High-Dimensional Random Walks

Walking on Matroids

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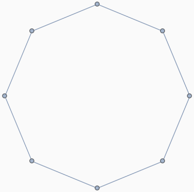
Motivation

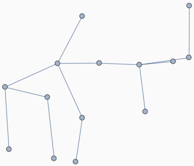
Random forests of a graph

Given an undirected graph $G = (V, E)$, want to count how many spanning forests there are in a certain graph G.

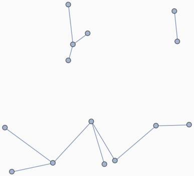
1. A **cycle** is a path that starts from a given vertex and ends at the same vertex.
2. A **tree** is a connected graph with no cycles.
3. A **forest** is a graph that has no cycles. Alternatively, it is a union of trees.



(a) Example of a cycle



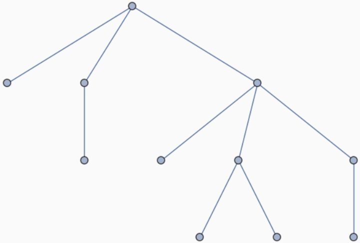
(b) Example of a tree



(c) Example of a forest

Random graphs

(a) Example $V = 11,\ E = 17$



(b) Corresponding spanning tree

Figure 2: Possible forest (in this case just one tree)

Figure 3: Random graph with $V = 11$, $E = 17$

Random graphs

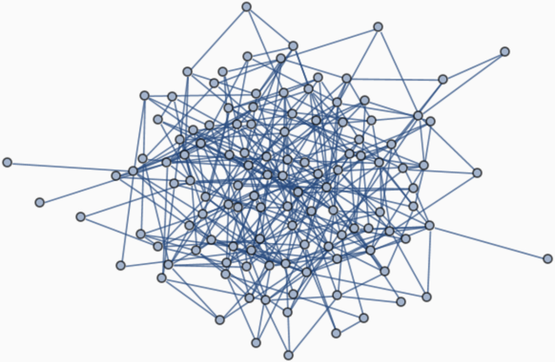
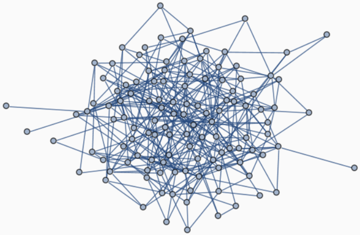
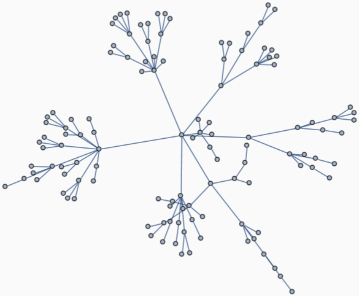


Figure 4: Random graph with $V = 120$, $E = 350$

Random graphs



(a) Example $V = 120$, $E = 350$



(b) Corresponding spanning tree

Figure 5: Possible forest (in this case just one tree)

Preliminaries

Matroids

A *matroid* M is a pair $M = (X, \mathcal{I})$ where X is a finite set and $\mathcal{I} \subseteq 2^X$ so that the following holds:

(i) *Non-emptyness*: $\emptyset \in \mathcal{I}$

(ii) *Monotonicity*: If $Y \in \mathcal{I}$ and $Z \subseteq Y$ then $Z \in \mathcal{I}$.

(iii) *Exchange property*: If $Y, Z \in \mathcal{I}$ and $|Y| < |Z|$, then for some $x \in Z \backslash Y$ we have $Y \cup \{x\} \in \mathcal{I}$

Definition (basis)
Let $M = (X, \mathcal{I})$ be a matroid. A maximal independent set $B \in \mathcal{I}$ is called a *basis* of X.

Graphic Matroids

We can think of a graph as a *graphic matroid*. For a graph $G = (V, E)$, we let $X = E$ and $\mathcal{I} = \{F \subseteq E | F \text{ acyclic}\}$. Then $M = (X, \mathcal{I}) = (E, \mathcal{I})$.

Random walks

Random walk on the bases of M

High-level idea: Graph $G \to$ Graphic Matroid $\to$ Random Walk
Procedure:

1. Start with graph $G = (V, E)$ with $|E| = n$.
2. Pick any maximal forest of size r in G. Forest is acyclic and of size $1 \leq r \leq n$.
3. This corresponds to basis element B of the graphic matroid of G (dimension r).
4. Drop a random element i from B. Pick j uniformly at random from $\{1, \ldots, n\}$, and try adding it to $B \setminus \{i\}$. Do it until we can.
5. Repeat step 4.

Random walk on the bases of M

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3. This corresponds to basis element B of the graphic matroid of G (dimension r).
4. Drop a random element i from B. Try adding an element $j \in \{1, \ldots, n\}$ to $B \setminus \{i\}$ uniformly at random. If we can't, try again.
5. Repeat step 4.

Theorem (Convergence of algorithm)
Given an error $0 < \epsilon < 1$ *and error probability* $0 < \delta < 1$, *after* $r \cdot \log(n^r/\epsilon) \cdot poly(\log(1/\delta))$ *steps, the algorithm will have seen all basis elements up to an error factor of* $1 \pm \epsilon$ *with probability at least* $1 - \delta$.

Random walks on bases of M

- The efficiency of the algorithm is due to [1, AKOV19]
- Ensuring we can go from sampling to counting is due to [2, JVV86]
- So we can sample an (almost) uniformly random basis element of M (a spanning forest of G) by running this Random Walk long enough.

Theorem (Convergence of algorithm)

Given an error $0 < \epsilon < 1$ and error probability $0 < \delta < 1$, after at most $r \cdot \log(n^r/\epsilon) \cdot poly(\log(1/\delta))$ steps, the algorithm will have seen all basis elements up to an error factor of $1 \pm \epsilon$ in the total number of basis elements with probability at least $1 - \delta$.

[1, AKOV19] and [2, JVV86]

- The n^r comes from the fact there are at most $\binom{n}{r} \leq n^r$ bases in an r-dimensional Matroid on the set $[n]$.

References

References

Anari, N., Liu, K., Gharan, S. O., Vinzant, C. (2019, June). *Log-concave polynomials II: High-dimensional walks and an FPRAS for counting bases of a matroid*. In Proceedings of the 51st Annual ACM SIGACT Symposium on Theory of Computing (pp. 1-12).

Mark R. Jerrum, Leslie G. Valiant, Vijay V. Vazirani, *Random generation of combinatorial structures from a uniform distribution*, Theoretical Computer Science, Volume 43, 1986 (pp. 169-188)