Dynamic α-DS mixture pricing in a market with bid-ask spreads

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Abstract

This paper faces the problem of pricing a European derivative contract inside a discrete-time market with frictions in the form of bid-ask spreads. To this aim, we use a Markov and time-homogeneous multiplicative binomial process under Dempster-Shafer uncertainty for modeling the bid price of a non-dividend paying stock. Next, by taking α-mixtures of bid-ask prices, where $\alpha \in [0,1]$ acts like a pessimism index, we propose a dynamic pricing rule consisting in the recursive one-step α-mixture of upper and lower conditional Choquet expectations. We provide a dynamic pricing rule that has a closed-form for monotonic contract functions. Finally, we perform a calibration procedure on market data, complying with the tuning of α.

Keywords. dynamic pricing rule, conditional α-DS mixture, DS-multiplicative binomial process, bid-ask spread

1. Introduction

The classical no-arbitrage pricing theory essentially relies on the absence of frictions in the market [22, 50], implying that buying and selling prices of securities coincide. Assuming a frictionless market leads to the linearity of the no-arbitrage pricing rule, that reduces to a discounted conditional expectation with respect to a risk-neutral probability measure. The latter is an "artificial" probability measure implied by the no-arbitrage condition which distinguishes from the "real-world" probability measure encoding market agents' beliefs. In this context, linearity means that the price of a linear combination of contract payoffs (a so-called *portfolio*) is the linear combination of their individual prices.

Nevertheless, the quoted hypothesis is often not reflected in the reality, since markets show frictions, particularly in the form of bid-ask spreads [1, 2]. Therefore, to develop a pricing theory in frictional markets and model bid-ask prices we necessarily need to depart from linearity, and switch to more general uncertainty frameworks than probability theory. As discussed in the seminal paper by Ellsberg [26], we have *ambiguity* in uncertainty measurement whenever we do not have a completely known probability measure, but we have instead a class of compatible probability measures (also known as credal set [47]) giving rise to non-additive measures through its envelopes (see [28, 31]). In particular, in this work we focus on Dempster-Shafer theory [23, 52] since it represents a good compromise between expressiveness and tractability, compared to more general uncertainty theories [57].

In line with the works [3, 5, 11–13, 17–19, 49], we aim at modeling bid-ask prices as discounted conditional Choquet expectations [15, 33], working inside Dempster-Shafer theory. For this, we refer to [17] where we defined a time-homogeneous Markov multiplicative binomial process that we called *DS-multiplicative binomial process* (where "DS" stands for Dempster-Shafer), which is characterized by a family of transition belief functions with respect to two strictly positive parameters b_u, b_d, satisfying $b_u + b_d \leq 1$. Such a process incorporates ambiguity in the probabilistic multiplicative binomial process used in the Cox-Ross-Rubinstein (CRR) pricing model [22]. In detail, for given evolution parameters $u > d > 0$, in each time period from $n-1$ to n, the initial value S_{n-1} can either go "up" to $S_n = uS_{n-1}$ with belief b_u, or "down" to $S_n = dS_{n-1}$ with belief b_d: the classical probabilistic model is recovered whenever $b_u + b_d = 1$.

A distinguishing feature of our approach is that we look for a global belief function, that generates all the local transition belief functions via the product (or geometric) conditioning rule [55]. The choice of the conditioning rule is motivated in [20]. Our notion of DS-multiplicative binomial process differs from other proposals that aim to introduce "imprecision" in a Markov process (see, e.g., [42, 44, 48, 54, 56]). The quoted proposals usually refer to more general non-additive uncertainty measures and to different notions of conditioning, and many of them only pay attention to local transition models. Hence, in a DS-multiplicative binomial process all local transitions are consistent with the global belief function which, in turn, is determined just by the parameters b_u, b_d, favoring scalability.

The quoted DS-multiplicative binomial process is used to express the bid price evolution of a non-dividend paying stock under a "real-world" belief function ν, where the term "real-world" is borrowed from the financial jargon. In turn, the stock is assumed to form a frictional market together with a frictionless risk-free bond with deterministic price process, that we use as numeraire. In such a market, we proved (see [17]) the existence of an equivalent belief function (also referred to as *risk-neutral belief function*), for which the process is still a DS-multiplicative binomial process under the risk-neutral parameters $\widehat{b_u}, \widehat{b_d}$, satisfying $\widehat{b_u} + \widehat{b_d} \leq 1$. Moreover, such $\widehat{\nu}$ makes the discounted stock bid price process a *one-step Choquet martingale* and a *global Choquet supermartingale*. In particular, the one-step Choquet martingale property is important since it allows to define a recursive one-step pricing rule, while the failure of the global Choquet martingale property tells us that working t-step-wise is not equivalent.

In the literature there are other proposals for addressing bid-ask prices in a dynamic setting: distinguished models are the axiomatic approach of [41], the time consistent pricing procedure of [7], and the conic market model of [9]. On the other hand, [25] considered dynamic pricing within the Choquet theory, though not coping with bid-ask spreads. The main characteristic of the model in [17] rests in the conditional Choquet expectation operator induced by a DS-multiplicative binomial process, which is a completely monotone conditional operator. Another important feature of the quoted model is its simple parameterization, where the two risk-neutral parameters $\widehat{b_u}, \widehat{b_d}$ can be interpreted as one-step "up" and "down" risk-neutral transition beliefs. This allows to quantify the extent of ambiguity in terms of the excessive weight to unity $\epsilon = 1 - (\widehat{b_u} + \widehat{b_d})$: as a special case, we obtain the probabilistic CRR pricing model [22] in absence of ambiguity, i.e., for $\epsilon = 0$. In turn, this has a direct impact on *calibration* tasks, that consist of fitting the pricing model to market data by minimizing a goodness-of-fit measure in a way to produce risk-neutral parameter estimates [34].

Referring to the introduced market model, in this paper we consider the problem of pricing a European derivative contract (i.e., a contingent claim without early exercise feature) on the stock that will be characterized by bid-ask prices. As is well-known, the standard approach in finance to deal with bid-ask prices is to consider their $\frac{1}{2}$-mixture (see, e.g., [46]). Inspired by this last approach, we introduce a dynamic pricing rule which consists in the one-step α-mixture of lower and upper conditional Choquet expectations, where $\alpha \in [0, 1]$ is a constant *pessimism index*. In particular, such conditional functional can be given an α-*maxmin* expression in the spirit of the *Hurwicz criterion* [35].

Here, we show that the resulting conditional operator is a conditional Choquet expectation with respect to a conditional version of an α-*DS mixture*, the latter being introduced in [49]. For a generic contract function, we provide a recursive one-step procedure for computing the α-*DS price process* of the derivative. Such a procedure reduces to a closed-form binomial pricing formula in case of a monotonic contract function. We also provide a normative justification in terms of a one-step generalized no-arbitrage condition in the spirit of [49].

Next, for a fixed pessimism index $\alpha \in [0, 1]$, we consider the issue of calibrating the introduced dynamic α-DS mixture pricing rule on market data. We show that such task can be carried out relying on a set of option prices for the chosen stock for a fixed maturity and a risk-free interest rate, through a least square calibration. Since the resulting optimization problem presents computational difficulties due to option payoffs, here we rely on the *particle swarm optimization (PSO)* technique [43]. Finally, since the choice of the pessimism index $\alpha \in [0, 1]$ is a delicate part of the model, we address a tuning procedure to choose an optimal value of α, which acts as a hyper-parameter influencing the calibration of the pricing model. As a by-product, the tuning of α allows to extract an indicator of the pessimism hidden in market option prices. In turn, such an indicator is shown to depend on the chosen stock, the valuation date, and the maturity.

We notice that the particular cases with $\alpha = 0$ and $\alpha = 1$ reduce to the bid and ask dynamic pricing rules analyzed in [17, 19]. There we motivated the recursive one-step definition of the dynamic pricing rule as a way to enforce *dynamic consistency* in the sense of [14], assuring that, at every valuation time, prices preserve inequalities between contract payoffs. Indeed, adopting a t-step definition we generally have a failure of dynamic consistency. Such a failure is a well-known problem in preference modeling and risk measurement (see, e.g., [14]): dynamic consistency is known to hold when the dynamic pricing rule can be expressed in terms of a closed and convex set $\mathcal{P}$ of probability measures, satisfying a suitable version of the tower property, called *rectangularity* in [27] or *consistency* in [51]. In this setting, conditioning is intended element-wise on $\mathcal{P}$ by relying on the classical Bayesian conditioning rule for probabilities. The issue of dynamic consistency has been analyzed also in an α-maxmin setting in [6]. Our recursive one-step definition of an α-DS mixture price process is again motivated by a constructive way to reach dynamic consistency.

The paper is structured as follows. Section 2 recalls the necessary material on Dempster-Shafer theory, conditioning and DS-multiplicative binomial processes. Section 3 introduces the dynamic α-DS mixture pricing rule for European derivatives and expresses it through a closed form binomial formula in case of a monotonic contract function, under a suitable change of parameters. Furthermore, here we provide a normative justification in terms of a generalized one-step no-arbitrage condition.

Next, Section 4 addresses the problem of calibrating the proposed dynamic pricing rule and of tuning the pessimism index α on real market data. Finally, Section 5 draws our conclusions and future perspectives. Proofs have been omitted due to the limited number of pages.

2. Preliminaries

2.1. Dempster-Shafer theory. Let $\Omega = \{\omega_1, \dots, \omega_d\}$ be a finite non-empty set of states of the world and $\mathcal{F} = \mathcal{P}(\Omega)$, where $\mathcal{P}(\Omega)$ stands for the power set of Ω. We denote by $\mathbf{R}^\Omega$ the set of all random variables on Ω.

The Dempster-Shafer theory [23, 52] encodes uncertainty through a non-additive measure, called *belief function*, that is a mapping $\nu : \mathcal{F} \to [0,1]$ satisfying:

(i) $\nu(\emptyset) = 0$ and $\nu(\Omega) = 1$;

(ii) for every $k \geq 2$ and every $A_1, \dots, A_k \in \mathcal{F}$,

$$\nu\left(\bigcup_{i=1}^k A_i\right) \geq \sum_{\emptyset \neq I \subseteq \{1,\dots,k\}} (-1)^{|I|+1} \nu\left(\bigcap_{i \in I} A_i\right).$$

Condition *(ii)* is called *complete monotonicity* and if *(ii)* holds as an equality, ν reduces to a probability measure (customarily denoted by P).

Every belief function is associated with a dual set function called *plausibility function* $\overline{\nu} : \mathcal{F} \to [0,1]$ defined, for all $A \in \mathcal{F}$, as $\overline{\nu}(A) = 1 - \nu(A^c)$.

Every belief function induces a non-empty, closed and convex set of probability measures on $\mathcal{F}$ called *core* (see, e.g., [33]) defined as

$$\mathbf{core}(\nu) = \{P \ : \ P \text{ is a probability measure}, \ P \geq \nu\},$$

for which it holds that, for every $A \in \mathcal{F}$,

$$\nu(A) = \min_{P \in \mathbf{core}(\nu)} P(A) \quad \text{and} \quad \overline{\nu}(A) = \max_{P \in \mathbf{core}(\nu)} P(A),$$

showing that ν and $\overline{\nu}$ are particular *(coherent) lower and upper probabilities* [57].

The notion of conditioning in Dempster-Shafer theory is still an open issue and several proposals have been given in the literature (see, e.g., [23, 53, 55]). In this work we refer to the *product (or geometric) conditioning rule* proposed by [55] (see also [21]). For every $E, H \in \mathcal{F}$ with $\nu(H) > 0$, we define

$$\nu(E|H) = \frac{\nu(E \cap H)}{\nu(H)}. \tag{1}$$

We have that, for every $H \in \mathcal{F}$ with $\nu(H) > 0$, $\nu(\cdot|H)$ is still a belief function, so, it induces a core denoted as $\mathbf{core}(\nu(\cdot|H))$.

Given $\nu(\cdot|H)$ and $X \in \mathbf{R}^\Omega$, we can introduce the *conditional Choquet expectation* of X with respect to $\nu(\cdot|H)$, defined through the Choquet integral [15]

$$\oint X \mathrm{d}\nu(\cdot|H) = \sum_{i=1}^d \left(X(\omega_{\sigma(i)}) - X(\omega_{\sigma(i+1)})\right) \nu(E_i^\sigma|H), \tag{2}$$

where σ is a permutation of Ω such that $X(\omega_{\sigma(1)}) \geq \cdots \geq X(\omega_{\sigma(d)})$, $E_i^\sigma = \{\omega_{\sigma(1)}, \dots, \omega_{\sigma(i)}\}$ for $i = 1, \dots, d$ and $X(\omega_{\sigma(d+1)}) = 0$. Moreover, the conditional Choquet expectation can be interpreted as a lower expectation locally on H, by referring to $\mathbf{core}(\nu(\cdot|H))$, since

$$\oint X \mathrm{d}\nu(\cdot|H) = \min_{P \in \mathbf{core}(\nu(\cdot|H))} \int X \mathrm{d}P. \tag{3}$$

Furthermore, if $\overline{\nu}(\cdot|H)$ is the dual plausibility function of $\nu(\cdot|H)$, it holds that

$$\begin{aligned} \oint X \mathrm{d}\overline{\nu}(\cdot|H) &= -\oint (-X) \mathrm{d}\nu(\cdot|H) \\ &= \max_{P \in \mathbf{core}(\nu(\cdot|H))} \int X \mathrm{d}P, \end{aligned} \tag{4}$$

that can be interpreted as an upper expectation locally on H.

Remark 2.1. The issue of updating ambiguous beliefs has been investigated, e.g., in [32], while [4] deals with updating in the context of pricing rules. In the case of Dempster-Shafer theory, as discussed in [17, 20], two other popular choices are the *Dempster's conditioning rule* [23] and the *Bayesian conditioning rule* [29]. In our pricing context, the choice of the product conditioning rule is motivated by the smaller dilation that it produces when computing the conditional Choquet integral, with respect to the Bayesian conditioning rule. On the other hand, no dominance relation holds between the product and the Dempster's conditioning rules [20]. ⋄

2.2. Bid-ask binomial market model under Dempster-Shafer uncertainty. We refer to the DS-multiplicative binomial process introduced and characterized in [17], in order to model the bid price evolution of a stock over a discrete set of times $\{0, \dots, T\}$, where $T \in \mathbf{N}$ is a finite time horizon.

Consider a discrete-time market model formed by a frictionless risk-free bond and a non-dividend paying stock with frictions, in the form of bid-ask spread. The price of the bond is expressed by the deterministic process $\{B_n\}_{n=0}^T$, where $B_0 = 1$ and, for $n = 1, \dots, T$,

$$B_n = (1+r)B_{n-1}, \tag{5}$$

with $1 + r > 0$, in which r is the risk-free interest rate over each period. On the other hand, the bid price of the stock is expressed by the process $\{S_n\}_{n=0}^T$ such that, for $n = 1, \dots, T$,

$$S_n = \begin{cases} uS_{n-1} & \text{if "up"}, \\ dS_{n-1} & \text{if "down"}, \end{cases} \tag{6}$$

where $S_0 = s_0 > 0$ and $u > d > 0$ are the "up" and "down" coefficients. Notice that u and d are constant and equal in every time period from $n-1$ to n.

The processes we consider are defined on a filtered measurable space $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{n=0}^T)$, where $\Omega = \{1, \dots, 2^T\}$ and $\mathcal{F}_n$ is the algebra generated by random variables $\{S_0, \dots, S_n\}$, for $n = 0, \dots, T$, with $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_T = \mathcal{P}(\Omega)$.

The trajectories of $\{S_n\}_{n=0}^T$ can be represented graphically on a recombining binomial tree and every state $\omega \in \Omega$ is identified with the path corresponding to the T-digit binary expansion of number $\omega - 1$. For $n = 1, \dots, T$, denoting $\mathcal{A}_n = \{a_k = u^k d^{n-k} \,:\, k = 0, \dots, n\}$, we write $As = \{a_k s \,:\, a_k \in A\}$, for every $s > 0$ and $A \in \mathcal{P}(\mathcal{A}_n)$. Therefore, each random variable S_n ranges in $\mathcal{S}_n = \mathcal{A}_n s_0$.

We assume that uncertainty is captured by a belief function $\nu : \mathcal{F} \to [0,1]$ singled out by a reference family of t-step transition belief functions $\{\beta_t\}_{t=1}^T$ determined by two parameters, b_u and b_d, such that $b_u, b_d \in (0,1)$ and $b_u + b_d \leq 1$, that can be interpreted as one-step "up" and "down" conditional beliefs. We refer to the family of t-step transition belief functions given, for $t = 1, \dots, T$ and for all $A \in \mathcal{P}(\mathcal{A}_t)$, by

$$\begin{aligned}\beta_t(A) = &\sum_{a_k \in A} \binom{t}{k} b_u^k b_d^{t-k} \\ &+ \sum_{\substack{[a_k, a_{k+j}] \subseteq A \\ j \geq 1}} \binom{t-j}{k} b_u^k b_d^{t-j-k} (1 - (b_u + b_d)),\end{aligned} \tag{7}$$

where, for $i \leq j$, $[a_i, a_j] = \{a_k \in \mathcal{A}_t \ : a_i \leq a_k \leq a_j\}$.

Our choice of (7) is due to its nice interpretation and parameterization. Indeed, the belief function β_t in (7) generalizes the binomial distribution by taking into account the contribution of intervals contained in A with a binomial-like weighting, deflated by the excessive weight to unity $1 - (b_u + b_d)$. In detail, $\beta_t(A)$ is the sum of binomial-like weights of all partial trajectories with decreasing length that support the evidence of having a final state of the process belonging to As_n. Furthermore, if $b_u + b_d = 1$, β_t reduces to the classical binomial distribution with parameters b_u and t. As will be shown in Section 4, the parameterization in (7) will play a crucial role in the model calibration.

As proved in [17], there exists a strictly positive belief function ν on $\mathcal{F}$ such that the process $\{S_n\}_{n=0}^T$ is Markov and time-homogeneous with transitions given by (7), namely a *DS-multiplicative binomial process*. Explicitly, this means that for every $0 \leq n \leq T-1$ and $1 \leq t \leq T-n$, $A \in \mathcal{P}(\mathcal{A}_t)$, and $s_0 \in \mathcal{S}_0, \dots, s_n \in \mathcal{S}_n$ it holds that

$$\begin{aligned}&\nu(S_{n+t} \in As_n | S_0 = s_0, \dots, S_n = s_n) \\ &\quad = \nu(S_{n+t} \in As_n | S_n = s_n) = \beta_t(A).\end{aligned} \tag{8}$$

Note that, there can be infinitely many belief functions that make the process a DS-multiplicative binomial process, and the entire family $\{\beta_t\}_{t=1}^T$ must be fixed to constrain a global ν (see [16] for a related discussion).

From now on we assume the "real-world" filtered belief space $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{n=0}^T, \nu)$ to be fixed. This allows us to define (see [17]), for every $X \in \mathbf{R}^\Omega$, the random variables $\mathbf{C}[X|\mathcal{F}_n]$ and $\mathbf{C}[X|S_0, \dots, S_n]$ by setting, for all $\omega \in \{S_0 = s_0, \dots, S_n = s_n\}$,

$$\begin{aligned}\mathbf{C}[X|\mathcal{F}_n](\omega) \ &:= \ \mathbf{C}[X|S_0, \dots, S_n](\omega) \\ &:= \ \oint X \mathrm{d}\nu(\cdot | S_0 = s_0, \dots, S_n = s_n),\end{aligned} \tag{9}$$

while the random variable $\mathbf{C}[X|S_n]$ is defined analogously by referring to $\nu(\cdot|S_n = s_n)$, for all $\omega \in \{S_n = s_n\}$. In what follows we refer to the operator $\mathbf{C}[\cdot|\mathcal{F}_n]$ as *conditional Choquet expectation* which, for every $0 \leq n \leq T-1$, $1 \leq t \leq T-n$, and every real-valued function of one real variable $\varphi(x)$ defined on the range of S_{n+t}, by (8) satisfies

$$\mathbf{C}[\varphi(S_{n+t})|\mathcal{F}_n] = \mathbf{C}[\varphi(S_{n+t})|S_n]. \tag{10}$$

As usual, taking the process $\{B_n\}_{n=0}^T$ as *numeraire*, we can define the discounted process $\{S_n^*\}_{n=0}^T$ setting, for $n = 0, \dots, T$,

$$S_n^* = \frac{S_n}{B_n} = \frac{S_n}{(1+r)^n}. \tag{11}$$

Theorem 2 in [17] shows that the condition $u > 1+r > d > 0$ is necessary and sufficient to the existence of an *equivalent one-step Choquet martingale belief function*, where we define a belief function $\hat{\nu}$ to be *equivalent* to ν if $\nu(A) = 0 \iff \hat{\nu}(A) = 0$, for every $A \in \mathcal{F}$. Such a belief function $\hat{\nu} : \mathcal{F} \to [0,1]$ is strictly positive and makes the process $\{S_n^*\}_{n=0}^T$ on the filtered belief space $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{n=0}^T, \hat{\nu})$ a DS-multiplicative binomial process and a *one-step Choquet martingale*, while the process is globally only a *Choquet super-martingale*. Denoting by $\widehat{\mathbf{C}}[\cdot|\mathcal{F}_n]$ the conditional Choquet expectation with respect to $\hat{\nu}$, the last two properties mean that, for every $0 \leq n \leq T-1$ and $1 \leq t \leq T-n$, it holds that

$$\widehat{\mathbf{C}}[S_{n+1}^*|\mathcal{F}_n] \ = \ S_n^*, \tag{12}$$

$$\widehat{\mathbf{C}}[S_{n+t}^*|\mathcal{F}_n] \ \leq \ S_n^*. \tag{13}$$

We stress that if $\hat{\nu}$ is not additive, then equation (12) does not imply that inequality (13) holds as an equality.

The belief function $\hat{\nu}$, completely specified by the parameters,

$$\widehat{b_u} = \frac{1+r-d}{u-d} \quad \text{and} \quad \widehat{b_d} \in (0, 1 - \widehat{b_u}], \tag{14}$$

is referred to as *risk-neutral belief function* in the following. Note that by varying $\widehat{b_d} \in (0, 1 - \widehat{b_u}]$ we actually have infinitely many risk-neutral belief functions. For fixed $\widehat{b_u}$ and $\widehat{b_d}$, $\hat{\nu}$ determines through the product conditioning rule a family of risk-neutral t-step transitions belief functions $\{\widehat{\beta}_t\}_{t=1}^T$ of the form (7) with parameters $\widehat{b_u}$ and $\widehat{b_d}$.

3. Dynamic α-DS mixture pricing

3.1. Conditional α-DS mixture theory. For a fixed conditioning event $H \in \mathcal{F}$ with $\nu(H) > 0$, the dual conditional measures $\nu(\cdot|H)$ and $\overline{\nu}(\cdot|H)$ introduced in Subsection 2.1, together with the related Choquet conditional expectations, implement a systematically pessimistic and optimistic attitudes towards uncertainty. Thus, a more realistic model is the one obtained by specifying a *pessimism index* $\alpha \in [0,1]$ to mix such extreme positions.

For a fixed $\alpha \in [0,1]$, in [49] we introduced unconditional *α-DS mixtures* (where "DS" stands again for Dempster-Shafer) which are capacities that can be expressed as the α-mixture of a belief function and its dual plausibility function. In turn, such capacities reveal to be particular *α-JP capacities* (where "JP" stands for Jaffray-Philippe) [39]. In this paper the following conditional version of α-DS mixtures will play a fundamental role.

Definition 3.1. Let $\alpha \in [0,1]$ and $H \in \mathcal{F}$. A mapping $\nu^\alpha : \mathcal{F} \times \{H\} \to [0,1]$ is called a **conditional α-DS mixture** if there exists a belief function $\nu : \mathcal{F} \to [0,1]$ with $\nu(H) > 0$ such that, for all $E \in \mathcal{F}$,

$$\begin{aligned}\nu^\alpha(E|H) &= \alpha\nu(E|H) + (1-\alpha)\overline{\nu}(E|H)\\ &= \alpha\nu(E|H) + (1-\alpha)(1-\nu(E^c|H)),\end{aligned}$$

where $\nu(\cdot|H)$ is computed as in (1) and $\overline{\nu}(\cdot|H)$ is its dual. The belief function ν is said to **represent** the conditional α-DS mixture $\nu^\alpha(\cdot|H)$.

Notice that, if $\nu(\cdot|H)$ is a probability measure, then $\nu^\alpha(\cdot|H)$ is a probability measure coinciding with $\nu(\cdot|H)$, independently of $\alpha \in [0,1]$. In particular, this holds for all $H \in \mathcal{F}$ with $\nu(H) > 0$, when the unconditional ν is a probability measure. Moreover, given $\alpha \in [0,1]$, from Proposition 4 in [49] we get that $\nu^\alpha(\cdot|H)$ satisfies the following properties:

(i) $\nu^\alpha(\emptyset|H) = 0$ and $\nu^\alpha(\Omega|H) = 1$;

(ii) $\nu^\alpha(A|H) \le \nu^\alpha(B|H)$, when $A \subseteq B$ and $A, B \in \mathcal{F}$;

(iii) $\nu^\alpha(A|H) = 1 - \nu^\alpha(A^c|H)$, for all $A \in \mathcal{F}$, if and only if $\nu^\alpha(\cdot|H)$ is a probability measure or $\alpha = \frac{1}{2}$;

(iv) $\nu^\alpha(A\cup B|H) \le \nu^\alpha(A|H) + \nu^\alpha(B|H)$, for all $A, B \in \mathcal{F}$ with $A \cap B = \emptyset$, if $\nu^\alpha(\cdot|H)$ is a probability measure or $\alpha \in \left[0, \frac{1}{2}\right]$.

As a direct application of Proposition 7 in [49] we get that the conditional Choquet expectation with respect to $\nu^\alpha(\cdot|H)$ can be given an Hurwicz-like expression [35, 37, 38] which shows the role of α as a pessimism index (see also [24]). Indeed, for all $X \in \mathbf{R}^\Omega$, we have

$$\begin{aligned}\oint X \mathrm{d}\nu^\alpha(\cdot|H) &= \alpha \oint X \mathrm{d}\nu(\cdot|H) + (1-\alpha)\oint X \mathrm{d}\overline{\nu}(\cdot|H)\\ &= \alpha \min_{P \in \mathbf{core}(\nu(\cdot|H))} \int X \mathrm{d}P\\ &\quad + (1-\alpha) \max_{P \in \mathbf{core}(\nu(\cdot|H))} \int X \mathrm{d}P.\end{aligned} \tag{15}$$

3.2. A dynamic α-DS mixture pricing rule. In this section we consider a DS-multiplicative binomial process $\{S_n\}_{n=0}^T$ defined on a risk-neutral filtered belief space $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{n=0}^T, \widehat{\nu})$, modeling the evolution of a non-dividend paying stock bid price. In turn, $\widehat{\nu}$ is characterized by the family of risk-neutral transition belief functions $\{\widehat{\beta}_t\}_{t=1}^T$ of the form (7), determined by the risk-neutral parameters $\widehat{b_u}, \widehat{b_d}$ satisfying (14). We assume the risk-free interest r to be fixed.

As usual, a *European derivative* on the stock is a financial contract whose payoff at the maturity T is a random variable $\varphi(S_T)$, where $\varphi : \mathcal{S}_T \to \mathbf{R}$ is the *contract function* characterizing the derivative. Notice that the payoff of a European derivative only depends on the value at maturity S_T of the underlying process, and the contract does not possess the *early exercise feature*, that is the payoff given by φ can be obtained only at time T.

Typical examples of European derivatives are *European call* and *put options* with strike price $K > 0$, whose contract functions are

$$\varphi_{call}(x) = \max\{x - K, 0\}, \tag{16}$$
$$\varphi_{put}(x) = \max\{K - x, 0\}. \tag{17}$$

Such contracts give to the holder the right, but not the obligation, to buy or sell, respectively, the underlying stock for the strike price K at the maturity T. We also notice that other European derivatives can be built by constructing payoffs which are linear combinations of European option payoffs: this is the case of *straddle*, *strip*, *strap*, and *strangle* contracts (see, e.g., [34]).

Remark 3.1. In the context of option contracts, the European adjective plays a crucial role. Indeed, *American call* and *put options* distinguish for having the early exercise feature: the payoff given by (16)-(17) can be obtained whenever the option is exercised prior the maturity [34].

Inspired by the usual practice in finance of taking $\frac{1}{2}$-mixtures of bid-ask prices (see, e.g., [46]), we introduce a recursive one-step pricing rule that takes α-mixtures of lower and upper conditional Choquet expectations, where $\alpha \in [0,1]$ is a fixed pessimism index. Notice that $1-\alpha$ can be interpreted as an optimism index, instead.

Definition 3.2. Let $\alpha \in [0,1]$ and $\varphi : \mathcal{S}_T \to \mathbf{R}$ be a contract function. The **dynamic α-DS mixture price** of a European derivative with payoff $Y_T^\alpha = \varphi(S_T)$ is the process $\{Y_n^\alpha\}_{n=0}^T$, where

$$Y_n^\alpha = \frac{1}{1+r}\left[\alpha\widehat{\mathbf{C}}[Y_{n+1}^\alpha|\mathcal{F}_n] - (1-\alpha)\widehat{\mathbf{C}}[-Y_{n+1}^\alpha|\mathcal{F}_n]\right], \tag{18}$$

for $n = 0, \ldots, T-1$.

Remark 3.2. Definition 3.2 assumes a deterministic and time-independent pessimism index. Such definition can be easily generalized by considering an adapted process $\{\alpha_n\}_{n=0}^{T-1}$, where each α_n ranges in $[0,1]$ and is $\mathcal{F}_n$-measurable. In this case, equation (18) generalizes to

$$Y_n^\alpha = \frac{1}{1+r}\left[\alpha_n\widehat{\mathbf{C}}[Y_{n+1}^\alpha|\mathcal{F}_n] - (1-\alpha_n)\widehat{\mathbf{C}}[-Y_{n+1}^\alpha|\mathcal{F}_n]\right].$$

A distinguished sub-case is obtained by taking a deterministic process $\{\alpha_n\}_{n=0}^{T-1}$, i.e., assuming that the pessimism index is a deterministic function of time. ⋄

By Definition 3.2 and equation (10) we get that

$$\widehat{\mathbf{C}}[Y_{n+1}^\alpha|\mathcal{F}_n] = \widehat{\mathbf{C}}[Y_{n+1}^\alpha|S_n], \tag{19}$$
$$\widehat{\mathbf{C}}[-Y_{n+1}^\alpha|\mathcal{F}_n] = \widehat{\mathbf{C}}[-Y_{n+1}^\alpha|S_n], \tag{20}$$

which imply that $Y_n^\alpha = \varphi_n^\alpha(S_n)$, where φ_n^α is a function on the range $\mathcal{S}_n$ of S_n, for $n = 0, \ldots, T-1$, and $\varphi_T^\alpha = \varphi$, that is all random variables Y_n^α's turn out to be functions of the corresponding random variables S_n's.

In turn, equations (19)-(20) and the linearity of the Choquet integral with respect to the integrating capacity (see, e.g., [33]) imply that, for all $\omega \in \{S_n = s_n\}$, it holds that

$$Y_n^\alpha(\omega) = \frac{1}{1+r}\oint Y_{n+1}^\alpha \mathrm{d}\widehat{\nu}^\alpha(\cdot|S_n = s_n), \tag{21}$$

where $\widehat{\nu}^\alpha(\cdot|S_n = s_n)$ is the conditional α-DS mixture represented by the risk-neutral belief function $\widehat{\nu}$, according to Definition 3.1.

The following proposition shows that (21) can be expressed in terms of the transition α-DS mixtures $\{\widehat{\beta}_t^\alpha\}_{t=1}^T$, induced by the risk-neutral transition belief functions $\{\widehat{\beta}_t\}_{t=1}^T$, where we define $\widehat{\beta}_t^\alpha = \alpha\widehat{\beta}_t + (1-\alpha)\overline{\widehat{\beta}}_t$ set-wise on the elements of $\mathcal{P}(\mathcal{A}_t)$.

Proposition 3.1. *Let $\alpha \in [0,1]$ and $\varphi : \mathcal{S}_T \to \mathbf{R}$ be a contract function, $Y_T^\alpha = \varphi(S_T)$ be the payoff of a European derivative, and let $\widehat{\nu}$ be a risk-neutral belief function with parameters $\widehat{b_u}, \widehat{b_d}$ satisfying (14). Let $\{Y_n^\alpha\}_{n=0}^T$ be an α-DS mixture price process according to (18). Then, for every $n = 0, \ldots, T-1$, for every $s_n \in \mathcal{S}_n$, and for every $\omega \in \{S_n = s_n\}$ it holds that*

$$\begin{aligned}Y_n^\alpha(\omega) &= \frac{1}{1+r}\oint_{\mathcal{A}_1} \varphi_{n+1}^\alpha(as_n)\mathrm{d}\widehat{\beta}_1^\alpha(a)\\ &= \tfrac{1}{1+r}\Big[\widehat{b_u}\varphi_{n+1}^\alpha(us_n) + \widehat{b_d}\varphi_{n+1}^\alpha(ds_n)\\ &\quad +(1-(\widehat{b_u}+\widehat{b_d}))\Big(\alpha \min_{a\in\mathcal{A}_1}\varphi_{n+1}^\alpha(as_n)\\ &\quad +(1-\alpha)\max_{a\in\mathcal{A}_1}\varphi_{n+1}^\alpha(as_n)\Big)\Big],\end{aligned} \tag{22}$$

where $\varphi_{n+1}^\alpha : \mathcal{S}_{n+1} \to \mathbf{R}$ satisfies $Y_{n+1}^\alpha = \varphi_{n+1}^\alpha(S_{n+1})$.

It is important to notice that, for a general contract function φ, Proposition 3.1 only provides a recursive algorithm to compute the α-DS mixture price process $\{Y_n^\alpha\}_{n=0}^T$ which, however, can be represented on a recombining binomial tree, as the following example shows.

Example 3.1. Take $T = 3$ periods and let $\{S_n\}_{n=0}^T$ be a DS-multiplicative binomial process modeling the bid price of a non-dividend paying stock with $S_0 = \$100$, $u = 2$, $d = \frac{1}{u}$. Figure 1 shows the recombining binomial tree representing the stock bid price.

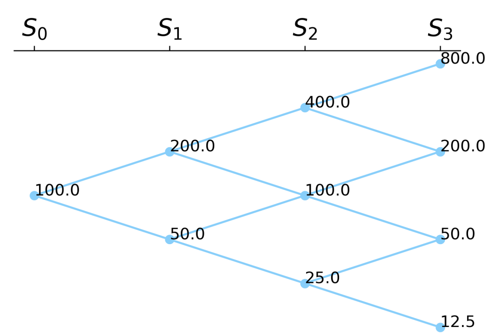

Figure 1. *Recombining binomial tree representing the stock bid price.*

Let $r = 5\%$ be the risk-free interest rate over every period. By (14) this implies that the risk-neutral parameters must satisfy $\widehat{b_u} = 0.36$ and $\widehat{b_d} \in (0, 1-\widehat{b_u}]$. In particular, we choose $\widehat{b_d} = 1 - \widehat{b_u} - 0.05$, thus the amount of ambiguity in the model can be quantified by the excessive weight to unity $\epsilon = 1 - (\widehat{b_u} + \widehat{b_d}) = 0.05$.

We consider a *strangle* contract [34], determined by the contract function

$$\varphi(x) = \max\{x - K_1, 0\} + \max\{K_2 - x, 0\},$$

where $K_1 = \$50$ and $K_2 = \$100$, whose graph is shown in Figure 2.

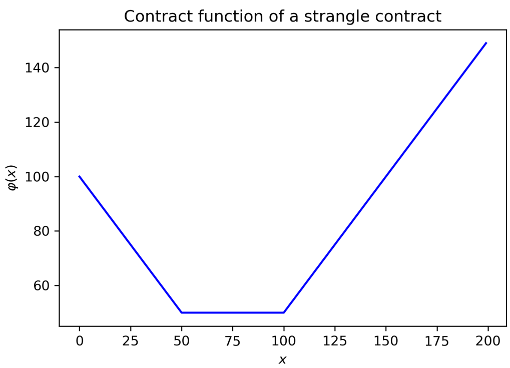

Figure 2. *Contract function of a strangle contract with $K_1 = \$50$ and $K_2 = \$100$.*

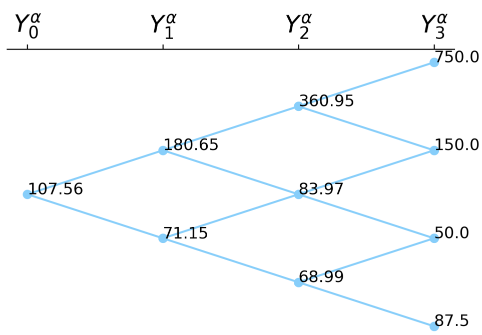

Figure 3. *Recombining binomial tree representing the α-DS mixture price process of the strangle contract.*

We choose $\alpha = 0.7$ that singles out a quite pessimistic agent. Applying Proposition 3.1 we get the α-DS mixture price process of the strangle contract depicted in Figure 3, from which we get that the current α-DS mixture price is $Y_0^\alpha = \$107.56$. $\diamond$

The following proposition shows that the α-DS mixture price process has a binomial formula expression when the contract function φ is monotonic, the latter seen as a real-valued function of one real variable.

Proposition 3.2. *Let $\alpha \in [0,1]$ and $\varphi : \mathcal{S}_T \to \mathbf{R}$ be a monotonic contract function, $Y_T^\alpha = \varphi(S_T)$ be the payoff of a European derivative, and let $\widehat{\nu}$ be a risk-neutral belief function with parameters $\widehat{b_u}, \widehat{b_d}$ satisfying (14). Let $\{Y_n^\alpha\}_{n=0}^T$ be an α-DS mixture price process according to (18). Then, for every $n = 0, \dots, T-1$, for every $s_n \in \mathcal{S}_n$, and for every $\omega \in \{S_n = s_n\}$ it holds that*

$$Y_n^\alpha(\omega) = \frac{1}{(1+r)^{T-n}} \sum_{k=0}^{T-n} \binom{T-n}{k} \varphi_k {\gamma^\alpha}^k (1-\gamma^\alpha)^{T-n-k}, \tag{23}$$

where we set $\varphi_k = \varphi(u^k d^{T-n-k} s_n)$ and

$$\gamma^\alpha = \begin{cases} \alpha\widehat{b_u} + (1-\alpha)(1-\widehat{b_d}) & \text{if } \varphi \text{ is non-decreasing,} \\ \alpha(1-\widehat{b_d}) + (1-\alpha)\widehat{b_u} & \text{if } \varphi \text{ is non-increasing.} \end{cases} \tag{24}$$

In particular, Proposition 3.2 applies to European call and put options, whose contract functions (16)-(17) are monotonic.

3.3. Normative justification of the α-DS mixture pricing rule. Given a fixed $\alpha \in [0,1]$, the choice of parameters in (14) can be justified through a one-step dynamic generalized notion of no-arbitrage that refers to the one for the single period case given in [49] (see also [18] for the case $\alpha = 1$). The quoted notion is based on the *partially resolving uncertainty* principle proposed by [36] and the *Hurwicz criterion* [35] (namely *α-PRU*).

At this aim, we consider a one-step no-arbitrage condition where, given the history $\{S_0 = s_0, \dots, S_n = s_n\}$, at time $n+1$ we define the two events $U(s_n) = \{S_{n+1} = us_n\}$ and $D(s_n) = \{S_{n+1} = ds_n\}$ that correspond to the two price movements “up” and “down”, respectively. Therefore, every random quantity X depending on S_{n+1} can be seen as a function on $\mathcal{W}(s_n) = \{U(s_n), D(s_n)\}$ from time n. The one-period market can be augmented by adding two artificial Arrow-Debreu securities with payoffs and prices, respectively,

$$A_{n+1}^u = \mathbf{1}_{U(s_n)}, \quad \text{and} \quad A_{n+1}^d = \mathbf{1}_{D(s_n)}, \tag{25}$$

$$A_n^u = \frac{\widehat{\beta}_1^\alpha(\{u\})}{1+r}, \quad \text{and} \quad A_n^d = \frac{\widehat{\beta}_1^\alpha(\{d\})}{1+r}, \tag{26}$$

where $\widehat{\beta}_1^\alpha(\{u\}) = \alpha\widehat{b_u} + (1-\alpha)(1-\widehat{b_d})$ and $\widehat{\beta}_1^\alpha(\{d\}) = \alpha\widehat{b_d} + (1-\alpha)(1-\widehat{b_u})$.

Given the history $\{S_0 = s_0, \dots, S_n = s_n\}$, by partially resolving uncertainty we mean that at time $n+1$ the market agent may not be able to determine which one between the two mutually exclusive events “up” and “down” for the stock has occurred. Thus, he/she needs to consider the set of all the possible pieces of information he/she may acquire once uncertainty is resolved at time $n+1$ that reduce to “up”, “down”, and “up or down”. In detail, we consider the set $\mathcal{U}(s_n) = \{U(s_n), D(s_n), U(s_n) \cup D(s_n)\}$ and, for any function X defined on $\mathcal{W}(s_n)$, we adopt the Hurwicz criterion of choice, i.e., we consider $[\![X]\!]^\alpha$ in place of X, where, for every $E \in \mathcal{U}(s_n)$,

$$[\![X]\!]^\alpha(E) = \alpha \min_{\substack{F \subseteq E \\ F \in \mathcal{W}(s_n)}} X(F) + (1-\alpha) \max_{\substack{F \subseteq E \\ F \in \mathcal{W}(s_n)}} X(F). \tag{27}$$

In the augmented one-period market over $[n, n+1]$, a portfolio is a vector $\boldsymbol{\delta}_n = (\delta_n^0, \delta_n^1, \delta_n^2, \delta_n^3)$, where the δ_n^i’s are $\mathcal{F}_n$-measurable random variables expressing, respectively, the number of units of bond, stock and Arrow-Debreu’s securities to buy (if positive) or short-sell (if negative) at time n up to time $n+1$.

Following [18], we define a generalized one-step arbitrage opportunity as a portfolio $\boldsymbol{\delta}_n = (\delta_n^0, \delta_n^1, \delta_n^2, \delta_n^3)$ that satisfies one of the following two conditions:

(a) $\tilde{\pi}_n^{\delta} < 0$ and $\tilde{\pi}_{n+1}^{\delta} \geq 0$ with $\tilde{\pi}_{n+1}^{\delta} = 0$ over $\mathcal{W}(s_n)$;

(b) $\tilde{\pi}_n^{\delta} \leq 0$ and $\tilde{\pi}_{n+1}^{\delta} \geq 0$ with $\tilde{\pi}_{n+1}^{\delta} \neq 0$ over $\mathcal{W}(s_n)$;

where $\tilde{\pi}_n^{\delta} = \delta_n^0 [\![B_n]\!]^\alpha + \delta_n^1 [\![S_n]\!]^\alpha + \delta_n^2 [\![A_n^u]\!]^\alpha + \delta_n^3 [\![A_n^d]\!]^\alpha$ and $\tilde{\pi}_{n+1}^{\delta} = \delta_n^0 [\![B_{n+1}]\!]^\alpha + \delta_n^1 [\![S_{n+1}]\!]^\alpha + \delta_n^2 [\![A_{n+1}^u]\!]^\alpha + \delta_n^3 [\![A_{n+1}^d]\!]^\alpha$ are the price and the payoff of the portfolio $\boldsymbol{\delta}_n$, under α-PRU. Notice that $\tilde{\pi}_n^{\delta}$ turns out to be a constant.

In *(a)* we have a portfolio for which we are paid at time n, that produces a non-negative payoff under α-PRU at time $n+1$, with no losses on those events where we have completely resolving uncertainty. Similarly, in *(b)* we have a portfolio for which we are paid or we pay nothing at time n, that produces a non-negative payoff under α-PRU at time $n+1$, with at least a gain on those events where we have completely resolving uncertainty. So, behaviorally, it seems natural to avoid such generalized one-step arbitrage opportunities.

As proved in [49], avoiding generalized one-step arbitrage opportunities is equivalent to the existence of a conditional belief function $\widehat{\nu}(\cdot|S_0 = s_0, \dots, S_n = s_n)$ defined on the ring generated by $\mathcal{W}(s_n)$ such that

(i) $\widehat{\nu}^\alpha(S_{n+1} = us_n | S_0 = s_0, \dots, S_n = s_n) = \widehat{\beta}_1^\alpha(\{u\})$,

(ii) $\widehat{\nu}^\alpha(S_{n+1} = ds_n | S_0 = s_0, \dots, S_n = s_n) = \widehat{\beta}_1^\alpha(\{d\})$,

(iii) it holds that

$$\frac{1}{1+r}\Big[\alpha \widehat{\mathbf{C}}[S_{n+1}|S_0 = s_0, \dots, S_n = s_n] - (1-\alpha)\widehat{\mathbf{C}}[-S_{n+1}|S_0 = s_0, \dots, S_n = s_n]\Big] = \alpha s_n + (1-\alpha)[(1-\widehat{b_d})us_n + \widehat{b_d} ds_n].$$

where $\widehat{b_u}$ and $\widehat{b_d}$ are as in (14). We point out that condition *(iii)* is the α-DS mixture price of the stock, where the term αs_n is due to the one-step Choquet martingale property of the stock bid price process under a risk neutral belief function $\widehat{\nu}$.

Notice that, if we assume completely resolving uncertainty, that is we work on $\mathcal{W}(s_n)$ in place of $\mathcal{U}(s_n)$, then *(a)* and *(b)* reduce to two standard one-step no-arbitrage opportunities (see [10]) as α-mixing is vacuous on singletons. In this case, the only possible choice is to take $\widehat{b_d} = 1 - \widehat{b_u}$ and conditions *(i)*–*(iii)* entirely characterize a global additive belief function $\widehat{\nu}$, that coincides with the equivalent martingale measure in the classical CRR model [22].

4. Calibration on market data

The goal of this section is to calibrate the pricing rule described in Section 3 to real market data. We refer to a frictionless risk-free bond and a non-dividend paying risky stock with frictions in the form of bid-ask spreads. All the calibration procedure is carried out in `Python` and the reference code is available on `GitHub`[1]. Data are retrieved from `Yahoo! finance`, relying on the `yfinance` library treating all options like European ones (see Remark 3.1).

We assume that the stock bid price process $\{S_n\}_{n=0}^T$ is a DS-multiplicative binomial process under an equivalent one-step Choquet martingale belief function $\widehat{\nu}$. This implies that $\widehat{\nu}$ gives rise to t-step transition belief functions $\{\widehat{\beta}_t\}_{t=1}^T$ of the form (7), with parameters $\widehat{b_u}$ and $\widehat{b_d}$.

Our normative model enforces the notion of one-step no-arbitrage under α-PRU discussed in Subsection 3.3, starting from given α-mixture prices of assets already quoted in the market. On the other hand, our model is not an equilibrium model, thus its purpose is not to explain the formation of α-mixture prices in the market. This approach is indeed in common with classical no-arbitrage pricing models.

The choice of the pessimism index $\alpha \in [0,1]$ is a delicate task that will be faced through a dedicated tuning procedure. For a fixed $\alpha \in [0,1]$, the parameters we need to estimate are r, u, d and $\widehat{b_d} \in (0, 1-\widehat{b_u}]$, where $\widehat{b_u}$ is automatically determined by r, u, d as in (14). It is important to notice that the classical historical estimation approach (see, e.g., [34]) is not applicable in our setting, due to the non-additivity of uncertainty measures. We propose a least square calibration procedure where r is fixed through a sovereign zero-coupon bond, while $u, d, \widehat{b_d}$ are implied by α-mixtures of bid-ask prices of European options on the stock.

Let $\alpha \in [0,1]$ be fixed. Let T be a fixed maturity, and $\mathcal{K}_{call}$ and $\mathcal{K}_{put}$ be the available sets of strike prices for European options on the stock. For every $K \in \mathcal{K}_{call}$, denote by $C_{0,M}^{K,\alpha}$ the market α-mixture of bid-ask prices of the call with strike K, while $C_0^{K,\alpha}$ stands for the theoretical α-DS mixture prices computed as in Proposition 3.2. Analogously, for every $K \in \mathcal{K}_{put}$, denote by $P_{0,M}^{K,\alpha}$ the market α-mixture of bid-ask prices of the put with strike K, while $P_0^{K,\alpha}$ stands for the theoretical α-DS mixture prices computed as in Proposition 3.2. Note that once r has been fixed, the theoretical bid-ask prices $C_0^{K,\alpha}$ and $P_0^{K,\alpha}$ are actually functions of $u, d, \widehat{b_d}$.

We define the *mean squared error* as a function of u, d, $\widehat{b_d}$, by setting

$$\mathrm{MSE}(u,d,\widehat{b_d}) = \frac{1}{N}\left[\sum_{K\in\mathcal{K}_{call}} (C_{0,M}^{K,\alpha} - C_0^{K,\alpha})^2 + \sum_{K\in\mathcal{K}_{put}} (P_{0,M}^{K,\alpha} - P_0^{K,\alpha})^2\right], \quad (28)$$

where $N = |\mathcal{K}_{call}| + |\mathcal{K}_{put}|$. Our goal is to solve the

[1] Public `GitHub` repository: `https://github.com/itsdavide/ISIPTA2025`.

following optimization problem

$$\begin{gathered}\text{minimize } \mathrm{MSE}(u, d, \widehat{b_d})\\ \text{subject to:}\\ \begin{cases} 0 < d < 1 + r < u, \\ 0 < \widehat{b_d} \leq 1 - \frac{1+r-d}{u-d}. \end{cases}\end{gathered} \tag{29}$$

Problem (29) is a constrained optimization problem where $\mathrm{MSE}(u, d, \widehat{b_d})$ is a non-linear objective function that contains maxima, due to the call and put contract functions appearing in $C_0^{K,\alpha}$ and $P_0^{K,\alpha}$ (see equations (16)-(17) and Proposition 3.2). A possible solution to this issue is to move the computation of $C_0^{K,\alpha}$ and $P_0^{K,\alpha}$ in the constraint section, by linearizing the maxima in the payoffs, through the introduction of binary variables. Nevertheless, such an approach makes the problem very difficult to solve, even for small values of T and a small number of options.

Therefore, here we face the problem by relying on the *particle swarm optimization (PSO)* technique, which is a stochastic incomplete method operating on a fixed number of candidate solutions [43]. For the PSO implementation we refer to the `PySwarm` library [45].

We identify the initial time $n = 0$ with the date 2025-01-24 and consider call and put options on the `AMZN` stock with maturities 2025-02-21, 2025-03-21, and 2025-05-16, that correspond to $T = 20$, 40 and 80 trading days (i.e., 1, 2 and 4 trading months, respectively).

Figure 4 shows the market bid-ask prices for the options on `AMZN` for the maturity $T = 20$.

Referring to a year composed of 250 trading days, we get r through a US T-bill maturing in 1 month from 2025-01-24, by setting $1 + r = 1.0445^{\frac{1}{250}}$.

We further add the `AMZN` stock market bid-ask prices $S_{0,M} = \$222.92$ and $\overline{S}_{0,M} = \$246.98$, as the bid-ask prices of a degenerate call option with strike price $K = 0$. Table 1 reports all information about the retrieved option datasets for the chosen maturities.

Maturity	T	N. calls	N. puts
2025-02-21	20	64	60
2025-03-21	40	55	53
2025-05-16	80	50	46

Table 1. *Information about the retrieved option datasets on `AMZN` for the maturities 2025-02-21, 2025-03-21, and 2025-05-16.*

We perform the calibration assuming that $d = \frac{1}{u}$, as is commonly done in the classical binomial calibration scheme: in this way the mean squared error reduces to a function of two variables $\mathrm{MSE}(u, \widehat{b_d}) := \mathrm{MSE}\left(u, \frac{1}{u}, \widehat{b_d}\right)$.

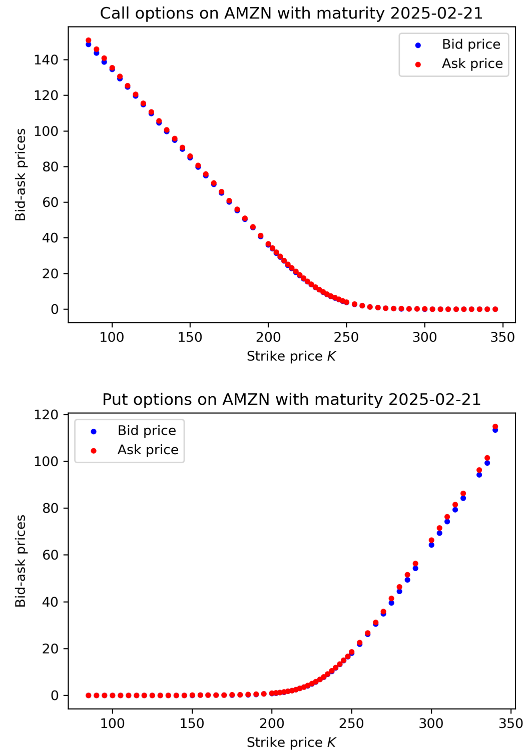

Figure 4. *Bid-ask prices of call and put options on `AMZN` with maturity 2025-02-21 retrieved on 2025-01-24.*

Under this assumption, we perform the PSO technique for 100 iterations relying on 100 particles.

Finally, the tuning of α is carried out by repeating the above procedure making it vary in the range $[0, 1]$ with a 0.02 step and seeking the minimum over α. For a sake of better comparison, Figure 5 shows the optimal *root mean squared error* $\mathrm{RMSE}(u, \widehat{b_d}) := \sqrt{\mathrm{MSE}(u, \widehat{b_d})}$ as a function of α for the three selected maturities.

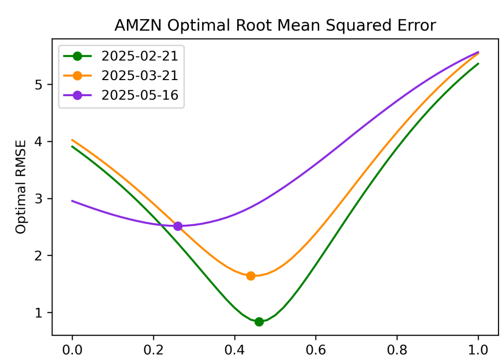

Figure 5. *Optimal* RMSE *for `AMZN` stock as a function of α for the maturities 2025-02-21, 2025-03-21, and 2025-05-16.*

Table 2 reports the values of α that assure the minimum RMSE for the three maturities. Such values highlight that at the fixed current time, for an increasing maturity, the optimal α for `AMZN` stock moves towards zero. This last fact can be interpreted as a decreasing pessimism in the market for larger maturities for this stock, since the closer to zero the more optimistic we are.

Maturity	T	α	u	$\widehat{b_d}$
2025-02-21	20	0.46	1.0217	0.3862
2025-03-21	40	0.44	1.0186	0.4368
2025-05-16	80	0.26	1.0177	0.4714

Table 2. *Optimal values of α, u, and $\widehat{b_d}$ for the maturities 2025-02-21, 2025-03-21, and 2025-05-16.*

Relying on the optimal values reported in Table 2 we can compute the theoretical α-DS mixture prices for the three maturities, using Proposition 3.2. Figure 6 reports the option price curves as functions of the strike price K.

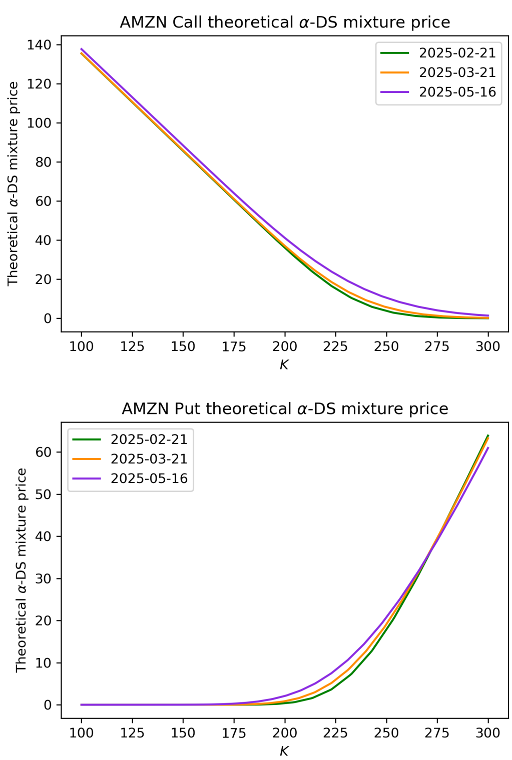

Figure 6. *Theoretical α-DS mixture prices of options on `AMZN` with maturities 2025-02-21, 2025-03-21, and 2025-05-16, computed on 2025-01-24.*

We notice that the optimal value of α we find with the tuning procedure is an index that depends on the underlying stock, the current date, and the maturity of the options. To see this, we repeat the calibration and tuning procedure choosing as $n = 0$ the date 2025-01-31 and considering options with maturity in 5, 10, 15, 20, 25, 35 trading days on the non-dividend paying stocks `AMZN`, `PYPL`, and `RDDT`. Figure 7 shows the optimal values of α highlighting that for the last maturity $T = 35$ the stocks `AMZN` and `RDDT` reach extreme values of α, i.e., complete pessimism and complete optimism, respectively.

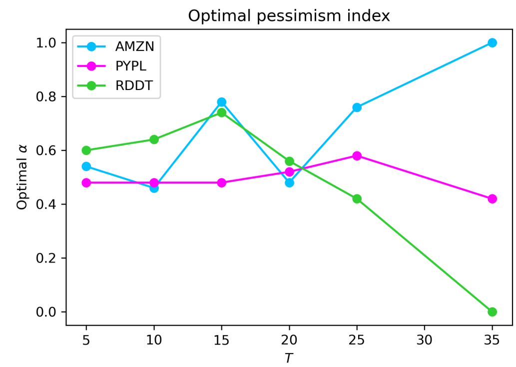

Figure 7. *Optimal values of α at the date 2025-01-31 for different maturities, for the stocks `AMZN`, `PYPL`, and `RDDT`.*

Besides operating a linear interpolation as in Figure 7, we could fit a smooth curve, in analogy with the bootstrapping construction of yield curves [40]. The resulting pessimism index function can be used to depict the evolution of the market sentiment and guide fixed-term investments on the underlying stock.

5. Conclusion

We propose a dynamic pricing rule that mixes one-step-wise the bid and ask prices of a European derivative, relying on a constant pessimism index $\alpha \in [0, 1]$. The quoted α-DS mixture pricing rule reduces to a closed-form binomial formula in case of a monotonic contract function, under a suitable reparameterization. We finally propose a market consistent calibration procedure based on option prices and an ensuing tuning of the pessimism index α. The estimated parameter α can be used as a measure of the pessimism hidden in the market option prices: such a measure reveals to depend on the chosen stock, the valuation date, and the maturity.

As aim of future research, we wish to generalize the notion of pessimism index in the line of Remark 3.2. Furthermore, an interesting line of development is the investigation of the convergence towards a continuous-time model. Restricting to the non-ambiguous case, the discrete-time CRR model [22] is known to converge to the continuous-time Black-Scholes model [8], relying on weak convergence of probability measures (see, e.g., [10]). A generalization of such result to our formulation under ambiguity would require recurring to Choquet weak convergence [30] in the context of belief functions.

Additional author information

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