Conditioning and AGM-like belief change in the Desirability-Indifference framework

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Abstract

We show how the AGM framework for belief change (expansion, revision, contraction) can be extended to deal with conditioning in the so-called Desirability-Indifference framework, based on abstract notions of accepting and rejecting options, as well as on abstract notions of events. This level of abstraction allows us to deal simultaneously with classical and quantum probability theory.

Keywords. AGM belief change, belief state, desirability, indifference, conditioning, Levi's identity, Harper's identity

1. The problem and its context

In an earlier paper [3], Gert de Cooman proposed a generalisation of the AGM framework for belief change — change of belief state — in propositional logic (and probability theory) [12] to what he called *belief models*, which are meant as a much more general and abstract representation for what may constitute a belief state.

More recently, he and co-authors introduced a theory of accepting and rejecting gambles [14], which provides a quite general, well-reasoned and operationalisable framework for decision-making under uncertainty, and provides a context for dealing with such notions as acceptability, desirability, and indifference of gambles, as well as the relationships between them.

In this paper, which is intended as a first exploration, we combine ideas from both papers to show how the AGM framework for belief change can be extended to deal with conditioning on events in the so-called Desirability-Indifference framework, based on abstract notions of accepting and rejecting options (uncertain rewards), as well as on abstract notions of events.

The paper is structured as follows. In Section 2, we introduce the general framework of *statement models* for abstract options, which are meant to represent uncertain rewards. We pay particular attention to a specific type of statement models, which define what we call the *Desirability-Indifference (or DI) framework*. A generalised notion of conditioning, based on abstract events that can be used to call off options, is briefly discussed in Section 3, and Section 4 has a succinct summary of the extension of the AGM framework for belief change to belief models. With this scaffolding in place, we show in Section 5 how statement models fit into the belief model framework, and how our abstract notion of conditioning can be seen as a form of belief change — belief revision — in the Desirability-Indifference framework, satisfying relevant generalised versions of the AGM axioms, as well as versions of the so-called Levi and Harper identities. Section 6 very briefly discusses two specific instances of the general framework, namely classical and quantum probabilistic inference, and in the Conclusion we point to some ways to further develop and refine our theory.

We have collected proofs for all the (arguably) less trivial claims we make in an Appendix.

2. Statement models

We consider an agent, called You, who is asked to make accept and/or reject statements about options. *Options* are abstract objects that represent uncertain rewards. When You accept an option, You agree to receiving an uncertain reward[1] that depends on something You are uncertain about. Rejecting an option is making the statement that accepting it is something You don't agree to. However, it may be that You don't have enough information to make any accept or reject statement about a given option; in that case You're allowed to remain uncommitted.

We'll assume that options u can be added and multiplied with real numbers, with all the usual properties, so that they live in some real linear space $\mathscr{U}$, called the *option space*. The null option 0 represents the status quo, and we'll assume that Your decision problem is *non-trivial* in the sense that $\mathscr{U} \neq \{0\}$. By talking about such abstract options rather than about the more specific cases of gambles [14] or quantum measurements [2, 11], we're able to establish a framework that encompasses both classical and quantum probability.

The set of those options You accept is denoted by $A_{\trianglerighteq}$

[1]What exactly it is that You pay or receive, can be expressed in utiles. We assume that having more utiles is always the desired outcome, and that this scales linearly.

and the set of options You reject by $A_{\lhd}$. Together, these sets form an *assessment* $A := \langle A_{\unrhd}; A_{\lhd} \rangle$ that describes Your behaviour regarding options You care to make statements about. We'll collect all possible assessments in the set $\mathbf{A} := \mathscr{P}(\mathscr{U}) \times \mathscr{P}(\mathscr{U})$. The options You are uncommitted about — the *unresolved* options — are given by $A_{\smile} := \mathrm{co}(A_{\unrhd} \cup A_{\lhd})$, where 'co' denotes the set complement. We can also identify Your set of *indifferent* options $A_{\equiv} := A_{\unrhd} \cap -A_{\unrhd}$ and Your set of *desirable* options $A_{\blacktriangleright} := A_{\unrhd} \cap -A_{\lhd}$:[2] You are indifferent about an option when You want to get it *and* give it away, and You find an option desirable when You want it *but* don't want to give it away.

If we define inclusion for assessments by $A \subseteq B \Leftrightarrow (A_{\unrhd} \subseteq B_{\unrhd}$ and $A_{\lhd} \subseteq B_{\lhd})$ for all $A, B \in \mathbf{A}$, then the structure $\langle \mathbf{A}, \subseteq \rangle$ is a complete lattice with meet and join defined by $A \cap B := \langle A_{\unrhd} \cap B_{\unrhd}; A_{\lhd} \cap B_{\lhd} \rangle$ and $A \cup B := \langle A_{\unrhd} \cup B_{\unrhd}; A_{\lhd} \cup B_{\lhd} \rangle$, top $\mathbf{1_A} := \langle \mathscr{U}; \mathscr{U} \rangle$ and bottom $\langle \emptyset; \emptyset \rangle$. If $A \subseteq B$, we say that A is *less resolved* than B.

2.1. Statement models. We'll be concerned with a special subset $\mathbf{M} \subseteq \mathbf{A}$ of assessments, called *statement models*. For an assessment to be called a statement model, four criteria have to be met:

M1. $0 \in M_{\unrhd}$; [indifference to status quo]

M2. $0 \notin M_{\lhd}$; [strictness]

M3. $M_{\unrhd}$ is a convex cone; [deductive closure]

M4. $\mathrm{sh}(M_{\lhd}) - M_{\unrhd} \subseteq M_{\lhd}$. [no limbo]

Here, $\mathrm{sh}(M_{\lhd})$ is the set of all (strictly) positive scalar multiples of the elements of $M_{\lhd}$. We see that $\langle \mathbf{M}, \subseteq \rangle$ is a *complete meet-semilattice*: $\mathbf{M}$ is closed under taking non-empty infima. This allows us to define a *closure operator* $\mathrm{cl}_{\mathbf{M}} : \mathbf{A} \to \mathbf{M} \cup \{\mathbf{1_A}\} : A \mapsto \bigcap\{M \in \mathbf{M} : A \subseteq M\}$ that maps any assessment to the *least resolved* statement model including it, if any. This condition, namely $\mathrm{cl}_{\mathbf{M}}(A) \in \mathbf{M}$, will be satisfied if and only if $\mathrm{posi}(A_{\unrhd}) \cap A_{\lhd} = \emptyset$, in which case we'll call A *deductively closable*. In that case also

$$\mathrm{cl}_{\mathbf{M}}(A) = \langle \mathrm{posi}(A_{\unrhd}); \mathrm{sh}(A_{\lhd}) \cup (\mathrm{sh}(A_{\lhd}) - \mathrm{posi}(A_{\unrhd})) \rangle, \tag{1}$$

where $\mathrm{posi}(B)$ is the smallest convex cone that includes the set $B \subseteq \mathscr{U}$. M1 to M4 are the minimal rationality requirements we'll impose on any accept-reject statements You make. For a thorough discussion of these and all other results and notions mentioned in this Section 2, we refer to [14].

2.2. About the background. Axiom M1, or equivalently, $V_o := \langle \{0\}; \emptyset \rangle \subseteq M$, requires that You should accept — even be indifferent to — the *status quo*, represented by the null option 0. V_o is an example of what we'll call a *background* (statement) *model*: a statement model that You always accept without any real introspection, regardless of any relevant information You might have. In specific cases, this background model, generically denoted by $V := \langle V_{\unrhd}; V_{\lhd} \rangle \in \mathbf{M}$, may be larger, but we'll always have $V_o \subseteq V$. We'll then require that the *background* V should be *respected*, which amounts to replacing M1 by

M1*. $V \subseteq M$. [background]

The set of all statement models that respect V is denoted by $\mathbf{M}(V) := \{M \in \mathbf{M} : V \subseteq M\}$; for the corresponding closure operator we have that $\mathrm{cl}_{\mathbf{M}(V)}(A) = \mathrm{cl}_{\mathbf{M}}(V \cup A)$ for all $A \in \mathbf{A}$. The background model V is the bottom of the complete meet-semilattice $\langle \mathbf{M}(V), \subseteq \rangle$, and it's also called the *vacuous* statement model.

2.3. The Desirability-Indifference framework. In the context of this paper, we'll focus on a special type of statement models:

Definition 2.1. A statement model $M \in \mathbf{M}$ is called a *Desirability-Indifference model*, or *DI model* for short, if it satisfies the Desirability-Indifference condition:

$$M_{\lhd} \subseteq -M_{\unrhd} \text{ and } M_{\unrhd} = M_{\blacktriangleright} \cup M_{\equiv}, \tag{DI}$$

which requires that You should only reject options that You want to give away, and that for any option You accept, You should be resolved about wanting or not wanting to give it away. An equivalent description is given by $M_{\blacktriangleright} = -M_{\lhd}$ and $M_{\equiv} = M_{\unrhd} \setminus -M_{\lhd}$. A DI model M can therefore be completely described by specifying its set of desirable options $M_{\blacktriangleright}$ and its set of indifferent options $M_{\equiv}$, as $M = \langle M_{\blacktriangleright} \cup M_{\equiv}; -M_{\blacktriangleright} \rangle$. We'll denote the set of all DI models by $\mathbf{M}_{\mathrm{DI}}$, and the set of all DI models that respect a given DI background model V by $\mathbf{M}_{\mathrm{DI}}(V)$, with $V \in \mathbf{M}_{\mathrm{DI}}(V_o)$.

For DI models, the rationality criteria M1* to M4 can be rewritten as follows in terms of the sets $M_{\blacktriangleright}$ and $M_{\equiv}$:

Theorem 2.1. *Consider any background $V \in \mathbf{M}_{\mathrm{DI}}(V_o)$, then an assessment M is a DI model that respects the background V if and only if*

DI1. *$V \subseteq M$;* [background]

DI2. *$0 \notin M_{\blacktriangleright}$;* [strictness]

DI3. *$M_{\blacktriangleright}$ is a convex cone and $M_{\equiv}$ is a linear space;* [deductive closedness]

DI4. *$M_{\blacktriangleright} + M_{\equiv} \subseteq M_{\blacktriangleright}$.* [compatibility]

2.4. The desirability framework. Now is a good moment to realign ourselves with the more familiar approach to dealing with sets of desirable options and sets of indifferent options in the imprecise probabilities literature — see for instance [4–6, 11] — and with the generic notation D for sets of desirable options and I for sets of indifferent options, rather than $M_{\blacktriangleright}$ and $M_{\equiv}$ respectively. With this change of notation, a reader familiar with that literature will no doubt recognise DI1 to DI4 as the typical conditions imposed on compatible sets of desirable and indifferent options.

[2] The minus sign before a set denotes the Minkowski additive inverse. For example, $-A_{\unrhd} := \{-u : u \in A_{\unrhd}\}$.

Something else that's also implicitly done in that literature, and which we'll adhere to here as well, is to focus solely on Your desirability assessments D, and keep all aspects of indifference I in the background model V. In this co-called *Desirability framework*, Your desirability assessment $A = \langle D \cup \{0\}\,;\, -D\,\rangle$ is then combined with a background $V = \langle V_{\blacktriangleright} \cup I\,;\, -V_{\blacktriangleright}\,\rangle \in \mathbf{M}_{\text{DI}}(V_o)$, where I is some given linear space, and $V_{\blacktriangleright}$ is some given convex cone for which $0 \notin V_{\blacktriangleright}$. The latter defines a strict vector ordering $\succ$ on $\mathcal{U}$ by

$$u \succ v \Leftrightarrow u - v \in V_{\blacktriangleright},\ \text{for all } u, v \in \mathcal{U},$$

which we'll call the *background ordering*. We'll also take $u \succeq 0$ to mean that '$u \succ 0$ or $u = 0$'. Observe that, conversely, the background ordering $\succ$ also uniquely determines the background cone $V_{\blacktriangleright}$, because

$$V_{\blacktriangleright} = \{u \in \mathcal{U} : u \succ 0\} =: \mathcal{U}_{\succ 0}.$$

That this background $V = \langle \mathcal{U}_{\succ 0} \cup I\,;\, -\mathcal{U}_{\succ 0}\,\rangle$ should be a DI model now only amounts to imposing the compatibility condition DI4:

$$\mathcal{U}_{\succ 0} + I \subseteq \mathcal{U}_{\succ 0}, \tag{2}$$

which, again, the reader conversant in the relevant literature will no doubt recognise. If we also assume, as is typically done in the literature, that Your set of desirable options D is *coherent*, so satisfies the conditions

D1. $\mathcal{U}_{\succ 0} \subseteq D$; [background]

D2. $0 \notin D$; [strictness]

D3. D is a convex cone, [deductive closedness]

then, since $V \cup A = \langle D \cup I\,;\, -D\,\rangle$ and

$$\begin{aligned}\operatorname{posi}(D \cup I) &= \operatorname{posi}(D) \cup \operatorname{posi}(I) \cup (\operatorname{posi}(D) + \operatorname{posi}(I))\\ &= D \cup I \cup (D + I) = I \cup (D + I),\end{aligned}$$

it becomes an easy exercise to find the least resolved model $\operatorname{cl}_{\mathbf{M}}(V \cup A)$ that includes Your set desirable options D as well as the background V, if it exists. The results in Section 2.1 tell us that the least resolved model exists if the $V \cup A$ is deductively closable, which is the case if and only if $\operatorname{posi}(D \cup I) \cap -D = \emptyset$, so if and only if

$$D \cap I = \emptyset,\ \text{or equivalently, } 0 \notin D + I. \tag{3}$$

The least resolved model we're after, is then given by

$$\operatorname{cl}_{\mathbf{M}}(V \cup A) = \langle\, (D + I) \cup I\,;\, -(D + I)\,\rangle, \tag{4}$$

and this is the DI model with set of desirable options $D + I$ and set of indifferent options I.

3. Events to condition on

What we're especially interested in, is the following situation: after You've stated Your preferences by means of some initial DI model $M := \langle\, (D + I) \cup I\,;\, -(D + I)\,\rangle$, You gain new information: some event e has occurred. How to change Your initial preferences in the presence of this newly acquired information?

For the sake of simplicity, we're going to assume that $I = \{0\}$, so $M = \langle D \cup \{0\}\,;\, -D\,\rangle \in \mathbf{M}_{\text{DI}}(V)$, with initial background model $V := \langle\, \mathcal{U}_{\succ 0} \cup \{0\}\,;\, -\mathcal{U}_{\succ 0}\,\rangle$. This, incidentally, is no real restriction, as we can always make sure of it by moving to the representation of $D + I$ in the quotient space $\mathcal{U}/I$, where the null option corresponds to I; see for instance the discussions in [4, 6].

3.1. Events and their properties. We'll consider an *event* e to be a special type of option that can be used to call off any option $u \in \mathcal{U}$, resulting in the *called-off option* $e * u$. We collect all events in the set $\mathscr{E}$ and call $e * \mathcal{U} := \{e * u : u \in \mathcal{U}\}$ the *called-off space*.

As before for options, we'll not specify *in concreto* the exact form of an event, nor what the exact result of calling off an option looks like, but we'll keep the discussion as abstract as possible by merely imposing a number of properties, E1 to E8, on the set of events $\mathscr{E}$ and the calling-off operation $* : \mathscr{E} \times \mathcal{U} \to \mathcal{U} : (e, u) \mapsto e * u$. In this way, we can keep the discussion general enough to encompass conditioning on events in *classical probability* and on observations in *quantum probability*, amongst others. These special instantiations of our general framework will be briefly discussed in Section 6, which may also serve a source of intuition for the more abstract discussion here.

E1. $e * (u + \lambda v) = e * u + \lambda(e * v)$ for $e \in \mathscr{E}, u, v \in \mathcal{U}$ and $\lambda \in \mathbb{R}$; [linearity]

E2. $e * u = e * (e * u)$ for all $e \in \mathscr{E}$ and all $u \in \mathcal{U}$; [idempotency]

E3. $u \succeq 0 \Rightarrow e * u \succeq 0$ for all $e \in \mathscr{E}$ and all $u \in \mathcal{U}$. [monotonicity]

By E1 and E2, the calling-off operation $e * \bullet$ is a linear projection, whose range is given by the called-off space $e * \mathcal{U}$ and whose kernel is given by

$$I_e := \{u \in \mathcal{U} : e * u = 0\}. \tag{5}$$

We'll furthermore assume that there's some *unit event* $1_{\mathcal{U}} \in \mathscr{E}$ such that

E4. $1_{\mathcal{U}} \succeq 0$ and $1_{\mathcal{U}} * u = u$ and $e * 1_{\mathcal{U}} = e$ for all $u \in \mathcal{U}$ and all $e \in \mathscr{E}$, [unit event]

and that there's some *null event* $0_{\mathcal{U}} \in \mathscr{E}$ such that

E5. $0_{\mathcal{U}} = 0$ and $0_{\mathcal{U}} * u = 0$ for all $u \in \mathcal{U}$. [null event]

The unit event is assumed to be special in that it allows for the following Archimedeanity property:

E6. for any $u \in \mathcal{U}$ there's some $\alpha \in \mathbb{R}$ such that $u + \alpha 1_{\mathcal{U}} \succeq 0$. [Archimedeanity]

It follows readily from these assumptions that

$$e * e = e \text{ and } e * 0_{\mathscr{U}} = 0 \text{ and } 0_{\mathscr{U}} \preceq e \text{ and } 0_{\mathscr{U}} \prec 1_{\mathscr{U}}, \text{ for all } e \in \mathscr{E} \quad (6)$$

and that

$$(\forall u \in \mathscr{U})(e_1 * u = e_2 * u) \Leftrightarrow e_1 = e_2, \text{ for all } e_1, e_2 \in \mathscr{E}, \quad (7)$$

which tells us that the calling-off operations $e * \bullet$ are in a one-to-one relation with the events e. We define the *event ordering* $\sqsubseteq$ on the set of events $\mathscr{E}$ as follows:

$$e_1 \sqsubseteq e_2 \Leftrightarrow (\forall u \in \mathscr{U})\big(e_1 * u = e_1 * (e_2 * u) = e_2 * (e_1 * u)\big), \text{ for all } e_1, e_2 \in \mathscr{E}. \quad (8)$$

Since the relation $\sqsubseteq$ is reflexive, antisymmetric and transitive, $\sqsubseteq$ is a partial order on $\mathscr{E}$. It has bottom $0_{\mathscr{U}}$ and top $1_{\mathscr{U}}$. We now impose a condition that links the event ordering $\sqsubseteq$ to the background ordering of options $\preceq$:

E7. $(\forall u \in \mathscr{U})(e_2 * u = 0 \Rightarrow e_1 * u \not\succ 0) \Rightarrow e_1 \sqsubseteq e_2$, for all $e_1, e_2 \in \mathscr{E}$. [ordering of events]

This assumption makes sure that the event ordering $\sqsubseteq$ can be related to the kernels of the relevant projections.

Proposition 3.1. *For all events $e_1, e_2 \in \mathscr{E}$, the following statements are equivalent:*

(i) $e_1 \sqsubseteq e_2$;
(ii) $I_{e_2} \subseteq I_{e_1}$;
(iii) $(\forall u \in \mathscr{U})(e_2 * u = 0 \Rightarrow e_1 * u \not\succ 0)$.

An interesting consequence of Proposition 3.1 is that the kernel I_e of an event e is in direct correspondence with the event itself. This result follows directly from the equivalence of statements (i) and (ii) in Proposition 3.1 and the fact that $\sqsubseteq$ is a partial order:

$$(\forall u \in \mathscr{U})(e_1 * u = 0 \Leftrightarrow e_2 * u = 0) \Leftrightarrow e_1 = e_2, \text{ for all } e_1, e_2 \in \mathscr{E}. \quad (9)$$

A further, and final, assumption concerns the existence of a *complementation operation* $\neg : \mathscr{E} \to \mathscr{E} : e \mapsto \neg e$ on events:

E8. for every $e \in \mathscr{E}$, there's some $\neg e \in \mathscr{E}$ such that

a. $e + \neg e \in \mathscr{E}$ and $(e + \neg e) * u = u$ for all $u \in \mathscr{U}$,
b. $\neg e * (e * u) = 0$ for all $u \in \mathscr{U}$.

Under these assumptions, it's easy to check that

$$e + \neg e = 1_{\mathscr{U}} \text{ and } \neg(\neg e) = e \text{ and } e * (\neg e) = (\neg e) * e = 0 \text{ for all } e \in \mathscr{E}, \quad (10)$$

so we find in particular that the *complement* $\neg e$ of an event e is unique.

The conditions E1 to E8 are internally consistent: as already mentioned above, we'll uncover in Section 6 two specific instances, namely events in classical probability and projections in quantum probability, where all these assumptions are satisfied.

3.2. Conditioning on an event. The effect of the occurrence of an event e is that You can no longer distinguish between any two options whose called-off options coincide: they've now become indifferent to You. This leads to a new linear space of indifferent options, given by the kernel $I_e = \{u \in \mathscr{U} : e * u = 0\}$ of the calling-off operation $e * \bullet$. It therefore seems perfectly reasonable to represent the new knowledge that the event e has occurred by the indifference assessment

$$M_e := \langle I_e ; \emptyset \rangle \in \mathbf{M}_{\mathrm{DI}}(V_o). \quad (11)$$

Which options should You find desirable after the occurrence of the event e? We're going to assume, in the spirit of discussions by de Finetti [7, 9], Williams [20] and Walley [18] that You now find desirable all options whose called-off versions were already desirable before the occurrence of the event, leading to the *conditional*, or *updated*, set of desirable options

$$D \,\|\, e := \{u \in \mathscr{U} : e * u \in D\}. \quad (12)$$

Even though it's guaranteed to satisfy D2 and D3, this new set of desirable options $D \,\|\, e$ won't necessarily be coherent, because not necessarily $\mathscr{U}_{\succ 0} \subseteq D \,\|\, e$: the original background ordering $\succ$ may not be compatible with the new information that e has occurred.

To see what may go wrong, let's find out what happens when we merge the new information in the shape of $M_e = \langle I_e ; \emptyset \rangle$ into the existing background, so when we combine V with the new background information M_e to get $V \cup \langle I_e ; \emptyset \rangle = \langle \mathscr{U}_{\succ 0} \cup I_e ; -\mathscr{U}_{\succ 0} \rangle$. This assessment is deductively closable only if $\mathscr{U}_{\succ 0} \cap I_e = \emptyset$, or equivalently, if

$$u \succ 0 \Rightarrow e * u \succ 0 \text{ for all } u \in \mathscr{U}, \quad (13)$$

in which case we'll call the event e *regular*. Interestingly, the statement in Equation (13) implies that $e = 1_{\mathscr{U}}$, so the only regular event is $1_{\mathscr{U}}$. In that case, E4 guarantees in a rather trivial manner that $D \,\|\, 1_{\mathscr{U}} = D$, $I_{1_{\mathscr{U}}} = \{0\}$ and $M_{1_{\mathscr{U}}} = \langle \{0\} ; \emptyset \rangle$; conditioning on the regular unit event $1_{\mathscr{U}}$ results in no change at all.

So we see that the more interesting events won't be regular, and since we're only taking into account called-off options, we must then *reduce* the background ordering to its restriction to the called-off space. This leads to a *revised* background ordering, denoted by $\succ_e$:

$$u \succ_e 0 \Leftrightarrow (u \in e * \mathscr{U} \text{ and } u \succ 0), \text{ for all } u \in \mathscr{U}, \quad (14)$$

so $\mathscr{U}_{\succ_e 0} = \mathscr{U}_{\succ 0} \cap e * \mathscr{U}$, by Equation (12). This ordering is non-empty if and only if there's some $u \in \mathscr{U}$ such that $e * u \succ 0$, or in other words, if $\mathscr{U}_{\succ 0} \cap e * \mathscr{U} \neq \emptyset$; we'll then call the event e *proper*. Interestingly, the only improper event is the null event $0_{\mathscr{U}} = 0$, for which $D \,\|\, 0_{\mathscr{U}} = \emptyset$, $I_{0_{\mathscr{U}}} = \mathscr{U}$ and $M_{0_{\mathscr{U}}} = \langle \mathscr{U} ; \emptyset \rangle$. For the regular (so proper) event $1_{\mathscr{U}}$, clearly $u \succ_{1_{\mathscr{U}}} 0 \Leftrightarrow u \succ 0$, so the revised background ordering $\succ_{1_{\mathscr{U}}}$ is the same as the original one.

Reducing $\succ$ to $\succ_e$ makes sure that Equation (13) now holds trivially, so the revised background assessment $\langle\, \mathscr{U}_{\succ_e 0} \cup I_e\, ;\, -\mathscr{U}_{\succ_e 0}\,\rangle$ is deductively closable. Taking its closure leads to the *revised* background DI model

$$V_e := \langle\, (\mathscr{U}_{\succ_e 0} + I_e) \cup I_e\, ;\, -(\mathscr{U}_{\succ_e 0} + I_e)\,\rangle, \quad e \in \mathscr{E}. \tag{15}$$

Since, trivially, $\mathscr{U}_{\succ_e 0} \subseteq D \,\|\, e$ and $D \,\|\, e + I_e = D \,\|\, e$, we infer from the discussion in Section 2.4, and in particular Equation (4), that the DI model

$$\langle\, D \,\|\, e \cup I_e\, ;\, -D \,\|\, e\,\rangle \in \mathbf{M}_{\mathrm{DI}}(V_e), \quad e \in \mathscr{E} \tag{16}$$

is the *least resolved statement model that combines the updated set of desirable options $D \,\|\, e$ with the revised background DI model V_e, or equivalently, with the new background information $M_e = \langle\, I_e\, ;\, \emptyset\,\rangle$*. This holds in particular for the regular unit event $1_{\mathscr{U}}$, which is necessarily proper because we've already shown that $1_{\mathscr{U}} \succ 0$ [see Equation (6)] and for which $V_{1_{\mathscr{U}}} = V$.

We see that the occurrence of an event e leads You to replace Your initial statement model with a new, *updated*, one. This is a form of belief change, and we are thus led to wonder if we can describe it using the ideas and tools introduced by Alchourrón, Gärdenfors, and Makinson (AGM) in [1]; see also the general discussion in [12] and the very thorough overview in [15].

It's not hard to see that we won't generally have that $D \subseteq D \,\|\, e$, nor that

$$V \subseteq V_e \text{ or } \langle\, D \cup \{0\}\, ;\, -D\,\rangle \subseteq \langle\, D \,\|\, e \cup I_e\, ;\, -D \,\|\, e\,\rangle.$$

Taking into account the extra information that an event e has occurred therefore doesn't necessarily lead to an increase in resolve: updating isn't necessarily a monotonic operation. Another way of saying this, is that updating typically can't be seen as belief *expansion*, but should rather be viewed as a form of belief *revision*.

But, as Alchourrón, Gärdenfors, and Makinson in [1] and Gärdenfors in [12] focused mainly on belief states that are sets of propositions, or that are probability measures, we need a way to generalise their ideas to the more abstract setting where the belief states are statement models. The tools for this were developed by one of us in a paper that was published quite some time ago [3], and we'll use the next section to briefly summarise the ideas there that are directly relevant to the present discussion.

4. Belief states and belief change

We'll consider abstract objects, called *belief models*, or *belief states*, and collect them in a set $\mathbf{B}$. We assume that the belief states b in $\mathbf{B}$ are partially ordered by a binary relation $\sqsubseteq$, and that they constitute a *complete lattice* $\langle \mathbf{B}, \sqsubseteq \rangle$. Denote its top as $\mathbf{1_B}$ and its bottom as $\mathbf{0_B}$, its meet as $\frown$ and its join as $\smile$.

We're particularly interested in a special non-empty subset $\mathbf{C} \subseteq \mathbf{B}$ of belief states, whose elements are called *coherent* and are considered to be 'more perfect' than the others. The inherited partial ordering $\sqsubseteq$ on $\mathbf{C}$ is interpreted as 'is more conservative than'.

Crucially, we'll assume that $\mathbf{C}$ is closed under arbitrary non-empty infima, so for every non-empty subset $C \subseteq \mathbf{C}$ we assume that $\inf C \in \mathbf{C}$. Additionally, we'll assume that the complete meet-semilattice $\langle \mathbf{C}, \sqsubseteq \rangle$ has no top — so definitely $\mathbf{1_B} \notin \mathbf{C}$ — implying that there's a smallest (most conservative) coherent belief state $\mathbf{0_C} = \inf \mathbf{C}$ but no largest (least conservative) one. We'll let the incoherent $\mathbf{1_B}$ represent contradiction. Of course, the set $\overline{\mathbf{C}} := \mathbf{C} \cup \{\mathbf{1_B}\}$ provided with the partial ordering $\sqsubseteq$ is a complete lattice. The corresponding triple $\langle \mathbf{B}, \mathbf{C}, \sqsubseteq \rangle$ is then called a *belief structure*.

That $\mathbf{C}$ is closed under arbitrary non-empty infima, leads us to define a closure operator as follows:

$$\mathrm{cl}_{\mathbf{C}} \colon \mathbf{B} \to \overline{\mathbf{C}} \colon b \mapsto \mathrm{cl}_{\mathbf{C}}(b) := \inf\{c \in \mathbf{C} \colon b \sqsubseteq c\},$$

so $\mathrm{cl}_{\mathbf{C}}(b)$ is the most conservative coherent belief state that is at least as committal as b — if any. The closure operator implements *conservative inference* in the belief structure $\langle \mathbf{B}, \mathbf{C}, \sqsubseteq \rangle$. Using this closure $\mathrm{cl}_{\mathbf{C}}$, we can now define notions of *consistency* and *closedness*: a belief state $b \in \mathbf{B}$ is *consistent* if $\mathrm{cl}_{\mathbf{C}}(b) \neq \mathbf{1_B}$, or equivalently, if $\mathrm{cl}_{\mathbf{C}}(b) \in \mathbf{C}$, and *closed* if $\mathrm{cl}_{\mathbf{C}}(b) = b$. Two belief states $b_1, b_2 \in \mathbf{B}$ are called *consistent* if their join $b_1 \smile b_2$ is. We see that $\overline{\mathbf{C}}$ is the set of all closed belief states, that the coherent belief states in $\mathbf{C}$ are the ones that are both consistent and closed, and that $\mathbf{1_B}$ represents inconsistency for the closed belief states.

The statement models $M \in \mathbf{M}(V_o)$ that respect indifference to the status quo, provided with the 'is at most as resolved as' ordering $\subseteq$, can be seen to constitute a special instance of these belief structures, with the correspondences identified in Table 1.

$\mathbf{B}$	$\rightsquigarrow$	$\mathbf{A}$
$\sqsubseteq$	$\rightsquigarrow$	$\subseteq$
$\mathbf{0_B}$	$\rightsquigarrow$	$\langle\, \emptyset\, ;\, \emptyset\,\rangle$
$\mathbf{1_B}$	$\rightsquigarrow$	$\langle\, \mathscr{U}\, ;\, \mathscr{U}\,\rangle$
inf	$\rightsquigarrow$	$\bigcap$
$\mathbf{C}$	$\rightsquigarrow$	$\mathbf{M}(V_o)$
$\overline{\mathbf{C}}$	$\rightsquigarrow$	$\overline{\mathbf{M}}(V_o) := \mathbf{M}(V_o) \cup \{\langle\, \mathscr{U}\, ;\, \mathscr{U}\,\rangle\}$
$\mathbf{0_C}$	$\rightsquigarrow$	V_o
$\mathrm{cl}_{\mathbf{C}}(\,\bullet\,)$	$\rightsquigarrow$	$\mathrm{cl}_{\mathbf{M}}(V_o \cup \bullet)$
consistent	$\rightsquigarrow$	deductively closable
coherent	$\rightsquigarrow$	statement model that respects V_o

Table 1. *Correspondences between belief states and sets of accept-reject statements*

When You're in a coherent belief state $c \in \mathbf{C}$, new information the form of some belief state $b \in \mathbf{B}$ — not necessarily coherent — can cause You to change Your belief state. Alchourrón, Gärdenfors, and Makinson [1] considered three important types of belief change: belief expansion, belief revision and belief contraction. They proposed a number of axioms for these operations, and studied and discussed their resulting properties.

De Cooman [3] argued that the axioms for belief expansion and belief revision can be elegantly translated to the setting of belief structures — the counterpart of belief contraction is more problematic. We'll not discuss these abstract axioms here, but propose to postpone listing their more concrete forms until the next section, where we consider belief change for Desirability-Indifference models caused by the occurrence of events.

5. Conditioning as belief change

As explained above, You start out with a DI model $M = \langle D \cup \{0\}\,;\, -D \rangle$ that respects a background $V = \langle \mathcal{U}_{>0} \cup \{0\}\,;\, -\mathcal{U}_{>0} \rangle$, and You gain new information in the form of the occurrence of an event e, which we've argued corresponds to new DI background information $M_e = \langle I_e\,;\, \emptyset \rangle$. We have already mentioned above that, due to (9), the sets of indifferent options I_e — and therefore also the statement models $M_e = \langle I_e\,;\, \emptyset \rangle$ — are in a one-to-one correspondence with the events e.

One way to combine Your statement model with such new information goes via a *belief expansion* operator [3]

$$\mathrm{E}\colon \mathbf{M}(V_o) \times \mathbf{A} \to \mathbf{A}\colon (M, A) \mapsto \mathrm{E}(M \mid A),$$

De Cooman [3] has argued that the AGM axioms for belief expansion [1, 12, 15] can be translated directly to the abstract setting of belief structures. His results lead us to conclude that the resulting expansion operator E must then be uniquely determined by the resulting model closure, in the sense that $\mathrm{E}(M \mid A) = \mathrm{cl}_{\mathbf{M}(V_o)}(M \cup A)$. In our conditioning setting, this leads to $\mathrm{E}(M \mid M_e) = \mathrm{cl}_{\mathbf{M}(V_o)}(M \cup M_e)$, where $M \cup M_e = \langle D \cup I_e\,;\, -D \rangle$. Since $\mathrm{posi}(D \cup I_e) = (D + I_e) \cup I_e$, we find after a few algebraic manipulations that $M \cup M_e$ is deductively closable if and only if $D \cap I_e = \emptyset$, which as we've already argued in Section 3 will only be the case if e is regular, and therefore only if $e = 1_{\mathcal{U}}$. Recalling that $I_{1_{\mathcal{U}}} = \{0\}$ and therefore $M_{1_{\mathcal{U}}} = \langle \{0\}\,;\, \emptyset \rangle$, we find that $\mathrm{cl}_{\mathbf{M}(V_o)}(M \cup M_e) = \mathrm{cl}_{\mathbf{M}(V_o)}(M) = M$. Hence,

$$\mathrm{E}(M \mid M_e) = \begin{cases} M = \langle D \cup \{0\}\,;\, -D \rangle & \text{if } e = 1_{\mathcal{U}} \\ \langle \mathcal{U}\,;\, \mathcal{U} \rangle & \text{otherwise.} \end{cases} \tag{17}$$

So, unless $e = 1_{\mathcal{U}}$, expanding M with M_e will lead to inconsistency; this is because D includes the original background cone $\mathcal{U}_{>0}$, which, as we've already mentioned in Section 3 will only be consistent with the indifferences in I_e provided that $I_e \cap \mathcal{U}_{>0} = \emptyset$, which only happens for the uniquely regular event $e = 1_{\mathcal{U}}$.

We're therefore led to consider a different type of belief change, based on a *belief revision* operator

$$\mathrm{R}\colon \mathbf{M}(V_o) \times \mathbf{A} \to \mathbf{A}\colon (M, A) \mapsto \mathrm{R}(M \mid A).$$

De Cooman [3] has argued that the AGM axioms for belief revision [1, 12] can be transported to the abstract setting of belief structures. His revision axioms, which correspond one to one with the AGM revision postulates K*1–K*8 in [12], are the following, when instantiated in the present context of statement models:

BR1. $\mathrm{R}(M \mid A) \in \overline{\mathbf{M}}(V_o)$ for M in $\mathbf{M}(V_o)$ and A in $\mathbf{A}$;

BR2. $A \subseteq \mathrm{R}(M \mid A)$ for M in $\mathbf{M}(V_o)$ and A in $\mathbf{A}$;

BR3. $\mathrm{R}(M \mid A) \subseteq \mathrm{E}(M \mid A)$ for M in $\mathbf{M}(V_o)$ and A in $\mathbf{A}$;

BR4. $\mathrm{E}(M \mid A) \subseteq \mathrm{R}(M \mid A)$ for M in $\mathbf{M}(V_o)$ and A in $\mathbf{A}$ such that A and M are consistent;

BR5. $\mathrm{R}(M \mid A)$ is inconsistent if and only if A is inconsistent, for M in $\mathbf{M}(V_o)$ and A in $\mathbf{A}$;

BR6. $\mathrm{R}(M \mid A) = \mathrm{R}(M \mid \mathrm{cl}_{\mathbf{M}(V_o)}(A))$ for M in $\mathbf{M}(V_o)$ and A in $\mathbf{A}$;

BR7. $\mathrm{R}(M \mid A_1 \cup A_2) \subseteq \mathrm{E}(\mathrm{R}(M \mid A_1) \mid A_2)$ for M in $\mathbf{M}(V_o)$ and A_1, A_2 in $\mathbf{A}$;

BR8. $\mathrm{E}(\mathrm{R}(M \mid A_1) \mid A_2) \subseteq \mathrm{R}(M \mid A_1 \cup A_2)$ for M in $\mathbf{M}(V_o)$ and A_1, A_2 in $\mathbf{A}$ such that $\mathrm{R}(M \mid A_1)$ and A_2 are consistent.

Belief revision is necessary when expansion leads to inconsistency [see BR3 and BR4], and works essentially by preserving A [see BR2] and reducing M.

Inspired by the discussion in Section 3.2, and in particular Equation (16), we propose the following form for a belief revision operator, whose first argument we'll restrict to DI models $M \in \mathbf{M}_{\mathrm{DI}}(V_o)$ of the form $M = \langle D \cup \{0\}\,;\, -D \rangle$ where D is a coherent set of desirable options, and whose second argument we'll restrict to the DI models $M_e = \langle I_e\,;\, \emptyset \rangle$, which are in a one-to-one correspondence with the events $e \in \mathscr{E}$:

$$\mathrm{R}(M \mid M_e) := \langle D \,\|\, e \cup I_e\,;\, -D \,\|\, e \rangle \in \mathbf{M}_{\mathrm{DI}}(V_e), \quad \text{for all } M \in \mathbf{M}_{\mathrm{DI}}(V_o) \text{ and all } e \in \mathscr{E}. \tag{18}$$

Remark that we never get to an inconsistency, because, as explained in the previous section, we've allowed ourselves to change the background model as a result of the conditioning operation.

Proposition 5.1. *When we restrict the belief revision operator $\mathrm{R}(M \mid M_e)$ as introduced in Equation (18) to DI models $M \in \mathbf{M}_{\mathrm{DI}}(V_o)$ of the form $M = \langle D \cup \{0\}\,;\, -D \rangle$ where D is any coherent set of desirable options, and to DI models $M_e = \langle I_e\,;\, \emptyset \rangle$ corresponding to events $e \in \mathscr{E}$, then it satisfies the correspondingly restricted BR1 to BR8.*

[3] The | symbol merely acts as a way to separate the first from the second argument here.

Alchourrón, Gärdenfors, and Makinson [1, 12] also discuss the notion of *belief contraction*, which in our context can be made to correspond to the action of a *belief contraction* operator

$$\mathrm{C}\colon \mathbf{M}(V_o)\times\mathbf{A}\to\mathbf{A}\colon (M,A)\mapsto \mathrm{C}(M\,|\,A),$$

for which the following direct counterparts of the AGM contraction postulates could be formulated as follows — the correspondence is again one to one with the AGM contraction postulates $\mathrm{K}^{-}1$–$\mathrm{K}^{-}8$ in [12]:

BC1. $\mathrm{C}(M\,|\,A)\in\overline{\mathbf{M}}(V_o)$ for M in $\mathbf{M}(V_o)$ and $A\in\mathbf{A}$;
BC2. $\mathrm{C}(M\,|\,A)\subseteq M$ for M in $\mathbf{M}(V_o)$ and $A\in\mathbf{A}$;
BC3. $\mathrm{C}(M\,|\,A)=M$ for M in $\mathbf{M}(V_o)$ and $A\in\mathbf{A}$ such that $\neg A$ and M are consistent;
BC4. if $A\subseteq\mathrm{C}(M\,|\,A)$ then $\neg A$ is inconsistent, for M in $\mathbf{M}(V_o)$ and $A\in\mathbf{A}$;
BC5. if $A\subseteq M$ then $M\subseteq\mathrm{E}(\mathrm{C}(M\,|\,A)\,|\,A)$, for M in $\mathbf{M}(V_o)$ and $A\in\mathbf{A}$;
BC6. $\mathrm{C}(M\,|\,A)=\mathrm{C}(M\,|\,\mathrm{cl}_{\mathbf{M}(V_o)}(A))$ for M in $\mathbf{M}(V_o)$ and $A\in\mathbf{A}$;
BC7. $\mathrm{C}(M\,|\,A_1)\cap\mathrm{C}(M\,|\,A_2)\subseteq\mathrm{C}(M\,|\,A_1\cup A_2)$ for M in $\mathbf{M}(V_o)$ and $A_1,A_2\in\mathbf{A}$;
BC8. $\mathrm{C}(M\,|\,A_1\cup A_2)\subseteq\mathrm{C}(M\,|\,A_1)$ for M in $\mathbf{M}(V_o)$ and $A_1,A_2\in\mathbf{A}$ such that $\neg A_1$ and $\mathrm{C}(M\,|\,A_1\cup A_2)$ are consistent.

Belief contraction aims at removing the consequences of an assessment A from a statement model M. For the axioms BC3, BC4 and BC8 to make sense at all, we need to be able to introduce a negation operator $\neg$ for accept-reject statement sets, which is far from obvious in the present context, where such statement sets needs't be propositional. It's for this reason that De Cooman refrained from discussing belief contraction in [3].

But when we restrict ourselves in the current context to belief contraction with statement models M_e, which are related to the occurrence of events e, contraction becomes a more reasonable proposition: for events e we have complemented events $\neg e$, so we'll define the *negated statement model*

$$\neg M_e := M_{\neg e}=\langle\, I_{\neg e}\,;\,\emptyset\,\rangle \tag{19}$$

as the one associated with the complemented event $\neg e$.

We propose the following form for a belief contraction operator, whose first argument we'll restrict to DI models $M\in\mathbf{M}_{\text{DI}}(V_o)$ of the form $M=\langle\, D\cup\{0\}\,;\,-D\,\rangle$ and whose second argument we'll restrict to the DI models $M_e=\langle\, I_e\,;\,\emptyset\,\rangle$, which are in a one-to-one correspondence with the events $e\in\mathscr{E}$:

$$\mathrm{C}(M\,|\,M_e) := M\cap\mathrm{R}(M\,|\,\neg M_e),\quad\text{for all } M\in\mathbf{M}_{\text{DI}}(V_o)\text{ and all } e\in\mathscr{E}, \tag{20}$$

leading to

$$\mathrm{C}(M\,|\,M_e)=\langle\,(D\cup\{0\})\cap(D\,\|\,\neg e\cup I_{\neg e})\,;\,-(D\cap D\,\|\,\neg e)\,\rangle,\quad\text{for all } M\in\mathbf{M}_{\text{DI}}(V_o)\text{ and all } e\in\mathscr{E}. \tag{21}$$

For the statement model M_e where $e=1_{\mathscr{U}}$, this contraction would result in removing the background V_o from the statement model M, leading to a violation of BC1. To avoid this, we'll restrict the DI models M_e to those corresponding to non-regular events $e\neq 1_{\mathscr{U}}$.

Proposition 5.2. *When we restrict the belief contraction operator $\mathrm{C}(M\,|\,M_e)$ as introduced in Equation (20) to DI models $M\in\mathbf{M}_{\text{DI}}(V_o)$ of the form $M=\langle\, D\cup\{0\}\,;\,-D\,\rangle$, where D is any coherent set of desirable options, and to DI models $M_e=\langle\, I_e\,;\,\emptyset\,\rangle$ corresponding to non-regular events $e\in\mathscr{E}\setminus\{1_{\mathscr{U}}\}$, then it satisfies the correspondingly restricted BC1 to BC6.*

Interestingly, we have a counterexample for BC7 in the case of classical probabilistic inference, which can also be used for quantum probabilistic inference. It would be interesting, but beyond the scope of the present paper, to check whether alternative versions of BC7 (and BC8), as they are for instance formulated in a monograph by Rott [15], would hold in our context.

To finalise the discussion, we mention that also in our present context, as in the propositional context covered by the AGM framework [1, 12], versions of Levi's and Harper's identities hold; our version of Harper's identity actually holds by definition.

Proposition 5.3 (Harper's identity)**.** *$M\in\mathbf{M}_{\text{DI}}(V_o)$ of the type $M=\langle\, D\cup\{0\}\,;\,-D\,\rangle$ and all $e\in\mathscr{E}$,*

$$\mathrm{C}(M\,|\,M_e)=M\cap\mathrm{R}(M\,|\,\neg M_e).$$

Proposition 5.4 (Levi's identity)**.** *For all $M\in\mathbf{M}_{\text{DI}}(V_o)$ of the type $M=\langle\, D\cup\{0\}\,;\,-D\,\rangle$ and all $e\in\mathscr{E}$,*

$$\mathrm{R}(M\,|\,M_e)=\mathrm{E}(\mathrm{C}(M\,|\,\neg M_e)\,|\,M_e).$$

6. Interesting Special Instances

Let's now identify two special instances of the general abstract option and event framework developed above.

6.1. Classical probabilistic inference. In a decision-theoretic context related to classical probabilistic reasoning, we consider a variable X that may assume values in some non-empty set $\mathscr{X}$, but whose actual value is unknown to You. With any bounded map $f\colon\mathscr{X}\to\mathbb{R}$, called *gamble*, there corresponds an uncertain reward $f(X)$, expressed in units of some linear utility scale. The set of all gambles, denoted by $\mathscr{G}(\mathscr{X})$, constitutes a real linear space under pointwise addition and pointwise scalar multiplication with real numbers.

So, we take as our option space the set $\mathscr{U}\leftsquigarrow\mathscr{G}(\mathscr{X})$ of all gambles, which You can express preferences between.

For the background ordering $\succ$, we take the *weak (strict) preference ordering* $>$ defined by

$$f > 0 \Leftrightarrow \underbrace{(\forall x \in \mathcal{X}) f(x) \geq 0}_{\text{this defines } f \geq 0} \text{ and } f \neq 0.$$

Here, *events* are subsets E of the possibility space $\mathcal{X}$, and they can (and will) be identified with special indicator gambles $\mathbb{I}_E$ that assume the value 1 on E and 0 elsewhere; so here, $\mathscr{E} \rightsquigarrow \{\mathbb{I}_E : E \subseteq \mathcal{X}\}$. The *unit event* corresponds to the constant gamble $1 = \mathbb{I}_{\mathcal{X}}$ and the *null event* to the constant gamble $0 = \mathbb{I}_{\emptyset}$. For the *called-off gambles*, we have that $\mathbb{I}_E * f \rightsquigarrow \mathbb{I}_E f$, and *complementation* corresponds to $\neg\mathbb{I}_E \rightsquigarrow 1 - \mathbb{I}_E$. It's a trivial exercise to show that the assumptions E1 to E8 are satisfied. The event ordering $\sqsubseteq$ corresponds to set inclusion: $\mathbb{I}_{E_1} \sqsubseteq \mathbb{I}_{E_2} \Leftrightarrow E_1 \subseteq E_2$. The only *regular event* is the unit event $1 = \mathbb{I}_{\mathcal{X}}$, and the *proper events* $\mathbb{I}_E$ correspond to the non-empty subsets $E \neq \emptyset$ of $\mathcal{X}$.

Updating a coherent set of desirable gambles D with a non-empty, and therefore proper, event $E \neq \emptyset$ to get to $D \,\|\, E$, is a well-established operation that leads to a generalisation of *Bayes's rule* for conditioning in probability theory; see for instance [16–19].

6.2. Quantum probabilistic inference. Consider, in a decision-theoretic context related to quantum probabilistic reasoning, a quantum system whose unknown state $|\Psi\rangle$ lives in a finite-dimensional state space $\mathcal{X}$, which is a complex Hilbert space whose dimension we'll denote by n. Options in this context are the Hermitian operators $\hat{A}$ on $\mathcal{X}$ corresponding to *measurements* on the system, where the outcome of a measurement $\hat{A}$ is interpreted as an uncertain reward, expressed in units of some linear utility scale, which You can get if you accept the measurement. The set of all such Hermitian operators $\hat{A}$ constitutes an n^2-dimensional real linear space $\mathscr{H}$, and this is Your option space for this quantum decision problem. We denote by $\operatorname{spec}(\hat{A})$ the set of all (real) eigenvalues — possible outcomes — of the measurement $\hat{A}$. For the background ordering, we take the strict vector ordering associated with *positive semidefiniteness*, defined by

$$\hat{A} > 0 \Leftrightarrow \underbrace{\min \operatorname{spec}(\hat{A}) \geq 0}_{\text{this defines } \hat{A} \geq 0} \text{ and } \hat{A} \neq 0.$$

Here, any *event* corresponds to a subspace $\mathscr{W}$ of the state space $\mathcal{X}$. It can be identified with a special measurement $\hat{P}_{\mathscr{W}} = \hat{P}_{\mathscr{W}}\hat{P}_{\mathscr{W}}$, the (linear and orthogonal) projection operator onto the subspace $\mathscr{W}$, with eigenvalue 1 associated with the eigenspace $\mathscr{W}$ and eigenvalue 0 associated with its orthogonal complement $\mathscr{W}^{\perp}$. This tells us that $\mathscr{E} \rightsquigarrow \{\hat{P}_{\mathscr{W}} : \mathscr{W} \text{ is a subspace of } \mathcal{X}\}$. The *unit event* corresponds to the identity measurement $\hat{I} = \hat{P}_{\mathcal{X}}$ and the *null event* to the zero measurement $\hat{0} = \hat{P}_{\{0\}}$. For the *called-off measurements*, we have that $\hat{P}_{\mathscr{W}} * \hat{A} \rightsquigarrow \hat{P}_{\mathscr{W}}\hat{A}\hat{P}_{\mathscr{W}}$, and *complementation* corresponds to $\neg\hat{P}_{\mathscr{W}} \rightsquigarrow \hat{I} - \hat{P}_{\mathscr{W}} = \hat{P}_{\mathscr{W}^{\perp}}$. It's a straightforward exercise to show that E1 to E6 and E8 hold. A proof for E7 is given in the Appendix. The event ordering $\sqsubseteq$ corresponds to the inclusion ordering of the subspaces:

$$\hat{P}_{\mathscr{W}_1} \sqsubseteq \hat{P}_{\mathscr{W}_2} \Leftrightarrow \hat{P}_{\mathscr{W}_1}\hat{P}_{\mathscr{W}_2} = \hat{P}_{\mathscr{W}_2}\hat{P}_{\mathscr{W}_1} = \hat{P}_{\mathscr{W}_1}$$
$$\Leftrightarrow \mathscr{W}_1 \subseteq \mathscr{W}_2.$$

The only regular event is the identity measurement $\hat{I} = \hat{P}_{\mathcal{X}}$, and the proper events $\hat{P}_{\mathscr{W}}$ correspond to the non-null subspaces $\mathscr{W} \neq \{0\}$ of $\mathcal{X}$.

Updating a coherent set of desirable measurements D with a non-null, and therefore proper, event $\hat{P}_{\mathscr{W}} \neq \hat{0}$ to get to $D \,\|\, \mathscr{W}$, is an operation that leads to a generalisation of *Lüders' conditioning rule* [10].

7. Conclusion

We were able to identify an abstract option and event structure that allows us to look at conditioning sets of desirable options in terms of belief change operators that are consistent with the AGM framework. This abstract account is general enough to cover conditioning both in classical and quantum probability settings.

We've formulated our arguments in the Desirability-Indifference framework, where the background is a DI model. Recent work [10], however, seems to suggest that it might be possible to find even more interesting results by allowing for (somewhat) more general background models, leading us to consider moving to the Accept-Desirability framework [14] for future work in this area. Also, more work is needed to fully relate our work to variants of the AGM framework that have been proposed in the literature, as they are for instance summarised in Rott's monograph [15].

A. Proofs

Proof of Equation (3). Since $\operatorname{posi}(D \cup I) = I \cup (D + I)$, we find that $\operatorname{posi}(D \cup I) \cap -D = \emptyset$ if and only if $0 \notin D + I + D = D + I$, or equivalently, $D \cap I = \emptyset$. ☐

Proof of Equation (4). Since we proved above that $D \cap I = \emptyset$ implies deductive closability, we find, using Equation (1), that for the accept part of $M := \mathrm{cl}_{\mathbf{M}}(V \cup A)$, $M_{\trianglerighteq} = \operatorname{posi}(D \cup I) = I \cup (D + I)$ and that for its reject part, $M_{\triangleleft} = -(D \cup (D + (I \cup (D + I)))) = -(D \cup (D + I) \cup (D + D + I)) = -(D + I)$. ☐

Proof of Equation (6). Consider any $e \in \mathscr{E}$. That $e * e = e$ follows from E2, by letting $u \rightsquigarrow 1_{\mathscr{U}}$ and recalling that $e * 1_{\mathscr{U}} = e$, by E4. That $e * 0_{\mathscr{U}} = 0$ follows from E1, by letting $v \rightsquigarrow u$ and $\lambda \rightsquigarrow -1$. That $0_{\mathscr{U}} \preceq e$ follows from E3 with $u \rightsquigarrow 1_{\mathscr{U}}$, taking into account E4. That, finally, $0_{\mathscr{U}} \prec 1_{\mathscr{U}}$ follows from E4 and E5 and our general non-triviality assumption that $\mathscr{U} \neq \{0\}$. ☐

Proof of Equation (7). It's clear that only the direct implication needs any attention. Letting $u \rightsquigarrow 1_{\mathscr{U}}$ on the left-hand side, and applying E4 leads directly to the right-hand side of the desired implication. □

Proof that $\sqsubseteq$ is a partial order on $\mathscr{E}$. The reflexivity of $\sqsubseteq$ follows from E2. For its antisymmetry, let's assume that $e_1 \sqsubseteq e_2$ and $e_2 \sqsubseteq e_1$, so we infer from Equation (8) that

$$(\forall u \in \mathscr{U})\begin{cases} e_1 * u = e_2 * (e_1 * u) \\ e_2 * u = e_2 * (e_1 * u). \end{cases}$$

If we now let $u \rightsquigarrow 1_{\mathscr{U}}$, then we infer from E4 that, indeed, $e_1 = e_2 * e_1 = e_2$. For transitivity, assume that $e_1 \sqsubseteq e_2$ and $e_2 \sqsubseteq e_3$, then we infer from Equation (8) that, for any $u \in \mathscr{U}$,

$$\begin{aligned} e_1 * u = e_1 * (e_2 * u) &= e_1 * (e_2 * (e_3 * u)) \\ &= e_1 * (e_3 * u) \end{aligned}$$

and similarly

$$\begin{aligned} e_1 * u = e_2 * (e_1 * u) &= e_3 * (e_2 * (e_1 * u)) \\ &= e_3 * (e_1 * u). \end{aligned}$$ □

Proof that the poset $\langle \mathscr{E}, \sqsubseteq \rangle$ has bottom $0_{\mathscr{U}}$ and top $1_{\mathscr{U}}$. Consider any $e \in \mathscr{E}$ and any $u \in \mathscr{U}$. It follows from E4 that $e * (1_{\mathscr{U}} * u) = e * u = 1_{\mathscr{U}} * (e * u)$, and this tells us that, indeed, $e \sqsubseteq 1_{\mathscr{U}}$. It also follows from E1 and E5 that $e * (0_{\mathscr{U}} * u) = e * 0 = 0 = 0_{\mathscr{U}} * u = 0_{\mathscr{U}} * (e * u)$, which tells us that, indeed, $0_{\mathscr{U}} \sqsubseteq e$. □

Proof of Proposition 3.1. We give a circular proof that (i)⇒(ii)⇒(iii)⇒(i). For (i)⇒(ii), assume that $e_1 \sqsubseteq e_2$ and consider any $u \in \mathscr{U}$ such that $e_2 * u = 0$. But then also, by assumption, $e_1 * u = e_1 * (e_2 * u) = e_1 * 0 = 0$, where the last equality follows from E1. That (ii)⇒(iii) is trivial, and that (iii)⇒(i) is the essence of E7. □

Proof of Equation (10). That $e + \neg e = 1_{\mathscr{U}}$ follows from E8a and E4, combined with the statement in Equation (7). This implies that $\neg e = 1_{\mathscr{U}} - e$, and therefore also $\neg(\neg e) = 1_{\mathscr{U}} - \neg e = 1_{\mathscr{U}} - (1_{\mathscr{U}} - e) = e$. That $(\neg e) * e = 0$ follows from E8b with $u \rightsquigarrow 1_{\mathscr{U}}$ and E4. If we now replace the event e by the event $\neg e$ in the previous statement, we find that $(\neg(\neg e)) * (\neg e) = 0$, and therefore that finally also $e * (\neg e) = 0$. □

Proof of Equation (13). We have to prove that

$$\mathscr{U}_{>0} \cap I_e = \emptyset \Leftrightarrow (\forall u \in \mathscr{U})(u > 0 \Rightarrow e * u > 0).$$

Since the converse implication is trivial, we focus on the direct implication. Assume that $\mathscr{U}_{>0} \cap I_e = \emptyset$, and consider any $u \in \mathscr{U}_{>0}$, then we have to prove that $e * u > 0$. It follows from $u > 0$ that $e * u \geq 0$, by E3. Since we know from the assumption that $u \notin I_e$ and therefore $e * u \neq 0$, this implies that, indeed, $e * u > 0$. □

Proof that $1_{\mathscr{U}}$ is the only regular event. It's clear from E4 that $1_{\mathscr{U}}$ satisfies the condition in Equation (13), and therefore is a regular event. Now consider any regular event e and observe that Equation (13) and E4 imply that $(\forall u \in \mathscr{U})(1_{\mathscr{U}} * u > 0 \Rightarrow e * u \neq 0)$, or equivalently, $(\forall u \in \mathscr{U})(e * u = 0 \Rightarrow 1_{\mathscr{U}} * u \not> 0)$. This implies that $1_{\mathscr{U}} \sqsubseteq e$, and therefore that $1_{\mathscr{U}} = e$, because we've proved above that $1_{\mathscr{U}}$ is the top of the poset $\langle \mathscr{E}, \sqsubseteq \rangle$. □

Proof that $0_{\mathscr{U}} = 0$ is the only improper event. First of all, E5 implies that $0_{\mathscr{U}} * \mathscr{U} = \{0\}$, which guarantees in turn that $\mathscr{U}_{>0} \cap (0_{\mathscr{U}} * \mathscr{U}) = \emptyset$, so the null event $0_{\mathscr{U}}$ isn't proper. Next, consider any improper event e, meaning that $\mathscr{U}_{>0} \cap e * \mathscr{U} = \emptyset$ and therefore $(\forall u \geq 0)\, e * u = 0$, by E3. But E4 then tells us that $1_{\mathscr{U}} \geq 0$ and therefore also $e = e * 1_{\mathscr{U}} = 0$. □

Proof of the statement involving Equation (16). We start by proving that $D \,\|\, e + I_e = D \,\|\, e$. It's clearly enough to show that $D \,\|\, e + I_e \subseteq D \,\|\, e$, so consider $u \in D \,\|\, e$ and $v \in I_e$, then $e * (u+v) = e * u + e * v = e * u \in D$, and therefore, indeed, $u + v \in D \,\|\, e$. Since this implies that $\operatorname{posi}(D \,\|\, e \cup I_e) = D \,\|\, e \cup I_e$, we see that the assessment $\langle\, D \,\|\, e \cup I_e \,;\, -D \,\|\, e \,\rangle$ is consistent because $D \,\|\, e \cap I_e = \emptyset$. Equation (1) and some basic algebraic manipulations now tell us that, indeed, $\mathrm{cl}_{\mathbf{M}(V_o)}(\langle\, D \,\|\, e \cup I_e \,;\, -D \,\|\, e \,\rangle) = \langle\, D \,\|\, e \cup I_e \,;\, -D \,\|\, e \,\rangle$. □

Proof of Proposition 5.1. It's clear that BR1 is satisfied, as $\mathrm{R}(M \,|\, M_e) \in \mathbf{M}_{\mathrm{DI}}(V_e) \subseteq \overline{\mathbf{M}}(V_o)$. Direct inspection based on Equations (11) and (18) also confirms that BR2 holds. For BR3, we first check the case that $e = 1_{\mathscr{U}}$. We recall from Section 3 that $D \,\|\, 1_{\mathscr{U}} = D$, $I_{1_{\mathscr{U}}} = \{0\}$ and $M_{1_{\mathscr{U}}} = \langle\, \{0\} \,;\, \emptyset \,\rangle \subseteq M$, so $\mathrm{R}(M \,|\, M_e) = \langle\, D \cup \{0\} \,;\, -D \,\rangle = M = \mathrm{E}(M \,|\, M_e)$. For all other events, trivially, $\mathrm{R}(M \,|\, M_e) \subseteq \langle\, \mathscr{U} \,;\, \mathscr{U} \,\rangle = \mathrm{E}(M \,|\, M_e)$. For BR4, recall from the discussion in Section 5 [belief expansion] that M and M_e are only consistent when $e = 1_{\mathscr{U}}$, and we've checked (above) that then $\mathrm{R}(M \,|\, M_e) = \mathrm{E}(M \,|\, M_e)$. Axiom BR5 is vacuously fulfilled because no model of the form $M_e = \langle\, I_e \,;\, \emptyset \,\rangle$ is inconsistent. Since $M_e = \mathrm{cl}_{\mathbf{M}(V_o)}(M_e)$, BR6 holds too. We'll tackle BR7 and BR8 together. The only case of interest is when $\mathrm{R}(M \,|\, M_{e_1})$ and M_{e_2} are consistent, or equivalently, after some algebraic manipulations, when $D \,\|\, e_1 \cap I_2 = \emptyset$. Because if not, then we'd have that $\mathrm{E}(\mathrm{R}(M \,|\, M_{e_1}) \,|\, M_{e_2}) = \langle\, \mathscr{U} \,;\, \mathscr{U} \,\rangle$, so BR7 holds trivially; and BR8 is satisfied vacuously. So, if $D \,\|\, e_1 \cap I_{e_2} = \emptyset$, we infer from $\mathscr{U}_{>0} \subseteq D$ that in particular $(\forall u \in \mathscr{U})(e_2 * \mathscr{U} = 0 \Rightarrow e_1 * u \not> 0)$. From E7 we then infer that $e_1 \sqsubseteq e_2$. Using the first two equivalences in Proposition 3.1, we see that $I_{e_2} \subseteq I_{e_1}$, and therefore $M_{e_2} \subseteq M_{e_1}$. This implies that, on the one hand, $M_{e_1} \cup M_{e_2} = M_{e_1}$ and $\mathrm{R}(M \,|\, M_{e_1} \cup M_{e_2}) = \mathrm{R}(M \,|\, M_{e_1})$. On the other hand, we find that $M_{e_2} \subseteq M_{e_1} \subseteq \mathrm{R}(M \,|\, M_{e_1})$ and therefore $\mathrm{E}(\mathrm{R}(M \,|\, M_{e_1}) \,|\, M_{e_2}) = \mathrm{R}(M \,|\, M_{e_1})$. This proves both BR7 and BR8. □

Proof of Proposition 5.2. We start by observing that both $D \| \neg e \cap D$ and $D \cap I_{\neg e}$ are convex cones, and that

$$\begin{aligned}&\mathrm{C}(M \mid M_e)\\&= \langle (D \| \neg e \cap D) \cup (D \cap I_{\neg e}) \cup \{0\}\, ;\, -(D \| \neg e \cap D) \rangle.\end{aligned}$$

BC1 holds because $\mathrm{C}(M \mid M_e)$ is defined in Equation (20) as a meet of two closed models M and $\mathrm{R}(M \mid \neg M_e)$ that respect V_o [$\mathrm{R}(M \mid \neg M_e) \in \overline{\mathbf{M}}(V_o)$ by Proposition 5.1]. BC2 holds because, clearly, $D \cap D \| \neg e \subseteq D$ and $D \cap I_{\neg e} \subseteq D$. We saw in Section 5 that $\neg M_e$ and M are only consistent when $\neg M_e$ corresponds to the regular event $1_{\mathscr{U}}$, so when $e = 0_{\mathscr{U}} = 0$. Since then $D \| \neg e = D \| 1_{\mathscr{U}} = D$ and $I_{\neg e} = I_{1_{\mathscr{U}}} = \{0\}$, we can come to the conclusion that $\mathrm{C}(M \mid M_e) = \mathrm{C}(M \mid M_{0_{\mathscr{U}}}) = \langle D \cup \{0\}\, ;\, -D \rangle = M$, so BC3 holds. The only event e for which $M_e \subseteq \mathrm{C}(M \mid M_e)$, and therefore $M_e \subseteq M$, is the regular unit event $1_{\mathscr{U}}$, which is outside the scope of the proposition, so BC4 holds vacuously. BC5 is also vacuously fulfilled, for the same reason. Since $\mathrm{cl}_{\mathbf{M}(V_o)}(M_e) = M_e$, BC6 is also obeyed. □

Proof of Proposition 5.4. First, observe that the assessment $\mathrm{C}(M \mid \neg M_e) \cup M_e = \langle (D \| e \cap D) \cup I_e\, ;\, -(D \| e \cap D) \rangle$ is consistent, because

$$\mathrm{posi}((D \| e \cap D) \cup I_e) = ((D \| e \cap D) + I_e) \cup I_e$$

and $D \| e \cap I_e = \emptyset$. This tells us that, using Equation (1), and after some algebraic manipulations,

$$\begin{aligned}&\mathrm{E}(\mathrm{C}(M \mid \neg M_e) \mid M_e)\\&\quad= \mathrm{cl}_{\mathbf{M}(V_o)}(\mathrm{C}(M \mid \neg M_e) \cup M_e)\\&\quad= \langle ((D \| e \cap D) + I_e) \cup I_e\, ;\, -(D \| e \cap D + I_e) \rangle,\end{aligned}$$

so it's enough to prove that $(D \| e \cap D) + I_e = D \| e$. On the one hand, $(D \| e \cap D) + I_e \subseteq D \| e + I_e = D \| e$. For the converse inclusion, consider that for all $u \in D \| e$, $u = e * u + (u - e * u)$, where $u - e * u \in I_e$ and $e * u \in D$. □

Proof that E7 holds in a quantum context. Assume that $(\forall \hat{A} \in \mathscr{H})(\hat{P}_{\mathscr{W}_2} \hat{A} \hat{P}_{\mathscr{W}_2} = \hat{0} \Rightarrow \hat{P}_{\mathscr{W}_1} \hat{A} \hat{P}_{\mathscr{W}_1} \not> \hat{0})$, then we must prove that $\hat{P}_{\mathscr{W}_1} = \hat{P}_{\mathscr{W}_2} \hat{P}_{\mathscr{W}_1} = \hat{P}_{\mathscr{W}_1} \hat{P}_{\mathscr{W}_2}$. Consider any $|\phi\rangle \in \bar{\mathscr{X}}$ and let $\mathscr{W} := \mathrm{span}(\{|\phi\rangle\})$, the linear space spanned by $|\phi\rangle$. Observe that for any $k \in \{1, 2\}$,

$$\hat{P}_{\mathscr{W}_k} \hat{P}_{\mathscr{W}} \hat{P}_{\mathscr{W}_k} \geq \hat{0} \text{ and } (\hat{P}_{\mathscr{W}_k} \hat{P}_{\mathscr{W}} \hat{P}_{\mathscr{W}_k} = \hat{0} \Leftrightarrow |\phi\rangle \in \mathscr{W}_k^{\perp}). \tag{22}$$

Now assume that $|\phi\rangle \in \mathscr{W}_2^{\perp}$, then we infer from the assumption and Equation (22) that $|\phi\rangle \in \mathscr{W}_1^{\perp}$, implying that $\mathscr{W}_2^{\perp} \subseteq \mathscr{W}_1^{\perp}$, or in other words,

$$\hat{P}_{\mathscr{W}_2^{\perp}} \hat{P}_{\mathscr{W}_1^{\perp}} = \hat{P}_{\mathscr{W}_1^{\perp}} \hat{P}_{\mathscr{W}_2^{\perp}} = \hat{P}_{\mathscr{W}_2^{\perp}}, \text{ with } \hat{P}_{\mathscr{W}_k^{\perp}} = \hat{I} - \hat{P}_{\mathscr{W}_k}.$$

A little algebra now leads to the desired equalities. □

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