Solution General Math Problems

G1

We choose to treat the string of 1's and 0's, without a loss of generality, as an ordered list of relationships (*opposite* or *same*) between consecutive digits. We find probability by enumerating the different satisfactory lists and dividing over the number of unique lists: 2^9.

(a) We need to place exactly 1 SAME into 9 digits. There are therefore $\binom{9}{1}$ satisfactory lists. The probability is $\frac{\binom{9}{1}}{2^9} = \frac{9}{2^9}$

(b) The reasoning is exactly the same as part (a). The probability is $\frac{\binom{9}{n}}{2^9}$.

It is worthy of note that each of our listing of relationship represents two original binary strings. A list of 9 *same*'s can be the string 1111111111 or 0000000000. But this redundancy is canceled out as we only divide by 2^9, not $2^1 0$.

G2

Clearly such a t exists when p is a perfect square. The question remains for non-perfect square p's, i.e. irrational $\sqrt{p}$'s. In this case, t cannot possibly be rational if we were to satisfy $t + \sqrt{p}$. Therefore, it must be the case that we can write $t = r - \sqrt{p}$, where r is some rational number. The first condition, $t + \sqrt{p}$ being rational, is satisfied with this definition of t.

Now we move on to examine the second condition. Plugging our definition in, we have

$$\begin{aligned}
\frac{1}{t} + \sqrt{p} &= \frac{1}{r - \sqrt{p}} + \sqrt{p} \\
&= \frac{r + \sqrt{p}}{r^2 - p} + \sqrt{p} \\
&= \frac{r + \sqrt{p} + r^2\sqrt{p} - p\sqrt{p}}{r^2 - p} \\
&= \frac{r + \sqrt{p}(1 + r^2 - p)}{r^2 - p}
\end{aligned}$$

Notice that the denominator is rational. So, the expression's rationality hinges on the numerator's rationality, which, as r is rational, equates to whether $\sqrt{p}(1 + r^2 - p)$ is rational. As the product of a rational and an irrational number is always irrational, the product of $\sqrt{p}$, an irrational square root, and $(1 + r^2 - p)$, a rational number (being the sum of rational parts), is irrational. Q.E.D. We have shown that for an irrational $\sqrt{p}$, there is no t that satisfies both conditions.

In conclusion, only perfect square p's satisfy the condition.

G3

(a) Let's refer to A as stage 0, B as stage 2, and any other vertices as stage 1. We realize that the first step the ant takes must lead to stage 1. Upon stage 1, the ant moves a second time and results in one of these three possibilities:

1. The ant lands on stage 2. (25%)
2. The ant lands on another stage 1 vertex. (50%)
3. The ant lands back on stage 0. (25%)

Given this, define E, the number of expected steps, recursively and solve for E.

$$E = 1 + \frac{1}{4}(E+1) + \frac{1}{2}E + \frac{1}{4} \times 1$$
$$E = \frac{3}{2} + \frac{3}{4}E$$
$$E = 6$$

Also, some fun quick code to check my math empirically:

```python
import random

def ant(latest=0, _cur_steps=0):
    if latest == 0:
        latest = 1
        _cur_steps += 1
        return ant(latest, _cur_steps)
    elif latest == 1:
        latest = random.choice([0, 1, 1, 2])
        _cur_steps += 1
        return ant(latest, _cur_steps)
    else:
        return _cur_steps

n = 10000
total = 0
for i in range(n):
    total += ant()

print(total/n)
```

(b) We treat this similarly. Let's refer to A as stage 0, any neighboring vertex of A as stage 1, any neighboring vertex of B as stage 2, and B as stage 3. Likewise, we observe that on stage 0, the only option is to move to stage 1; on stage 2, $\frac{1}{3}$ of the time the ant moves to stage 0 and $\frac{2}{3}$ of the time the ant moves to stage 2; on stage 3, $\frac{1}{3}$ of the time the ant moves to stage 3 and finishes and $\frac{2}{3}$ of the time the ant returns to stage 1.

Then define E_n as the number of expected steps if the ant were on stage n.

$$\begin{aligned} E_0 &= 1 + E_1 \\ E_1 &= 1 + \frac{1}{3}E_0 + \frac{2}{3}E_2 \\ E_2 &= 1 + \frac{2}{3}E_1 \end{aligned}$$

Let's solve this set of equations.

$$\begin{aligned} E_1 &= 1 + \frac{1}{3} + \frac{1}{3}E_1 + \frac{2}{3}(1 + \frac{2}{3}E_1) \\ \frac{2}{9}E_1 &= 2 \\ E_1 &= 9 \\ E_0 &= 1 + E_1 = 10 \end{aligned}$$

Therefore the expected value is 10.
Likewise, some quick code to check the math:

```
import random

def ant(latest=0, _cur_steps=0):
    if latest == 0:
        latest = 1
        _cur_steps += 1
        return ant(latest, _cur_steps)
    elif latest == 1:
        latest = random.choice([0, 2, 2])
        _cur_steps += 1
        return ant(latest, _cur_steps)
    elif latest == 2:
        latest = random.choice([1, 1, 3])
        _cur_steps += 1
        return ant(latest, _cur_steps)
    elif latest == 3:
        return _cur_steps

n = 10000
total = 0
for i in range(n):
    total += ant()

print(total/n)
```

G4

(a) Rephrasing this problem, we need to find the smallest n such that n or $n + 1$ divides all integers between 1 and 8, but not 9. It is trivial to see that this is equivalent to asking for dividing 5, 6, 7, 8, as a number that divides 8 already divides 2 and 4, etc. Since only one of n and $n + 1$ is odd, the even one of them must have all the even factors (namely, 6 and 8) and therefore must be a multiple of 24. Then either the odd one is a multiple of 35, or the even one has one or both of 5 and 7 as factors. By enumerating the multiples $24k$ of 24 and checking the pairs $(24k, 24k + 1)$ and $(24k - 1, 24k)$, the smallest such pair that satisfies the requirement is $(119, 120)$ (i.e. $n = 119$).

(b) Factoring 2018, we get two prime numbers 2 and 1009. The requirement that 2018 is not a factor means that there one of n and $n + 1$ contains the factor 1009 but not 2 -- hence it must be odd one. Let's call the odd one n_0 and the even one n_1 . Again, like part (a), we only need n_0 and n_1 to factor all of $[1009, 2017]$, since all integer factors smaller than 1009 are already included in that range. Clearly we need n_1 to factor all the even numbers in $[1009, 2017]$. In fact, we can let n_1 be the LCM of all integers in $[1010, 2017]$.
Since 1009 is relatively prime to n_1 , there is an integer j, k such that $1009j = kn_1 - 1$, i.e. there must be a $k \in [1, 1008]$ such that $\frac{kn_1}{1009}$ has the remainder 1. As k varies in $[1, 1008]$, the remainder gets unrepeatedly mapped onto $[1, 1008]$, since a remainder of 1009 is impossible and a remainder of 0 only happens if $k = 1008$ or its multiples. j also must be odd, since kn_1 is even. Therefore, neither j nor 1009 is a multiple of 2018. Let n be $kn_1 - 1$. n factors all integers in $[1010, 2017]$ but clearly not 2018, since it does not factor 1009, while $n + 1$ factors 1009 but not 2018, since it is odd.

(c) I meant to come back to this problem later (admittedly also because I couldn't make good headway with this), but unfortunately this is currently 11:04 PM on December 1st, 2018, with the problem set due in less than an hour. So...I'm sorry! I've been preoccupied with too many other things lately. I hope the other solutions are still satisfactory!

G5

We assume perfect players, or else it's kind of moot since the game doesn't always force one side to win no matter what the players do: take $n = 6$, Alice could either take 2 and lose the next round as Bob takes any amount to leave Alice with a prime number, or she could take 3 and win.

(a) For $n = 6$ and $n = 8$, Alice can just take stone away and force a prime on Bob. So Alice wins. For $n = 10$, although Alice can just take 5 and force another prime on Bob, it should be noted that Alice also wins if she takes 1 stone. In fact, she will win **no matter how Bob responds** if she takes 1 stone in this move. See this by enumerating the two possible moves for Bob: with 9 stones left, he could take either 1 or 3; in both cases Alice wins by taking another 1.

(b) Not gonna lie: I first noticed the pattern by hardcoding the result out with some code:

```
import math

I_win = [False, False]
# if you were to have i stones left:
# if I_win[i-1], then the person currently playing will win
# if not, then the other person will win

```

```python
def divisors(n) :
    l1 = []
    l2 = []
    for i in range(1, int(math.sqrt(n) + 1)) :

        if (n % i == 0) :
            if (n / i == i) :
                l1.append(i)
            else :
                l1.append(i)
                l2.append(int(n / i))
    return l1 + l2[::-1][:-1]

n = 100000
for i in range(3,n):
    print("{}: {}".format(i-1, "Alice" if I_win[i-2] else "Bob"))
    ds = divisors(i)
    if len(ds) == 1: # if it's a prime:
        I_win.append(False)
        continue
    for d in ds:
        if not I_win[i-d-1]:
            I_win.append(True)
            break
    if len(I_win) < i:
        I_win.append(False)
```

(The result is that for any $n \geq 3$, Alice wins all even n games and loses all odd n games)

Let's prove this statement with induction. Let $P(n)$ be the property that if any player ends up with an even n stones in his/her round, he/she wins, and if that player has an odd n stones in his/her round, he/she loses.

For the base cases of $n = 3$ and $n = 4$, the player loses automatically for $n = 3$, and the player can take 1 stone and make the other person lose.

Then, for the inductive case, assume that for a certain $n + 1$, $P(i)$ holds for $i \in [3, n]$.

If $n + 1$ is even, then our current player can take away 1 stone (since 1 is a divisor of every number), leaving the opponent with n stones. Since $P(n)$ holds and n is odd, the opponenet must lose, which is equivalent to our current player winning.

If $n + 1$ is odd, then our current player can only remove an odd i stones, since odd numbers only have odd divisors. So whatever what our player does, the opponent is left with an even number of stones. Also note that $i \leq \frac{n+1}{3}$, since the only odd factor that's possibility biggert than $\frac{n+1}{3}$ is going to be $n + 1$ itself, which is not allowed. This means that for $n + 1 > 3$, $n + 1 - i > 3$. As $P(n + 1 - i)$ holds and $P(n + 1 - i)$ is even, the opponent wins no matter the choice of i, so our player must lose.

Applying the Principle of Mathematical Induction, this property holds for all $n \geq 3$, and we conclude that Alice wins for all even n and loses for all odd n.

G6

(a) A quick brute forcing to check for possible pairs of perfect square yields the pair $(338^2, 335^2)$. Luckily both 338 and 335 are composite. Here's the code:

```
import math

def test(n):
    return math.sqrt(abs(2019-n**2))

for i in range(400):
    n = test(i)
    if n-int(n) < 0.001:
        print(i, n)
```

Then we simply set all the powers to be 2, and write that

$$\begin{aligned} 338^2 - 335^2 &= 2019 \\ 2^2 169^2 - 5^2 67^2 &= 2019 \end{aligned}$$

Therefore our matrix is found

$$\begin{bmatrix} 2^2 & 5^2 \\ 67^2 & 169^2 \end{bmatrix}$$

(b) We prove the existence of such matrices by conforming the determinant expression with a Pell equation and solving it.

Note that any x^n where $n > 3$ can be rewritten as $(x^{\frac{n}{2}})^2$ (even n) or as $x(x^{\frac{n-1}{2}})^2$ (odd n). As x^2 is itself an integer, we only need to consider the possibilities of $n = 2$ or $n = 3$ (the rest can be simplified into these two cases by setting $x' = x^{\frac{n}{2}}$ or $x' = x^{\frac{n-1}{2}}$).

To conform the equation $a^x d^w - b^y c^z = 1$ (for positive integers a, b, c, d, x, y, z, w) with a Pell's equation ($m^2 - Dn^2 = 1$, where D is not a perfect square), just let x, y, w all be 2 and z be 3. Then, let $m = ad, n = bc, D = c$. Since in general, for a Pell's equation, there are infinitely many solutions (provided that D is not a perfect square), we will show that for a particular choice of D, infinitely many members of the set of infinitely many solutions have the properties:

1. n divides D
2. both m and n are composite

Let $D = 2$. Simple enumeration shows that the fundamental solution is $m = 3, n = 2$. Clearly this fundamental solution fails property 2. However, all the subsequent solutions generated by the recurrence formula

$$\begin{aligned} m_{k+1} &= m_1 m_k + D n_1 n_k \\ n_{k+1} &= m_1 n_k + n_1 m_k \end{aligned}$$

which, when we plug in our specific values, becomes

$$\begin{aligned} m_{k+1} &= 3m_k + 4n_k \\ n_{k+1} &= 2m_k + 3n_k \end{aligned}$$

Since $n_1 = 2$ is even, clearly all n_k where $k > 1$ is even and therefore composite. Though $m_1 = 3$ is prime, and certainly many m_k's are prime, but clearly there are infinitely composite m_k's. If the number of composite m_k's were finite, we have just come up with an efficient way to generate prime numbers: compute m_k using the recursive formula and declare it a prime if it's not in that finite list of composite numbers.

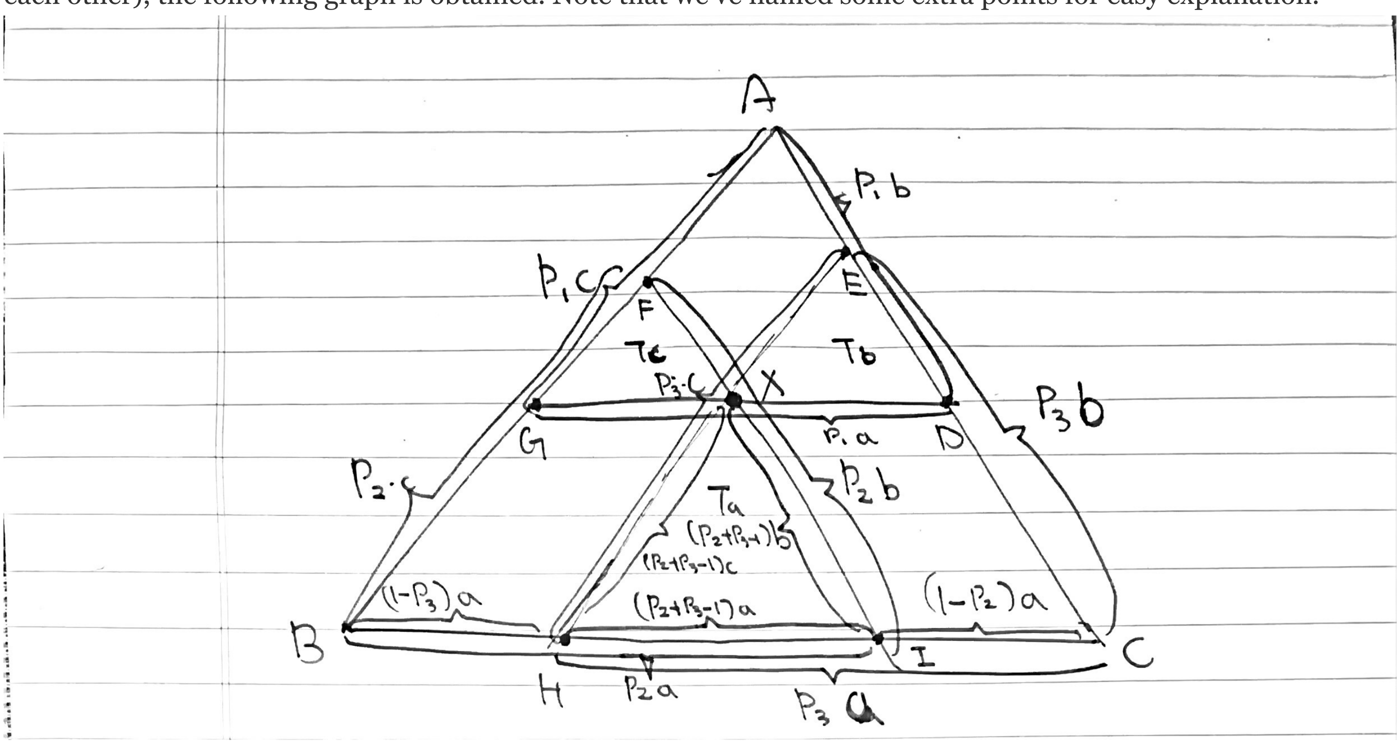
Therefore, we conclude that there are infinitely many m, n that satisfy properties 1 and 2. Let $c = 2, b = \frac{n}{2}$, and a and d to be any two non-one integer factors of m such that $ad = m$. The resulting infinitely matrices in the form

$$\begin{bmatrix} a^2 & b^2 \\ c^3 & d^2 \end{bmatrix}$$

have determinant 1.

G7

Let's define P_1 to be the proportion of the line crossing X parallel to $\overline{BC}$ to the line $\overline{BC}$ (in length). Define P_2 to be that of $\overline{AC}$ and P_3 to be that of $\overline{AB}$. After noticing a series of proportions (as all the triangles are similar to each other), the following graph is obtained. Note that we've named some extra points for easy explanation.



The length of $\overline{HI}$ is computed by adding the bottom two lines $\overline{BI}$ and $\overline{HC}$ (they overlap on $\overline{HI}$) and subtracting $\overline{BC}$. So the length of $\overline{BI}$ is $(P_2 + P_3 - 1)a$. Likewise, by properties of similar triangles, the other two sides of T_a

($\overline{HX}$ and $\overline{IX}$)are clearly $(P_2 + P_3 - 1)b$ and $(P_2 + P_3 - 1)c$. This same reasoning can be applied to T_b and T_c, and we conclude that T_b has proportion $(P_1 + P_3 - 1)$ and T_c has proportion $(P_1 + P_2 - 1)$.

We also observe the following equations:

$$\begin{aligned} 0 &< P_1 < 1 \\ 1 - P_1 &< P_2 < 1 \\ P_3 &= 2 - P_1 - P_2 \end{aligned}$$

The first equation obviously holds, since P_1 is a proportion. The second equation holds by observing that after we decide P_1 and lay down $\overline{GD}$, we must place $\overline{FI}$ (of length $P_2 b$) such that it crosses $\overline{GD}$ by setting $P_2 > 1 - P_1$. The third equation holds by noticing that $\overline{BH}$ and $\overline{IC}$ are the same as $\overline{GX}$ and $\overline{XD}$ respectively by parallelograms. This means that

$$\begin{aligned} &(1 - P_3)a + (1 - P_2)a \\ = \overline{BH} + \overline{IC} = \overline{GX} + \overline{XD} &= \overline{GD} \\ &= P_1 a \end{aligned}$$

which is equivalent to our third equation after some simple algebra.

(a) Since the total area of the triangle $\triangle ABC$ is given by Heron's Formula

$$\begin{aligned} s &= \frac{a+b+c}{2} \\ A &= \sqrt{s(s-a)(s-b)(s-c)} \end{aligned}$$

we only need to compute the possible range of ratio of area of the three smaller triangles to the area of the whole triangle.

By placing X extremely close to one vertex, the total area of the three triangles (in particular, one of the three) approaches the area of $\triangle ABC$. So the maximum is infinitely close to the area of the whole triangle. Taking the average (by dividing 3), we get $S < \frac{A}{3}$.

To find the minimum area, we need to write an expression for the total area of T_1, T_2, T_3 and try to minimize that expression with optimal choices of P_1 and P_2 (that still satisfy the earlier constraints on these values).All three of these triangles are similar to $\triangle ABC$. Let's take T_a for an example of how to compute the ratio of its area to $\triangle ABC$'s area. By similarity, its area ratio is simply $(\frac{(P_2+P_3-1)a}{a})^2 = (P_2 + P_3 - 1)^2$. An analogous reasoning shows that the ratio of T_b and T_c are respectively $(P_1 + P_3 - 1)^2$ and $(P_1 + P_2 - 1)^2$. Therefore the total area of these three triangles is

$$(P_2 + P_3 - 1)^2 + (P_1 + P_3 - 1)^2 + (P_2 + P_1 - 1)^2$$

After substituting $P_3 = 2 - P_1 - P_2$ and some algebraic manipulations, this equates to

$$3 + 2{P_1}^2 + 2{P_2}^2 + 2P_1P_2 - 4P_1 - 4P_2$$

For ease of expression, let's define

$$f(x, y) = 3 + 2x^2 + 2y^2 + 2xy - 4x - 4y$$

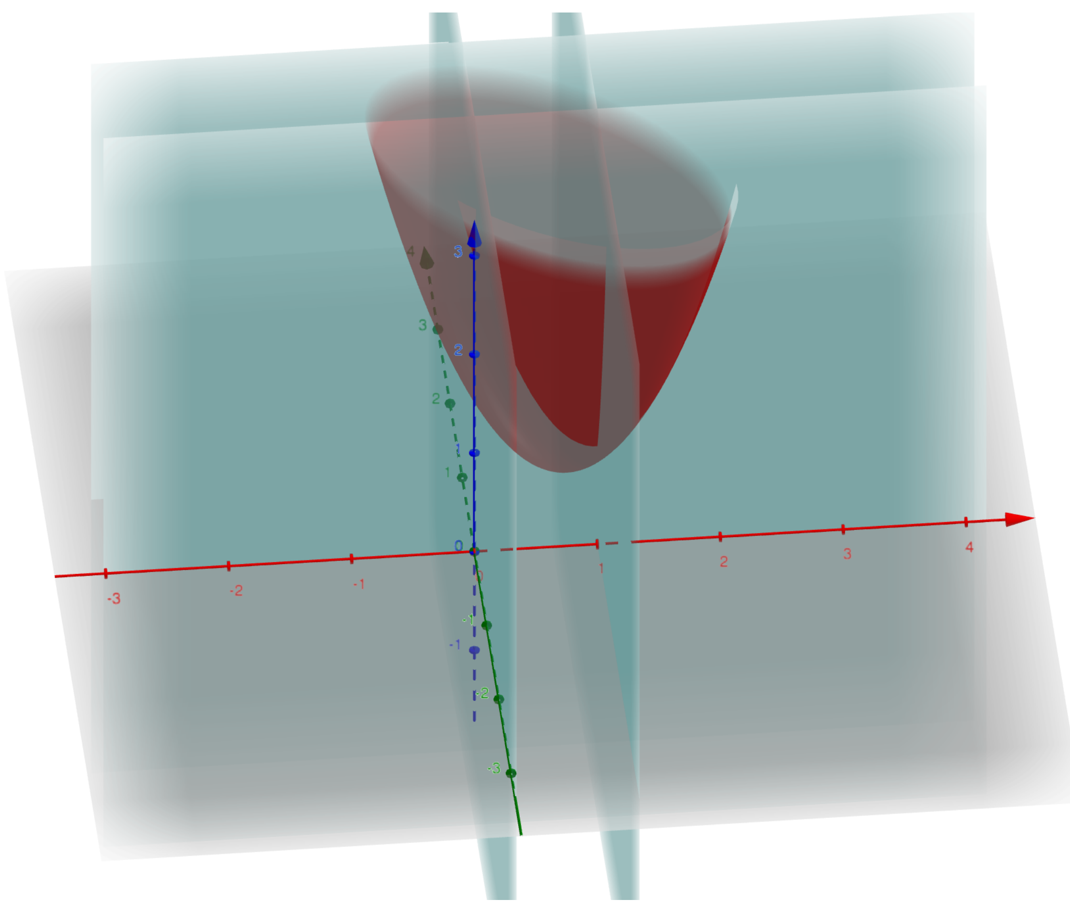
Then, to find the minimum possible ratio, let's find f's critical points.

$$f_x(x, y) = 4x + 2y - 4 = 0$$
$$f_y(x, y) = 4y + 2x - 4 = 0$$

Solving the set of equations, we get the critical point $(\frac{2}{3}, \frac{2}{3})$. By the second derivative test

$$f_{xx}(x, y) = 4 > 0$$
$$f_{yy}(x, y) = 4$$
$$f_{xy}(x, y) = 2$$
$$(f_{xx} \times f_{yy} - {f_{yy}}^2)(\frac{2}{3}, \frac{2}{3}) = 12 > 0$$

this point is a local minimum. Graphing f confirms that it is the minimum within the box of $x \in (0, 1)$ and $y \in (0, 1)$.



Therefore, evaluating at the point gives and taking the average (by dividing 3)

$$\frac{f(\frac{2}{3}, \frac{2}{3})}{3} = \frac{1}{9}$$

So
$\frac{A}{9} < s < \frac{A}{3}$, where A is the area of $\triangle ABC$ as defined earlier.

(b) Note that

$$\begin{aligned}
&P_1 + P_2 + P_3 \\
=&P_1 + P_2 + 2 - P_1 - P_2 \\
=&2
\end{aligned}$$

The perimeter of all three triangles added up is

$$\begin{aligned}
3p = &(P_1 + P_2 - 1)(a + b + c) \\
&+ (P_2 + P_3 - 1)(a + b + c) \\
&+ (P_1 + P_3 - 1)(a + b + c) \\
&= (2P_1 + 2P_2 + 2P_3 - 3)(a + b + c) \\
&= (4 - 3)(a + b + c) \\
&= a + b + c
\end{aligned}$$

The slightly surprising conclusion

$$p = \frac{a + b + c}{3}$$

means that p is the same no matter what where X is placed.