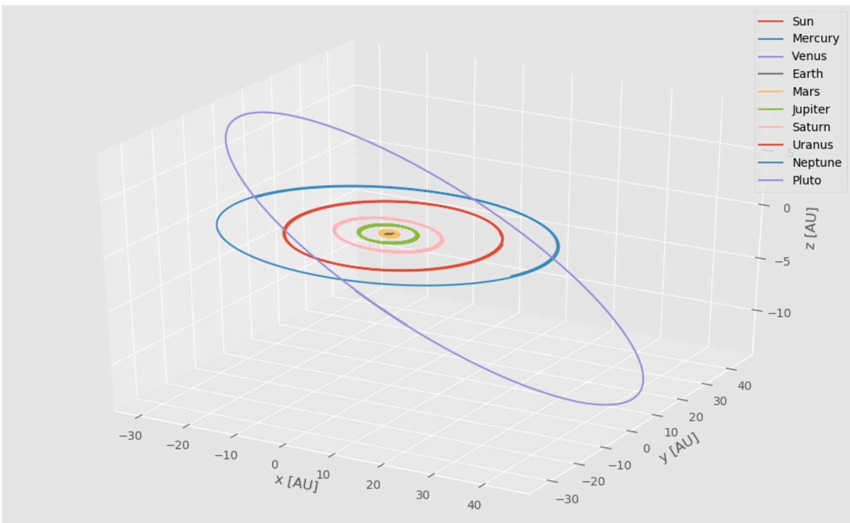

FIGURE 4.8. A simulation of all eight planets and Pluto over 250 years.

how a system might behave. The Velocity Verlet method comes at the expense of only a few CPU cycles more, but in return it's numerical precission is of many orders of magnitude better, making it the method of choice for simulations of this type, where conserving energy is of importance.

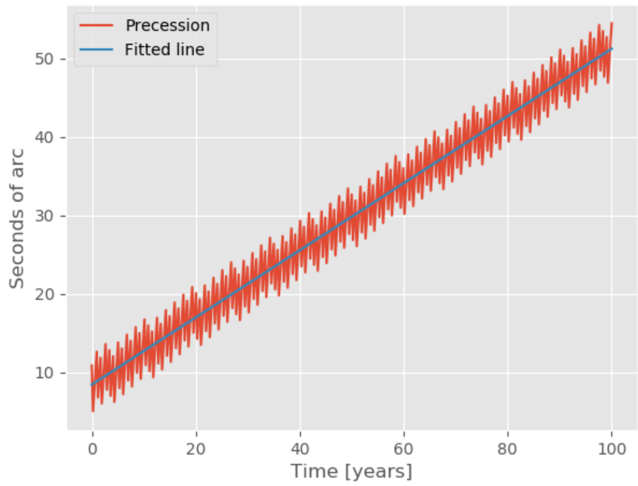

FIGURE 4.9. Change in Mercury's perihelion angle.

5.2. **Escape Velocities.** Experimenting to find the escape velocity for the Earth to escape the Sun, is not a very practical exercise. Through the derivation of the analytical expression for the escape velocity in equation 10, we are dealing with infinities, and they are not exactly well represented on a computer. To experimentally find the escape velocity, one must define infinity to be some finite value and also check whether the final velocity is zero or not. In out results, the Earth distance to the Sun after 10 000 years about 5500 AU, and it's velocity is still not zero, but for our purposes we have deemed is a good enough result.

5.3. **The Three Body Problem.** All though Jupiter is the second most massive object in the solar system, it does not effect the earth much over small time scales. If the simulation had been run over tens of thousands of years, we would presumably see a bigger difference in Earths orbit, since the gravitational forces between the Earth and Jupiter are still at work, however small they may be. Increasing Jupiter's mass serves the purpose of better demonstrating what effects the planets have on each others orbits. The fact is that the planets do not orbit the sun directly, but rather the center of mass of all the masses combined. Increasing the mass of one planet will displaces the center of mass, and the effects of this are clearly demonstrated in the results. When Jupiter's mass is increased to same the magnitude as of the Sun, the results show that this would wreak havoc on the solar system and could send the Earth smashing into Jupiter.

5.4. **Perihelion Precession of Mercury.** Running the simulation of the Mercury-Sun system with the relativistic correctional factor added to the gravitational force

15, shows that comparing our simulated values for the perihelion precession to the observed values serves as a good test of the general theory of relativity.

6. Conclusion

Comparing the results produced by Euler's method and Velocity Verlet, it is clear that Verlet is the preferred method for solving ODE's, due to it's impeccable numerical precission and relatively low CPU demand.

With our object oriented code, we can easily apply new characteristics and properties to all the planets in our system, such as adding relativistic effects to our calculations for the acceleration, with just a few lines of code. Solving many-body systems such as this without generalizing, would be an absolute hassle. If we were to add all the moons of our planets and perhaps comets and asteroids, this would be virtually impossible without writing object oriented code.

We have also seen that the perihelion precession of Mercury can serve as a good test of the theory of general relativity, where our numerical results were only off by 0.17 arcseconds after 100 years.

Playing around with the mass of Jupiter has given insights into how the planets effect each others orbits, and in the extreme case demonstrated what would happen in the unlikely event that a massive interstellar object came close to our solar system.

References

[1] Department of Physics, University of Oslo, Norway (2018), *"Computational Physics Lectures: Ordinary Differential Equations"*, pages 6-8, `http://compphysics.github.io/ComputationalPhysics/doc/pub/ode/pdf/ode-print.pdf`

[2] NASA Jet Propulsion Laboratory, `https://ssd.jpl.nasa.gov/horizons.cgi`

7. Appendix

The program code for this project can be found on GitHub.