Algorithms (6470) HW01

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Answers

Q1

(a) Psuedocode for finding frequency of a number in a given array A:

```
def count(Array A):
    counter = 0
    value = someNumber
    for i = 1 to A.Length:
        if A[i] == value:
            counter = counter + 1
    return counter
```

Use Loop Invariant Technique

(b) Initalization ("Loop Start"): i = 1

(1) Calling count on A gives the freq of value: counter = count(A[1...i])

(2) When i = 1, counter = count(A[1...1]) = count(A[1])

(3) count(A[1]) returns 1 if A[1] == value and 0 otherwise.

(4) True prior to first iteration. **Intialization checked.**

(c) Maintenance: step = i

(1) Assume "counter for i-1" = count(A[1...i-1])

(2) For step i, counter = count(A[1...i-1], A[i]) = count(A[1...i])

(3) True at step i, given step i-1. **Maintenance checked.**

(d) Termination ("Loop End"): i = n+1

(1) Assume "counter for i-1" = count(A[1...i-1])

(2) For step n+1, counter = count(A[1...n+1-1]) = count(A[1...n])

(3) True at step n+1. **Termination checked.**

Q2

(a) Prove: $\lfloor an \rfloor + \lceil (1-a)*n \rceil =$ n.

(1) $\lceil (1-a)*n \rceil = \lceil n - an \rceil$

(2) $\lceil n - an \rceil =$ n + $\lceil -an \rceil$

(3) $\lceil -an \rceil =$ - $\lfloor an \rfloor$, so n + $\lceil -an \rceil =$ n - $\lfloor an \rfloor$

(4) Sub above back into original equation: $\lfloor an \rfloor +$ n - $\lfloor an \rfloor =$ n.

(5) The $\lfloor an \rfloor$ terms cancel out, and: n = n. **Proof complete.**

Q3

(a) Given $f(n) \in O(g(n))$, show $g(n) \in \Omega(f(n))$

(1) Assumption: $f(n) \leq c * g(n)$

(2) Prove: $g(n) \geq k * f(n)$

(3) $f(n) \leq c * g(n)$ is equivalent to $\frac{1}{c} * f(n) \leq g(n)$

(4) $\frac{1}{c} * f(n) \leq g(n)$ can be flipped: $g(n) \geq \frac{1}{c} * f(n)$

(5) So, we've now got the form of Big Theta: $g(n) \geq \frac{1}{c} * f(n)$

(6) If we assume: $\frac{1}{c} = k$, then $g(n) \geq k * f(n)$ and $g(n) \in \Omega(f(n))$.

(7) So, if $k = 1$, then $g(n) \in \Omega(f(n))$ for $n \geq 1$.

(8) This follows from the transpose symmetry in the textbook. **Proof complete.**.

(b) Given $p(n) \in O(f(n))$ then $p(n)f(n) \notin O(f(n))$

(1) Assumption: $p(n) \leq c * f(n)$

(2) Prove this is not possible: $p(n)f(n) \leq c * f(n)$

(3) Multiply each side of the assumption by f(n): $f(n)*p(n) \leq c*f(n)^2$

(4) So, this leaves 1. $f(n)*p(n) \leq c*f(n)^2$ and 2. $p(n)f(n) \leq c*f(n)$

(5) I believe that given $p(n) \leq c*f(n)$, we can now see that $p(n)f(n) \in O(f(n)^2)$ and $p(n)f(n) \notin O(f(n))$

(6) I think those 2 expressions are contradictory, just proving the original statement. **Proof complete.**.

(c) Show: $max(f(n), g(n)) \in O(f(n) + g(n))$

The max of 2 functions will return either function 1 or function 2. So we need to check the math for the case where f(n) > g(n) and the case where g(n) > f(n).

(1) Assumption: $max(f(n), g(n)) = f(n)$ or $g(n)$

(2) Prove: $max(f(n), g(n)) \leq c * [f(n) + g(n)]$

(3) $c * [f(n) + g(n)] = c * f(n) + c * g(n)$

(4) Prove Case 1: $f(n) \leq c * [f(n) + g(n)]$

(5) So, $f(n) \leq c * f(n) + c * g(n)$ rearrange: $f(n) * \frac{(1-c)}{c} \leq g(n)$

(6) $f(n)$ and $g(n)$ are non negative, when $c \geq 1$, this inequality will be true for all n>0: $f(n) * \frac{(1-c)}{c} \leq g(n)$

(7) **So for Case 1: Proof complete.**

(8) Prove Case 2: $g(n) \leq c * [f(n) + g(n)]$

(9) So, $g(n) \leq c * f(n) + c * g(n)$ rearrange: $g(n) * \frac{(1-c)}{c} \leq f(n)$

(10) $f(n)$ and $g(n)$ are non negative, when $c \geq 1$, this inequality will be true for all n>0: $g(n) * \frac{(1-c)}{c} \leq f(n)$

(11) **So for Case 2: Proof complete.**

(d) Show: $n! \in O(n^n)$

(1) n! = n*(n-1)*(n-2)*...*(1)

(2) n^n = n*n*n*n*n*...*n

(3) So, $n! \leq c * n^n$ for c = 1 and $n \geq 1$.

(4) $n * (n-1) * ... * (1) \leq c * n * n * ... * n$

(5) Given n^n is greater at each step after n=1, $n! \in O(n^n)$ **Proof complete.**

Q4

(a) Prove: $f(n) = 4n^2 - 50n + 10 \in o(n^3)$

(1) Need to show that $\lim_{n\to\infty} \frac{f(n)}{n^3} = 0$

(2) Break up the limit into 3 parts.

(3) Part 1: $\lim_{n\to\infty} \frac{4n^2}{n^3} = \lim_{n\to\infty} \frac{4}{n} = 0$

(4) Part 2: $\lim_{n\to\infty} \frac{50n}{n^3} = \lim_{n\to\infty} \frac{50}{n^2} = 0$

(5) Part 3: $\lim_{n\to\infty} \frac{10}{n^3} = 0$

(6) So, $\lim_{n\to\infty} \frac{4n^2}{n^3} + \lim_{n\to\infty} \frac{50n}{n^3} + \lim_{n\to\infty} \frac{10}{n^3} = 0$

(7) Since this limit goes to zero, $f(n) = 4n^2 - 50n + 10 \in o(n^3)$

(b) Prove: $f(n) = n^3 - 5n^2 - 5 \in \omega(n^2)$

(1) Need to show that $\lim_{n\to\infty} \frac{f(n)}{n^2} = \infty$

(2) Break up the limit into 3 parts.

(3) Part 1: $\lim_{n\to\infty} \frac{n^3}{n^2} = \lim_{n\to\infty} n = \infty$

(4) Part 2: $\lim_{n\to\infty} \frac{5n^2}{n^2} = \lim_{n\to\infty} 5 = 5$

(5) Part 3: $\lim_{n\to\infty} \frac{-5}{n^2} = 0$

(6) So, $\lim_{n\to\infty} \frac{f(n)}{n^2} = \infty + 5 + 0 = \infty$

(7) Since this limit goes to ∞, $f(n) = n^3 - 5n^2 - 5 \in \omega(n^2)$

Q5

(a) Substitution Method: $T(n) = T(n/2) + n^3$

(1) a = 1, b = 2, d = 3. This means we use Master's Theorem Guess of $O(n^3)$ because $a < b^d = 1 < 2^3$

(2) $T(n) \leq cn^3$

(3) $T(n/2) \leq c * (\frac{n}{2})^3$

(4) $T(n) = c * (\frac{n}{2})^3 + n^3 = n^3 + \frac{c*n^3}{8}$

(5) We now need to get to the form: $T(n) = cn^3 - (something)$

(6) $\frac{c*n^3}{8} = cn^3 - \frac{7c*n^3}{8}$

(7) $n^3 + \frac{c*n^3}{8} = cn^3 + n^3 - \frac{7c*n^3}{8}$

(8) $cn^3 + n^3 - \frac{7c*n^3}{8} = cn^3 + n^3 * (1 - \frac{7c}{8}) = cn^3 - n^3 * (\frac{7c}{8} - 1)$

(9) So, finally: something $= (\frac{7c}{8} - 1) > 0$

(10) Solve for c and get: $c > \frac{8}{7}$, for n $>$ 0

(b) Substitution Method: $T(n) = 3T(n/3) + n$

(1) a = 3, b = 3, d = 1. This means we use Master's Theorem Guess of $O(nlogn)$ because $a = b^d = 3 = 3^1$

(2) $T(n) \leq c * n * logn$

(3) Show that $T(n) = c * n * logn - (something)$

(4) $T(n/3) \leq c * \frac{n}{3} log \frac{n}{3}$

(5) $T(n) = 3 * (c\frac{n}{3} log \frac{n}{3}) + n = cn * (logn - log3) + n$

(6) $T(n) = cnlogn + n - cnlog3 = cnlogn - n(clog3 - 1)$

(7) Show that, something: $(clog3 - 1) > 0 = c = \frac{1}{log3}$, for n $>$ 2

(c) Substitution Method: $T(n) = 3T(n-1) + 1$

(1) a = 3, b = 1, d = 0. This means we use Master's Theorem Guess of $O(3^n)$ because $a > b^d = 3 > 3^0$

(2) $T(n) \leq c * 3^n$

(3) Show that $T(n) = c * 3^n - (something)$

(4) $T(n-1) \leq c * 3^{n-1}$

(5) $T(n) = 3 * c * 3^{n-1} + 1 = c3^n + 1$

(6) So, we've reached a point where we need to make a new guess given we cannot make this into the form: $c * 3^n - (something)$. We look at our last step and use that as our new guess, given its $c3^n - lowerOrderTerms$

(7) New Guess: $T(n) \leq c * 3^n - 1$

(8) $T(n) = 3 * (c * 3^{n-1} - 1) + 1 = c3^n - 3 + 1 = c3^n - 2$

(9) $T(n) = c3^n - 2$ for c>1 and n>1.

Q6

(a) Show Substitution Method Failure and Lower Order Reguess: $T(n) = 8T(n/2) + n^2$

(1) Trying to show that $T(n) \leq cn^3$ fails but $T(n) \in O(n^3)$

(2) $T(n) \leq cn^3$

(3) $T(n/2) \leq \frac{cn^3}{8}$

(4) $T(n) = 8 * \frac{cn^3}{8} + n^2 = cn^3 + n^2$

(5) There is no way to make $n^2 < 0$ so, we need to reguess.

(6) Reguess: $T(n) \leq cn^3 - kn^2$

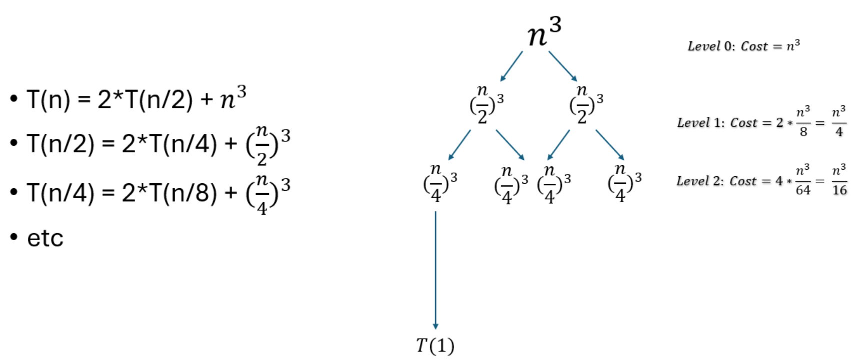
(7) $T(n/2) \leq \frac{cn^3}{8} - \frac{kn^2}{4}$

(8) $T(n) = 8 * \frac{cn^3}{8} - 8 * \frac{kn^2}{4} + n^2 = cn^3 - 4kn^2 + n^2 = cn^3 - n^2(4k - 1)$

(9) So, now $(4k - 1) > 0$ or $k > \frac{1}{4}$ and n>1.

Q7

(a) The following shows how you'd use a recurrence tree to gather the information needed to solve the recurrence relationship for $T(n) = 2T(n/2) + n^3$



(b) Now we can express the number of levels "i" by solving $\frac{n}{4^i} = 1$ and $i = log_4 n$

(c) Next, we need to determine the cost at each level. Given the tree, it appears to be a geometic series where a = 1 and r = 1/4.

(d) $\sum_{i=0}^{logn} n^3 * \frac{1}{4^i} = n^3 * \sum_{i=0}^{logn} \frac{1}{4^i} = n^3 * \frac{1}{1-1/4} = \frac{4n^3}{3}$

(e) So, $T(n) \in O(n^3)$

Q8

(a) Calculate the Expected Value of 2 fair dice.

(1) You can treat these as independent events, x_i and x_j

(2) $E[x_i] = 1 * \frac{1}{6} + 2 * \frac{1}{6} + 3 * \frac{1}{6} + 4 * \frac{1}{6} + 5 * \frac{1}{6} + 6 * \frac{1}{6}$

(3) $E[x_i] = frac1 + 2 + 3 + 4 + 5 + 66 = 3.5$

(4) $E[x_j] = 1 * \frac{1}{6} + 2 * \frac{1}{6} + 3 * \frac{1}{6} + 4 * \frac{1}{6} + 5 * \frac{1}{6} + 6 * \frac{1}{6}$

(5) $E[x_j] = frac1 + 2 + 3 + 4 + 5 + 66 = 3.5$

(6) So, $E[x_i + x_j] = E[x_i] + E[x_j] = 3.5 + 3.5 = 7$

(7) $E[x_i + x_j] = 7$

(b) Calculate the Expected Value of 2 weighted dice.

(1) So, $E[x_i + x_j] = E[x_i] + E[x_j]$

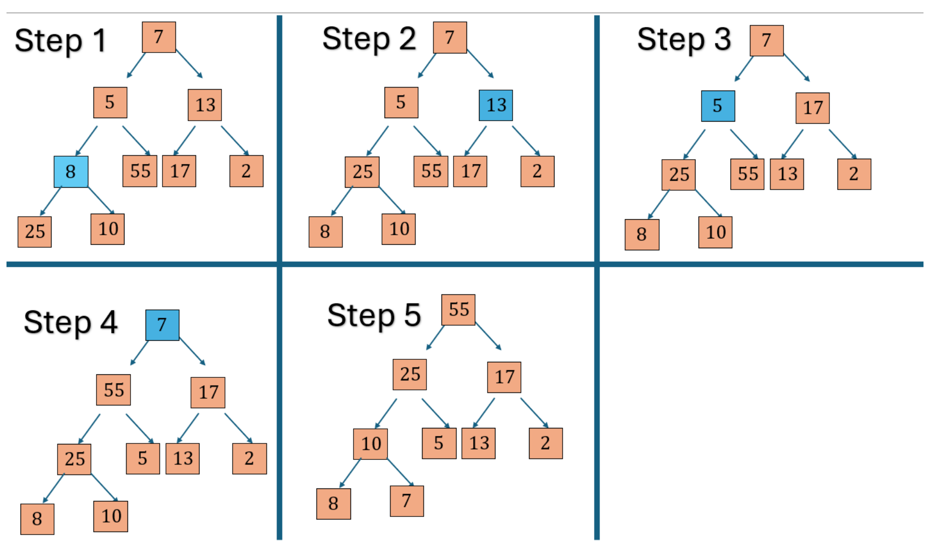
(2) $E[x_i] = 1 * (.11) + 2 * (.04) + 3 * (.07) + 4 * (.30) + 5 * (.33) + 6 * (.15) = 4.15$

(3) $E[x_j] = 1 * (.17) + 2 * (.28) + 3 * (.21) + 4 * (.07) + 5 * (.03) + 6 * (.24) = 3.23$

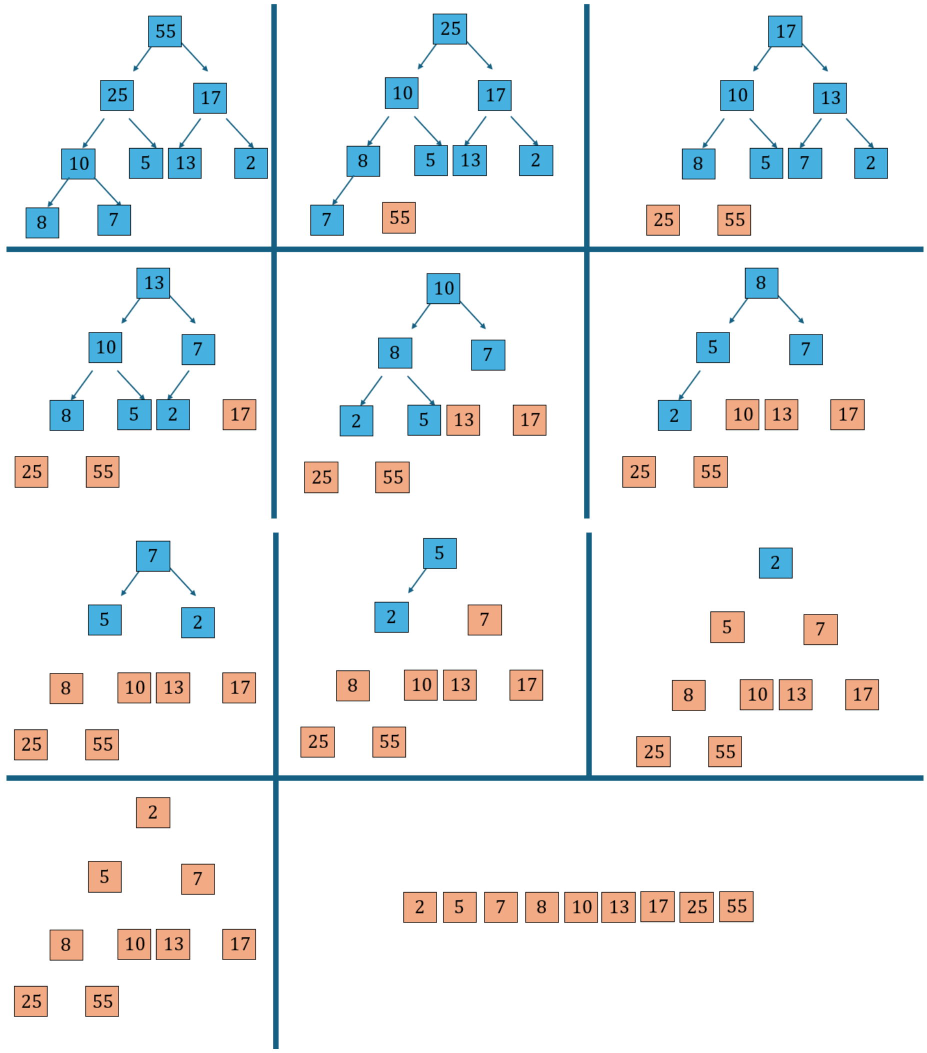
(4) $E[x_i + x_j] = 4.15 + 3.23 = 7.38$

Q9

(a) BUILD-MAX-HEAP for A = <7, 5, 13, 8, 55, 17, 2, 25, 10>



(b) Demonstrate HeapSort on the Array A:



Q10

(a) Assume you have k sorted lists and total n items and you want to merge into 1 sorted list. Psuedocode and use a heap.

(b) Answer Description: I would first build a heap using the lists, and then call heap sort on the heap.

```
    def mergeLists(Array A):

        A_count = 1
        A.heap-size = n
        // Assign the values of the lists to an array
```

```
    for i = 1 to k:
        for j = 1 to list_i.Length:
            A[A_count] = lists[i][j]
            A_count = A_count + 1

    // Make the array into a heap
    for i = floor(n/2) to 1:
        MAX-HEAPIFY(A,i)

    // Sort the Array with heapsort
    for i = n to 2:
        swap A[1] with A[i]
        A.heap-size = A.heap-size - 1
        MAX-HEAPIFY(A,1)
```

(c) I believe the time complexity to complete with algorithm will be O(n) for building the array, O(n) for making the array a heap, then sorting alone takes nlog(n). So total time complexity is n+n+nlogn or O(nlogn).

Q1 (Graduate students only)

(a) Use the definition of ω and Sterlings approximation given in the book to prove that $n! = \omega(2n)$.

(1) Stirling's Approximation: $n! = \sqrt[2]{2\pi n} * (\frac{n}{e})^n * e^{a_n}$

(2) For ω, we need to show $\lim_{n\to\infty} \frac{n!}{2^n} = \infty$

(3) $\lim_{n\to\infty} \frac{n!}{2^n} = \lim_{n\to\infty} \frac{\sqrt{2\pi n}*(\frac{n}{e})^n*e^{a_n}}{2^n}$

(4) e^{a_n} goes to 1 as $\frac{1}{12n+1} < a < \frac{1}{12n}$. As n approaches ∞, $\frac{1}{12n}$ goes to 0 and $e^0 = 1$.

(5) So we're left with: $\lim_{n\to\infty} \frac{\sqrt{2\pi n}*(\frac{n}{e})^n}{2^n}$

(6) $\lim_{n\to\infty} \frac{\sqrt{2\pi n}*(\frac{n}{e})^n}{2^n} = \lim_{n\to\infty} \sqrt{2\pi n} * (\frac{n}{2e})^n$

(7) $\sqrt{2\pi n}$ goes to ∞ as n goes to ∞.

(8) So, the final term to evaluation is $(\frac{n}{2e})^n$. 2e is a constant, so $\frac{n}{2e}$ is greater than one when n is sufficiently large (over about 5).

(9) This means, $\frac{n}{2e}$ approaches ∞ as well!

(10) Thus, $\lim_{n\to\infty} \frac{n!}{2^n} = \infty * \infty * 1 = \infty$.

(11) $n! = \omega(2n)$. **Proof complete.**

Q2 (Graduate students only)

(a) Let $T(n) = aT(n/b) + f(n)$. Prove that if f(n) is a polynomial and $f(n) = \Omega(n^{log_b(a+e)})$ where e>0, then Master's Theorem applies for all $a \geq 1$ and $b \geq 1$

(1) Assumption: $f(n) \geq c * n^{log_b(a+e)}$

(2) Assumption: $a \geq 1$ and $b \geq 1$

(3) Assumption: f(n) is asymptotically positive and a polynomial.

(4) Unfortunately need to give up on this problem. I'm going to attempt to talk it out, but I can't say I know how this works. I believe that, regardless of the values of a or b, we can show that all 3 cases of the Master Theorem work and properly bound T(n). I have attached a picture that I attempted to create for some explanation using a recurrence tree, but I don't think that is what you're looking for.