Supplementary Material: Proofs for Function-Coherent Gambles

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1 Introduction

In this supplementary document we provide detailed proofs for the key properties and representation results underlying the framework of *function-coherent gambles* as presented in the main paper.

In our framework, a set of acceptable gambles is defined on a domain X by means of a strictly increasing and continuous utility function

$$u : X \to \mathbb{R},$$

with the normalization

$$u(0) = 0.$$

The acceptance set is defined as

$$\mathbb{D} = \{f \in X : u(f) \geq 0\}.$$

We assume the following axioms for function-coherence:

(F1) **Avoid Partial Losses:** If $f < 0$ (i.e., $f(s) < 0$ for every state s), then $f \notin \mathbb{D}$.

(F2) **Monotonicity:** If $f \geq g$ (pointwise) and $g \in \mathbb{D}$, then $f \in \mathbb{D}$.

(F3) **u-Convexity:** For any $f, g \in \mathbb{D}$ and any nonnegative scalars λ, μ for which

$$h = u^{-1}\Big(\lambda\, u(f) + \mu\, u(g)\Big)$$

is well defined, we have $h \in \mathbb{D}$.

In addition, we assume:

(F3a) The utility function $u : X \to u(X) \subseteq V$ is a strictly increasing and continuous bijection onto its image.

(F3b) The image $u(X)$ is convex.

The results below establish fundamental properties of $\mathbb{D}$ (non-triviality, monotonicity/upward closure, convexity in the transformed space, invariance under strictly increasing transformations, a representation theorem, and closure under limits).

2 Non-Triviality and Consistency

Theorem 2.1 (Non-Triviality and Consistency)**.** *Under axioms (F1) and (F2), with acceptance defined by $u(f) \geq 0$ (and $u(0) = 0$), the acceptance set $\mathbb{D}$ is nonempty and contains no sure losses. Moreover, every gamble f satisfying $u(f) \geq 0$ is acceptable.*

Proof. **Non-Triviality:** Consider the constant gamble 0 defined by $0(s) = 0$ for every state s. By normalization, $u(0) = 0$, so

$$u(0) \geq 0,$$

and thus $0 \in \mathbb{D}$. Hence, $\mathbb{D}$ is nonempty.

Avoidance of Sure Loss: Axiom (F1) stipulates that if $f < 0$ then $f \notin \mathbb{D}$. Because u is strictly increasing, any gamble f with $f(s) < 0$ for all s will satisfy $u(f) < u(0) = 0$, ensuring $f \notin \mathbb{D}$.

Consistency: By definition, a gamble f is acceptable if $u(f) \geq 0$. Moreover, axiom (F2) guarantees that if $f \in \mathbb{D}$ and $g(s) \geq f(s)$ for every state s, then $u(g) \geq u(f) \geq 0$ so that $g \in \mathbb{D}$. ☐

3 Upward Closure

Theorem 3.1 (Upward Closure). *Let $f \in \mathbb{D}$ and let $g \in X$ be any gamble such that $g(s) \geq f(s)$ for every state s. Then $g \in \mathbb{D}$.*

Proof. Since $f \in \mathbb{D}$, we have $u(f) \geq 0$. Because $g(s) \geq f(s)$ for all s and u is strictly increasing, it follows that

$$u(g) \geq u(f) \geq 0.$$

Thus, by the definition of $\mathbb{D}$, we conclude $g \in \mathbb{D}$. $\square$

4 u-Convexity in the Transformed Space (Transform Convexity)

Theorem 4.1 (Transform Convexity). *Under axiom (F3), the u-transformed acceptance set*

$$U(\mathbb{D}) := \{u(f) : f \in \mathbb{D}\}$$

is a convex cone. That is, for any $x, y \in U(\mathbb{D})$ and any nonnegative scalars λ, μ (with $\lambda\, x + \mu\, y$ lying in the range of u), we have

$$\lambda\, x + \mu\, y \in U(\mathbb{D}).$$

Proof. Let $x, y \in U(\mathbb{D})$. By definition, there exist $f, g \in \mathbb{D}$ such that

$$x = u(f) \quad \text{and} \quad y = u(g).$$

For any nonnegative scalars λ, μ such that $\lambda\, u(f) + \mu\, u(g)$ lies in the range of u, define

$$h = u^{-1}\Big(\lambda\, u(f) + \mu\, u(g)\Big).$$

Axiom (F3) implies $h \in \mathbb{D}$, so that

$$u(h) = \lambda\, u(f) + \mu\, u(g) \in U(\mathbb{D}).$$

Thus, $U(\mathbb{D})$ is closed under nonnegative linear combinations and is therefore a convex cone. $\square$

5 Invariance Under Strictly Increasing Transformations

Theorem 5.1 (Transform Invariance). *Let $\phi : \mathbb{R} \to \mathbb{R}$ be any strictly increasing function with $\phi(0) = 0$ and define $\tilde{u} = \phi \circ u$. Then*

$$\{f \in X : \tilde{u}(f) \geq 0\} = \{f \in X : u(f) \geq 0\} = \mathbb{D}.$$

Proof. For any $f \in X$, since ϕ is strictly increasing and $\phi(0) = 0$, we have

$$u(f) \geq 0 \quad \Longleftrightarrow \quad \phi\big(u(f)\big) \geq \phi(0) = 0.$$

That is,

$$u(f) \geq 0 \quad \Longleftrightarrow \quad \tilde{u}(f) \geq 0.$$

Hence, the acceptance set defined via $\tilde{u}$ coincides with $\mathbb{D}$. $\square$

6 Representation Theorem

Suppose that X is a real vector space of gambles and that V is a locally convex, Hausdorff topological vector space. Let $u : X \to V$ be a strictly increasing and continuous function with $u(0) = 0$. Assume that the acceptance set is given by

$$\mathbb{D} = \{f \in X : u(f) \geq 0\},$$

and that the following regularity conditions hold:

(R1) The u-transformed set $U(\mathbb{D}) = \{u(f) : f \in \mathbb{D}\}$ is closed in V.

(R2) $U(\mathbb{D})$ has nonempty interior in V.

Theorem 6.1 (Representation Theorem)**.** *Under the above assumptions and regularity conditions, there exists a continuous linear functional $\ell : V \to \mathbb{R}$ (unique up to multiplication by a positive scalar) such that for every $f \in X$,*

$$f \in \mathbb{D} \quad \Longleftrightarrow \quad \ell\big(u(f)\big) \geq 0.$$

Proof. Since $U(\mathbb{D})$ is a closed convex cone with nonempty interior in V (by (F3) together with conditions (R1) and (R2)), a standard separation theorem (such as the Hahn–Banach theorem) guarantees the existence of a nonzero continuous linear functional $\ell : V \to \mathbb{R}$ such that

$$\ell(x) \geq 0 \quad \text{for all } x \in U(\mathbb{D}).$$

By definition, for any $f \in X$,

$$f \in \mathbb{D} \quad \Longleftrightarrow \quad u(f) \in U(\mathbb{D}) \quad \Longleftrightarrow \quad \ell\big(u(f)\big) \geq 0.$$

Defining the evaluation (or risk) functional $\rho : X \to \mathbb{R}$ by

$$\rho(f) := \ell\big(u(f)\big),$$

we obtain the desired representation. Uniqueness of ℓ up to a positive scalar follows from the properties of the separating hyperplane. □

7 Closure Under Limits

Assume that X is endowed with a topology that makes it a topological vector space and that $u : X \to \mathbb{R}$ is continuous. Recall that

$$\mathbb{D} = \{f \in X : u(f) \geq 0\}.$$

Theorem 7.1 (Closure Under Limits)**.** *Let $\{f_n\}$ be a sequence in $\mathbb{D}$ that converges to some $f \in X$. Then $f \in \mathbb{D}$.*

Proof. Since $f_n \in \mathbb{D}$ for all n, we have

$$u(f_n) \geq 0 \quad \text{for all } n.$$

By the continuity of u, it follows that

$$\lim_{n \to \infty} u(f_n) = u(f).$$

Since the limit of nonnegative numbers is nonnegative, we have $u(f) \geq 0$. Hence, by definition, $f \in \mathbb{D}$. □