Random walks on graphs with interval weights as a model of reversible imprecise Markov chains

Damjan Škulj[1]

[1]Faculty of Social Sciences, University of Ljubljana, Slovenia

Abstract

We consider random walks on weighted graphs where the edge weights are interval-valued, reflecting uncertainty in the relationships between vertices. We study this model in the framework of reversible imprecise Markov chains by viewing them as sets of precise inhomogeneous Markov chains. We define and analyze the notion of reversibility for such sets by extending classical reversibility concepts to the imprecise setting. These concepts are then applied to interval-weighted random walks, where the individual weight functions may not be symmetric but their sets exhibit symmetry. Our approach provides a basis for analyzing random walks in environments with uncertain or incomplete information.

Keywords. random walk, weighted graph, interval weight, imprecise Markov chain, reversibility, inhomogeneous Markov chain, uncertainty

1. Introduction

A *Markov chain* is said to be *reversible* if the probability of any sequence of states is equal to the probability of the time-reversed sequence, provided that the process is in its stationary distribution. Random walks on weighted graphs provide a natural and widely used model for reversible Markov chains. In this model, the vertices of the graph represent the states of the chain, and the edge weights directly determine the transition probabilities: The probability of transition from one vertex to another is proportional to the weight of the connecting edge [1, 6, 8, 17]. The well-developed theory of random walks on graphs provides a set of analytical tools to study reversible Markov chains. Reversible Markov chains are indispensable for many Monte Carlo methods, especially for Markov Chain Monte Carlo (MCMC), as they ensure that the stationary distribution of the chain matches the desired target distribution for the sample [9, 11, 12]. Random walks on graphs have found widespread applications in various fields, including network analysis [5, 15, 16, 24], social networks [3, 14, 23] and web recommender systems [7].

Modelling real-world systems with Markov chains often requires the estimation of numerous parameters. Even with an abundance of data, these estimates are often subject to significant uncertainty, which can lead to unreliable results if not adequately accounted for. The need to explicitly represent and manage uncertainty in probabilistic models has led to the development of the theory of *imprecise probabilities*, which provides a framework for working with probabilities that are not precisely specified [2, 22]. In particular, the theory of *imprecise Markov chains* has been developed for both discrete [4, 10, 19] and continuous cases [13, 20].

In this paper, we extend the classical model of random walks on weighted graphs to allow for *interval-valued* weights. The model of random walks on weighted undirected graphs is closely related to the model of reversible Markov chains. Consequently, interval-valued random walks are naturally related to imprecise Markov chains. Since reversibility in this context has not been fully explored in the literature, we first develop an approach for reversible imprecise Markov chains and show that random walks on graphs with interval weights fit naturally into this theory, which supports our approach.

A random walk on an interval-valued weighted graph is a stochastic process over the vertices of a graph in which the next vertex is chosen with a transition probability proportional to an arbitrary weight function consistent with the given interval weights. This model builds on an earlier approach [21] that allowed interval weights but retained precise marginal distributions. Our model serves as a prototype for general reversible imprecise Markov chains, and several of the results presented here apply to this broader class of models.

The paper is organised as follows. In Section 2 we introduce random walks on weighted graphs. In Section 3 we analyse inhomogeneous Markov chains from the point of view of reversibility. In Section 4 we introduce the modelling of reversibility with joint distribution matrices. The basic theory of reversibility for sets of Markov processes is developed in Section 5 and finally applied to random walks with interval weights in Section 6.

2. Random walk on weighted graph

Let $\mathcal{X}$ denote the set of vertices of an *undirected weighted graph*, with edges weighted by a weight function w, where $w(x, y)$ represents the weight of the edge (x, y). A *random walk* is then a stochastic process $(X_n)_{n\in\mathbb{N}}$ whose state space is $\mathcal{X}$ and is governed by the probabilistic rule that the next state is chosen among the neighbors of the current state, with probabilities proportional to the weights of the edges connecting them. Additionally we assume the graph is connected, meaning that for every two vertices u and v a sequence of vertices $u = x_0, x_1, \dots, x_n = v$ exists such that $w(x_i, x_{i+1}) > 0$ for each $i = 0, \dots, n-1$.

2.1. Random walk as Markov chain. We relate random walks on weighted graphs to the model of Markov chains, which we now briefly explain. Consider a finite set $\mathcal{X}$ (we use the same notation for the set of states as for the set of vertices because they will have the same role in the analysed processes). A (finite or infinite) sequence of random variables $(X_n)_{n\in\mathbb{N}}$ taking values in $\mathcal{X}$ is called a *discrete-time Markov chain* if it satisfies the following *Markov property*:

$$\begin{aligned} P(X_{n+1} = y \mid X_n = x, X_{n-1} = x_{n-1}, \dots, X_0 = x_0) && (1)\\ = P(X_{n+1} = y \mid X_n = x) && (2)\\ = P_n(x, y), && (3) \end{aligned}$$

where $P_n(x, y)$ is the transition probability. If P_n is the same for all n, we say that the chain is *time-homogeneous*. (Time homogeneity is assumed to the end of this section.) The *marginal distributions* of a Markov chain are denoted by q^i. That is, $q^i(x) = P(X_i = x)$, where q^1 is a given initial distribution. We have that $q^i = q^1P^{i-1}$.

A random walk can be viewed as a Markov chain with transition probabilities given by

$$P(x, y) = \frac{w(x, y)}{w(x)}, \tag{4}$$

where $w(x) = \sum_{y\in\mathcal{X}} w(x, y)$ is the total weight of edges connected to x. Since $w(x, y) = 0$ indicates the absence of an edge, we can assume that the graph is complete by assigning zero weights to nonexistent edges.

We also assume that the resulting Markov chain is *ergodic*, which means it is irreducible and aperiodic (see e.g., [18]). Ergodicity implies the existence of a unique *stationary distribution* π, that satisfies $\pi P = \pi$. A random walk on a connected graph is ergodic if the greatest common divisor of the lengths of all *closed walks* containing each vertex is 1. The stationary distribution of an ergodic random walk is

$$\pi(x) = \frac{w(x)}{W}, \tag{5}$$

where $W = \sum_{y\in\mathcal{X}} w(y)$ is the total weight of all edges in the graph. Since the graph is undirected, each edge weight is counted twice, once for each possible direction.

2.2. Reversibility of a random walk. For convenience, we introduce the notation $X_{m:n} = (X_m, \dots, X_n)$ and analogous. If $n < m$, this notation will simply denote the decreasing sequence $(X_m, X_{m-1}, \dots, X_n)$. Thus, we will only observe chains in a finite number of steps $(X_{m:n})$. In most cases, where there is no specific reason to start at a particular time, we will simply assume that the chain starts at time 1, in which case we will have chains of the form $(X_{1:N})$ for some $N \in \mathbb{N}$.

Let $X_{1:N}$ be a time-homogeneous ergodic Markov chain with transition matrix P. The sequence $X_{N:1}$ is then a Markov chain as well, often referred to as a *reversed chain*. The transition matrix of the reversed chain equals

$$P^*(x, y) = \frac{\pi(y)P(y, x)}{\pi(x)}, \tag{6}$$

or in matrix form $P^* = \mathrm{diag}(\pi)^{-1}P^t\mathrm{diag}(\pi)$.

The reversed chain is also irreducible and its stationary distribution is also π.

If the reversed transition probabilities are the same as those for the original chain, the chain is then said to be *reversible*. Thus, in a reversible Markov chain we have that $P^*(x, y) = P(x, y)$. From (6), it follows that for a reversible Markov chain the following *detailed balance* condition holds:

$$\pi(x)P(x, y) = \pi(y)P(y, x) \qquad \text{for all } x, y \in \mathcal{X}, \tag{7}$$

or $\mathrm{diag}(\pi)P = P^t\mathrm{diag}(\pi)$. A Markov chain is reversible if and only if it satisfies the detailed balance condition.

The following is a well known result (see e.g., [1]).

Theorem 2.1. *Let $(X_{1:N})$ be a stationary reversible Markov chain and $(X_{N:1})$ its reversed chain. Then*

$$P(X_{1:N} = x_{1:N}) = P(X_{N:1} = x_{1:N}). \tag{8}$$

It is straightforward to verify that a random walk is reversible. Since $w(x, y) = w(y, x)$ for an undirected graph, it follows that

$$\begin{aligned} \pi(x)P(x, y) &= \frac{w(x)}{W}\frac{w(x, y)}{w(x)}\\ &= \frac{w(y)}{W}\frac{w(y, x)}{w(y)}\\ &= \pi(y)P(y, x), \end{aligned}$$

which confirms the detailed balance condition and thus reversibility.

An extension of this model to reversible imprecise frameworks was proposed in [21], where the model is generalized to allow interval weights. In that approach, each weight $w(x, y)$ is replaced by an interval $[\underline{w}(x, y), \overline{w}(x, y)]$, enabling the representation of uncertainty in edge weights. We will discuss this generalized model later.

3. Inhomogeneous Markov chains

To extend random walks to graphs with interval weights, we need to consider Markov chains, whose transition probabilities are not necessarily constant in time. Such chains are known as *inhomogeneous Markov chains*. A finite inhomogeneous Markov chain $(X_{1:N})$ is generally described by an initial distribution q^1 and the sequence of transition probabilities $P_{1:N-1}$. We say that the tuple $\Gamma = (q^1, P_{1:N-1})$ specifies the *transition law* of the chain $(X_{1:N})$.

The marginal distributions q^k of the variables X_k under a transition law Γ are given by

$$q^k = q^1 P_1 \dots P_{k-1}.$$

In this paper, marginal distributions are not a primary focus, as the processes under study are generally stationary, implying that the marginal distributions remain constant over time. Instead, our main objective is to examine the joint distribution of the process $(X_{1:N})$. Its probability mass function is given by:

$$\begin{aligned} p_\Gamma(x_{1:N}) &= P(X_{1:N} = x_{1:N}) && (9)\\ &= P(X_1 = x_1)P(X_2 = x_2 | X_1 = x_1) \dots \times && (10)\\ &\quad \times P(X_N = x_N | X_{N-1} = x_{N-1}) && (11)\\ &= q^1(x_1)P_1(x_1, x_2) \cdots P_{N-1}(x_{N-1}, x_N). && (12) \end{aligned}$$

3.1. Time reversal for inhomogeneous Markov chains. Reversing a non-stationary Markov chain results in transition matrices that depend on the initial distribution. The following definition presents the basic operation of reversal.

Definition 3.1. Let P be a transition matrix and q any probability mass function on $\mathcal{X}$. Then let

$$P^*_q = \mathrm{diag}(qP)^{-1} P^t \mathrm{diag}(q)$$

and we denote P^*_q a *q-reverse* of P.

The q-reverse of a matrix P is a transition matrix of a chain where instead of a stationary distribution the chain starts from a distribution q. Note that in case π is a stationary distribution for P, π-reverse of P is exactly the usual reversed transition matrix P^*.

The following proposition shows that the q-reverse of a stochastic matrix is also a stochastic matrix and can therefore be used to model transitions of the reversed chain. In subsequent text we often use the notation 1_A to denote the characteristic function of any set $A \subseteq \mathcal{X}$, i.e. $1_A(x) = 1$ if $x \in A$ and $1_A(x) = 0$ if $x \in \mathcal{X} \backslash A$. In particular, $1_\mathcal{X}$ provides a convenient notation for a vector of ones with length $|\mathcal{X}|$.

Proposition 3.1. *Let P be a $|\mathcal{X}| \times |\mathcal{X}|$ stochastic matrix and q a probability mass function on $\mathcal{X}$. Then the q-reverse P^*_q is also a stochastic matrix.*

Proof. We need to prove that the row sums of P^*_q equal 1, or equivalently that $P^*_q 1^t_\mathcal{X} = 1^t_\mathcal{X}$:

$$\begin{aligned} P^*_q 1^t_\mathcal{X} &= \mathrm{diag}(qP)^{-1} P^t \mathrm{diag}(q) 1^t_\mathcal{X} \\ &= \mathrm{diag}(qP)^{-1} P^t q^t = \mathrm{diag}(qP)^{-1} (qP)^t = 1^t_\mathcal{X}. \end{aligned}$$

□

Proposition 3.2. *Let P be an $|\mathcal{X}| \times |\mathcal{X}|$ stochastic matrix and q a probability mass function on $\mathcal{X}$. Then*

(i) $qPP^*_q = q$;

(ii) $(P^*_q)^*_{qP} = P$.

Thus, the qP-reverse of the q-reverse of P is again P.

Proof. Let q and P be as assumed. Then we have that

$$\begin{aligned} qPP^*_q &= (qP)\mathrm{diag}(qP)^{-1} P^t \mathrm{diag}(q) \\ &= 1_\mathcal{X} P^t \mathrm{diag}(q) = (P 1^t_\mathcal{X})^t \mathrm{diag}(q) = 1_\mathcal{X} \mathrm{diag}(q) = q, \end{aligned}$$

which confirms (i).

To see (ii), we use (i) to obtain

$$\begin{aligned} (P^*_q)^*_{qP} &= \mathrm{diag}(qPP^*_q)^{-1} (P^*_q)^t \mathrm{diag}(qP) \\ &= \mathrm{diag}(q)^{-1} (\mathrm{diag}(qP)^{-1} P^t \mathrm{diag}(q))^t \mathrm{diag}(qP) \\ &= \mathrm{diag}(q)^{-1} \mathrm{diag}(q) P \mathrm{diag}(qP)^{-1} \mathrm{diag}(qP) = P. \end{aligned}$$

□

The q-reverse is the transition matrix when two consecutive states of a Markov chain are reversed. The following definition generalizes the notion of a reversed transition matrix to a reversed sequence of transition matrices, along with the initial distribution, which together form what we call a transition law.

Definition 3.2. Let $\Gamma = (q^1, P_{1:N-1})$ denote a transition law of an inhomogeneous Markov chain. Then we define $\Gamma^* = (q^N, P^*_{N-1:1})$, where $P^*_k = P^*_{q^k}$, i.e. $P^*_k = \mathrm{diag}(q_{k+1})^{-1} P^t_k \mathrm{diag}(q_k)$ and $q^k = q^1 P_1 \cdots P_{k-1}$, to be the *reverse transition law* to Γ.

Remark 3.1. Note that if $N = 2$, then $\Gamma = (q, P)$ and its reverse transition law is $\Gamma^* = (qP, P^*_q)$.

Theorem 3.1. *Let $(X_{1:N})$ be an inhomogeneous Markov chain with the transition law $\Gamma = (q^1, P_{1:N-1})$. Then its time reversal is the chain $(X_{N:1})$ and its transition law is the reversed transition law Γ^*.*

Proof. Take a sequence of states $x_{1:N}$ and first calculate the transition probabilities

$$\begin{aligned} P^*_{k-1}(x_k, x_{k-1}) &= P(X_{k-1} = x_{k-1} | X_k = x_k) \\ &= \frac{P(X_{k-1} = x_{k-1}, X_k = x_k)}{P(X_k = x_k)} \end{aligned}$$

$$
\begin{aligned}
&= \frac{P(X_k = x_k|X_{k-1} = x_{k-1})P(X_{k-1} = x_{k-1})}{P(X_k = x_k)} \\
&= P_{k-1}(x_{k-1}, x_k)\frac{q^{k-1}(x_{k-1})}{q^k(x_k)} \\
&= P^*_{q^{k-1}}(x_k, x_{k-1}).
\end{aligned}
$$

$$
\begin{aligned}
P(X_{N:1} = x_{N:1}) = P(X_N = x_N)\times \\
\times P(X_{N-1} = x_{N-1}|X_N = x_N)\cdots\times \\
\times P(X_1 = x_1|X_2 = x_2) \\
= q^N(x_N)P^*_{N-1}(x_N, x_{N-1})\cdots P^*_1(x_2, x_1).
\end{aligned}
$$

Thus, the chain with transition law $\Gamma^* = (q^N, P^*_{N-1:1})$ represents the same process with time reversed. □

Remark 3.2.
1. Note that the transition probability matrices P^*_k for the reversed chain depend both on the initial distribution q^1 and the transition matrices P_i for $i = 1, \dots, k$.
2. Even if the transition law is homogeneous with a transition matrix P, the reversed law is in general inhomogeneous with transition matrices $P^*_k = P^*_{q^{k-1}}$, unless it starts from the stationary distribution π corresponding to P. In that case, the reversed transition law is homogeneous with the usual reversed transition matrix $P^* = P^*_\pi$.

4. Joint distribution matrices

The analysis of reversed inhomogeneous Markov chains will be simplified by introducing the *joint distribution matrix*. It will denote the joint distribution for two consecutive terms of a Markov chain. While the pair of an initial distribution and transition matrix corresponds to a marginal and conditional distribution, which together induce a joint distribution, the joint distribution matrix will then implicitly contain all this information in one matrix. However, the main reason for switching to joint distribution matrices is that when taking sets of those, the reversed process is obtained by the simple operation of matrix transpose, which preserves desirable properties such as convexity.

Definition 4.1. Let q be a probability mass function over $\mathcal{X}$ and P a transition matrix. Then we denote by $Q_{q,P}$ the *joint distribution matrix* to be the probability mass function over $\mathcal{X}^2$ corresponding to the chain (X_1,X_2) with initial distribution q, i.e. q is the distribution of X_1, and transition matrix P, thus corresponding to the conditional distribution $P(X_2|X_1)$.

When it is clear from the context, we omit q and P and just write Q for a joint distribution matrix.

Proposition 4.1. *Let (q, P) be a transition law corresponding to a Markov chain (X_1,X_2), and $Q = Q_{q,P}$ the joint distribution matrix. Further, let (qP, P^*_q) be the reversed transition law. The following propositions hold:*

(i) $Q = \operatorname{diag}(q)P$.
(ii) $P = \operatorname{diag}(q)^{-1}Q$.
(iii) $q = (Q1^t_{\mathcal{X}})^t$.
(iv) $qP = 1_{\mathcal{X}}Q$.
(v) The joint distribution corresponding to the reversed transition law (qP, P^*_q) is Q^t.

Proof. Statements (i) and (ii) follow directly from the definition of the joint distribution matrix:

$$
Q(x, y) = P(X_1 = x, X_2 = y) = q(x)P(x, y).
$$

Statements (iii) and (iv) simply tell that q and qP are marginals corresponding to Q and therefore its row and column sums respectively.

Now let Q^* be the joint distribution matrix corresponding to the reversed process (X_2,X_1). We use the fact that the marginal distribution of X_2 is qP and the definition of P^*_q to obtain

$$
\begin{aligned}
Q^*(x, y) &= P(X_2 = x, X_1 = y) \\
&= P(X_2 = x)P^*_q(x, y) \\
&= (qP)(x)P(y, x)\frac{q(y)}{(qP)(x)} \\
&= q(y)P(y, x) \\
&= Q(y, x) = Q^t(x, y),
\end{aligned}
$$

which confirms (v). □

There is actually a one-to-one correspondence between joint distribution matrices and pairs of transition matrices and distribution vectors over $\mathcal{X}$. That is, for any joint distribution matrix Q, a unique transition matrix P, given by (ii) of the last proposition, and distribution vector q given by (iii) of the same proposition exist, satisfying relation (i).

The following simple result follows directly from the basic properties of reversible Markov chains.

Proposition 4.2. *Let P be a transition matrix with a unique stationary distribution and π probability mass function. Then the following proposition are equivalent:*

(i) π is a unique stationary distribution for P and P is reversible.
(ii) $Q_{\pi,P}$ is symmetric, i.e. $Q^t_{\pi,P} = Q_{\pi,P}$.

Proof. While (i) $\implies$ (ii) follows directly from the definition of reversibility, symmetry of $Q = Q_{\pi,P}$ implies that $\pi = (Q1^t_{\mathcal{X}})^t = 1_{\mathcal{X}}Q = \pi P$, which shows that π is indeed a stationary distribution of P. Thus, we have a stationary chain, whose reversed joint distribution coincides with the forward distribution. Thus (ii) $\implies$ (i) holds as well. □

Proposition 4.3. *Let $\Gamma = (q^1, P_{1:N-1})$ denote a transition law of an inhomogeneous Markov chain. Then the joint probability mass function corresponding to the chain $(X_{1:N})$ is*

$$p_\Gamma(x_{1:N}) = \frac{\prod_{k=1}^{N-1} Q_k(x_k, x_{k+1})}{\prod_{k=2}^{N-1} q^k(x_k)}$$

where $q^k = q^1 P_1 \dots P_k$ and $Q_k = Q_{q^k, P_k}$.

Moreover, the following relations hold:

$$q^k = (Q_k 1_{\mathcal{X}}^t)^t \quad \text{and} \quad q^{k+1} = 1_{\mathcal{X}} Q_k. \tag{13}$$

Proof. Note that $Q_k = \mathrm{diag}(q^k)P_k$ implies that $P_k = \mathrm{diag}(q^k)^{-1}Q_k$, whence the assertion of the theorem follows from direct calculation:

$$\begin{aligned} p_\Gamma(x_{1:N}) &= q^1(x_1)P_1(x_1, x_2) \dots P_{N-1}(x_{N-1}, x_N) \\ &= q_1(x_1)\frac{Q_1(x_1, x_2)}{q^1(x_1)} \cdots \frac{Q_{N-1}(x_{N-1}, x_N)}{q^{N-1}(x_{N-1})} \\ &= \frac{\prod_{k=1}^{N-1} Q_k(x_k, x_{k+1})}{\prod_{k=2}^{N-1} q^k(x_k)}. \end{aligned}$$

The relations $q^k = (Q_k 1_{\mathcal{X}}^t)^t$ and $q^{k+1} = 1_{\mathcal{X}} Q_k$ follow from Proposition 4.1 (iii) and (iv). □

Corollary 4.1. *Let $\Gamma^* = (q^N, P^*_{N-1:1})$ denote a reversed transition law for a Markov chain $(X_{1:N})$. Then the corresponding joint probability mass function is*

$$p_{\Gamma^*}(x_{1:N}) = \frac{\prod_{k=1}^{N-1} Q_k^t(x_k, x_{k+1})}{\prod_{k=2}^{N-1} q^k(x_k)},$$

where $q^k = q^1 P_1 \dots P_k$ and $Q_k = Q_{q^k, P_k}$ and the relations (13) hold.

Proof. Let

$$\begin{aligned} Q_k^* &= \mathrm{diag}(q^{k+1})P_k^* \\ &= \mathrm{diag}(q^{k+1})\mathrm{diag}(q^{k+1})^{-1}P_k^t \mathrm{diag}(q^k) \\ &= (\mathrm{diag}(q^k)P_k)^t = Q_k^t. \end{aligned}$$

The assertion now follows by applying Proposition 4.3. □

Notice that the joint probability matrices $Q_{1:N-1}$ in Proposition 4.3 and Corollary 4.1 satisfy the condition that $q^{i+1} = 1_{\mathcal{X}} Q_i = (Q_{i+1} 1_{\mathcal{X}}^t)^t$. That is, the row sums of the first matrix corresponds to the column sums of the second one. We introduce the following definition.

Definition 4.2. Let Q_1 and Q_2 be joint probability matrices such that $1_{\mathcal{X}} Q_1 = (Q_2 1_{\mathcal{X}}^t)^t$. Then we say that Q_1 is *marginally compatible* with Q_2. That is, the marginal distribution of Q_1 of the second variable matches the marginal distribution of Q_2 of the first variable.

Notice that marginal compatibility is not a symmetric relation.

Corollary 4.2. *To every transition law of a Markov chain given in terms of $\Gamma = (q^1, P_{1:N-1})$, a sequence $Q_{1:N-1}$ of joint probability matrices can be assigned such that Q_i is marginally compatible with Q_{i+1} for every $i \in \{1, \dots, N-2\}$; and conversely, every sequence $Q_{1:N-1}$ of joint probability matrices such that Q_i is marginally compatible with Q_{i+1} for every $i \in \{1, \dots, N-2\}$ corresponds to a unique transition law $\Gamma = (q^1, P_{1:N-1})$.*

Proof. The first part follows directly from Proposition 4.3. To see the converse correspondence, take some sequence $Q_{1:N-1}$ satisfying the conditions of corollary. Then define $q^i = (Q_i 1_{\mathcal{X}}^t)^t$ and $P_i = \mathrm{diag}(q^i)^{-1}Q_i$ for every $i \in \{1, \dots, N-1\}$. Note that by Proposition 4.1, each joint probability matrix defines a unique pair q^i and P_i and vice versa, each pair q^i and transition matrix P_i induces a unique joint probability matrix Q_i. Now, by marginal compatibility of Q_i and Q_{i+1}, we have that $1_{\mathcal{X}} Q_i = q^{i+1}$, which implies

$$q^i P_i = q^i \mathrm{diag}(q^i)^{-1} Q_i = 1_{\mathcal{X}} Q_i = q^{i+1}.$$

Thus, the transition law $\Gamma = (q^1, P_{1:N})$ coincides with the law induced by the sequence of joint probability matrices. □

Corollary 4.3. *Let $\Gamma = (q^1, P_{1:N-1})$ be a transition law that is equivalent to the law given by a sequence of joint probability matrices $Q_{1:N-1}$. Then the reversed transition law $\Gamma^* = (q^N, P^*_{N-1:1})$ is equivalent to the law given by the sequence $Q^t_{N-1:1}$.*

Let $Q_{1:N-1}$ be a sequence of joint distribution matrices such that Q_i is marginally compatible with Q_{i+1} for every $i \in \{1, \dots, N-1\}$. Then we will write $\Gamma_{Q_{1:N-1}}$ to denote the transition law for a sequence of random variables $X_{1:N}$ that induces the same joint distribution as $Q_{1:N-1}$.

5. Reversibility for Sets of Inhomogeneous Markov Chains

Sets of inhomogeneous Markov chains provide a framework for imprecise Markov chains under the *strong independence interpretation*. Under the strong independence interpretation, an imprecise Markov chain with a transition set $\mathcal{T}$ is a process in which transition matrices are chosen independently at each step. Under this interpretation, a finite imprecise Markov chain is a process $X_{1:N}$ whose transition law is only known to belong to a set of transition laws, denoted by $\mathcal{G}$. Assume that $\mathcal{G}$ *consists of* transition laws of the form $\Gamma = (q^1, P_{1:N-1})$, where $q^1 \in \mathcal{C}^1$ and $P_k \in \mathcal{T}$, where $\mathcal{C}^1$ is a set of probability mass functions on $\mathcal{X}$. Existing models of imprecise Markov chains usually assume that the set corresponding to the joint distribution of $X_{1:N}$ contains all possible transition laws of this form. However, the reversed process often fails to remain in the same set.

Consider a general imprecise Markov chain starting from an initial distribution $\mathcal{C}^1$ and a set of transition matrices $\mathcal{T}$. The marginal distributions corresponding to X_k are then given by $\mathcal{C}^k = \mathcal{C}^1\mathcal{T}^{k-1} = \{q^1P_1 \dots P_{k-1} : q^1 \in \mathcal{C}^1, P_i \in \mathcal{T}, \forall i \in \{1, \dots, k-1\}\}$. By analogy with precise processes, one would expect the existence of a reversed set of transition matrices, denoted by $\mathcal{T}^*$, such that $\mathcal{C}^{N-k} = \mathcal{C}^N(\mathcal{T}^*)^{k-1}$. However, such a set does not exist in general, as the following example shows.

Example 5.1. Let us model the imprecision in transition matrix by the convex set generated by the following transition matrices:

$$P_1 = \begin{pmatrix} 0.2 & 0.8 \\ 0.7 & 0.3 \end{pmatrix}, P_2 = \begin{pmatrix} 0.6 & 0.4 \\ 0.5 & 0.5 \end{pmatrix}.$$

Let $\mathcal{T}$ be the convex hull of $\{P_1, P_2\}$.

Since the set $\mathcal{T}$ only contains 2 by 2 matrices, each matrix from $\mathcal{T}$ is individually reversible. Let Π be the unique stationary set for $\mathcal{T}$, i.e. $\Pi\mathcal{T} = \Pi$. As the transition matrices are regular, the uniqueness of Π is guaranteed by the results in [19]. An imprecise Markov chain starting with the initial set of distributions Π is therefore stationary, and one might expect it to be also reversible, because all transition matrices in $\mathcal{T}$ are reversible. Let us show that this is not the case.

First we explicitly construct the bounds for Π. Clearly, Π must contain the stationary distributions of the individual members of $\mathcal{T}$, P_1 and P_2, which are $\pi_1 = (7/15, 8/15)$ and $\pi_2 = (5/9, 4/9)$, respectively. We have that $\pi_1 P_2 = (41/75, 34/75)$ and $\pi_2 P_1 = (19/45, 26/45)$. Since $\pi_1 P_2$ is a convex combination of π_1 and π_2, it follows that $\pi_1 P_{i_1} \dots P_{i_N}$ remains a convex combination of π_1 and π_2 for all sequences $(P_{i_1}, \dots, P_{i_N})$, where all $P_k \in \{P_1, P_2\}$. Since $\pi_2 P_1$ is not a convex combination of π_1 and π_2, we further calculate $\pi_2 P_1^2 = (22/45, 23/45)$, which is a convex combination of $\pi_2 P_1$ and π_2; and $\pi_2 P_1 P_2 = (122/225, 103/225)$, which is also a convex combination of $\pi_2 P_1$ and π_2. Setting

$$u = \pi_2 P_1 = (19/45, 26/45) \approx (0.55, 0.44) \text{ and}$$
$$v = \pi_2 = (5/9, 4/9) \approx (0.42, 0.58),$$

we deduce that all vectors in the convex hull $\text{co}\{u, v\}$ remain in that convex hull when multiplied by any sequence of matrices from $\{P_1, P_2\}$. Moreover, both u and v are contained in the stationary set, since they are obtained by multiplying the stationary vectors of the transition matrices by the transition matrices. Therefore, the stationary set Π corresponding to $\mathcal{T}$ is contained in $\text{co}\{u, v\}$.

If the stationary chain with initial set Π and transition set $\mathcal{T}$ were to be reversible, all q-reverses of matrices $P \in \mathcal{T}$ and distributions $q \in \Pi$ would need to be contained in $\mathcal{T}$. However, as we show next this is generally not true. First we calculate the u-reverse of P_1, which is

$$(P_1^*)_u = \begin{pmatrix} \frac{19}{110} & \frac{91}{110} \\ \frac{76}{115} & \frac{39}{115} \end{pmatrix}.$$

Note that $(P_1^*)_u$ is clearly not in $\mathcal{T}$. Furthermore, the stationary set of distributions Π is not also stationary for the set of reversed transition matrices, as we show next. For instance, we have $w = uP_1 = (22/45, 23/45)$, and $w \cdot (P_1^*)_u = u$, as required. However, $v \cdot (P_1^*)_u \approx (0.3897, 0.6103)$, which is clearly outside the convex hull $\text{co}\{u, v\} \supseteq \Pi$. Therefore, Π, which is stationary for $\mathcal{T}$, is clearly not stationary for $\mathcal{T}^* = \{P_q^* : P^* \in \mathcal{T}, q \in \Pi\}$.

This example shows that even in the simplest case of 2 by 2 matrices, the natural candidate for the reversed process not only is not the same as the forward process, but does not even have the same stationary set of distributions. Recall that in the precise case the reversed process always has the same stationary distribution as the forward one, even if its joint distribution is different.

We continue with defining reversible processes in terms of their sets of joint distributions rather than transition sets. Let $\text{distr}(X_{1:N})$ denote the set of joint distributions corresponding to the chain $X_{1:N}$. In the case of sets of inhomogeneous Markov chains, we have that

$$\text{distr}(X_{1:N}) = \{p_\Gamma : \Gamma \in \mathcal{G}\},$$

where $\mathcal{G}$ is a set of transition laws and p_Γ is defined by (9).

In the case of a single precise distribution, the common criterion for reversibility is the detailed balance condition (7), which is not straightforward to generalize to sets of distributions. Therefore, we use Theorem 2.1, which equivalently defined reversibility in terms of a joint distribution. In the case of a set of distributions, we consider a set of joint distributions.

Definition 5.1. An imprecise Markov chain $X_{1:N}$ is *reversible* if $\text{distr}(X_{1:N}) = \text{distr}(X_{N:1})$.

The following proposition follows immediately.

Proposition 5.1. *An imprecise Markov chain $X_{1:N}$, corresponding to a set of transition laws $\mathcal{G}$, is reversible if and only if $\mathcal{G}^* = \{\Gamma^* : \Gamma \in \mathcal{G}\} = \mathcal{G}$.*

A more operational criterion for reversibility is obtained using the two-dimensional joint probability matrices. Recall that, by Proposition 4.1, a two-dimensional joint probability matrix of a chain (X_1, X_2) with a probability law $\Gamma = (q, P)$ is $Q_\Gamma = \mathrm{diag}(q)P$. Consider a chain (X_1, X_2), whose set of transition laws is $\mathcal{G}$, containing transition laws of the form $\Gamma = (q, P)$.

Definition 5.2. Let $\mathcal{G}$ be a set of transition laws corresponding to an imprecise Markov chain (X_1, X_2). Then let $\mathcal{Q}_\mathcal{G} = \{Q_\Gamma : \Gamma \in \mathcal{G}\}$.

It follows directly from Proposition 4.1 (v) that the set of all transposed matrices in $\mathcal{Q}_\mathcal{G}$, which we naturally denote $\mathcal{Q}^t_\mathcal{G}$ corresponds to the set of reversed chains corresponding to (X_2, X_1). Therefore, the two-step chain is reversible if and only if $\mathcal{Q}^t_\mathcal{G} = \mathcal{Q}_\mathcal{G}$, in which case we say that the set is *symmetric*.

In the previous section, we showed that the reversed imprecise Markov chain loses the structure given by a set of transition matrices, where each member can be multiplied by a transition matrix from a given set. Here, we present an alternative way of modeling, utilizing joint distribution matrices.

In what follows, $\mathcal{Q}$ will denote a set of joint probability matrices. We associate two sets of marginal probability distributions:

$$\mathcal{C}_\mathcal{Q} = (\mathcal{Q}1^t_\mathcal{X})^t \tag{14}$$

$$\mathcal{C}^*_\mathcal{Q} = 1_\mathcal{X}\mathcal{Q} \tag{15}$$

and the set of transition matrices

$$\mathcal{T}_\mathcal{Q} = \{P_Q = \mathrm{diag}((Q1^t_\mathcal{X})^t)^{-1}Q : Q \in \mathcal{Q}.\}$$

The sets $\mathcal{C}_\mathcal{Q}$ and $\mathcal{C}^*_\mathcal{Q}$ are the sets of all row and column sums, respectively. It is readily verified that

$$\mathcal{C}^*_\mathcal{Q} = \mathcal{C}_{\mathcal{Q}^t},$$

which implies that a symmetric set $\mathcal{Q}$ induces the same marginal sets.

Proposition 5.2. *Let $\mathcal{C}^1$ be an initial set of distributions and $\mathcal{T}$ a set of transition matrices. Let $\mathcal{C}^k = \mathcal{C}^1\mathcal{T}^{k-1}$ for $k \in \{2, \dots, N\}$. Then there exists a sequence of sets of joint probability matrices $\mathcal{Q}_{1:N-1}$ such that for every transition law $\Gamma = (q^1, P_{1:N})$, where $q^1 \in \mathcal{C}^1$ and $P_k \in \mathcal{T}$, $k \in \{1, \dots, N-1\}$, there is a sequence of joint probability matrices $Q_{1:N-1}$. In this sequence, Q_i is marginally compatible with Q_{i+1} for every $i \in \{1, \dots, N-2\}$, and the joint distribution induced by the sequence is equal to that of Γ.*

Proof. The proposition follows directly by applying Corollary 4.2 to every transition law Γ compatible with the imprecise transition law. □

The sets of joint distribution matrices in the above proposition are of the form

$$\mathcal{Q}_i = \mathcal{Q}_{\mathcal{C}^i,\mathcal{T}} := \{\mathrm{diag}(q)P : q \in \mathcal{C}^i, P \in \mathcal{T}\}. \tag{16}$$

Remark 5.1. Note that the converse of Proposition 5.2 does not hold in the imprecise case. More precisely, a sequence of sets of joint distributions induces a sequence of marginal sets of distributions and a sequence of sets of transition matrices. However, it is not guaranteed that the products of the form q^iP_i are all contained in $\mathcal{C}^{i+1}$ when $q^i \in \mathcal{C}^i$ and $P_i \in \mathcal{T}$, as demonstrated by the example below. Thus, the joint probability distribution matrix model is more general than the conventional model for imprecise Markov chains.

Example 5.2. Consider $\mathcal{Q} = \mathrm{co}\{Q_1, Q_2\}$, where

$$Q_1 = \begin{pmatrix} 0.1 & 0.2 \\ 0.3 & 0.4 \end{pmatrix}, \qquad Q_2 = \begin{pmatrix} 0.2 & 0.2 \\ 0.3 & 0.3 \end{pmatrix}.$$

The corresponding left marginal set is $\mathrm{co}\{l_1, l_2\}$, where $l_1 = (0.3, 0.7)$ and $l_2 = (0.4, 0.6)$ are the row-sum vectors, and the right marginal set is $\mathrm{co}\{r_1, r_2\}$, where $r_1 = (0.4, 0.6)$ and $r_2 = (0.5, 0.5)$ are the column-sum vectors. The transition matrices corresponding to Q_1 and Q_2 are:

$$P_1 = \mathrm{diag}(l_1)^{-1}Q_1 = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{3}{7} & \frac{4}{7} \end{pmatrix},$$

$$P_2 = \mathrm{diag}(l_2)^{-1}Q_2 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

However,

$$l_2P_1 = \left(\frac{41}{105}, \frac{64}{105}\right) \approx (0.3905, 0.6095),$$

which clearly lies outside the right marginal set.

We will from now on stipulate that the sets $\mathcal{Q}_i$ are constant in time. In principle, it would be possible to study imprecise Markov chains whose sets of transition matrices or joint distribution matrices vary over time. However, it is more common to require that these sets are constant and to consider only the individual elements that vary over time. We now show that under the assumption of stationarity, a constant set of transition matrices implies that the corresponding sets of joint distribution matrices are also constant.

Proposition 5.3. *Let $\mathcal{C}$ be an initial set of distributions and $\mathcal{T}$ a set of transition matrices, such that $\mathcal{C}$ is the unique stationary set of $\mathcal{T}$, i.e., $\mathcal{C}\mathcal{T} = \mathcal{C}$. Then the corresponding sequence of joint distribution matrices consists of a sequence of identical sets.*

Conversely, if the sequence of joint distribution matrices corresponding to $\mathcal{C}^1$ and $\mathcal{T}$ is constant, the corresponding chain is stationary.

Proof. Immediate consequence of relations (16), (14) and (15). □

However, even if we have a stationary imprecise Markov chain, it will hardly ever be reversible, as Example 5.1 shows. To make a chain reversible, we therefore need to abandon the usual assumption, which is that every distribution can be transformed by any transition matrix in $\mathcal{T}$, because this assumption is too restrictive to allow for symmetry of the joint distribution matrix. Instead, we focus on symmetric joint distribution matrices, which result in reversibility.

Theorem 5.1. *Let $(X_{1:N})$ be an imprecise Markov chain whose transition law is induced by a set of joint distribution matrices $\mathcal{Q}$. Then the chain is stationary with stationary set $\mathcal{C}_{\mathcal{Q}} = \mathcal{C}^*_{\mathcal{Q}}$ and reversible if and only if $\mathcal{Q}$ is symmetric.*

Proof. First suppose $\mathcal{Q}$ is symmetric. The set of processes compatible with the assumptions contains all processes with joint distributions of the form:

$$p_\Gamma(x_{1:N}) = \frac{\prod_{k=1}^{N-1} Q_k(x_k, x_{k+1})}{\prod_{k=2}^{N-1} q^k(x_k)},$$

where $Q_k = Q_{q^k,P_k} \in \mathcal{Q}$ and $q^k = 1_{\mathcal{X}} Q_k$. The joint distribution of the reversed process is

$$p_{\Gamma^*}(x_{1:N}) = \frac{\prod_{k=1}^{N-1} Q_k^t(x_k, x_{k+1})}{\prod_{k=2}^{N-1} q^k(x_k)},$$

and since $\mathcal{Q}$ is symmetric, the transposed matrices $Q_k^t \in \mathcal{Q}$, too. Thus, the reversed process is of the same form as the forward one, and is therefore compatible with the imprecise transition law. This implies that $\mathrm{distr}(X_{1:N})^* = \mathrm{distr}(X_{1:N})$, which makes it reversible.

Recall the observation after Definition 5.2 that an imprecise Markov chain of length 2 is reversible exactly if the corresponding set of joint distribution matrices is symmetric. Moreover, it is clear that if a chain of length N is reversible, then so is each subchain of length 2. Hence, reversibility implies symmetry of the set $\mathcal{Q}$, which proves the opposite implication of the theorem. □

The following proposition states that every imprecise Markov chain can be extended to a reversible one.

Proposition 5.4. *Let $\mathcal{T}$ be a transition set and Π its stationary set of distributions. Let $\mathcal{Q}_{\Pi,\mathcal{T}}$ be the corresponding set of joint distribution matrices. Then the set $\hat{\mathcal{Q}}_{\Pi,\mathcal{T}} = \mathcal{Q}_{\Pi,\mathcal{T}} \cup \mathcal{Q}^t_{\Pi,\mathcal{T}}$ is the smallest set of joint distribution matrices inducing a reversible imprecise Markov chain.*

Moreover, the corresponding reversible process $X_{1:N}$ is stationary with distribution set Π.

Proof. The set $\hat{\mathcal{Q}}_{\Pi,\mathcal{T}}$ is clearly the smallest symmetric set containing $\mathcal{Q}_{\Pi,\mathcal{T}}$ and therefore the smallest inducing a reversible chain. Transposing a joint distribution matrix swaps the left and right marginals, both contained in Π. Thus, adding transposed matrices does not increase the set of marginals. □

Next, we show that every symmetric set of joint distributions can be extended to a convex set, preserving symmetry. Convexity is desirable, as it ensures that the set of possible models is closed under convex combinations, which is useful for computational purposes.

Proposition 5.5. *Let $\mathcal{Q}$ be a symmetric set of joint distributions with the stationary set of distributions Π. Then $\mathrm{co}(\mathcal{Q})$ is a convex and symmetric set of joint distributions with the stationary set $\mathrm{co}(\Pi)$.*

Proof. To see that $\mathrm{co}(\mathcal{Q})$ is symmetric, take a convex combination $\alpha Q + \beta Q'$, where $0 \le \alpha, \beta \le 1$ and $\alpha + \beta = 1$. We have that $(\alpha Q + \beta Q')^t = \alpha Q^t + \beta Q'^t$, which is a convex combination of the transposed matrices. By symmetry, Q^t and Q'^t are also in $\mathcal{Q}$, so $(\alpha Q + \beta Q')^t \in \mathrm{co}(\mathcal{Q})$.

Clearly, symmetry of $\mathcal{Q}$ implies that the corresponding unique stationary set Π equals $\mathcal{C}_{\mathcal{Q}} = \mathcal{C}^*_{\mathcal{Q}}$, and clearly $\mathcal{C}_{\mathrm{co}(\mathcal{Q})} = \mathrm{co}(\mathcal{C}_{\mathcal{Q}}) = \mathrm{co}(\Pi)$.

□

6. Random walk on graph with interval weights

6.1. Random walks on undirected graphs with interval weights. The basic idea to extend random walks on a graph to allow for imprecise weights is to replace a fixed weight function w with a set of weights $\mathcal{W}$. One such approach was presented in [21], where the set of weights was given in terms of intervals $(\underline{w}, \overline{w})$, which induce $\mathcal{W} = \{w : \underline{w} \le w \le \overline{w}\}$. We stick to the connectivity assumption, and additionally, we assume that either $\underline{w}(x,y) > 0$ or $\overline{w}(x,y) = 0$ for each pair of vertices $x, y \in \mathcal{X}$. While the weights form a convex set, the induced sets of transition matrices are not convex in general, as demonstrated by the following simple example.

Example 6.1. Take the following weight functions:

$$W_1 = \begin{pmatrix} 1 & 3 \\ 3 & 2 \end{pmatrix} \qquad W_2 = \begin{pmatrix} 1 & 5 \\ 5 & 2 \end{pmatrix}.$$

The induced transition matrices are respectively

$$P_1 = \begin{pmatrix} \frac{1}{4} & \frac{3}{4} \\ \frac{3}{5} & \frac{2}{5} \end{pmatrix} \qquad P_2 = \begin{pmatrix} \frac{1}{6} & \frac{5}{6} \\ \frac{5}{7} & \frac{2}{7} \end{pmatrix}.$$

However, the convex combination $W = \frac{1}{2}W_1 + \frac{1}{2}W_2$ induces transition matrix

$$P = \begin{pmatrix} \frac{1}{5} & \frac{4}{5} \\ \frac{4}{6} & \frac{2}{6} \end{pmatrix},$$

which is not a linear combination of P_1 and P_2, which suggests that the set of transition matrices induced by interval set of weights is not convex in general.

Convexity is essential for efficient estimation of expectations using linear programming. To ensure convexity, [21] restricted the set of possible weights such that the marginal distributions were precise (i.e., the sum of weights of edges incident to each node was fixed). This restriction results in convex sets of transition matrices, but they lack the property of having *separately specified rows*. This means that the transition probabilities $P(\cdot|x)$ for different values of x cannot be selected independently. Consequently, the marginal sets of distributions at later times are non-convex, preventing the use of standard linear programming techniques for estimating bounds.

Here we propose a different model that does not require restricting to precise marginals. To ensure convexity we will model the imprecise random walk process using sets of joint distribution matrices instead of transition matrices. Recall that a precise random walk on a weighted graph is assumed to start from a stationary distribution given by (5). The following proposition provides a joint distribution matrix corresponding to a random walk.

Proposition 6.1. *Let w be a weight function on a graph with vertices $\mathcal{X}$. Then the joint distribution corresponding to a two step random walk starting from a stationary distribution is given by*

$$Q_w(x, y) = \frac{w(x, y)}{W},$$

where W denotes the total sum of weights (counted twice for each edge, as an incoming and an outgoing weight).

Let $\mathcal{W}$ be a convex set of weights. We associate a set of joint distribution matrices $\mathcal{Q}_{\mathcal{W}} = \{Q_w : w \in \mathcal{W}\}$. In contrast to the corresponding transition set, the induced set of joint distribution matrices is convex as shown by the following proposition.

Proposition 6.2. *Let $\mathcal{W}$ be a convex set of weight functions and let $\mathcal{Q}$ be the corresponding set of joint distribution matrices. Then $\mathcal{Q}$ is a convex set.*

Proof. Let w be a weight function. Take some weights $w, w' \in \mathcal{W}$ and some positive constants α, β summing to 1. For every pair $x, y \in \mathcal{X}$ we have

$$\begin{aligned} Q_{\alpha w+\beta w'}(x, y) &= \frac{\alpha w(x, y) + \beta w'(x, y)}{\alpha W + \beta W'} \\ &= \frac{\alpha W}{\alpha W + \beta W'}\frac{w(x, y)}{W} \\ &\quad + \frac{\beta W'}{\alpha W + \beta W'}\frac{w'(x, y)}{W'} \\ &= AQ_w(x, y) + BQ_{w'}(x, y), \end{aligned}$$

where $A = \frac{\alpha W}{\alpha W+\beta W'}$ and $B = \frac{\beta W'}{\alpha W+\beta W'}$. Clearly both A and B are non-negative and sum to 1, whence $Q_{\alpha w+\beta w'}$ is a convex combination of Q_w and $Q_{w'}$. □

As discussed in previous sections, random processes corresponding to a set of joint distribution matrices are induced by sequences $Q_{1:N-1}$ of joint distribution matrices in $\mathcal{Q}_{\mathcal{W}}$ that are pairwise marginally compatible. However, the left and right marginals of every joint distribution matrix Q induced by a weight function on an *undirected* graph are identical. Therefore, the induced sequences can only consist of matrices with equal marginals, which is a rather restricted set of random walks. To induce a richer set of random walks, we relax the condition that the weight functions are symmetric and consider directed graphs.

6.2. Random walks on directed weighted graphs with interval weights. We now consider convex sets $\mathcal{W}$ of not necessarily symmetric weight functions defined on a complete graph with vertex set $\mathcal{X}$. For a weight function $w \in \mathcal{W}$, the sum of all outgoing weights from any vertex equals the sum of all incoming weights (since each directed edge weight is counted exactly once in the graph), and we denote this total by W. For each weight function w, we define the joint distribution matrix $Q_w = w/W$. The left marginal distribution is given by $\frac{w_O}{W}$, where $w_O(x) = \sum_{y\in\mathcal{X}} w(x, y)$ represents the total weight of all outgoing edges from vertex x, while the right marginal distribution is $\frac{w_I}{W}$, where $w_I(y) = \sum_{x\in\mathcal{X}} w(x, y)$ is the total weight of all incoming edges to vertex y. A random walk on a directed graph with weight function w is a random process $(X_{1:N})$ taking values in $\mathcal{X}$, where the probability of transitioning from x to y is $\frac{w(x,y)}{w_O(x)}$. Let $\mathcal{Q}_{\mathcal{W}} = \{Q_w : w \in \mathcal{W}\}$ denote the set of all joint distribution matrices induced by the weight functions in $\mathcal{W}$.

Definition 6.1. Let $\mathcal{W}$ be a set of weight functions on a complete graph with vertex set $\mathcal{X}$, and let $\mathcal{Q}_{\mathcal{W}}$ be the corresponding set of joint distribution matrices. A *random walk* on a weighted graph with weight set $\mathcal{W}$ is an imprecise Markov chain whose set of transition laws is given by

$$\mathcal{G}_{\mathcal{W}} = \{\Gamma_{Q_{1:N-1}} : Q_i \in \mathcal{Q}_{\mathcal{W}}\},$$

where, in all sequences, the sequential joint matrices are marginally compatible.

The following proposition establishes a connection between the symmetry of the set of weight functions and the symmetry of the corresponding set of joint distribution matrices.

Proposition 6.3. *Let $\mathcal{W}$ be a convex set of weight functions such that for every $w \in \mathcal{W}$, the transposed weight function w^t, defined by $w^t(x, y) = w(y, x)$, also belongs to $\mathcal{W}$. Then the set $\mathcal{Q}_\mathcal{W} = \{Q_w : w \in \mathcal{W}\}$ is symmetric, and consequently, the corresponding random walk is reversible.*

Proof. The result follows directly from the observation that $Q^t_w = Q_{w^t}$. □

A symmetric set of weights $\mathcal{W}$ can be viewed as an imprecise generalization of a symmetric weight function. Although individual weight functions in $\mathcal{W}$ may not be symmetric – meaning the weights on edges can depend on the direction of traversal – the set as a whole exhibits symmetry. This concept is analogous to an imprecise time-homogeneous Markov chain, which comprises a set of non-homogeneous processes but remains homogeneous when considered as an imprecise process.

7. Conclusions

In this paper, we propose a model of reversible imprecise Markov chains. An imprecise Markov chain is represented as a collection of precise, inhomogeneous processes compatible with imprecise estimates of transition probabilities and marginal distributions. We apply reversal to the individual processes within these imprecise bounds, which requires the development of methods to reverse inhomogeneous precise Markov chains, as inhomogeneous processes have rarely been studied in the context of invertibility in the existing literature.

Reversible Markov chains are closely related to random walks on graphs, a connection uniquely established in the precise case through weighted undirected graphs. In the imprecise case, we introduce a natural relaxation by replacing fixed weights with intervals, leaving open the question of whether individual weights remain symmetric. Assuming symmetry for each weight function results in a relatively restricted set of random walks. However, by relaxing this assumption, we obtain a richer and more flexible family of random walks that retain reversibility when symmetry is imposed at the level of the entire set of processes.

This article lays out the basic concepts of the model, but leaves some key questions unresolved. These open questions, which we intend to address in future work, include the development of practical methods for computing probability distributions and for calculating return and hitting times.

Additional author information

Acknowledgements. The author acknowledges the financial support from the Slovenian Research Agency (research core funding No. P5-0168).

References

[1] David Aldous and James Allen Fill. *Reversible Markov Chains and Random Walks on Graphs*. Unfinished monograph, recompiled 2014, available at http://www.stat.berkeley.edu/~aldous/RWG/book.html. 2002.

[2] Thomas Augustin, Frank PA Coolen, Gert de Cooman, and Matthias CM Troffaes. *Introduction to imprecise probabilities*. John Wiley & Sons, 2014. DOI: 10.1002/9781118763117.

[3] Lars Backstrom and Jure Leskovec. "Supervised random walks: predicting and recommending links in social networks". In: *Proceedings of the Fourth ACM International Conference on Web Search and Data Mining*. WSDM '11. Hong Kong, China: Association for Computing Machinery, 2011, pp. 635–644. DOI: 10.1145/1935826.1935914.

[4] Gert de Cooman, Filip Hermans, and Erik Quaeghebeur. "Imprecise Markov chains and their limit behavior". In: *Probability in the Engineering and Informational Sciences* 23.4 (2009), pp. 597–635. DOI: 10.1017/S0269964809990039.

[5] Don Coppersmith, Peter Doyle, Prabhakar Raghavan, and Marc Snir. "Random walks on weighted graphs and applications to on-line algorithms". In: *J. ACM* 40.3 (July 1993), pp. 421–453. DOI: 10.1145/174130.174131.

[6] Uriel Feige. "A tight lower bound on the cover time for random walks on graphs". In: *Random Structures & Algorithms* 6.4 (1995), pp. 433–438.

[7] Francois Fouss, Alain Pirotte, Jean-michel Renders, and Marco Saerens. "Random-Walk Computation of Similarities between Nodes of a Graph with Application to Collaborative Recommendation". In: *IEEE Transactions on Knowledge and Data Engineering* 19.3 (2007), pp. 355–369. DOI: 10.1109/TKDE.2007.46.

[8] F. Gobel and A.A. Jagers. "Random walks on graphs". In: *Stochastic processes and their applications* 2.4 (1974), pp. 311–336. DOI: 10.1016/0304-4149(74)90001-5.

[9] Peter J. Green. "Reversible jump Markov chain Monte Carlo computation and Bayesian model determination". In: *Biometrika* 82.4 (Dec. 1995), pp. 711–732. DOI: 10.1093/biomet/82.4.711.

[10] Darald J. Hartfiel. *Markov Set-Chains*. Berlin: Springer-Verlag, 1998. DOI: 10.1007/BFb0094586.

[11] W Keith Hastings. "Monte Carlo sampling methods using Markov chains and their applications". In: *Biometrika* 57.1 (1970), pp. 97–109. DOI: 10.1093/biomet/57.1.97.

[12] Mark Jerrum and Alistair Sinclair. "The Markov chain Monte Carlo method: an approach to approximate counting and integration". In: *Approximation algorithms for NP-hard problems* (1996), pp. 482–520.

[13] Thomas Krak, Jasper De Bock, and Arno Siebes. "Imprecise continuous-time Markov chains". In: *International Journal of Approximate Reasoning* 88 (2017), pp. 452–528. DOI: 10.1016/j.ijar.2017.06.012.

[14] Rong-Hua Li, Jeffrey Xu Yu, and Jianquan Liu. "Link prediction: the power of maximal entropy random walk". In: *Proceedings of the 20th ACM international conference on Information and knowledge management*. ACM. 2011, pp. 1147–1156. DOI: 10.1145/2063576.2063741.

[15] Yuan Lin and Zhongzhi Zhang. "Mean first-passage time for maximal-entropy random walks in complex networks". In: *Scientific reports* 4 (2014). DOI: 10.1038/srep05365.

[16] Hongyu Liu and Evangelos Milios. "Probabilistic models for focused web crawling". In: *Computational Intelligence* 28.3 (2012), pp. 289–328. DOI: 10.1111/j.1467-8640.2012.00411.x.

[17] László Lovász. "Random walks on graphs". In: *Combinatorics, Paul erdos is eighty* 2.1-46 (1993), p. 4.

[18] J. Norris. *Markov Chains*. Cambridge: Cambridge University Press, 1997. DOI: 10.1017/CBO9780511810633.

[19] Damjan Škulj. "Discrete time Markov chains with interval probabilities". In: *International Journal of Approximate Reasoning* 50.8 (Sept. 2009), pp. 1314–1329. DOI: 10.1016/j.ijar.2009.06.007.

[20] Damjan Škulj. "Efficient computation of the bounds of continuous time imprecise Markov chains". In: *Applied Mathematics and Computation* 250.0 (Jan. 2015), pp. 165–180. DOI: 10.1016/j.amc.2014.10.092.

[21] Damjan Škulj. "Random walks on graphs with interval weights and precise marginals". In: *International Journal of Approximate Reasoning* 73 (2016), pp. 76–86. DOI: 10.1016/J.IJAR.2016.02.008.

[22] Peter Walley. *Statistical Reasoning with Imprecise Probabilities*. London, New York: Chapman and Hall, 1991.

[23] Zhijun Yin, Manish Gupta, Tim Weninger, and Jiawei Han. "A unified framework for link recommendation using random walks". In: *Advances in Social Networks Analysis and Mining (ASONAM), 2010 International Conference on*. IEEE. 2010, pp. 152–159. DOI: 10.1109/ASONAM.2010.27.

[24] Zhongzhi Zhang, Huan Li, and Yibin Sheng. "Effects of reciprocity on random walks in weighted networks". In: *Scientific reports* 4 (2014). DOI: 10.1038/srep07460.