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Problem 1

a)

The conditions for a $M/M/1$ queue is satisfied because the interarrival times are independent and identically distributed with distribution $Exp(\lambda)$. The treatment times are independent and identically distributed with distribution $Exp(\mu)$. They only treat one patient at a time, and the treatment time is independent of the arrival time. Hence, the UCC is a $M/M/1$ queue.

The stochastic process $\{X(t) : t \leq 0\}$ can be viewed as a birth and death process, because the arrival rates and treatment times are independent and the treatment time (sojourn time) is an exponential distribution $Exp(\mu)$. The 'birth' rates is the arrival rate $\lambda_i = \lambda$, and the 'death' rate is the treatment time $\mu_i = \mu$.

Determining the average time a patient spent in the UCC:

$$\Theta_0 = 1, \quad \Theta_k = \Pi_{i=1}^{k} \frac{\lambda_{i-1}}{\lambda_i} = (\frac{\lambda}{\mu})^k, \quad k = 1, 2, \dots$$

$$\sum_{k=0}^{\infty} \Theta_k = \sum_{k=0}^{\infty} (\frac{\lambda}{\mu})^k = \begin{cases} \frac{1}{1-\lambda/\mu}, & \lambda < \mu \\ \infty, & \text{else} \end{cases}$$

we then get that the limit is,

$$\pi_k = \frac{\Theta_k}{\sum_{j=0}^{\infty} \Theta_j} = (\frac{\lambda}{\mu})^k (1 - \frac{\lambda}{\mu}), \quad k = 0, 1, \dots$$

which is a geometric distribution, but 0-indexed. Let $X(t)$ be the number of patients in the UCC. $X(t) + 1$ is then a geometric distribution with probability $p = (1 - \frac{\lambda}{\mu})$.

The average time a patient will spend in the UCC is then,

$$L = E[X + 1] - 1 = \frac{1}{1 - \frac{\lambda}{\mu}} - 1 = \frac{\lambda}{\mu - \lambda}$$

Using Little's law,

$$W = \frac{L}{\lambda} = \frac{1}{\mu - \lambda}$$

b)

We assume that $\lambda = 5$ patients per hour and $\mu = 6$ treatments per hour. These values give an average time spent in the UCC, $W = 1$ hours.

Estimator for the expected time a patient spends in the UCC. First estimate the long-run mean number of patients in the UCC, then using Little's law to find an estimate for the expected time,

$$\hat{L} = \frac{\overline{x \cdot S}}{\sum_j S_j}$$

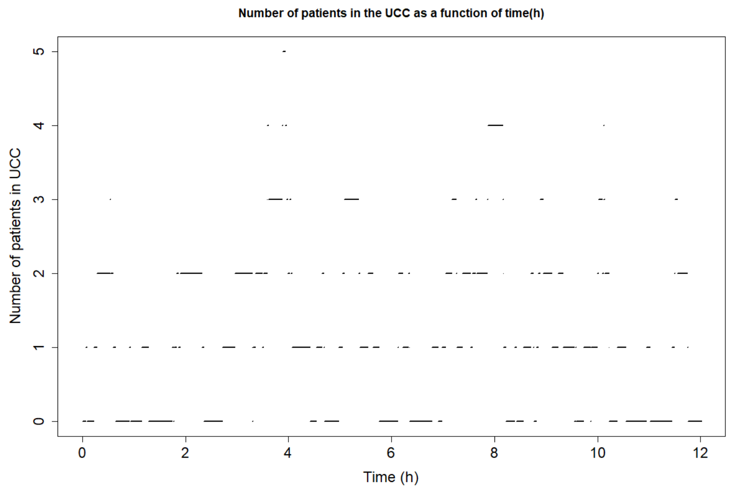

Figure 1: Total patients in the UCC

$$\hat{W} = \frac{\hat{L}}{\lambda}$$

, where S is the sojourn times for the corresponding states in x. Finding a 95% confidence interval. Using an estimator for $\hat{\sigma}^2 = S^2 = \frac{1}{n-1}\sum_j(\overline{x} - x_j)^2$,

$$T = \frac{\overline{x} - L}{S/\sqrt{30}} \sim \text{t-distribution}(\upsilon = 29)$$

$$\overline{x} - t_{0.025,29}\frac{S}{\sqrt{n}} < L < \overline{x} + t_{0.025,29}\frac{S}{\sqrt{n}}$$

We then get the confidence interval:

$$\hat{W} = [0.9897728, 1.0999]$$

We notice that the analytic solution for $W = 1$, is in the interval.

c)

Arrival rate of urgent patient is now $\lambda_U = p\lambda$, and arrival rate of normal patient is $\lambda_N = (1-p)\lambda$, where p is the probability that arrived patient is urgent.

Using same calculations as previous, one can derive that the long-run mean number of urgent patients in the UCC is,

$$L_U = \frac{p\lambda}{\mu - p\lambda}$$

d)

$\{N(t) : t \geq 0\}$ does not behave as an M/M/1 queue because the service times depends on the arrival of urgent patients. Hence, the service times are dependent on the arrival process meaning this is not an M/M/1 queue.

Determining the long-run mean number of normal patients in the UCC. We know that the long run mean number of urgent and normal patients is equal to the long run number of total patients in the UCC, hence:

$$L_N = L - L_U = \frac{\lambda}{\mu - \lambda} - \frac{p\lambda}{\mu - p\lambda} = \frac{(1-p)\lambda\mu}{(\mu - \lambda)(\mu - p\lambda)}$$

e)

Using Little's law one can derive the expected time for urgent and normal patients in the UCC,

$$W_U = \frac{L_U}{p\lambda} = \frac{1}{\mu - p\lambda}$$

$$W_N = \frac{L_N}{(1-p)\lambda} = \frac{\mu}{(\mu - \lambda)(\mu - p\lambda)}$$

f)

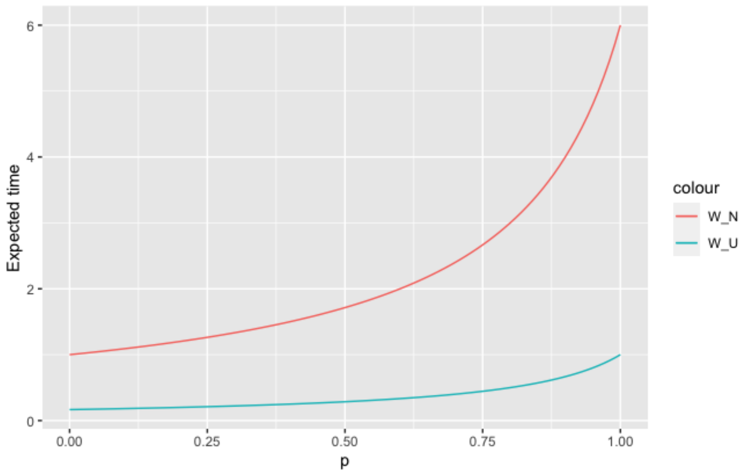

Figure 2: W_U and W_N as functions of p

When p $\approx$ 1, almost all patients arriving the UCC are classified as urgent. On the other hand, when p $\approx$ 0, there are more or less only normal patients arriving. In addition, as there are no urgent patients, we can classify $\{N(t) : t \geq 0\}$ as a $M/M/1$ queue. Moreover, as $p \to 0$, we get that $W_N \to 1$. Similarly, as $p \to 1$, we get that $W_N \to 6$. Furthermore, we want to calculate p when we know that the expected time spent at the UCC for a normal patient is 2 hours:

$$W_N = \frac{\mu}{(\mu - \lambda)(\mu - p\lambda)} \implies p = \frac{\mu}{\lambda} - \frac{\mu}{W_N \lambda(\mu - \lambda)} = \frac{6}{5} - \frac{6}{2 \cdot 5(6-5)} = 0.6$$

Now we want to plot W_N and W_U as functions of p. The plots are shown in Figure 2.

g)

Assuming that $\lambda = 5$ patients per hour, $\mu = 6$ treatments per hour, and probability for urgent patient is $p = 0.80$. Hence the expected time in the UCC for urgent and normal patients is $W_U = 0.5$ hours and $W_N = 3$ hours.

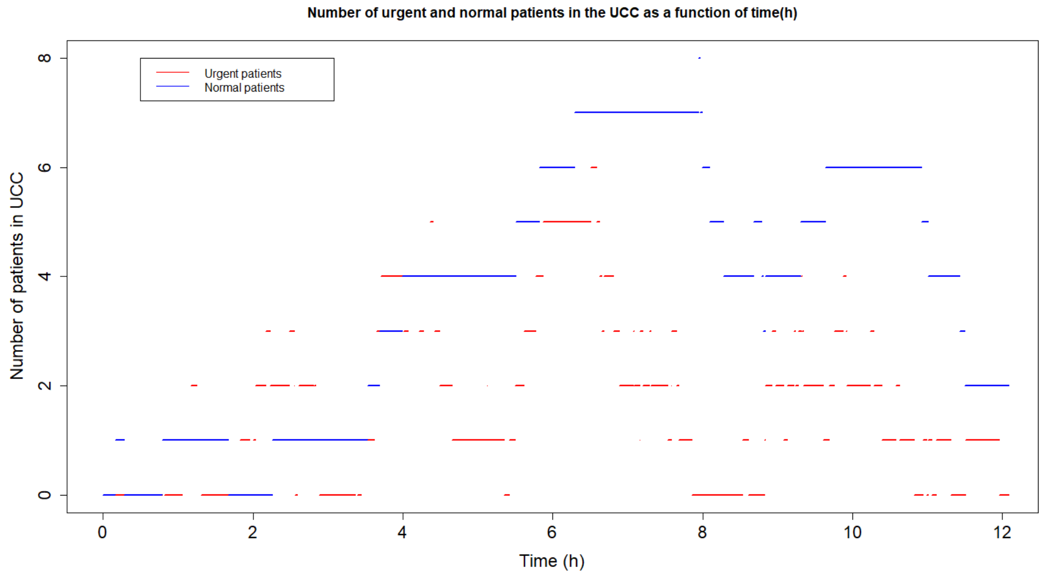

Figure 3: Urgent and normal patients in the UCC

Using same method as previous to estimate the expected time in the UCC for urgent and normal patients, and calculating the 95% confidence interval.

We get the confidence intervals:

$$\hat{W_U} = [0.4943141, 0.5217088]$$
$$\hat{W_N} = [2.626687, 3.058269]$$

We notice that both the analytic solutions $W_U = 0.5$ and $W_N = 3$ are in the intervals.

Problem 2

a)

Let's define the regular grid of the parameter θ as

$$\boldsymbol{\theta}_R = [\theta_{R,i}]_{i=1}^{51} = [0.25 + 0.005i]_{i=1}^{51}, \tag{1}$$

with corresponding mean vector given by

$$\boldsymbol{\mu}_R = [E[Y(\theta_{R,j}]]_{i=1}^{51} = [0.50]_{i=1}^{51} \tag{2}$$

The scientists' observed values are given by

$$\boldsymbol{\theta}_O = [\theta_{O,i}]_{i=1}^{5} = [0.30, 0.35, 0.39, 0.41, 0.45]^T \tag{3}$$

$$\boldsymbol{y}_O = [y_{O,i}(\theta_{O,i})]_{i=1}^{5} = [0.50, 0.32, 0.40, 0.35, 0.60]^T \tag{4}$$

with corresponding mean vector given by

$$\boldsymbol{\mu}_O = [E[Y_{O,i}(\theta_{O,i})]]_{i=1}^{5} = [0.50]_{i=1}^{5} \tag{5}$$

If given two vectors of stochastic variables $\boldsymbol{Y}_A(\theta_A)$ with length m and $\boldsymbol{Y}_B(\theta_B)$ with length n from our Gaussian process model, we can calculate their covariance matrix as

$$\Sigma_{AB} = \Big[\mathrm{Corr}[Y_{A,i}, Y_{B,j}]{\cdot}0.5^2\Big]_{i,j=1}^{m,n} = \Big[(1{+}15|\theta_{A,i}{-}\theta_{B,j}|)\exp(-15|\theta_{A,i}{-}\theta_{B,j}|){\cdot}0.5^2\Big]_{i,j=1}^{m,n}, \tag{6}$$

where we have used

$$\mathrm{Var}[Y_{A,i}(\theta_{A,i})] = \mathrm{Var}[Y_{B,i}(\theta_{B,i})] = 0.5^2 \tag{7}$$

Using equation 6 with θ values given by the grid 1 and the observations 3, we can calculate the following covariance matrices

$$\Sigma_{RR}, \Sigma_{OO}, \Sigma_{RO}, \Sigma_{OR} \tag{8}$$

Now with the covariance matrices 8, the mean vectors 2 and5 and the observed $y(\theta)$ values 4 it is possible to calculate the conditional mean and covariance matrix as

$$\boldsymbol{\mu}_C = \boldsymbol{\mu}_R + \Sigma_{RO}\Sigma_{OO}^{-1}(\boldsymbol{y}_O - \boldsymbol{\mu}_O) \tag{9}$$

$$\Sigma_C = \Sigma_{RR} - \Sigma_{RO}\Sigma_{OO}^{-1}\Sigma_{OR} \tag{10}$$

This allows simulations of $\boldsymbol{Y}_R|\boldsymbol{Y}_O$. Below is 100 such realizations. Using the mean

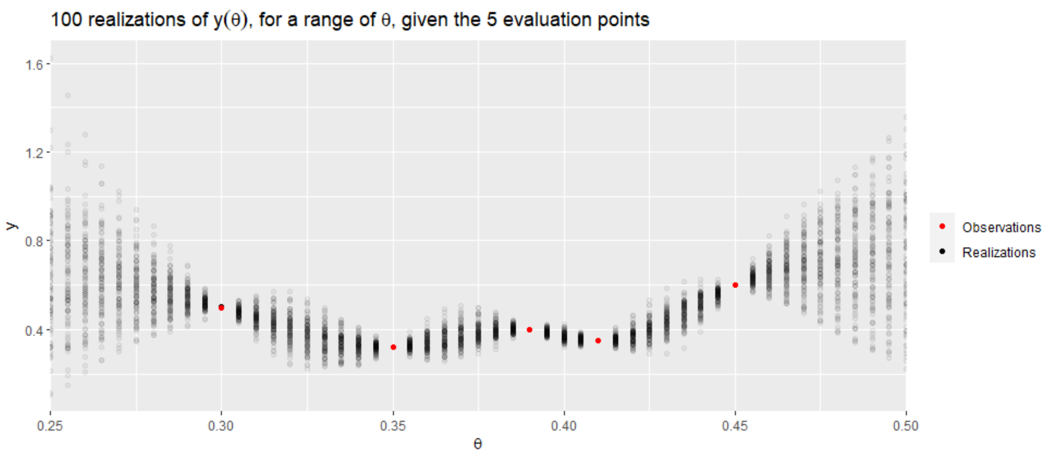

Figure 4: 100 realizations of $\boldsymbol{Y}_R|\boldsymbol{Y}_O$ for θ given by 1

vector $\boldsymbol{\mu}_C$ 9 and the variance given by the diagonal of Σ_C 10 a 90% prediction interval can be given by

$$\boldsymbol{\mu}_C - 1.64\sqrt{\mathrm{diag}(\Sigma_C)} \leq \boldsymbol{\mu}_C \leq \boldsymbol{\mu}_C + 1.64\sqrt{\mathrm{diag}(\Sigma_C)}, \tag{11}$$

plotted below

b)

Since we are assuming that the process is Gaussian, i.e. we have $\boldsymbol{Y}_R|\boldsymbol{Y}_O \sim \mathcal{N}_m(\boldsymbol{\mu}_C, \Sigma_C)$, we can find

$$Pr[Y(\theta_{R,j})|\boldsymbol{Y}_O < 0.30]$$

$$Pr\left[\sqrt{\mathrm{diag}(\Sigma_C)}_j Z_j + \mu_{C,j} < 0.30\right]$$

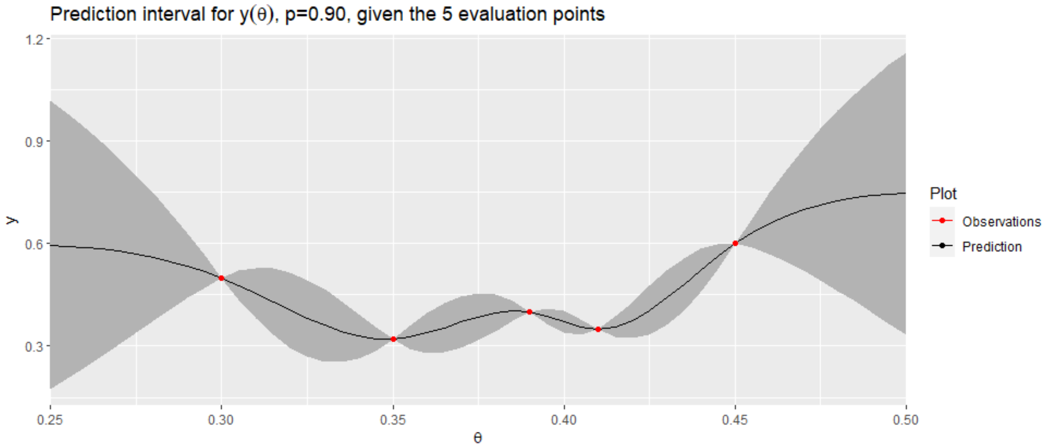

Figure 5: The 90% prediction interval for $\boldsymbol{Y}_R|\boldsymbol{Y}_O$

using the cdf of the normal distribution at every value of θ with

$$Pr\left[Z_j < \frac{0.30 - \mu_{C,j}}{\sqrt{\text{diag}(\Sigma_C)}}\right], \tag{12}$$

the probability that a given a $(\theta, y(\theta))$ pair has $y(\theta) < 0.30$, as plotted below.

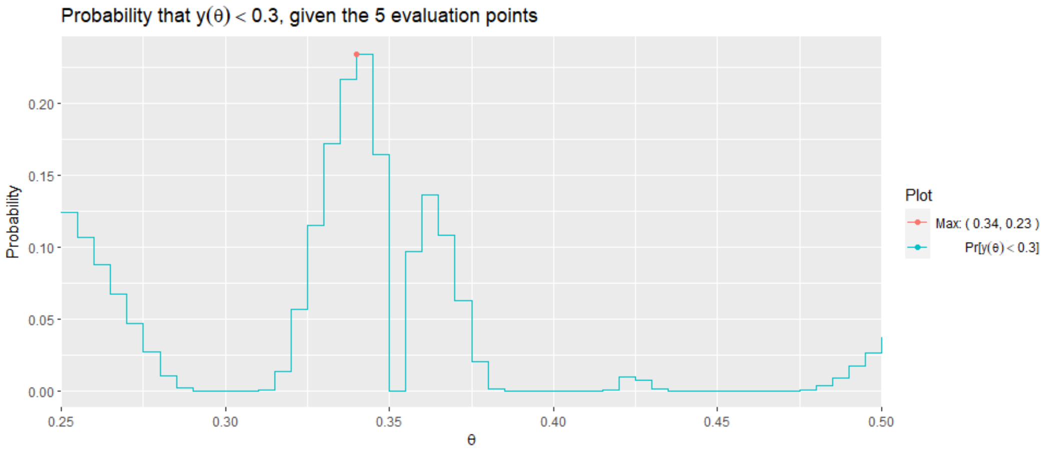

Figure 6: $Pr[Y(\theta_{R,j})|\boldsymbol{Y}_O < 0.30]$

c)

With the new data point $(0.33, 0.40)$ it is possible two use the same methods as discussed in 2a) and 2b) with the observed vector updated, i.e. $(\boldsymbol{\theta}_O^*, \boldsymbol{Y}_O^*)$. This gives the following plots as seen below. We would recommend the scientists to use the value of $\theta = 0.36$ since it has the highest probability, as seen in 9.

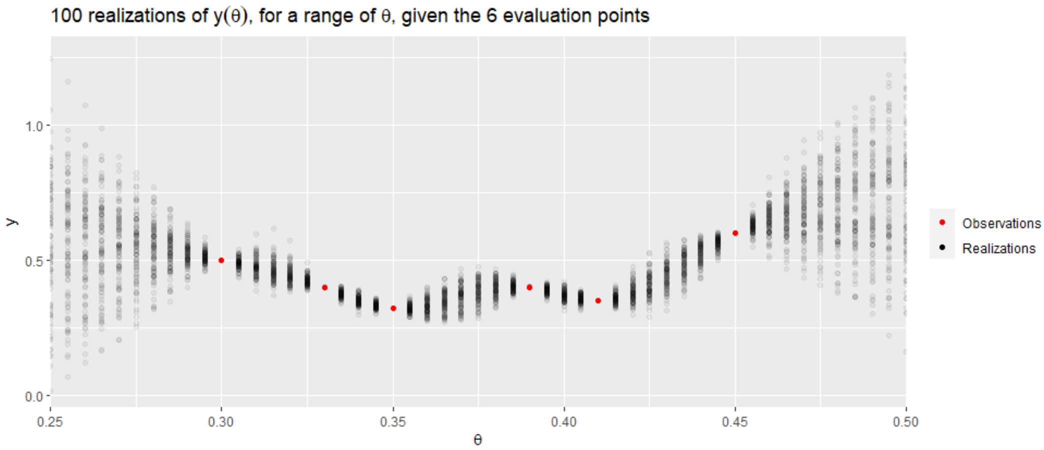

Figure 7: 100 realizations of $\boldsymbol{Y}_R|\boldsymbol{Y}_O^*$

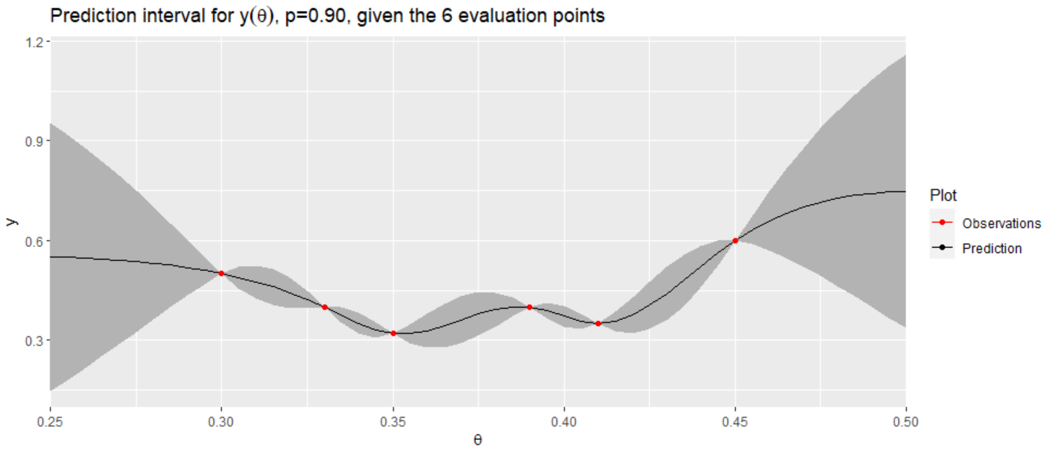

Figure 8: The 90% prediction interval for $\boldsymbol{Y}_R|\boldsymbol{Y}_O^*$

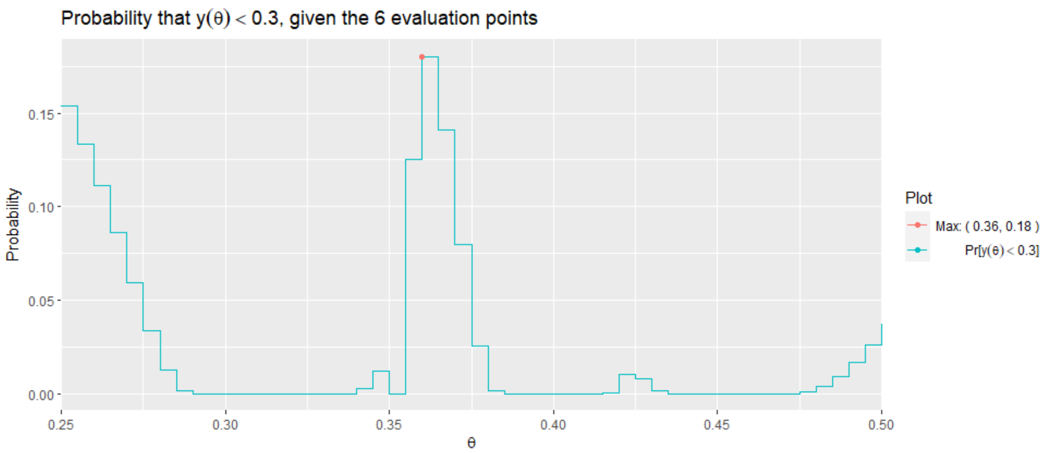

Figure 9: $Pr[Y(\theta_{R,j})|\boldsymbol{Y}_O^* < 0.30]$