On the closure of aggregation rules for imprecise probabilities

Enrique Miranda[1] **Ignacio Montes**[1]

[1]Department of Statistics and O.R., University of Oviedo, Spain

ABSTRACT

We consider the problem of aggregating a number of imprecise probability models into a joint one, and compare four aggregation rules: conjunction, disjunction, mixture and Pareto. We investigate for which particular cases of imprecise probability models these operators are closed, meaning that the output belongs to the same family as the inputs. Specifically, we analyse this problem for comparative probability models, 2-monotone capacities, probability intervals, belief functions, p-boxes and minitive measures.

Keywords. belief pooling, Choquet capacities, p-boxes, comparative probabilities, probability intervals

1. INTRODUCTION

The problem of belief aggregation appears naturally in contexts where the information is coming from different sources, or when different experts provide their input on the uncertainty. The second of these scenarios is often linked to subjective assessments, and in those cases it may be more reliable to model the uncertainty by means of an imprecise model. This leads to the problem of aggregating imprecise probabilities, that has some additional layers of difficulty with respect to the classical case.

Building on the work carried out in [1, 5, 6, 18, 22, 25], in [14] we tackled the problem of aggregating coherent lower previsions. We compared the conjunction, disjunction, mixture, Pareto, conjunction/disjunction and maximal consistent subsets rules, and our comparison was made in terms of twenty different rationality criteria, based on those considered in the aforementioned works.

While coherent lower previsions are undoubtedly interesting, they are not the only mathematical model that has been proposed to deal with imprecise information; we may for instance consider comparative probabilities [4, 27], Choquet capacities [2], belief functions [20], probability intervals [3] or possibility and necessity measures [7]. These models differ in their mathematical properties and, in some cases, in the underlying interpretation attached.

Should we decide to model imprecision by means of one of these models, the problem of aggregation would have to be reconsidered, and some of the rationality axioms we discussed in [14] may have to be adapted. For instance, in [14] we considered as a basic requirement that the output of the aggregation of a finite number of coherent lower previsions should again be a coherent lower prevision; but if we model our uncertainty in terms of belief functions, it would make sense to impose that their aggregation should be a belief function too. There are two main reasons for this: on the one hand, there are some specific mathematical advantages of some of the models, that we may want to preserve through the aggregation; but also from a conceptually point of view, it for instance we work in the realm of belief functions and give an evidential interpretation to our uncertainty, this interpretation should carry on to the aggregated model.

In this paper, we focus on the conjunction, disjunction, mixture and Pareto rules, whose definition we recall in Section 3, and investigate in which cases these aggregation rules are closed for a number of imprecise probability models: comparative probabilities (Section 4), 2-monotone capacities (Section 5), probability intervals (Section 6), belief functions (Section 7), p-boxes (Section 8) and minitive measures (Section 9), and in some cases provide sufficient conditions for this to be the case. A brief discussion of our results and some additional comments will be provided in the conclusions.

2. IMPRECISE PROBABILITY MODELS

In this paper we consider finite possibility spaces $\mathcal{X} = \{x_1, \dots, x_k\}$. We denote by $\mathcal{L}(\mathcal{X})$ the set of gambles, by $\underline{P}$ and $\overline{P}$ conjugate lower and upper previsions and by $\mathcal{M}(\underline{P})$ their associated credal set. Also, $\mathbb{P}(\mathcal{X})$, $\underline{\mathbb{P}}(\mathcal{X})$ and $\underline{\mathbb{P}}'(\mathcal{X})$ are used to denote the sets of linear previsions, lower previsions and coherent lower previsions defined on $\mathcal{L}(\mathcal{X})$. In particular, a lower prevision whose domain are indicator gambles of subsets of $\mathcal{X}$ is called a lower probability. In that case, we use for simplicity $\underline{P}(A)$ instead of $\underline{P}(I_A)$. Whenever a (coherent) lower prevision is defined in a subset of $\mathcal{L}(\mathcal{X})$ we assume that it extends to $\mathcal{L}(\mathcal{X})$ using natural extension.

We consider the following particular cases of (coherent) lower previsions:

Comparative probabilities [4, 27]: We consider a set $\mathcal{L} \subseteq \mathcal{X} \times \mathcal{X}$ where $(x_i, x_j) \in \mathcal{L}$ means that x_i is at least as probable as x_j.We refer to the lower envelope of the credal set:

$$\mathcal{M}(\mathcal{L}) = \\ \{P \in \mathbb{P}(\mathcal{X}) \mid (\forall (x_i, x_j) \in \mathcal{L})(P(\{x_i\}) \geq P(\{x_j\}))\}$$

as a *comparative probability* determined by $\mathcal{L}$. Note that in this paper we consider only the scenario where the comparisons are made on singletons, as in [12].

2-monotone capacities [2]: A lower probability $\underline{P}$ is 2-monotone when $\underline{P}(A \cup B) + \underline{P}(A \cap B) \geq \underline{P}(A) + \underline{P}(B)$ for any $A, B \subseteq \mathcal{X}$. They are also called Choquet capacities of order 2 or fuzzy measures [10] in the literature.

Probability intervals [3]: A family of non-empty intervals $\mathcal{I} = \{[l_i, u_i]\}_{i=1,\dots,k}$, where $[l_i, u_i] \subseteq [0,1]$, determines the following credal set:

$$\mathcal{M}(\mathcal{I}) = \{P \in \mathbb{P}(\mathcal{X}) \mid (\forall i = 1, \dots, k)(l_i \leq P(\{x_i\}) \leq u_i)\}.$$

Its lower envelope $\underline{P}$ is called *probability interval* determined by $\mathcal{I}$. In [3], necessary and sufficient conditions for the non-emptiness of $\mathcal{M}(\mathcal{I})$ were given, and also for its the lower $\underline{P}$ and its conjugate upper $\overline{P}$ probabilities to satisfy $\underline{P}(\{x_i\}) = l_i$ and $\overline{P}(\{x_i\}) = u_i$ for any $i = 1, \dots, k$. When that is the case, they also established that the probability interval is a 2-monotone capacity in addition to being coherent.

Belief functions [20]: A lower probability is characterised by its Möbius inverse $m : \mathcal{P}(\mathcal{X}) \to \mathbb{R}$, given by $m(A) = \sum_{B \subseteq A} (-1)^{|A \setminus B|} \underline{P}(B)$ for any $A \subseteq \mathcal{X}$. Then, the lower probability can be expressed as $\underline{P}(A) = \sum_{B \subseteq A} m(B)$. Whenever $m(A) \in [0,1]$ for any $A \subseteq \mathcal{X}$, $m(\emptyset) = 0$ and $\sum_{A \subseteq \mathcal{X}} m(A) = 1$, the lower probability is called a *belief function*, and it is in particular 2-monotone. The events A satisfying $m(A) > 0$ are called *focal elements*, and the set of all the focal elements is denoted by $\mathcal{F}$.

P-boxes [9]: Assume that $\mathcal{X}$ is ordered such that $x_1 < \dots < x_k$, and let $\underline{F}, \overline{F} : \mathcal{X} \to [0,1]$ be two cumulative distribution functions (cdfs, for short) such that $\underline{F} \leq \overline{F}$. These determine the following credal set:

$$\mathcal{M}(\underline{F}, \overline{F}) = \{P \in \mathbb{P}(\mathcal{X}) \mid \underline{F} \leq F_P \leq \overline{F}\},$$

where F_P denotes the cdf associated with P. Its lower envelope on events, denoted by $\underline{P}_{(\underline{F},\overline{F})}$, is called a p-box, and it is in particular a belief function whose focal elements are ordered intervals. Moreover, it satisfies $\underline{P}_{(\underline{F},\overline{F})}(\{x_1, \dots, x_i\}) = \underline{F}(x_i)$ and $\overline{P}_{(\underline{F},\overline{F})}(\{x_1, \dots, x_i\}) = \overline{F}(x_i)$ for any $i = 1, \dots, k$. We refer to [9, 23] for some seminal studies on p-boxes.

Minitive measures [7]: A lower probability $\underline{P}$ satisfying $\underline{P}(A \cap B) = \min\{\underline{P}(A), \underline{P}(B)\}$ for any $A, B \subseteq \mathcal{X}$ is called *minitive measure*, and its conjugate upper probability $\overline{P}$ is called *maxitive measure*. Any minitive measure is a belief function whose focal elements are ordered by set inclusion.

3. Summary of the Aggregation Rules

Consider a group of n experts, each of them modelling the uncertainty about the experiment using a coherent lower prevision on $\mathcal{L}(\mathcal{X})$. An aggregation procedure aims at finding another lower prevision on $\mathcal{L}(\mathcal{X})$ summarising the global opinion of the group.

Definition 3.1. An *aggregation rule on coherent lower previsions* is a map $\mathcal{A} : \left(\underline{\mathbb{P}}'(\mathcal{X})\right)^n \to \underline{\mathbb{P}}(\mathcal{X})$.

The aggregation rules we shall analyse in this paper are the following:

Conjunction ($\mathcal{A}_C$) It corresponds [5, 25] to the natural extension of the lower prevision $\max\{\underline{P}_1, \dots, \underline{P}_n\}$, or, equivalently, to the lower envelope of $\cap_{i=1}^n \mathcal{M}(\underline{P}_i)$.

Disjunction ($\mathcal{A}_D$) It is given [5, 25] by $\mathcal{A}_D(f) = \min\{\underline{P}_1(f), \dots, \underline{P}_n(f)\}$ for any $f \in \mathcal{L}(\mathcal{X})$. This rule is called *convex pooling* in [22]. Its credal set coincides with $\mathcal{CH}(\mathcal{M}(\underline{P}_1) \cup \dots \cup \mathcal{M}(\underline{P}_n))$ where $\mathcal{CH}$ stands for the convex hull.

Mixture ($\mathcal{A}_M$) Given fixed non-negative $\alpha_1, \dots, \alpha_n$ satisfying $\alpha_1 + \dots + \alpha_n = 1$, the mixture aggregation rule gives [5, 25] $\mathcal{A}_M(f) = \sum_{i=1}^n \alpha_i \underline{P}_i(f)$ for any $f \in \mathcal{L}(\mathcal{X})$. This rule is called *linear opinion pooling* in [22], and its credal set is given by $\alpha_1 \mathcal{M}(\underline{P}_1) + \dots + \alpha_n \mathcal{M}(\underline{P}_n)$.

Pareto ($\mathcal{A}_P$) The Pareto rule [25] considers a gamble desirable when it is desirable for at least one member of the group and it is not undesirable for any other member. It leads to the lower prevision

$$\mathcal{A}_P(f) = \min\left\{\max_{j=1,\dots,n} \underline{P}_j(f), \min_{j=1,\dots,n} \overline{P}_j(f)\right\}, \forall f \in \mathcal{L}(\mathcal{X}).$$

We refer to [1, 6, 18, 22, 25] for a deeper discussion of the rationality behind these rules. Some of them are more appropriate from the perspective of probabilistic opinion pooling, while others are more interesting from the point of view of consensus seeking. See also [14] for a thorough discussion of these rules in the context of coherent lower previsions. In particular, in this reference it was shown that the Pareto rule need not give a coherent lower prevision as an output, that the conjunction rule does so if and only if the maximum of the inputs avoids sure loss and that the other rules always provide a coherent lower prevision. With respect to the conjunction, it follows from [26, Sect. 3.1.1] that if $\max_{i=1,\dots,n} \underline{P}_i$ does not avoid sure loss, then its natural extension takes the value $+\infty$ on any gamble f, and is therefore not coherent. In the remainder of this paper, we shall investigate if they are a closed operator when the input belongs to the specific type of coherent lower previsions enumerated in Section 2.

One important remark is that even though the above rules can be applied on any finite number of models, here we shall focus on the particular case of $n = 2$. This is no restriction in what concerns the conjunction and disjunction rules, because they are associative [14]; it can also be checked that the majority of the results generalise immediately to the case of an arbitrary n.

4. Comparative probabilities

We begin our analysis considering comparative probabilities. Before starting our analysis, we give the following useful result.

Lemma 4.1. *Let $\underline{P}$ be a comparative probability determined by $\mathcal{L}$. If $(x_i, x_j) \notin \mathcal{L}$, then $\underline{P}(I_{x_i} - I_{x_j}) < 0$.*

Proof. If $(x_i, x_j) \notin \mathcal{L}$, then the set $H(x_j)$ of predecessors of x_j in the graph associated with $\mathcal{L}$ [12] contains x_j but not x_i. Since the uniform distribution P on $H(x_j)$ belongs to $\mathcal{M}(\mathcal{L})$ by [12, Thm. 1], we deduce that $\underline{P}(I_{x_i} - I_{x_j}) \leq P(I_{x_i} - I_{x_j}) = P(\{x_i\}) - P(\{x_j\}) = 0 - \frac{1}{|H(x_j)|} < 0$. □

4.1. Conjunction.

We start proving that comparative probabilities are preserved under conjunction.

Proposition 4.1. *Consider two comparative probabilities determined by $\mathcal{L}_1$ and $\mathcal{L}_2$. Then, their conjunction is again a comparative probability and it is determined by $\mathcal{L}_1 \cap \mathcal{L}_2$.*

Proof. This follows immediately from the definition of the credal sets $\mathcal{M}(\mathcal{L}_1), \mathcal{M}(\mathcal{L}_2)$ and $\mathcal{M}(\mathcal{L}_1 \cup \mathcal{L}_2)$. □

4.2. Disjunction.

In contrast, the disjunction rule is not closed within the family of comparative probabilities.

Example 4.1. Consider a three element possibility space $\mathcal{X} = \{x_1, x_2, x_3\}$ and the comparative probabilities determined by $\mathcal{L}_1 = \{(x_1, x_2)\}$ and $\mathcal{L}_2 = \{(x_3, x_2)\}$. Applying the disjunction rule, we obtain

$$\mathcal{CH}\big(\mathcal{M}(\mathcal{L}_1) \cup \mathcal{M}(\mathcal{L}_2)\big) = \{P \in \mathbb{P}(\mathcal{X}) \mid P(\{x_2\}) \leq 0.5\},$$

that cannot be expressed as a comparative probability. ♦

Even though the answer is negative, we can characterise the conditions under which the disjunction of two comparative probability models is again a comparative probability.

Proposition 4.2. *Consider two comparative probabilities determined by $\mathcal{L}_1$ and $\mathcal{L}_2$. Then, their disjunction is a comparative probability if and only if $\mathcal{CH}\big(\mathcal{M}(\mathcal{L}_1) \cup \mathcal{M}(\mathcal{L}_2)\big) = \mathcal{M}(\mathcal{L}_1 \cap \mathcal{L}_2)$. In such a case, the disjunction is determined by $\mathcal{L}_1 \cap \mathcal{L}_2$.*

Proof. Since the direct implication is trivial, we only need to prove the converse. Assume that the disjunction is a comparative probability induced by some set of comparisons $\mathcal{L}^{\cup} \subseteq \mathcal{X} \times \mathcal{X}$, meaning that $\mathcal{CH}\big(\mathcal{M}(\mathcal{L}_1) \cup \mathcal{M}(\mathcal{L}_2)\big) = \mathcal{M}(\mathcal{L}^{\cup})$, and let us prove that the comparison set $\mathcal{L}^{\cup}$ coincides with $\mathcal{L}_1 \cap \mathcal{L}_2$: $\mathcal{L}^{\cup} = \mathcal{L}_1 \cap \mathcal{L}_2$.

On the one hand, if $(x_i, x_j) \in \mathcal{L}^{\cup}$ then $P(\{x_i\}) \geq P(\{x_j\})$ for any $P \in \mathcal{CH}\big(\mathcal{M}(\mathcal{L}_1) \cup \mathcal{M}(\mathcal{L}_2)\big)$. In particular, this implies that $P(\{x_i\}) \geq P(\{x_j\})$ for any $P \in \mathcal{M}(\mathcal{L}_1) \cup \mathcal{M}(\mathcal{L}_2)$, whence $(x_i, x_j) \in \mathcal{L}_1 \cap \mathcal{L}_2$. This implies that $\mathcal{L}^{\cup} \subseteq \mathcal{L}_1 \cap \mathcal{L}_2$.

On the other hand, if $(x_i, x_j) \in \mathcal{L}_1 \cap \mathcal{L}_2$, then $P(\{x_i\}) \geq P(\{x_j\})$ for any $P \in \mathcal{M}(\mathcal{L}_1) \cup \mathcal{M}(\mathcal{L}_2)$. Given $P \in \mathcal{CH}\big(\mathcal{M}(\mathcal{L}_1) \cup \mathcal{M}(\mathcal{L}_2)\big)$, there are $\alpha \in (0,1)$, $P_1 \in \mathcal{M}(\mathcal{L}_1)$ and $P_2 \in \mathcal{M}(\mathcal{L}_2)$ such that $P = \alpha P_1 + (1-\alpha) P_2$. Thus, $P(\{x_i\}) = \alpha P_1(\{x_i\}) + (1-\alpha) P_2(\{x_i\}) \geq \alpha P_1(\{x_j\}) + (1-\alpha) P_2(\{x_j\}) = P(\{x_j\})$, meaning that $(x_i, x_j) \in \mathcal{L}^{\cup}$. Therefore, $\mathcal{L}_1 \cap \mathcal{L}_2 \subseteq \mathcal{L}^{\cup}$. □

Following the terminology in [16, 17], the lower envelope of $\mathcal{M}(\mathcal{L}_1 \cap \mathcal{L}_2)$ is the unique undominated outer approximation in the family of comparative probabilities of the disjunction, meaning that any other comparative probability $\mathcal{M}(\mathcal{L})$ that outer approximates $\mathcal{CH}\big(\mathcal{M}(\mathcal{L}_1) \cup \mathcal{M}(\mathcal{L}_2)\big)$ satisfies $\mathcal{M}(\mathcal{L}) \supseteq \mathcal{M}(\mathcal{L}_1 \cap \mathcal{L}_2)$.

Corollary 4.1. *Consider two comparative probabilities determined by $\mathcal{L}_1$ and $\mathcal{L}_2$. There is a unique undominated outer approximation in the family of comparative probabilities and it is determined by $\mathcal{L}_1 \cap \mathcal{L}_2$.*

Proof. Let $\mathcal{M}(\mathcal{L}^*)$ be the credal set of an outer approximation of the disjunction, and consider $(x_i, x_j) \in \mathcal{L}^*$. Then $P(\{x_i\}) \geq P(\{x_j\})$ for any $P \in \mathcal{M}(\mathcal{L}^*)$, whence $P(\{x_i\}) \geq P(\{x_j\})$ for any $P \in \mathcal{M}(\mathcal{L}_1)$ and any $P \in \mathcal{M}(\mathcal{L}_2)$. Hence, $(x_i, x_j) \in \mathcal{L}_1 \cap \mathcal{L}_2$. We conclude that $\mathcal{L}^* \subseteq \mathcal{L}_1 \cap \mathcal{L}_2$, and consequently the credal set of any comparative probability that outer approximates the disjunction must be included in $\mathcal{M}(\mathcal{L}_1 \cap \mathcal{L}_2)$. □

4.3. Convex mixtures.

Using Lemma 4.1 we can give the form of the set of comparisons when the convex mixture is a comparative probability.

Proposition 4.3. *Consider two comparative probabilities determined by $\mathcal{L}_1$ and $\mathcal{L}_2$, and $\alpha \in (0,1)$. If their convex mixture is a comparative probability, then it is determined by $\mathcal{L}^{\alpha} = \mathcal{L}_1 \cap \mathcal{L}_2$.*

Proof. On the one hand, if $(x_i, x_j) \in \mathcal{L}_1 \cap \mathcal{L}_2$, then $P(\{x_i\}) \geq P(\{x_j\})$ for any $P \in \mathcal{M}(\mathcal{L}_1) \cup \mathcal{M}(\mathcal{L}_2)$. Then, for any $P \in \alpha\mathcal{M}(\mathcal{L}_1) + (1-\alpha)\mathcal{M}(\mathcal{L}_2)$ there are $P_1 \in \mathcal{M}(\mathcal{L}_1)$ and $P_2 \in \mathcal{M}(\mathcal{L}_2)$ such that $P = \alpha P_1 + (1-\alpha) P_2$, and $P(\{x_i\}) = \alpha P_1(\{x_i\}) + (1-\alpha) P_2(\{x_i\}) \geq \alpha P_1(\{x_j\}) + (1-\alpha) P_2(\{x_j\}) = P(\{x_j\})$, meaning that $(x_i, x_j) \in \mathcal{L}^{\alpha}$. This implies $\mathcal{L}_1 \cap \mathcal{L}_2 \subseteq \mathcal{L}^{\alpha}$.

On the other hand, if $(x_i, x_j) \in \mathcal{L}^\alpha$, this implies that

$$0 = \mathcal{A}_M(I_{x_i} - I_{x_j}) \\ = \alpha \underline{P}_1(I_{x_i} - I_{x_j}) + (1-\alpha)\underline{P}_2(I_{x_i} - I_{x_j}). \quad (1)$$

If $(x_i, x_j) \notin \mathcal{L}_1 \cap \mathcal{L}_2$, then at least one of $\underline{P}_1(I_{x_i} - I_{x_j})$ and $\underline{P}_2(I_{x_i} - I_{x_j})$ is negative from Lemma 4.1, while the other will be lower than or equal to zero, meaning that Eq. (1) would be violated. We conclude that $\mathcal{L}^\alpha \subseteq \mathcal{L}_1 \cap \mathcal{L}_2$. □

This allows us to establish that the mixture is a comparative probability just in very particular scenarios.

Corollary 4.2. *Consider two comparative probabilities determined by $\mathcal{L}_1$ and $\mathcal{L}_2$. Their convex mixture is a comparative probability if and only if $\mathcal{L}_1 = \mathcal{L}_2$.*

Proof. The direct implication is trivial. For the converse, if $\alpha\mathcal{M}(\mathcal{L}_1) + (1-\alpha)\mathcal{M}(\mathcal{L}_2)$ is the credal set of a comparative probability, then it is induced by the set of comparisons $\mathcal{L}_1 \cap \mathcal{L}_2$, from Proposition 4.3. Also, we have the following inclusions:

$$\mathcal{M}(\mathcal{L}_1 \cap \mathcal{L}_2) \supseteq \mathcal{CH}\big(\mathcal{M}(\mathcal{L}_1) \cup \mathcal{M}(\mathcal{L}_2)\big) \\ \supseteq \alpha\mathcal{M}(\mathcal{L}_1) + (1-\alpha)\mathcal{M}(\mathcal{L}_2) = \mathcal{M}\big(\mathcal{L}_1 \cap \mathcal{L}_2\big)$$

where the first inclusion follows from Corollary 4.1. We deduce that the convex hull of the disjunction coincides with the convex mixture, but this is only possible if $\mathcal{M}(\mathcal{L}_1) = \mathcal{M}(\mathcal{L}_2)$ or, equivalently, if $\mathcal{L}_1 = \mathcal{L}_2$. □

4.4. Pareto. As discussed before, the conjunction of comparative probabilities is coherent, and therefore according to [14, Prop.7.1] the Pareto rule is given by $\mathcal{A}_P(f) = \max\{\underline{P}_1(f), \underline{P}_2(f)\}$ for any $f \in \mathcal{L}(\mathcal{X})$. From this condition we obtain the set of comparisons determining its credal set when the Pareto rule gives rise to a comparative probability.

Proposition 4.4. *Consider two comparative probabilities determined by $\mathcal{L}_1$ and $\mathcal{L}_2$. The Pareto rule is a comparative probability determined by $\mathcal{L}^P$ if and only if $\mathcal{L}^P = \mathcal{L}_1 \cup \mathcal{L}_2$.*

Proof. The direct implication is trivial. To see the converse, assume that $\mathcal{A}_P$ is a comparative probability associated with $\mathcal{L}^P$. If $(x_i, x_j) \in \mathcal{L}^P$, then $0 = \mathcal{A}_P(I_{x_i} - I_{x_j}) = \max_{\ell=1,2} \underline{P}_\ell(I_{x_i} - I_{x_j})$, which implies that $(x_i, x_j) \in \mathcal{L}_\ell$ for some $\ell \in \{1, 2\}$, using Lemma 4.1.

Conversely, if $(x_i, x_j) \in \mathcal{L}_1 \cup \mathcal{L}_2$, we get that $\underline{P}_\ell(I_{x_i} - I_{x_j}) = 0$ whenever $(x_i, x_j) \in \mathcal{L}_\ell$ and it is negative otherwise. Thus, $\mathcal{A}_P(I_{x_i} - I_{x_j}) = \max_{\ell=1,2} \underline{P}_\ell(I_{x_i} - I_{x_j}) = 0$, meaning that $(x_i, x_j) \in \mathcal{L}^P$. □

From this result and Proposition 4.4 we deduce that the Pareto rule gives rise to a comparative probability if and only if it coincides with the conjunction. Moreover, in that case $\mathcal{A}_P$ gives a coherent lower prevision.

5. 2-MONOTONE CAPACITIES

We turn next our attention to 2-monotone lower probabilities.

5.1. Conjunction. In general, the conjunction of two 2-monotone capacities need not be 2-monotone: it suffices to observe that any probability measure is 2-monotone and take two different probability measures $P_1 \neq P_2$ (recall that if the maximum of a number of coherent lower previsions does not avoid sure loss then its natural extension is constant on $+\infty$, and therefore incoherent). For an example where $\max\{\underline{P}_1, \underline{P}_2\}$ avoids sure loss and therefore the conjunction produces a coherent lower probability, consider the following example:

Example 5.1. Consider $\mathcal{X} = \{x_1, x_2, x_3, x_4\}$ and let $\underline{P}_1$ and $\underline{P}_2$ be the lower envelopes on events of the credal sets

$$\mathcal{M}_1 = \{(1/4 + \alpha, 1/4 + \beta, 1/4 - \alpha, 1/4 - \beta) \mid \alpha, \beta \leq 1/4\} \\ \mathcal{M}_2 = \{(1/4 + \alpha, 1/4 + \beta, 1/4 - \beta, 1/4 - \alpha) \mid \alpha, \beta \leq 1/4\}.$$

It can be verified that both $\underline{P}_1, \underline{P}_2$ are 2-monotone. Their conjunction $\mathcal{A}_C$ coincides with $\max\{\underline{P}_1, \underline{P}_2\}$, and satisfies $\mathcal{A}_C(\{x_1, x_2, x_3\}) + \mathcal{A}_C(\{x_3\}) = 0.75 < 1 = \mathcal{A}_C(\{x_1, x_3\}) + \mathcal{A}_C(\{x_2, x_3\})$, and as a consequence it is not 2-monotone. ♦

In spite of this, and inspired by [15, Prop. 7], it is possible to obtain a sufficient condition for the conjunction to preserve 2-monotonicity:

Proposition 5.1. *Let $\underline{P}_1, \underline{P}_2$ be 2-monotone lower probabilities. If $\mathcal{M}(\underline{P}_1) \cup \mathcal{M}(\underline{P}_2)$ is convex, then $\mathcal{A}_C = \max\{\underline{P}_1, \underline{P}_2\}$ and it is 2-monotone.*

Proof. It suffices to establish [24, Cor. 6.5] that for any pair of events $A \subseteq B$ there exists some $P \geq \mathcal{A}_C$ such that $P(A) = \mathcal{A}_C(A)$ and $P(B) = \mathcal{A}_C(B)$. Consider then such events A, B. Since $\underline{P}_1$ is 2-monotone, there exists some $P_1 \in \mathcal{M}(\underline{P}_1)$ such that $P_1(A) = \underline{P}_1(A)$ and $P_1(B) = \underline{P}_1(B)$; similarly, there is some $P_2 \in \mathcal{M}(\underline{P}_2)$ such that $P_2(A) = \underline{P}_2(A)$ and $P_2(B) = \underline{P}_2(B)$. Since $\mathcal{M}(\underline{P}_1) \cup \mathcal{M}(\underline{P}_2)$ is convex, it follows from [28, Thm.6] that there is some $\alpha \in [0, 1]$ such that $P = \alpha\underline{P}_1 + (1-\alpha)\underline{P}_2$ belongs to $\mathcal{M}(\underline{P}_1) \cap \mathcal{M}(\underline{P}_2)$ and therefore dominates $\mathcal{A}_C$. On the other hand, $P(A) \leq \max\{P_1(A), P_2(A)\} \leq \mathcal{A}_C(A)$ and $P(B) \leq \max\{P_1(B), P_2(B)\} \leq \mathcal{A}_C(B)$, whence $P(A) = \mathcal{A}_C(A)$ and $P(B) = \mathcal{A}_C(B)$. Thus, $\mathcal{A}_C$ is 2-monotone and $\mathcal{A}_C(A) = \max\{\underline{P}_1(A), \underline{P}_2(A)\}$ for any event $A \subseteq \mathcal{X}$. □

This sufficient condition is not necessary:

Example 5.2. On the possibility space $\mathcal{X} = \{x_1, x_2, x_3\}$ consider the following coherent lower probabilities:

B	$\{x_1\}$	$\{x_2\}$	$\{x_3\}$	$\{x_1, x_2\}$	$\{x_1, x_3\}$	$\{x_2, x_3\}$
$\underline{P}_1(B)$	0.16	0	0	0.4	0.4	0
$\underline{P}_2(B)$	0.15	0	0	0.5	0.3	0
$\mathcal{A}_C(B)$	0.16	0	0	0.5	0.4	0

and taking the values 0, 1 on $\emptyset$ and $\mathcal{X}$, respectively. Since $\max\{\underline{P}_1, \underline{P}_2\}$ is coherent, by [14, Prop. 7] it coincides with the conjunction of $\underline{P}_1, \underline{P}_2$. Moreover, since we have cardinality three, $\underline{P}_1, \underline{P}_2$ and $\max\{\underline{P}_1, \underline{P}_2\}$ are all 2-monotone.

To see however that $\mathcal{M}(\underline{P}_1) \cup \mathcal{M}(\underline{P}_2)$ is not convex, note that $P_1 = (0.16, 0.24, 0.6) \in \mathcal{M}(\underline{P}_1)$, $P_2 = (0.15, 0.35, 0.5) \in \mathcal{M}(\underline{P}_2)$ but $P = 0.5P_1 + 0.5P_2 \notin \mathcal{M}(\underline{P}_1) \cup \mathcal{M}(\underline{P}_2)$: we have that $P(\{x_1\}) = 0.155 < \underline{P}_1(\{x_1\})$ and $P(\{x_1, x_2\}) = 0.45 < \underline{P}_2(\{x_1, x_2\})$. ♦

5.2. Disjunction. That 2-monotonicity is not preserved by disjunction follows from the fact that coherent lower probabilities, that are the lower envelopes of probability measures (which are in particular 2-monotone) need not be 2-monotone. It is nevertheless possible to establish the following sufficient condition:

Proposition 5.2. *Let $\underline{P}_1$ be a 2-monotone lower probability and let $\underline{P}_2$ be the minitive measure given by:*

$$\underline{P}_2(A) = \begin{cases} 1 & \text{if } C \subseteq A \\ 0 & \text{otherwise,} \end{cases}$$

where $C \subseteq \mathcal{X}$ is a fixed event. Then, the restriction to events of $\mathcal{A}_D = \min\{\underline{P}_1, \underline{P}_2\}$ is a 2-monotone capacity.

Proof. It follows from the definition that $\mathcal{A}_D(A) = \underline{P}_1(A)$ if $C \subseteq A$ and is equal to 0 otherwise. As a consequence, given events $A, B \subseteq \mathcal{X}$, there are two possibilities:

- If for instance $C \nsubseteq A$, then $\mathcal{A}_D(A \cup B) + \mathcal{A}_D(A \cap B) - \mathcal{A}_D(A) - \mathcal{A}_D(B) = \mathcal{A}_D(A \cup B) - \mathcal{A}_D(B) \geq 0$, where the inequality follows from the monotonicity of $\mathcal{A}_D$. A similar result can be established if $C \nsubseteq B$.
- If on the other hand $C \subseteq A$ and $C \subseteq B$, then also $C \subseteq A \cap B$, whence $\mathcal{A}_D(A \cup B) + \mathcal{A}_D(A \cap B) - \mathcal{A}_D(A) - \mathcal{A}_D(B) = \underline{P}_1(A \cup B) + \underline{P}_1(A \cap B) - \underline{P}_1(A) - \underline{P}_1(B) \geq 0$, taking into account that $\underline{P}_1$ is 2-monotone. □

5.3. Convex mixtures. Like coherence [26, Thm.2.6.4], it is immediate to establish that 2-monotonicity is preserved by convex mixtures:

Proposition 5.3. *Let $\underline{P}_1, \underline{P}_2$ be two 2-monotone lower probabilities. Then, the convex mixture is also 2-monotone.*

Proof. Let $A, B \subseteq \mathcal{X}$. Then $\underline{P}(A \cup B) + \underline{P}(A \cap B) = \alpha \underline{P}_1(A \cup B) + (1-\alpha)\underline{P}_2(A \cup B) + \alpha \underline{P}_1(A \cap B) + (1-\alpha)\underline{P}_2(A \cap B) = \alpha\big(\underline{P}_1(A \cup B) + \underline{P}_1(A \cap B)\big) + (1-\alpha)\big(\underline{P}_2(A \cup B) + \underline{P}_2(A \cap B)\big) \geq \alpha\big(\underline{P}_1(A) + \underline{P}_1(B)\big) + (1-\alpha)\big(\underline{P}_2(A) + \underline{P}_2(B)\big) = \underline{P}(A) + \underline{P}(B)$, meaning that $\underline{P}$ is 2-monotone. □

5.4. Pareto. From [14, Prop.7], when $\max\{\underline{P}_1, \underline{P}_2\}$ avoids sure loss the Pareto rule coincides with this lower probability. However, this need not be coherent (that is, it need not coincide with the result of the conjunction rule), and as a consequence it need not be 2-monotone either. For an explicit example on a space of cardinality three, we refer to [14, Ex.3].

6. Probability intervals

In this section we analyse the aggregation of probability intervals, which are particular cases of 2-monotone capacities. Note that some results concerning the conjunction and disjunction were already given in [3].

6.1. Conjunction. The same comments made for the conjunction of 2-monotone capacities allow us to deduce that the family of probability intervals is not closed under conjunction, simply because the conjunction may not be coherent (i.e., the intersection of their respective credal sets may be empty); for this, it suffices to take into account that probability measures are particular cases of probability intervals, and take two different probability measures $P_1 \neq P_2$. From [3, Prop.7], given two probability intervals $\mathcal{I}$ and $\mathcal{I}'$, they are compatible (i.e., the intersection of their respective credal sets is non-empty) if and only if for every $i = 1, \dots, k$ it holds that $l_i \leq u'_i, l'_i \leq u_i$ and $\sum_{i=1}^{k} \max\{l_i, l'_i\} \leq 1 \leq \sum_{i=1}^{k} \min\{u_i, u'_i\}$. In that case, their conjunction is again a probability interval. Notice the difference with the situation depicted in Example 5.1: while the conjunction is not a closed rule in the class of 2-monotone capacities, it is closed if we focus on the subclass of probability intervals.

Proposition 6.1. *[3, Prop.7] Let $\underline{P}_1$ and $\underline{P}_2$ be two compatible coherent probability intervals determined by $\mathcal{I}_1$ and $\mathcal{I}_2$. Then, their conjunction is again a probability interval where for every $i = 1, \dots, k$:*

$$\mathcal{A}_C(\{x_i\}) = \max\Big\{l_i, l'_i, 1 - \sum_{j \neq i} \min\{u_i, u'_i\}\Big\}, \text{ and}$$
$$\overline{\mathcal{A}}_C(\{x_i\}) = \min\Big\{u_i, u'_i, 1 - \sum_{j \neq i} \max\{l_i, l'_i\}\Big\}.$$

6.2. Disjunction. De Campos et al. also analysed the disjunction of two coherent probability intervals. In this case, they show that the disjunction is not a probability interval in general [3, Ex.1]. They also showed [3, Prop.9] that there is a unique undominated outer approximation of the disjunction using a probability interval, given by $\mathcal{I} = \{[\min\{l_i, l'_i\}, \max\{u_i, u'_i\}] \mid i = 1, \dots, k\}$. See also [16, Prop.5] for a similar result.

6.3. Convex mixtures. Given two coherent probability intervals determined by $\mathcal{L}_1$ and $\mathcal{L}_2$, and a parameter $\alpha \in (0, 1)$, we can define another probability interval by:

$$[l_i, u_i] = \big[\alpha l_i^1 + (1-\alpha) l_i^2, \alpha u_i^1 + (1-\alpha) u_i^2\big] \qquad (2)$$

for any $i = 1, \dots, k$. The following result is almost immediate and its proof is omitted.

Proposition 6.2. *Let $\underline{P}_1$ and $\underline{P}_2$ be two coherent probability intervals determined by $\mathcal{I}_1$ and $\mathcal{I}_2$, and $\alpha \in (0,1)$. Define $\mathcal{I}^\alpha = \{[l_i, u_i]\}_{i=1,\dots,k}$ where $[l_i, u_i]$ are given by Eq. (2). Then $\mathcal{I}^\alpha$ determines a coherent probability interval and $\alpha\mathcal{M}(\mathcal{I}_1) + (1-\alpha)\mathcal{M}(\mathcal{I}_2) \subseteq \mathcal{M}(\mathcal{I}^\alpha)$.*

Perhaps surprisingly, the convex mixture of two probability interval does not coincide in general with the probability interval determined by $\mathcal{L}^\alpha$, as we show in the following example. In other words, the mixture need not be determined by its lower and upper probabilities on the singletons, and the inclusion in the previous proposition may be strict.

Example 6.1. Consider the coherent probability intervals determined by $\mathcal{I}_1$, $\mathcal{I}_2$, and $\mathcal{I}^\alpha$ for $\alpha = 0.5$:

	x_1	x_2	x_3	x_4
l_i^1	0.1	0.2	0.2	0.2
u_i^1	0.2	0.3	0.3	0.3
l_i^2	0.2	0.2	0.1	0.1
u_i^2	0.4	0.4	0.3	0.35
l_i	0.15	0.2	0.15	0.15
u_i	0.3	0.35	0.3	0.325

Taking $A = \{x_1, x_2\}$, it holds that:

$$\underline{P}_1(A) = \max\{0.1 + 0.2, 1 - 0.3 - 0.3\} = 0.4.$$
$$\underline{P}_2(A) = \max\{0.2 + 0.2, 1 - 0.3 - 0.35\} = 0.4.$$
$$\mathcal{A}_M(A) = \max\{0.15 + 0.2, 1 - 0.3 - 0.325\} = 0.375.$$

Hence, $\mathcal{A}_M(A) \neq \alpha\underline{P}_1(A) + (1-\alpha)\underline{P}_2(A)$, meaning that the convex mixture $\mathcal{A}_M$ is not a probability interval. ♦

The previous example, together with Proposition 6.2, allows us to deduce that $\mathcal{L}^\alpha$ determines an undominated outer approximation of the convex mixture.

Corollary 6.1. *Let $\underline{P}_1$ and $\underline{P}_2$ be two coherent probability intervals determined by $\mathcal{L}_1$ and $\mathcal{L}_2$, and $\alpha \in (0,1)$. Then, $\mathcal{L}^\alpha$ determines an undominated outer approximation of $\mathcal{A}_M$ in the family of probability intervals.*

6.4. Pareto rule. Recall that, by [14, Prop.7], when $\max\{\underline{P}_1, \underline{P}_2\}$ avoids sure loss it agrees with $\mathcal{A}_P$. This produces the following lower and upper probabilities on the singletons:

$$\mathcal{A}_P(\{x_i\}) = \max\{l_i^1, l_i^2\}, \quad \overline{\mathcal{A}}_P(\{x_i\}) = \min\{u_i^1, u_i^2\}$$

for any $i = 1, \dots, k$. Our next example shows that, even in that case, the Pareto rule need be a probability interval.

Example 6.2. Consider the possibility space $\mathcal{X} = \{x_1, x_2, x_3, x_4\}$ and the coherent probability intervals $\underline{P}_1$ and $\underline{P}_2$ determined by $\mathcal{L}_1$ and $\mathcal{L}_2$, where:

	x_1	x_2	x_3	x_4
l_i^1	0.05	0.15	0.1	0.3
u_i^1	0.2	0.4	0.2	0.5
l_i^2	0.1	0.1	0.1	0.1
u_i^2	0.3	0.3	0.3	0.3

It holds that $\max\{\underline{P}_1, \underline{P}_2\}$ avoids sure loss because the probability associated with the mass function $P = (0.2, 0.3, 0.2, 0.3)$ dominates $\max\{\underline{P}_1, \underline{P}_2\}$. Therefore, $\mathcal{A}_P = \max\{\underline{P}_1, \underline{P}_2\}$, but it is not coherent. Indeed, $\mathcal{A}_P(\{x_3\}) = \max\{\underline{P}_1(\{x_3\}), \underline{P}_2(\{x_3\})\} = 0.1$ but $\mathcal{M}(\mathcal{A}_P) = \{P\}$, meaning that its lower envelope $\underline{P}$ satisfies $\underline{P}(\{x_3\}) = P(\{x_3\}) = 0.2$. ♦

On the other hand, when $\max\{\underline{P}_1, \underline{P}_2\}$ is coherent, then the Pareto rule coincides with the conjunction, and then it produces a probability interval, as shown in Proposition 6.1.

Finally, when $\max\{\underline{P}_1, \underline{P}_2\}$ does not avoid sure loss, the Pareto rule on two probability intervals need not produce another probability interval.

Example 6.3. Consider a four-element possibility space $\mathcal{X} = \{x_1, x_2, x_3, x_4\}$ and the coherent probability intervals $\underline{P}_1$ and $\underline{P}_2$ determined by $\mathcal{L}_1$ and $\mathcal{L}_2$ given by:

	x_1	x_2	x_3	x_4
l_i^1	0.1	0.1	0.1	0.1
u_i^1	0.3	0.3	0.3	0.3
l_i^2	0.1	0.1	0.1	0.4
u_i^2	0.2	0.2	0.2	0.5

It can be easily seen that $\max\{\underline{P}_1, \underline{P}_2\}$ does not avoid sure loss. Applying the Pareto rule we get $\mathcal{A}_P(\{x_1\}) = \mathcal{A}_P(\{x_2\}) = \mathcal{A}_P(\{x_3\}) = 0.1$, $\mathcal{A}_P(\{x_4\}) = 0.3$ and $\overline{\mathcal{A}}_P(\{x_1\}) = \overline{\mathcal{A}}_P(\{x_2\}) = \overline{\mathcal{A}}_P(\{x_3\}) = 0.4$, $\overline{\mathcal{A}}_P(\{x_4\}) = 0.7$. If $\mathcal{A}_P$ were a probability interval, we would have $\mathcal{A}_P(\{x_1, x_2\}) = 0.2$. However, it can be checked that $\mathcal{A}_P(\{x_1, x_2\}) = 0.4$, whence $\mathcal{A}_P$ is not a probability interval. ♦

7. Belief functions

We focus next on another particular case of 2-monotone capacities: belief functions.

7.1. Conjunction. It is quite straightforward to observe that the conjunction of two belief functions is not necessarily a belief function, simply by considering two different probability measures. More interestingly, even if their associated credal sets are compatible (i.e., if $\max\{\underline{P}_1, \underline{P}_2\}$ avoids sure loss), the conjunction need not produce a belief function. For a counterexample, consider the lower probabilities $\underline{P}_1, \underline{P}_2$ in Example 5.1, that are belief functions in addition to being 2-monotone.

In Proposition 5.1, we established a sufficient condition for the conjunction of 2-monotone capacities to be 2-monotone. The result does not generalise to belief functions, as our next example shows:

Example 7.1. Let $\mathcal{X} = \{x_1, x_2, x_3\}$ and consider the belief functions $\underline{P}_1, \underline{P}_2$ given in the following table, where we also show $\underline{P} = \max\{\underline{P}_1, \underline{P}_2\}$:

	$\underline{P}_1$	m_1	$\underline{P}_2$	m_2	$\underline{P}$	m
$\{x_1\}$	0	0	0	0	0	0
$\{x_2\}$	0	0	0	0	0	0
$\{x_3\}$	0	0	0	0	0	0
$\{x_1, x_2\}$	0.5	0.5	0.5	0.5	0.5	0.5
$\{x_1, x_3\}$	0	0	0.5	0.5	0.5	0.5
$\{x_2, x_3\}$	0.5	0.5	0	0	0.5	0.5
$\mathcal{X}$	1	0	1	0	1	-0.5

While $\underline{P}_1$ and $\underline{P}_2$ are belief functions, $\underline{P}$ is not since $m(\mathcal{X}) < 0$. However, $\mathcal{M}(\underline{P}_1) \cup \mathcal{M}(\underline{P}_2)$ is a convex set, and it coincides with the convex hull of $\{(0,1,0), (0,0,1), (0.5,0.5,0), (0.5,0,0.5)\}$. Applying Proposition 5.1, we obtain that $\underline{P}$ is 2-monotone and agrees with the conjunction of $\underline{P}_1, \underline{P}_2$. ♦

7.2. Disjunction. Clearly, the disjunction of two belief functions does not produce a belief function in general: if it did, any coherent lower probability would be a belief function, and we know that this is not the case.

On the other hand, we shall see in Proposition 9.1 that the minimum of two minitive measures is again minitive, and as a consequence also a belief function. We may then wonder if the minimum of a belief function and a minitive measure is again a belief function. Our next example shows that this is not the case:

Example 7.2. Let $\mathcal{X} = \{x_1, x_2, x_3, x_4\}$ and $\underline{P}_1, \underline{P}_2$ be the belief functions determined by the following Möbius inverses:

$$m_1(A) = \begin{cases} 0.1 & \text{if } |A| = 1, 4. \\ 0.05 & \text{if } |A| = 2, 3. \end{cases}$$

$$m_2(\{x_1\}) = m_2(\{x_1, x_2\}) = m_2(\{x_1, x_2, x_3\}) = m_2(\mathcal{X}) = 0.25,$$

and m_2 is zero elsewhere. Then $\mathcal{A}_D = \min\{\underline{P}_1, \underline{P}_2\}$ is not a belief function because its Möbius inverse satisfies $m_D(\{x_1, x_3, x_4\}) < 0$. ♦

In fact, the disjunction $\mathcal{A}_D$ in the above example is not 2-monotone. On the other hand, the counterexample cannot be established for smaller cardinalities:

Proposition 7.1. *Let $\mathcal{X} = \{x_1, x_2, x_3\}$, and let $\underline{P}_1, \underline{P}_2$ be two belief functions. If $\underline{P}_2$ is minitive, then $\mathcal{A}_D$ is a belief function.*

Proof. If $\underline{P}_2$ is minitive, then its focal elements are nested. We may then assume without loss of generality that $\mathcal{F}_2 \subseteq \{\{x_1\}, \{x_1, x_2\}, \mathcal{X}\}$. This implies that $\mathcal{A}_D(\{x_2\}) = \mathcal{A}_D(\{x_3\}) = \mathcal{A}_D(\{x_2, x_3\}) = 0$, and that the Möbius inverse of $\mathcal{A}_D$ satisfies

$$\begin{aligned} m_D(\{x_1\}) &= \mathcal{A}_D(\{x_1\}), \\ m_D(\{x_1, x_2\}) &= \mathcal{A}_D(\{x_1, x_2\}) - \mathcal{A}_D(\{x_1\}), \\ m_D(\{x_1, x_3\}) &= \mathcal{A}_D(\{x_1, x_3\}) - \mathcal{A}_D(\{x_1\}), \\ m_D(\mathcal{X}) &= 1 - m_D(\{x_1\}) - m_D(\{x_1, x_2\}) - m_D(\{x_1, x_3\}), \end{aligned}$$

and that it is zero otherwise. The coherence of $\mathcal{A}_D$ allows us to deduce that $m_D(\{x_1\}), m_D(\{x_1, x_2\})$ and $m_D(\{x_1, x_3\})$ are non-negative. On the other hand, $m_D(\mathcal{X}) = 1 + \mathcal{A}_D(\{x_1\}) - \mathcal{A}_D(\{x_1, x_2\}) - \mathcal{A}_D(\{x_1, x_3\}) \geq 0$, taking into account that any coherent lower probability on a space of cardinality three is 2-monotone [24] and applying the 2-monotonicity condition to $A = \{x_1, x_2\}$ and $B = \{x_1, x_3\}$. □

In spite of Example 7.2, it is possible to obtain some (albeit restrictive) condition for the minimum between a belief function and a minitive measure to be a belief function.

Proposition 7.2. *Let $\underline{P}_1$ be a belief function and $\underline{P}_2$ be a degenerate probability measure on a fixed $x \in \mathcal{X}$. Then, the restriction to events of $\mathcal{A}_D$ is a belief function.*

Proof. It follows from the definition that $\mathcal{A}_D(A) = \underline{P}_1(A)$ if $x \in A$ and is 0 otherwise. From this it follows that the Möbius inverse of $\mathcal{A}_D$ is given by

$$m_D(A) = \begin{cases} m_1(A) + m_1(A \setminus \{x\}) & \text{if } x \in A. \\ 0 & \text{otherwise.} \end{cases}$$

As a consequence, $\underline{P}$ is a belief function. □

In other words, the result shows that the combination of a belief function with a degenerate probability produces again a belief function. A somewhat related result in the context of multi-valued mappings was established in [11].

7.3. Convex mixtures. It is almost immediate to establish that belief functions are preserved by convex mixtures, and moreover to give a connection between their focal elements:

Proposition 7.3. *Let $\underline{P}_1, \underline{P}_2$ be two belief functions with respective families of focal elements $\mathcal{F}_1, \mathcal{F}_2$, and let $\mathcal{A}_M$ be their convex mixture for some $\alpha \in (0,1)$. Then $\mathcal{A}_M$ is a belief function and its family of focal elements is $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$.*

Proof. For any event A, it follows from the expression of the Möbius inverse m_M of $\mathcal{A}_M$ that $m_M(A) = \alpha m_1(A) + (1-\alpha) m_2(A)$. Since $\underline{P}_1, \underline{P}_2$ are belief functions, this allows us to establish that $m_M(A) \geq 0$ for every A, whence $\mathcal{A}_M$ is also a belief function. Also, it follows that $m_M(A) > 0$ if and only if $\max\{m_1(A), m_2(A)\} > 0$. Thus, $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$. □

7.4. Pareto. It is easy to verify that the lower probabilities in [14, Ex. 3], for which the Pareto rule does not produce a coherent lower probability, are in particular belief functions. As a consequence, this very same counterexample allows us to show that the family of belief functions is not closed under this rule.

8. P-BOXES

Next we continue with the analysis of the aggregation of p-boxes, which are particular cases of belief functions whose focal elements are nested.

8.1. Conjunction. We begin by analysing the conjunction. It is not difficult to prove a condition somewhat similar to the one established for probability intervals.

Proposition 8.1. *Let $\underline{P}_{(\underline{F}_1,\underline{F}_1)}$ and $\underline{P}_{(\underline{F}_2,\underline{F}_2)}$ be two p-boxes. Their conjunction is a p-box if and only if $\mathcal{M}(\underline{F}_1,\overline{F}_1) \cap \mathcal{M}(\underline{F}_2,\overline{F}_2) \neq \emptyset$. In that case, $\mathcal{M}(\mathcal{A}_C) = \mathcal{M}(\underline{F},\overline{F})$ where $\underline{F} = \max\{\underline{F}_1,\underline{F}_2\}$ and $\overline{F} = \min\{\overline{F}_1,\overline{F}_2\}$.*

Proof. Necessity is trivial; for sufficiency, if the intersection of the credal sets is non-empty, we get $\mathcal{M}(\underline{F}_1,\overline{F}_1) \cap \mathcal{M}(\underline{F}_2,\overline{F}_2) = \mathcal{M}(\underline{F},\overline{F})$, where $\underline{F} = \max\{\underline{F}_1,\underline{F}_2\}$ and $\overline{F} = \min\{\overline{F}_1,\overline{F}_2\}$. □

8.2. Disjunction. Regarding disjunction, the next example shows that the disjunction of two p-boxes is not a p-box in general.

Example 8.1. Consider an ordered possibility space $\mathcal{X} = \{x_1, x_2, x_3\}$ and the p-boxes determined by

	x_1	x_2	x_3
$[\underline{F}_1,\overline{F}_1]$	$[0,0]$	$[0,0.5]$	$[1,1]$
$[\underline{F}_2,\overline{F}_2]$	$[0.5,1]$	$[1,1]$	$[1,1]$

Their credal sets can be expressed as:

$$\mathcal{M}(\underline{F}_1,\overline{F}_1) = \{P = (0,\alpha,1-\alpha) \mid \alpha \in [0,0.5]\}.$$
$$\mathcal{M}(\underline{F}_2,\overline{F}_2) = \{P = (1-\beta,\beta,0) \mid \beta \in [0,0.5]\}.$$

The convex hull of their union is

$$\mathcal{CH}\Big(\mathcal{M}(\underline{F}_1,\overline{F}_1) \cup \mathcal{M}(\underline{F}_2,\overline{F}_2)\Big) = \{P \mid P(\{x_2\}) \leq 0.5\}.$$

It is not a p-box, because otherwise it must coincide with $\mathcal{M}(\underline{F},\overline{F})$ where $[\underline{F}(x_1),\overline{F}(x_1)] = [\underline{F}(x_2),\overline{F}(x_2)] = [0,1]$ and $[\underline{F}(x_3),\overline{F}(x_3)] = [1,1]$, meaning that $\mathcal{M}(\underline{F},\overline{F}) = \mathbb{P}(\mathcal{X})$, but it clearly differs from $\{P \mid P(\{x_2\}) \leq 0.5\}$. ♦

Nevertheless, and similarly to the case of probability intervals, we can easily determine the unique optimal outer approximation of the disjunction of two p-boxes $\underline{P}_{(\underline{F}_1,\overline{F}_1)}, \underline{P}_{(\underline{F}_2,\overline{F}_2)}$: it will be the p-box determined by $\big(\min\{\underline{F}_1,\underline{F}_2\}, \max\{\overline{F}_1,\overline{F}_2\}\big)$.

8.3. Convex mixtures. We can use Proposition 7.3 to characterise in which cases p-boxes are preserved by mixtures. For this aim, recall that a p-box is a belief function whose focal elements are ordered intervals.

Proposition 8.2. *Let $\underline{P}_{(\underline{F}_1,\overline{F}_1)}$ and $\underline{P}_{(\underline{F}_2,\overline{F}_2)}$ be two p-boxes, and denote by $\mathcal{F}_1$ and $\mathcal{F}_2$ their respective set of focal elements. Then, their mixture is a p-box if and only if $\mathcal{F}_1 \cup \mathcal{F}_2$ is formed by ordered intervals.*

Proof. By Proposition 7.3, the family of focal elements of the mixture is $\mathcal{F}_1 \cup \mathcal{F}_2$, and a belief function is a p-box if and only if the focal events are ordered intervals. □

Using this result, it is easy to give a counterexample where the convex mixture of two p-boxes is not a p-box.

Example 8.2. Consider a three-element ordered possibility space $\mathcal{X}$, and the p-boxes determined by

	x_1	x_2	x_3
$[\underline{F}_1,\overline{F}_1]$	$[0.2,0.2]$	$[0.5,0.5]$	$[1,1]$
$[\underline{F}_2,\overline{F}_2]$	$[0,1]$	$[0,1]$	$[1,1]$

On the one hand, $\mathcal{M}(\underline{F}_1,\overline{F}_1) = \{(0.2,0.3,0.5)\}$, and on the other hand $\mathcal{M}(\underline{F}_2,\overline{F}_2) = \mathbb{P}(\mathcal{X})$. Then, the convex mixture is a linear vacuous model (a convex combination of a probability measure and a vacuous model). Explicitly, taking $\alpha = 0.5$ we obtain:

A	$\{x_1\}$	$\{x_2\}$	$\{x_3\}$	$\{x_1,x_2\}$	$\{x_1,x_3\}$	$\{x_2,x_3\}$
$\mathcal{A}_M$	0.1	0.15	0.25	0.25	0.35	0.4

$\mathcal{A}_M$ is not only coherent but also a belief function. It determines the lower and upper cdfs $(\underline{F}_M,\overline{F}_M)$ given by $[\underline{F}_M(x_1),\overline{F}_M(x_1)] = [0.1,0.6]$, $[\underline{F}_M(x_2),\overline{F}_M(x_2)] = [0.25,0.75]$ and $[\underline{F}_M(x_3),\overline{F}_M(x_3)] = [1,1]$. However, the cdf associated with $P = (0.5,0,0.5)$ is bounded by $\underline{F}_M$ and $\overline{F}_M$, but $P \notin \mathcal{M}(\underline{P})$. Hence, $\mathcal{A}_M$ is not a p-box. ♦

8.4. Pareto. The Pareto rule applied to p-boxes may give rise to a non-coherent lower probability, that is therefore not associated with a p-box.

Example 8.3. Consider the possibility space $\mathcal{X} = \{x_1, x_2, x_3\}$ and the p-boxes determined by

	x_1	x_2	x_3
$[\underline{F}_1,\overline{F}_1]$	$[0.3,0.6]$	$[0.3,0.7]$	$[1,1]$
$[\underline{F}_2,\overline{F}_2]$	$[0,0.2]$	$[0.1,0.5]$	$[1,1]$

They are given by

	$\{x_1\}$	$\{x_2\}$	$\{x_3\}$	$\{x_1,x_2\}$	$\{x_1,x_3\}$	$\{x_2,x_3\}$
$\underline{P}_{(\underline{F}_1,\overline{F}_1)}$	0.3	0	0.3	0.3	0.6	0.4
$\overline{P}_{(\underline{F}_1,\overline{F}_1)}$	0.6	0.4	0.7	0.7	1	0.7
$\underline{P}_{(\underline{F}_2,\overline{F}_2)}$	0	0	0.5	0.1	0.5	0.8
$\overline{P}_{(\underline{F}_2,\overline{F}_2)}$	0.2	0.5	0.9	0.5	1	1
$\mathcal{A}_P$	0.2	0	0.5	0.3	0.6	0.7

The table above also shows the Pareto rule $\mathcal{A}_P$, that is not coherent: $\mathcal{A}_P(\{x_1,x_3\}) < \mathcal{A}_P(\{x_1\}) + \mathcal{A}_P(\{x_3\})$. ♦

In this example, the maximum of the lower probabilities avoids sure loss and as a consequence it coincides with the output by the Pareto rule, but it is not coherent. When the maximum of the lower probabilities is coherent, then the Pareto rule agrees with the conjunction, and therefore produces a coherent lower probability that is moreover a probability box, by Proposition 8.1.

9. Minitive measures

We conclude our analysis with another particular case of belief functions: minitive measures.

9.1. Conjunction. The study of the conjunction of minitive measures was the subject of the work carried out in [15], considering their conjugate maxitive measures:

Theorem 9.1. *[15] Let $\overline{P}_1,\overline{P}_2$ be two maxitive measures, $\overline{P} = \max\{\overline{P}_1,\overline{P}_2\}$ and let $\overline{E}$ the natural extension of $\overline{P}$.*

(i) $\overline{P}$ is maxitive $\Leftrightarrow$ $\min_{i=1,2}(\max_{j=1,2}\pi_i(x_j)) = \max_{j=1,2}(\min_{i=1,2}\pi_i(x_j))$ for all $\{x_1,x_2\} \subseteq \mathcal{X}$.

(ii) If $\overline{P}$ is maxitive, then so is $\overline{E}$.

(iii) If $\overline{P}_1,\overline{P}_2$ are strictly positive on singletons, then $\overline{E}$ is maxitive if and only if $\overline{P}$ is.

It was also established in [15, Ex. 5] that the minimum of two maxitive measures may not be maxitive, even if it is coherent. Next we reformulate the example so as to show that the family of minitive measures is not closed under conjunction:

Example 9.1. Consider $\mathcal{X} = \{x_1,x_2,x_3\}$ and let $\underline{P}_1,\underline{P}_2$ be the minitive measures determined by the Möbius inverses given by $m_1(\{x_1\}) = m_1(\mathcal{X}) = 0.5$, and $m_2(\{x_2\}) = m_2(\{x_1,x_2\}) = 0.5$ and zero elsewhere. Then $\max\{\underline{P}_1,\underline{P}_2\}$ is coherent because it is a probability measure (and as a consequence it coincides with the Pareto rule), but it is not minitive since is gives strictly positive lower probability to the disjoint events $\{x_1\}$ and $\{x_2\}$. ♦

9.2. Disjunction. Concerning the disjunction, it is quite straightforward to see that the minimum of two minitive measures is again minitive; it is also possible to establish some relationship between their focal elements:

Proposition 9.1. *Let $\underline{P}_1$ and $\underline{P}_2$ be two minitive measures. Then, their disjunction $\mathcal{A}_D$ is minitive and $\mathcal{F}_1 \cap \mathcal{F}_2 \subseteq \mathcal{F}_D$.*

Proof. The first part follows immediately from the definition. To see the second part, observe that if A is a focal element of both $\underline{P}_1,\underline{P}_2$ then $\underline{P}_1(A) > \max\{\underline{P}_1(B) \mid B \subset A\}$ and $\underline{P}_2(A) > \max\{\underline{P}_2(B) \mid B \subset A\}$. This implies that $\mathcal{A}_D(A) = \min\{\underline{P}_1(A),\underline{P}_2(A)\} > \max\{\underline{P}(B) \mid B \subset A\}$ and therefore that A is a focal element of $\mathcal{A}_D$. □

The inclusion between the focal elements in the above proposition may be strict: if for instance $\underline{P}_1,\underline{P}_2$ are the minitive measures on $\mathcal{X} = \{x_1,x_2,x_3\}$ with respective focal elements $\mathcal{F}_1 = \{x_1\}$ and $\mathcal{F}_2 = \{x_2\}$, then $\mathcal{A}_D = \min\{\underline{P}_1,\underline{P}_2\}$ is the minitive measure whose only focal element is $\{x_1,x_2\}$.

9.3. Convex mixtures. While belief functions are preserved under mixtures, it is quite easy to establish that minitive measures are not. For this, observe that the mixture of two minitive measures will not be minitive as soon as there are two different singletons with strictly positive lower probability, since this prevents the focal elements from being nested. This said, our previous results allow us to immediately characterise in which cases the mixture of two minitive measures is again minitive:

Corollary 9.1. *Let $\underline{P}_1$ and $\underline{P}_2$ be two minitive measures and consider their mixture $\mathcal{A}_M$ for some $\alpha \in (0,1)$. $\mathcal{A}_M$ is a minitive measure if and only if its family of focal elements $\mathcal{F}_1 \cup \mathcal{F}_2$ is totally ordered by means of set inclusion.*

9.4. Pareto rule. That minitivity is not preserved by the Pareto rule can already be seen in Example 9.1, given that in that case the Pareto rule agrees with the conjunction. In our next example the Pareto rule does not agree with the conjunction, even if the latter produces a coherent lower probability. This implies immediately that the output of the Pareto rule avoids sure loss but is not coherent, let alone minitive.

Example 9.2. Consider $\mathcal{X} = \{x_1,x_2,x_3\}$ and let $\underline{P}_1,\underline{P}_2$ be the minitive measures given in the following table, where we also show the Pareto rule:

A	$\{x_1\}$	$\{x_2\}$	$\{x_3\}$	$\{x_1,x_2\}$	$\{x_1,x_3\}$	$\{x_2,x_3\}$
$\underline{P}_1$	0.3	0	0	0.3	0.5	0
$\underline{P}_2$	0	0.4	0	0.4	0	0.6
$\mathcal{A}_P$	0.3	0.4	0	0.4	0.5	0.6

Thus, the Pareto rule is not a minitive measure, since $\mathcal{A}_P(\{x_1\}) = 0.3 \neq \min\{\mathcal{A}_P(\{x_1,x_2\}),\mathcal{A}_P(\{x_1,x_3\})\}$. ♦

10. Conclusions

In this work we have performed an analysis of the conjunction, disjunction, convex mixtures and the Pareto rules as aggregation methods for some particular imprecise probability models, focusing on whether the output

	Conj. $\mathcal{A}_C$	Disj. $\mathcal{A}_D$	Mixt. $\mathcal{A}_M$	Pareto $\mathcal{A}_P$
Comparative probabilities	✓ (Prop. 4.1)	✓ (Ex. 4.1, Prop. 4.2)	✓ (Cor. 4.2)	✓ (Prop. 4.4)
2-monotone capacities	✓ (Ex. 5.1,Prop. 5.1)	✓ (Prop. 5.2)	✓ (Prop. 5.3)	✗ ([14, Ex. 3])
Probability intervals	✓ ([3, Prop.7])	✗ ([3, Ex.1])	✗ (Ex. 6.1)	✗ (Ex. 6.2)
Belief functions	✗ (Ex.7.1)	✓ (Ex. 7.2, Prop 7.2)	✓ (Prop. 7.3)	✗ ([14, Ex. 3])
Probability boxes	✓ (Prop.8.1)	✗ (Ex. 8.1)	✗ (Ex. 8.2)	✗ (Ex. 8.3)
Minitive measures	✓ ([15])	✓ (Prop. 9.1)	✓ (Cor. 9.1)	✗ (Ex. 9.2)

Table 1. *Summary of the models preserved by each aggregation rule. ✓means that the rule is closed, and ✓that it is closed under some conditions.*

belongs to the same type of imprecise probabilities as the input. Table 1 summarises our results.

While all the aggregation rules save for Pareto preserve coherence as soon as there exists a probability measure that is compatible with all the models, they are not necessarily closed once we focus on some particular subfamilies of coherent lower previsions. Although in this paper we have established some sufficient conditions for the aggregated model to be in the same family as the sources, a much deeper study is needed in order to completely characterise these rules, similarly to what was done in [15] for the conjunction of possibility measures.

When the aggregated model does not belong to the same family, two main avenues appear before us: we may on the one hand look for (possibly unique) inner [13] or outer [16, 17] approximations of the aggregated model; or we may also propose other rules that are tailor-made for that specific family. As an illustration, in the case of belief functions we may look for a rule that distributes the mass among all possible focal elements, using aggregation rules for precise probabilities.

An essential assumption in this paper has been the finiteness of both the possibility space $\mathcal{X}$ and the number of models to the aggregated. The extension to infinite spaces is relatively simple, although in some cases, such as belief functions, we should take into account some continuity properties [21]. A more delicate matter that should be addressed carefully would be the aggregation of an infinity of models. Here issues of compactness and continuity should be considered.

A topic we have not discussed in this paper due to the space limitations are the other rationality criteria that were considered in [14]. In this respect, we should investigate (a) whether some of the properties that a rule does not satisfy in general hold when we restrict its domain to a particular subfamily of coherent lower previsions; and (b) if the rationality criteria may be suitably reformulated for such subfamilies taking into account their specific features. Another interesting study would be to consider other models such as comparative probabilities in a wider sense, where the comparisons are done between events and not only between singleton. For this purpose, the results in [8] may be of interest. And finally, some suitable modifications of conjunction that may allow to achieve coherence, for instance based on distortions of credal sets [19], may be useful as a tool to address the conflict between imprecise probability models. We intend to report about our findings in these problems in future work.

Additional Author Information

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