The marginal problem for sets of desirable gamble sets

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Abstract

We study the marginal problem for sets of desirable gamble sets (SoDGSes), which is equivalent to studying this problem for choice functions. More specifically, given a number of marginal SoDGSes on overlapping domains, we establish conditions under which they are compatible in the sense that they can be derived from a common joint SoDGS. We do so for SoDGSes that admit a concrete finite representation. Our main result is that such SoDGSes are compatible if they are pairwise compatible and if a running intersection property is satisfied.

Keywords. sets of desirable gamble sets, choice functions, marginal problem, representation

1. Introduction

Consider a probability mass function p_1 describing an agent's beliefs about the uncertain variables X_1 and X_2, and a probability mass function p_2 describing beliefs about the uncertain variables X_2 and X_3. When are they *compatible*, in the sense that they are derived from a common joint p about X_1, X_2 and X_3? This problem is called the 'marginal problem': the compatibility of a number of marginal uncertainty models with a joint model. The difficulty here is that p_1 and p_2 both represent the variable X_2: their respective domains 'overlap'. So one necessary condition for a positive answer to the marginal problem, is that p_1 and p_2 marginalise to a common probability mass function for X_2. The marginal problem has been studied extensively for probabilities in the past [4, 17] and more recently [23, and references therein].

Our inspiration for this paper is the work of Miranda and Zaffalon [23], who studied the marginal problem for sets of desirable gambles, and obtained sufficient conditions that guarantee a positive answer. We study the marginal problem for sets of desirable gamble sets ('SoDGSes' – singular 'SoDGS'), thereby generalising some of their results. SoDGSes attribute desirability to *sets of gambles* rather than to gambles. When an agent finds a set $\{f, g\}$ desirable – preferred over the status quo indicated by 0 – this means that one of f or g is preferred over 0, but she might not be able to identify which of f or g is preferred over 0. As such, SoDGSes are capable of modelling disjunctions of preference statements [11], and this makes them among the most expressive uncertainty models in the literature of imprecise probabilities. They are equivalent to imprecise-probabilistic choice functions, which were introduced by Kadane, Schervish, and Seidenfeld [18] and Seidenfeld, Schervish, and Kadane [29]. Choice functions and SoDGSes are gaining popularity, and various investigations of foundational aspects have recently been carried out [3, 8, 11, 14, 16, 31, 32].

One advantage of SoDGSes and choice functions is that they are easy to work with from a theoretical point of view, thanks to their representation in terms of a collection of partial preference orders – a collection of sets of desirable gambles. In this paper we solve the marginal problem for an interesting subclass of SoDGSes, namely those that admit a finite representation.

In the study of the marginal problem for sets of desirable gambles by Casanova, Kohlas, and Zaffalon [6], they followed a different approach than the one by Miranda and Zaffalon [23]: they solved the marginal problem for any *valuation algebra*, and have shown that coherent sets of desirable gambles form such an algebra. In our current study of the marginal problem for SoDGSes, we do not follow this approach, as it would not lead to a representation of the SoDGS compatible with the given SoDGSes in terms of a collection of sets of desirable gambles. Instead, we follow a more direct approach, which, as we will see, will lead to a representation of the compatible joint.

We first recall the theory of SoDGSes (Section 2). This is followed by our contributions concerning the finite representation (Section 3). Then we discuss multivariate SoDGSes in the by then established context (Section 4) in preparation of our formulation and solution of the marginal problem (Section 5).

2. Sets of desirable gamble sets

2.1. Preliminaries. Consider an uncertain variable X, taking values in a finite possibility space $\mathcal{X}$. A gamble is a real-valued function on $\mathcal{X}$. A gamble f commits its owner to the following transaction: first, X's real outcome x in $\mathcal{X}$ is revealed, and then the owner receives $f(x)$ units of utile on a predetermined utility scale. This value $f(x)$ may be negative, so gambles can be regarded as risky transactions.

We denote the set of all gambles on $\mathcal{X}$ by $\mathcal{L}(\mathcal{X})$, and sometimes also simply by $\mathcal{L}$ if it is unambiguous from the context what the possibility space $\mathcal{X}$ is. $\mathcal{L}$ is an $|\mathcal{X}|$-dimensional linear space under the pointwise addition and scalar multiplication of gambles. We collect the 'positive' gambles in $\mathcal{L}_{>0}(\mathcal{X}) := \{f \in \mathcal{L}(\mathcal{X}) : f > 0\}$, and the 'non-positive' ones in $\mathcal{L}_{\leq 0}(\mathcal{X}) := \{f \in \mathcal{L}(\mathcal{X}) : f \leq 0\}$, or $\mathcal{L}_{>0}$ and $\mathcal{L}_{\leq 0}$ when unambiguous.[1]

2.2. Sets of desirable gambles. An agent might have beliefs about the uncertain variable X, which can lead her to prefer some gambles over others. She might prefer a gamble f over the status quo 0, in which case we call f *desirable*. The agent's *set of desirable gambles* $D \subseteq \mathcal{L}$ collects all the gambles that she finds desirable. They have been introduced by Williams [36] and Seidenfeld, Schervish, and Kadane [27], and studied extensively by Walley [33, 34], De Cooman and Quaeghebeur [10], De Cooman and Miranda [9] and Quaeghebeur [25], among others.

Definition 2.1 (Coherent set of desirable gambles)**.** A set of desirable gambles $D \subseteq \mathcal{L}$ is called *coherent* if:

D_1. $0 \notin D$;

D_2. $\mathcal{L}_{>0} \subseteq D$;

D_3. if $f, g \in D$ and $(\lambda, \mu) > 0$, then $\lambda f + \mu g \in D$.

We collect all the coherent sets of desirable gambles in $\overline{\mathcal{D}}(\mathcal{X}) \subseteq \mathcal{P}(\mathcal{L})$, or $\overline{\mathcal{D}}$ for short, where $\mathcal{P}$ denotes the powerset of its argument.

We say that the set of desirable gambles D_1 is at least as informative (or committal, precise) as the set of desirable gambles D_2 when $D_2 \subseteq D_1$, simply because an agent with D_1 finds more gambles desirable. The partially ordered set $(\overline{\mathcal{D}}, \subseteq)$ is a complete meet-semilattice. De Cooman [7] showed that such a structure is strong enough to yield a closure operator

$$\mathrm{cl}_{\overline{\mathcal{D}}} : \mathcal{P}(\mathcal{L}) \to \overline{\mathcal{D}} \cup \{\mathcal{L}\} : A \mapsto \mathrm{cl}_{\overline{\mathcal{D}}}(A) := \bigcap\nolimits_{A \subseteq D \in \overline{\mathcal{D}}} D$$

that maps any partially specified set of desirable gambles $A \subseteq \mathcal{L}$ that is *consistent* – has a coherent superset – to its unique least informative coherent extension $\mathrm{cl}_{\overline{\mathcal{D}}}(A)$. The sets of desirable gambles that are not consistent are mapped to $\mathcal{L}$. We call $\mathrm{cl}_{\overline{\mathcal{D}}}(A)$ the *natural extension* of the assessment A. So, the natural extension is the unique set of desirable gambles that is the consequence of finding every gamble in A desirable and taking into account the rationality axioms D_1–D_3, but nothing else.

There is a useful characterisation of $\mathrm{cl}_{\overline{\mathcal{D}}}$ in terms of the positive linear hull operator 'posi', defined for all $B \subseteq \mathcal{L}$ as follows

$$\mathrm{posi}(B) := \Big\{ \sum\nolimits_{f \in F} \lambda_f f : F \Subset B, \lambda \in (\mathbb{R}^F)_{>0} \Big\},$$

where $\Subset$ denotes the finite subset relation.

Theorem 2.1 ([10, Theorem 1])**.** *Consider any assessment $A \subseteq \mathcal{L}$. Then A is consistent if and only if $0 \notin \mathrm{posi}(\mathcal{L}_{>0} \cup A)$. If this is the case, then $\mathrm{cl}_{\overline{\mathcal{D}}}(A) = \mathrm{posi}(\mathcal{L}_{>0} \cup A)$.*

The smallest coherent set of desirable gambles – the *vacuous* set of desirable gambles – can be obtained from the empty assessment. We call it $D_{\mathrm{v}} := \mathrm{cl}_{\overline{\mathcal{D}}}(\emptyset) = \mathcal{L}_{>0}$.

2.3. Sets of desirable gamble sets. The agent might have beliefs that allow her to state 'gamble f is desirable or gamble g is desirable', but she might not have sufficient information to decide which of f and g are desirable. In other words, she knows that the set $\{f, g\}$ contains a desirable gamble, in which case we call $\{f, g\}$ a *desirable gamble set*.

More generally, a *gamble set* on $\mathcal{X}$ is a finite set of gambles on $\mathcal{X}$. We denote the set of all gamble sets on $\mathcal{X}$ by $\mathcal{Q}(\mathcal{X})$, or $\mathcal{Q}$ when unambiguous. A gamble set is *desirable* when it contains at least one desirable gamble. We collect an agent's desirable gamble sets in her *set of desirable gamble sets (SoDGS) $K \subseteq \mathcal{Q}$*.

De Bock and De Cooman [12] gave a well-justified definition and axiomatisation of *coherent* SoDGSes – SoDGSes of rational agents.

Definition 2.2 (Coherent SoDGS)**.** An SoDGS $K \subseteq \mathcal{Q}$ is called *coherent* if:

K_0. $\emptyset \notin K$;

K_1. $F \in K \Rightarrow F \setminus \{0\} \in K$;

K_2. $\{f\} \in K$ for all f in $\mathcal{L}_{>0}$;

K_3. if $F, G \in K$ and $(\lambda^{f,g}, \mu^{f,g}) > 0$ for every pair (f, g) in $F \times G$, then $\{\lambda^{f,g} f + \mu^{f,g} g : f \in F, g \in G\} \in K$;

K_4. if $F \in K$ and $F \subseteq G$, then $G \in K$.

We collect all the coherent SoDGSes in the collection $\overline{\mathcal{K}}(\mathcal{X})$, or $\overline{\mathcal{K}}$ when unambiguous.

Similarly as we did for sets of desirable gambles, we say that the SoDGS K_1 is at least as informative as the SoDGS K_2 when $K_2 \subseteq K_1$, simply because an agent with K_1 finds more gamble sets desirable. The partially ordered set $(\overline{\mathcal{K}}, \subseteq)$ is again a complete meet-semilattice, so it, too, yields a closure operator

$$\mathrm{cl}_{\overline{\mathcal{K}}} : \mathcal{P}(\mathcal{Q}) \to \overline{\mathcal{K}} \cup \{\mathcal{Q}\} : \mathcal{A} \mapsto \mathrm{cl}_{\overline{\mathcal{K}}}(\mathcal{A}) := \bigcap\nolimits_{\mathcal{A} \subseteq K \in \overline{\mathcal{K}}} K$$

that maps any partially specified SoDGS $\mathcal{A} \subseteq \mathcal{Q}$ that is *consistent* – has a coherent superset – to its unique least informative coherent extension $\mathrm{cl}_{\overline{\mathcal{K}}}(\mathcal{A})$. We call $\mathrm{cl}_{\overline{\mathcal{K}}}(\mathcal{A})$ the *natural extension* of the assessment $\mathcal{A}$. Here too, the natural extension is the consequence of finding every gamble set in $\mathcal{A}$ desirable and the rationality axioms K_0–K_4, and nothing else.

De Bock and De Cooman [12] showed that there is a useful characterisation of $\mathrm{cl}_{\overline{\mathcal{K}}}$ in terms of the positive

[1] For all elements u and v of a finite vector space, we write $u \leq v$ if $u(z) \leq v(z)$ (or $u_z \leq v_z$) for all arguments (indices) z, and $u < v$ if $u \leq v$ but $u \neq v$. These definitions are used in this paper for gambles, but also for sequences of coefficients.

linear hull operator 'posi' lifted to all $\mathcal{B} \subseteq \mathcal{Q}$ as follows

$$\operatorname{Posi}(\mathcal{B}) := \Big\{\Big\{\textstyle\sum_{F\in\mathcal{F}} \lambda_F^h h_F : h \in H\Big\} : \mathcal{F} \Subset \mathcal{B},\\ (\forall h \in H := \textstyle\bigtimes_{F\in\mathcal{F}} F)\lambda^h \in (\mathbb{R}^{\mathcal{F}})_{>0}\Big\}.$$

We also use a version of $\mathcal{L}_{>0}(\mathcal{X})$ lifted to gamble sets, namely $\mathcal{L}^{\mathrm{s}}_{>0}(\mathcal{X}) := \{\{f\} : f \in \mathcal{L}_{>0}(\mathcal{X})\}$, or $\mathcal{L}^{\mathrm{s}}_{>0}$ when unambiguous, and use the transformation $\operatorname{Rs} : \mathcal{P}(\mathcal{Q}) \to \mathcal{P}(\mathcal{Q}) : \mathcal{B} \mapsto \operatorname{Rs}(\mathcal{B}) := \{F \in \mathcal{Q} : (\exists G \in \mathcal{B}) G \setminus \mathcal{L}_{\leq 0} \subseteq F\}$, which is the abbreviation of 'Remove negative gambles from gamble sets in its input and take supersets'.

Theorem 2.2 ([12, Theorem 10]). *Consider any assessment $\mathcal{A} \subseteq \mathcal{Q}$. Then $\mathcal{A}$ is consistent if and only if $\emptyset \notin \mathcal{A}$ and $\{0\} \notin \operatorname{Posi}(\mathcal{L}^{\mathrm{s}}_{>0} \cup \mathcal{A})$. If this is the case, then $\operatorname{cl}_{\overline{\mathcal{K}}}(\mathcal{A}) = \operatorname{Rs}(\operatorname{Posi}(\mathcal{L}^{\mathrm{s}}_{>0} \cup \mathcal{A}))$.*

The smallest coherent SoDGS – the *vacuous* SoDGS – is obtained from the empty assessment. We call it $K_{\mathrm{v}} := \operatorname{cl}_{\overline{\mathcal{K}}}(\emptyset) = \operatorname{Rs}(\mathcal{L}^{\mathrm{s}}_{>0}) = \{F \in \mathcal{Q} : (\exists f \in \mathcal{L}_{>0}) f \in F\}$.

If the agent has a set of desirable gambles D, then she finds desirable all the gamble sets $\{f\}$ for f in D. In other words, her SoDGS that corresponds to D is the natural extension of the assessment $\mathcal{A} := \{\{f\} : f \in D\}$, which is given by $\operatorname{cl}_{\overline{\mathcal{K}}}(\mathcal{A})$. Whenever D is coherent, it follows from Van Camp and Miranda [32, Proposition 5] that $\operatorname{cl}_{\overline{\mathcal{K}}}(\mathcal{A})$ is coherent and equal to $K_D := \{F \in \mathcal{Q} : F \cap D \neq \emptyset\}$.

We use the notation K_D for $\{F \in \mathcal{Q} : F \cap D \neq \emptyset\}$ even when D is not coherent. De Bock and De Cooman [13, Proposition 8] showed that K_D is coherent if and only if D is. We call an SoDGS K *binary* if there is a set of desirable gambles D such that $K_D = K$, because K is then characterised by the preference order determined by D, which is a *binary* order. The vacuous SoDGS K_{v} is binary, and $D_{\mathrm{v}} = \mathcal{L}_{>0}$ is the set of desirable gambles that determines it: $K_{\mathrm{v}} = \{F \in \mathcal{Q} : F \cap \mathcal{L}_{>0} \neq \emptyset\} = K_{D_{\mathrm{v}}}$.

Conversely, if the agent has an SoDGS K, then $D_K := \{f : \{f\} \in K\}$ is a set of desirable gambles for her. Moreover, De Bock and De Cooman [12, Lemma 13] showed that D_K is coherent whenever K is.

2.4. Choice functions. An SoDGS can be equivalently represented by a choice function, which is a map

$$C : \mathcal{Q} \to \mathcal{Q} : F \mapsto C(F) \subseteq F.$$

The idea is that C identifies the non-rejected (or choiceworthy) gambles $C(F)$ in any gamble set F, which represents a finite decision problem. Choice functions are intimately related to the fundamental problem of decision theory: how to make decisions from a set of available options. Von Neumann and Morgenstern [24] introduced rationality axioms for choice functions based on binary comparisons, which was generalised to more general choice functions by Arrow [1] and Rubin [26], among others. Kadane, Schervish, and Seidenfeld [18] and Seidenfeld, Schervish, and Kadane [29] have introduced imprecise-probabilistic choice functions, which have the characteristic feature that an agent can be indecisive between two gambles f and g, without being necessarily indifferent between f and g.

A choice function C determines an SoDGS as follows. When a gamble set F satisfies $0 \notin C(\{0\} \cup F)$ – in other words, when the status quo is rejected from $\{0\} \cup F$ – then we call F *desirable*, and collect all the desirable gamble sets in $K_C := \{F \in \mathcal{Q} : 0 \notin C(\{0\} \cup F)\}$. The idea is that, if 0 is not choiceworthy in $\{0\} \cup F$, then F must contain a gamble preferred to 0.

The other way around, an SoDGS K determines a choice function as follows. For any gamble set F, let $C_K(F) := \{f \in F : \{g - f : g \in F\} \notin K\}$, which defines the corresponding choice function.

Choice function have rationality axioms similar to Axioms K_1–K_4 [12, Definition 4]. They can also be ordered: If $C_1(F) \subseteq C_2(F)$ for all F in $\mathcal{Q}$, then we call C_1 more informative than C_2.

Under very mild conditions, much weaker than coherence,[2] these two constructions commute ($K_{C_K} = K$ and $C_{K_C} = C$) and preserve the order and coherence, so choice functions and SoDGSes are equivalent representations of the same information. In this paper, we will for notational reasons stick to SoDGSes, but due to this equivalence, *all the results can be translated to choice functions*.

3. Finite representation

Coherent SoDGSes have an interesting representation in terms of collections of sets of desirable gambles.

Theorem 3.1 (Representation [13, Theorem 9]). *For any SoDGS K, the following two expressions are equivalent:*

(i) K is coherent;

(ii) there is a non-empty set $\mathcal{D} \subseteq \overline{\mathcal{D}}$ of coherent sets of desirable gambles such that $K = K_{\mathcal{D}} := \bigcap\{K_D : D \in \mathcal{D}\}$, in which case we say that $\mathcal{D}$ represents K.

Moreover, K's (unique) largest representing set is $\mathcal{D}_K := \{D \in \overline{\mathcal{D}} : K \subseteq K_D\}$.

Note that $\mathcal{D}_K$ is an *upset*: if $D_1 \in \mathcal{D}_K$ and $D_1 \subseteq D_2$, then $D_2 \in \mathcal{D}_K$, for any D_1 and D_2 in $\overline{\mathcal{D}}$. In other words, $\mathcal{D}_K = \uparrow\mathcal{D}_K$, where $\uparrow\mathcal{D} := \{D \in \overline{\mathcal{D}} : (\exists D' \in \mathcal{D}) D' \subseteq D\}$ is the smallest upset containing $\mathcal{D}$, for any $\mathcal{D} \subseteq \overline{\mathcal{D}}$.

Theorem 3.1 also allows for a simpler expression for the natural extension:

Theorem 3.2 (De Bock & De Cooman, private communication). *An assessment $\mathcal{A} \subseteq \mathcal{Q}$ is consistent if and only*

[2] The condition for SoDGSes K is: $F \in K \Leftrightarrow \{0\} \cup F \in K$, for all F in $\mathcal{Q}$. The condition for choice functions C is: $C(F + \{f\}) = C(F) + \{f\}$ for all F in $\mathcal{Q}$ and f in $\mathcal{L}$.

if there is some D in $\overline{\mathcal{D}}$ such that $\mathcal{A} \subseteq K_D$. In that case $\mathrm{cl}_{\overline{\mathcal{K}}}(\mathcal{A}) = \bigcap\{K_D : D \in \overline{\mathcal{D}}$ and $\mathcal{A} \subseteq K_D\}$.

The representation of coherent K in terms of $\mathcal{D}_K \subseteq \overline{\mathcal{D}}$ in Theorem 3.1 has been crucial for deriving several theoretical properties of SoDGSes, among which the irrelevant natural extension [32], independent natural extension [31] and marginal extension [22].

Despite the success of the representation of Theorem 3.1, we here focus our attention, for technical reasons,[3] on a special subclass of coherent SoDGSes: the ones that admit a *finite representation*.

Definition 3.1 (Finite representation)**.** Consider a coherent SoDGS K. We say that K has a *finite representation* if there is a finite subset $\mathcal{D} \subseteq \overline{\mathcal{D}}$ that represents K.

An interesting characterisation of an SoDGS K with a finite representation involves the existence of minimal elements in the poset $(\mathcal{D}_K, \subseteq)$ of K's largest representation. In order to discuss it, for every $\mathcal{D} \subseteq \overline{\mathcal{D}}$, define $\min \mathcal{D} := \{D \in \mathcal{D} : (\forall D' \in \mathcal{D})(D' \subseteq D \Rightarrow D' = D)\}$ as $\mathcal{D}$'s minimal elements.

Theorem 3.3. *For any coherent SoDGS K, we have that $\min \mathcal{D}_K \neq \emptyset$ so the poset $(\mathcal{D}_K, \subseteq)$ has minimal elements. Moreover, $\mathcal{D}_K = \uparrow \min \mathcal{D}_K$. As a consequence $K = K_{\min \mathcal{D}_K}$ so $\min \mathcal{D}_K$ represents K.*

Proof. We will first show that $\min \mathcal{D}_K \neq \emptyset$. To this end, consider any non-empty chain $\mathcal{C} \subseteq \mathcal{D}_K$, meaning that $\mathcal{C}$ is totally ordered by $\subseteq$. By Zorn's Lemma it suffices to show that $\mathcal{C}$ has a lower bound in $\mathcal{D}_K$. We will show that the lower bound $D_\star := \bigcap \mathcal{C}$ of $\mathcal{C}$ belongs to $\mathcal{D}_K$. In order to do so, note already that $D_\star$ is a coherent set of desirable gambles because every element of $\mathcal{C}$ is coherent. To show that $D_\star$ belongs to $\mathcal{D}_K$ – or equivalently, that $K \subseteq K_{D_\star}$ – consider any F in $\mathcal{Q}$ such that $F \notin K_{D_\star}$, implying that $F \cap D_\star = \emptyset$, and we will show that $F \notin K$. From $F \cap D_\star = \emptyset$, it follows that $F \subseteq D_\star^c = \bigcup_{D \in \mathcal{C}} D^c$, so for every gamble f in F there is some set of desirable gambles D_f in $\mathcal{C}$ such that $f \notin D_f$. Defining $D^\star := \bigcap_{f \in F} D_f$, we infer that $F \cap D^\star = \emptyset$, whence $F \notin K_{D^\star}$. But since F is finite and $\mathcal{C}$ is a chain, $D^\star = \bigcap_{f \in F} D_f$ simply is the smallest element of the finite $\{D_f : f \in F\} \subseteq \mathcal{C}$, which therefore belongs to $\mathcal{C} \subseteq \mathcal{D}_K$. This implies that $F \notin K_{\mathcal{D}_K} = K$, where the equality follows from Theorem 3.1. So $\mathcal{C}$ has a lower bound $D_\star$ that belongs to $\mathcal{D}_K$. Therefore, since the choice of $\mathcal{C} \subseteq \mathcal{D}_K$ was arbitrary, using Zorn's Lemma we know that $\min \mathcal{D}_K$ is non-empty.

We now use this to show that $\mathcal{D}_K = \uparrow \min \mathcal{D}_K$. Because $\min \mathcal{D}_K \subseteq \mathcal{D}_K$, and the fact that $\mathcal{D}_K$ is an upset, we already know that $\mathcal{D}_K \supseteq \uparrow \min \mathcal{D}_K$, so let us prove the converse set inclusion $\mathcal{D}_K \subseteq \uparrow \min \mathcal{D}_K$. To this end, consider any $D^\star$ in $\mathcal{D}_K$, and assume *ex absurdo* that $D' \nsubseteq D^\star$ for every D' in $\min \mathcal{D}_K$. This would imply that the non-empty $\downarrow D^\star := \{D \in \mathcal{D}_K : D \subseteq D^\star\}$ has no minimal element. Indeed, if $\downarrow D^\star$ had a minimal element $D_\star$, then $D_\star$ would also be a minimal element of $\mathcal{D}_K$, because otherwise there would be an element $D' \subset D_\star$ in $\mathcal{D}_K$ which would then also belong to $\downarrow D^\star$.

We will apply Zorn's Lemma to the poset $(\downarrow D^\star, \subseteq)$, so consider any non-empty chain $\mathcal{C} \subseteq \downarrow D^\star$. Then $D_\star := \bigcap \mathcal{C}$ is a coherent set of desirable gambles, and by a similar argument as above we find that $K \subseteq K_{D_\star}$. This implies that $D_\star$ belongs to $\downarrow D^\star$, so the chain $\mathcal{C}$ has a lower bound in $\downarrow D^\star$. But since the choice of the chain $\mathcal{C} \subseteq \downarrow D^\star$ was arbitrary, Zorn's Lemma tells us that then $\downarrow D^\star$ has at least one minimal element, contradicting that $\downarrow D^\star$ has no minimal elements. We conclude that it is impossible that $D' \nsubseteq D^\star$ for every D' in $\min \mathcal{D}_K$, and hence indeed $D^\star \in \uparrow \min \mathcal{D}_K$.

We now turn to the last statement, that $K = K_{\min \mathcal{D}_K}$. Using Theorem 3.1 we know that $K = K_{\mathcal{D}_K}$, and since $\min \mathcal{D}_K \subseteq \mathcal{D}_K$ we know that $K = K_{\mathcal{D}_K} \subseteq K_{\min \mathcal{D}_K}$, so it suffices to show the converse set inclusion $K_{\mathcal{D}_K} \supseteq K_{\min \mathcal{D}_K}$. To this end, consider any F in $K_{\min \mathcal{D}_K}$, meaning that $F \cap D \neq \emptyset$ for every D in $\min \mathcal{D}_K$, and hence also for every D in $\uparrow \min \mathcal{D}_K$. Since we have shown above that $\uparrow \min \mathcal{D}_K = \mathcal{D}_K$, we infer that $F \cap D \neq \emptyset$ for every D in $\mathcal{D}_K$, whence indeed $F \in K_{\mathcal{D}_K}$. □

Theorem 3.3 tells us that every coherent SoDGS K is represented by an antichain – a set with the property that no two distinct elements are ordered by $\subseteq$ – namely $\min \mathcal{D}_K$, but there might be several different antichains that represent the same K. However, we will establish in Proposition 3.2 later on that any coherent SoDGS with a finite representation has a unique finite antichain that represents it.

Lemma 3.1. *Consider any coherent SoDGS K with a finite representation $\mathcal{D}$. Then $\min \mathcal{D}$ is also a representation of K.*

Proof. Since $(\mathcal{D}, \subseteq)$ is a *finite* poset, its set of minimal elements $\min \mathcal{D}$ is non-empty, and every D in $\mathcal{D}$ dominates – is a superset of – a minimal element in $\min \mathcal{D}$. Consider any gamble set F and infer that

$$\begin{aligned} F \in K &\Leftrightarrow (\forall D \in \mathcal{D}) F \cap D \neq \emptyset \\ &\Leftrightarrow (\forall D \in \min \mathcal{D}) F \cap D \neq \emptyset \Leftrightarrow F \in K_{\min \mathcal{D}}, \end{aligned}$$

where the first equivalence follows since $\mathcal{D}$ represents K, and the second equivalence since every element of $\mathcal{D}$ is a superset of an element of $\min \mathcal{D}$. Since the choice of F was arbitrary, this implies that $K = K_{\min \mathcal{D}}$, and hence, indeed, $\min \mathcal{D}$ represents K. □

Proposition 3.1. *Consider any coherent SoDGS K. Then K has a finite representation if and only if $\min \mathcal{D}_K$ is finite.*

[3] More specifically, our proof of Proposition 4.2 depends crucially on this assumption, on which Theorem 5.1 builds.

Proof. Sufficiency follows at once from the fact that $\min \mathcal{D}_K$ is a representation of K, established in Theorem 3.3.

For necessity, assume that K has a finite representation $\mathcal{D}$. Then by Lemma 3.1 $\min \mathcal{D}$, which is a subset of $\mathcal{D}$ and hence finite, also represents K. We show that $\min \mathcal{D}_K \subseteq \min \mathcal{D}$, ensuring that the former is finite. To do so, consider any D in $\overline{\mathcal{D}}$ such that $D \notin \min \mathcal{D}$, and we will show that $D \notin \min \mathcal{D}_K$. If $D \supseteq D'$ for some $D' \in \min \mathcal{D} \subseteq \mathcal{D}_K$, then $D \neq D'$ and hence D is not a minimal element of the poset $(\mathcal{D}_K, \subseteq)$, so $D \notin \min \mathcal{D}_K$ and we are done. So assume $(\forall D' \in \min \mathcal{D}) D' \nsubseteq D$, so $(\forall D' \in \min \mathcal{D})(\exists f_{D'} \in D') f_{D'} \notin D$. Collect all these $f_{D'} \in D' \setminus D$ in the set $F := \{f_{D'} : D' \in \min \mathcal{D}\}$ which is finite and therefore a valid gamble set. Then $F \cap D' \neq \emptyset$ for every $D' \in \min \mathcal{D}$, so $F \in K$ because $\min \mathcal{D}$ represents K, as established earlier. But $F \cap D = \emptyset$, so $F \notin K_D$ and hence $K \nsubseteq K_D$, whence $D \notin \mathcal{D}_K$, and therefore in particular $D \notin \min \mathcal{D}_K$, indeed. □

Example 3.1. SoDGSes with a finite representation are still fairly general. For instance, consider any set of probability mass functions $\mathcal{M}$. This set $\mathcal{M}$ might for instance correspond to a non-additive measure, a belief function, or a lower probability. Denoting the p-expectation by E_p, then the coherent SoDGS based on $\mathcal{M}$ using *Sen–Walley maximality* [30, 33] is

$$K^{\mathrm{m}}_{\mathcal{M}} := K_{\mathrm{v}} \cup \{F \in \mathcal{Q} : (\exists f \in F)(\forall p \in \mathcal{M}) E_p(f) > 0\},$$

which is a binary SoDGS, and hence has a finite – singleton – representation.

Moreover, the coherent SoDGS based on $\mathcal{M}$ using E-*admissibility* [18, 20, 29] is

$$K^{\mathrm{E}}_{\mathcal{M}} := K_{\mathrm{v}} \cup \{F \in \mathcal{Q} : (\forall p \in \mathcal{M})(\exists f \in F) E_p(f) > 0\},$$

which is represented by $\{D_p : p \in \mathcal{M}\}$ [see 31, Lemma 5], where $D_p := \{f \in \mathcal{L} : E_p(f) > 0\} \cup \mathcal{L}_{>0}$ is the smallest coherent set of desirable gambles that contains all the gambles with positive p-expectation. So if $\mathcal{M}$ is finite, then $K^{E}_{\mathcal{M}}$ has a finite representation.

More generally, any Archimedean SoDGS based on a finite number of positive superlinear bounded real functionals [15] has a finite representation.

The main importance, for our purpose, of SoDGSes with a finite representation lies in the following property.

Proposition 3.2. *Consider two coherent SoDGSes K_1 and K_2 with finite representation $\mathcal{D}_1$ and $\mathcal{D}_2$, respectively. Then $K_1 = K_2$ if and only if $\min \mathcal{D}_1 = \min \mathcal{D}_2$.*

Proof. For necessity, assume that $\min \mathcal{D}_1 \neq \min \mathcal{D}_2$, and we will show that $K_1 \neq K_2$. That $\min \mathcal{D}_1 \neq \min \mathcal{D}_2$ implies that $\min \mathcal{D}_1 \nsubseteq \min \mathcal{D}_2$ or $\min \mathcal{D}_1 \nsupseteq \min \mathcal{D}_2$ – we will assume that $\min \mathcal{D}_1 \nsubseteq \min \mathcal{D}_2$; the other case will follow by a very similar argument. This implies that there is some D_1 in $\min \mathcal{D}_1$ such that $D_1 \neq D_2$ for all D_2 in $\min \mathcal{D}_2$.

If there is some D_2 in $\min \mathcal{D}_2$ such that $D_2 \subseteq D_1$, then also $D_2 \subset D_1$, and hence also $D'_1 \nsubseteq D_2$ for all $D'_1 \in \min \mathcal{D}_1$ since $\min \mathcal{D}_1$ is an antichain. So for every D'_1 in $\min \mathcal{D}_1$ there is a gamble $f_{D'_1}$ in $D'_1 \setminus D_2$. Collect all these gambles in $F := \{f_{D'_1} : D'_1 \in \min \mathcal{D}_1\}$ which is finite and hence a valid gamble set. Then $F \cap D'_1 \neq \emptyset$ for every D'_1 in $\min \mathcal{D}_1$, so $F \in K_1$ because $\min \mathcal{D}_1$ represents K_1 by Lemma 3.1. Also, $F \cap D_2 = \emptyset$, so $F \notin K_2$ because $\mathcal{D}_2$ represents K_2, and hence $K_1 \neq K_2$ and we are done.

So assume that $D_2 \nsubseteq D_1$ for all D_2 in $\mathcal{D}_2$. Then for every D_2 in $\min \mathcal{D}_2$ there is a gamble f_{D_2} in $D_2 \setminus D_1$. Collect all these gambles in $F := \{f_{D_2} : D_2 \in \min \mathcal{D}_2\}$ which is finite and hence a valid gamble set. Then $F \cap D_2 \neq \emptyset$ for every D_2 in $\min \mathcal{D}_2$, so $F \in K_2$ because $\min \mathcal{D}_2$ represents K_2 by Lemma 3.1. Also, $F \cap D_1 = \emptyset$, so $F \notin K_1$ because $\mathcal{D}_1$ represents K_1 and hence, indeed, $K_1 \neq K_2$.

For sufficiency, assume that $\min \mathcal{D}_1 = \min \mathcal{D}_2$. Since $\mathcal{D}_1$ (finitely) represents K_1 and $\mathcal{D}_2$ (finitely) represents K_2, use Lemma 3.1 to infer that also $\min \mathcal{D}_1$ represents K_1, and, similarly, $\min \mathcal{D}_2$ represents K_2. But since $\min \mathcal{D}_1 = \min \mathcal{D}_2$ we find that, indeed, $K_1 = K_2$. □

Proposition 3.2 implies that the coherent SoDGSes with finite representation are *in a one-to-one relation with the finite antichains in $\overline{\mathcal{D}}$.*

Lemma 3.1, and therefore also Proposition 3.2, fail to hold for infinite representations $\mathcal{D}$, as the following example shows.

Example 3.2. Consider the vacuous SoDGS K_{v}. We will show that each of the three sets $\mathcal{D}_1 := \overline{\mathcal{D}}$, $\mathcal{D}_2 := \{D_{\mathrm{v}}\}$ and $\mathcal{D}_3 := \overline{\mathcal{D}} \setminus \{D_{\mathrm{v}}\}$ represents K_{v}. To see this, observe that $\mathcal{D}_2$ represents K_{v} by definition, and then since $\mathcal{D}_1 = \uparrow\mathcal{D}_2$, also $\mathcal{D}_1$ represents K_{v}. To show that $\mathcal{D}_3$ also represents K_{v}, since $\mathcal{D}_3 \subseteq \mathcal{D}_1$ it suffices to show that $K_{\mathcal{D}_3} \subseteq K_{\mathrm{v}}$. To this end, consider any $F \notin K_{\mathrm{v}}$, and we will infer that $F \notin K_D$ for some D in $\mathcal{D}_3$. That $F \notin K_{\mathrm{v}}$ implies $F \cap \mathcal{L}_{>0} = \emptyset$. If $F \subseteq \mathcal{L}_{\leq 0}$ then F cannot intersect any coherent set of desirable gambles D: if it did, then there is some $f \in D$ such that $f \leq 0$. By Axiom D_1, $f \neq 0$, so $-f > 0$ and therefore $-f \in D$ by Axiom D_2, whence $0 = f - f \in D$ by Axiom D_3, contradicting Axiom D_1. So $F \notin K_D$ for any coherent D and we are done, so assume that F contains at least one gamble in $\mathcal{L} \setminus (\mathcal{L}_{>0} \cup \mathcal{L}_{\leq 0})$. The idea is now to consider the gamble (or one of them) $f \in F$ whose ray is closest to $\mathcal{L}_{>0}$, which exists since F is finite. Then $f(x) < 0$ for some x, and by letting $\epsilon := f(x)/2$ we find that $f^\star := f + \epsilon \notin \mathcal{L}_{>0}$. $f^\star$ is contained on a ray that is (strictly) closer to $\mathcal{L}_{>0}$ than the ray through any gamble in F. Consider now $D := \mathrm{cl}_{\overline{\mathcal{D}}}(\{f^\star\}) \in \mathcal{D}_3$, whose rays all are closer to $\mathcal{L}_{>0}$ than the rays of gambles in F. This implies that $F \cap D = \emptyset$, whence, indeed, $F \notin K_D$.

So we see that the representation of K_{v} is not unique: in this case, as $\mathcal{D}_2 = \mathcal{D}_1 \setminus \mathcal{D}_3$, there is even a partition $\{\mathcal{D}_2, \mathcal{D}_3\}$ of representations. The representation $\mathcal{D}_3$ does not have minimal elements, demonstrating that the requirement that K has a finite representation in Lemma 3.1 is not superfluous.

Note that K_{v} has a finite representation, so Proposition 3.2 predicts that it has a unique finite antichain that represents it, which in this case is $\{D_{\mathrm{v}}\}$.

4. Multivariate sets of desirable gamble sets

This section introduces the notation and concepts necessary for multivariate SoDGSes. It is heavily based on the work of Van Camp, Blackwell, and Konek [31, Section 5]. Along the way, we add some specialised results for SoDGSes that admit a finite representation, needed for Theorem 5.1.

4.1. Preliminaries. Consider $n \in \mathbb{N}$ uncertain variables $X_1, \dots, X_n$, each assuming values in the finite possibility spaces $\mathcal{X}_1, \dots, \mathcal{X}_n$, respectively. For brevity, we will use the notation $N := \{1, \dots, n\}$ for the global index set. To express beliefs – sets of desirable gambles or SoDGSes – about the uncertain variables $X_1, \dots, X_n$ together, we consider gambles on the Cartesian product $\mathcal{X} := \times_{k=1}^{n} \mathcal{X}_k$, which belong to the $\prod_{k=1}^{n} |\mathcal{X}_k|$-dimensional linear space $\mathcal{L}(\mathcal{X})$.

For any subset $I \subseteq N$ of indices, we let X_I be the tuple of uncertain variables that assumes values in $\mathcal{X}_I := \times_{k \in I} \mathcal{X}_k$. If $I = \emptyset$, then $\mathcal{X}_\emptyset$ contains only one element: the empty map. In this case, there is no uncertainty about the variable $X_\emptyset$.

For any subset $I \subseteq N$, any gamble f on $\mathcal{X}$ and any $x_I \in \mathcal{X}_I$, we can regard the partial map $f(x_I, \bullet)$ as a gamble on $\mathcal{X}_{I^c}$, where we let $I^c := N \setminus I$ be the indices outside I. Conversely, it will be useful to relate a gamble f on $\mathcal{X}_I$ to a gamble on $\mathcal{X}$.

Definition 4.1 (Cylindrical extension)**.** Given two disjoint subsets I and J of N and any gamble f on $\mathcal{X}_I$, we let its *cylindrical extension* f^J to $\mathcal{X}_{I \cup J}$ be defined by $f^J(x_I, x_J) := f(x_I)$ for all x_I in $\mathcal{X}_I$ and x_J in $\mathcal{X}_J$. Similarly, given any set of gambles $F \subseteq \mathcal{L}(\mathcal{X}_I)$, we let its *cylindrical extension* $F^J \subseteq \mathcal{L}(\mathcal{X}_{I \cup J})$ be defined as $F^J := \{f^J : f \in F\}$.

Formally, f belongs to $\mathcal{L}(\mathcal{X}_I)$ while f^J belongs to $\mathcal{L}(\mathcal{X}_{I \cup J})$. However, for any x_I in $\mathcal{X}_I$, the partial map $f^J(x_I, \bullet) \in \mathcal{L}(\mathcal{X}_J)$ is actually constant, so f^J depends only on the value of X_I. So we see that f and f^J are indistinguishable from a behavioural point of view.

Remark 4.1. We do *not* notationally distinguish between f and f^J, and identify a gamble f on $\mathcal{X}_I$ with its cylindrical extension f^J [9, 31].

Using this convention allows us to regard $\mathcal{L}(\mathcal{X}_I)$ as a subset of $\mathcal{L}(\mathcal{X}_{I \cup J})$ and, similarly, $\mathcal{Q}(\mathcal{X}_I)$ as a subset of $\mathcal{Q}(\mathcal{X}_{I \cup J})$.

4.2. Marginalisation. Assume that we have an SoDGS $K \subseteq \mathcal{Q}(\mathcal{X})$ that models the agent's beliefs about the uncertain variable X_N. We are interested in the information about X_S present in K, where $S \subseteq N$. This information can be obtained by collecting the gamble sets that belong to K but depend only on X_S.

Definition 4.2 (Marginalisation [32])**.** For any SoDGS $K \subseteq \mathcal{Q}(\mathcal{X})$ and any $S \subseteq N$, its *S-marginal* $\operatorname{Marg}_S K \subseteq \mathcal{Q}(\mathcal{X}_S)$ is defined as

$$\operatorname{Marg}_S K := K \cap \mathcal{Q}(\mathcal{X}_S).$$

Note that marginalisation preserves the order: if $K_1 \subseteq K_2$ then $\operatorname{Marg}_S K_1 \subseteq \operatorname{Marg}_S K_2$ [32, Section 4.2]. This definition of marginalisation generalises the one for sets of desirable gambles, in that $\operatorname{Marg}_S K_D = K_{\operatorname{marg}_S D}$ [32, Proposition 10], where $\operatorname{marg}_S D := D \cap \mathcal{L}(\mathcal{X}_S)$ for all $D \subseteq \mathcal{L}(\mathcal{X})$, as defined by De Cooman and Miranda [9]. It will be convenient to lift the operator marg_S on $\mathcal{P}(\mathcal{L}(\mathcal{X}))$ to a version on $\mathcal{P}(\mathcal{P}(\mathcal{L}(\mathcal{X})))$, defined by $\operatorname{marg}_S \mathcal{D} := \{\operatorname{marg}_S D : D \in \mathcal{D}\}$ for every $\mathcal{D} \subseteq \mathcal{P}(\mathcal{L}(\mathcal{X}))$.

Proposition 4.1 ([31, Proposition 11])**.** *Consider any coherent SoDGS K, any representation $\mathcal{D}$ of it, and any $S \subseteq N$. Then $\operatorname{Marg}_S K$ is coherent. Moreover, $\operatorname{Marg}_S K$ is represented by $\operatorname{marg}_S \mathcal{D}$, meaning that $\operatorname{Marg}_S K = K_{\operatorname{marg}_S \mathcal{D}}$.*

Theorem 3.1 and Proposition 4.1 imply that $\operatorname{marg}_S \mathcal{D}_K$ is a representation of $\operatorname{Marg}_S K$, but it does not imply that $\operatorname{marg}_S \mathcal{D}_K$ is equal to the unique largest representation $\mathcal{D}_{\operatorname{Marg}_S K}$ of $\operatorname{Marg}_S K$. However, if K has a finite representation then this turns out to be the case, which will be a useful property further on. This is what we set out to do for the rest of this section.

Lemma 4.1. *Consider any $S \subseteq N$, any coherent set of desirable gambles $D \subseteq \mathcal{L}(\mathcal{X})$ and any coherent set of desirable gambles $D_S \subseteq \mathcal{L}(\mathcal{X}_S)$ such that $\operatorname{marg}_S D \subseteq D_S$. Then $D^\star := \operatorname{posi}(D \cup D_S)$ is a coherent set of desirable gambles, that marginalises to D_S, in the sense that $\operatorname{marg}_S D^\star = D_S$.*

Proof. For the first statement, it suffices to show that $0 \notin \operatorname{posi}(D \cup D_S)$, taking into account Theorem 2.1 and the fact that $\mathcal{L}_{>0}(\mathcal{X}) \subseteq D$. To this end, assume *ex absurdo* that $0 \in \operatorname{posi}(D \cup D_S)$, so we would find f in D and g in D_S such that $f + g = 0$, taking into account the coherence of D and D_S. But then $f = -g$, and hence f would belong to $\mathcal{L}(\mathcal{X}_S)$, since g belongs to $\mathcal{L}(\mathcal{X}_S)$. So we would find that $f \in D \cap \mathcal{L}(\mathcal{X}_S) = \operatorname{marg}_S D$, and hence, since $\operatorname{marg}_S D \subseteq D_S$, also that $-g = f \in D_S$. But also $g \in D_S$, contradicting D_S's coherence: Axiom D_3 implies that $0 = g - g \in D_S$, which contradicts Axiom D_1.

For the second statement, note that $D_S \subseteq D^\star$ by construction, whence $D_S \subseteq \operatorname{marg}_S D^\star$, so it suffices to show that $\mathrm{marg}_S\, D^\star \subseteq D_S$. To this end, consider any f in $\mathrm{marg}_S\, D^\star$, so that $f = g + g_S$ for some $g \in D \cup \{0\}$ and $g_S \in D_S \cup \{0\}$, taking the coherence of D and D_S into account. Since both f and g_S belong to $\mathcal{L}(\mathcal{X}_S)$, so does g, and hence $g \in \mathrm{marg}_S\, D \cup \{0\} \subseteq D_S \cup \{0\}$. So we find that $f = g + g_S$ for some g and g_S in $D_S \cup \{0\}$, so by coherence [more specifically, Axiom D_3] we infer that $f \in D_S \cup \{0\}$, and, taking the coherence of $D^\star$ into account, which we have established above, even that $f \in D_S$, indeed. □

Lemma 4.2. *Consider any coherent SoDGS $K \subseteq \mathcal{Q}(\mathcal{X})$, any $S \subseteq N$, and any coherent set of desirable gambles $D_S \subseteq \mathcal{L}(\mathcal{X}_S)$. Then $D_S \in \mathrm{marg}_S\, \mathcal{D}_K \Leftrightarrow (\exists D \in \mathcal{D}_K)\, \mathrm{marg}_S\, D \subseteq D_S$.*

Proof. For necessity, assume that $D_S \in \mathrm{marg}_S\, \mathcal{D}_K$, implying that $\mathrm{marg}_S\, D = D_S$ – and hence indeed $\mathrm{marg}_S\, D \subseteq D_S$ – for some $D \in \mathcal{D}_K$.

For sufficiency, assume that $\mathrm{marg}_S\, D \subseteq D_S$ for some $D \in \mathcal{D}_K$, and let $D^\star := \mathrm{posi}(D \cup D_S)$. Then Lemma 4.1 tells us that $D^\star$ is coherent and marginalises to D_S. Moreover, $D^\star \supseteq D$ by construction, so $D^\star$ also belongs to $\mathcal{D}_K$. So we have found $D^\star$ in $\mathcal{D}_K$ such that $\mathrm{marg}_S\, D^\star = D_S$, whence, indeed, $D_S \in \mathrm{marg}_S\, \mathcal{D}_K$. □

Proposition 4.2. *Consider any coherent SoDGS $K \subseteq \mathcal{Q}(\mathcal{X})$ that has a finite representation, and any $S \subseteq N$. Then $\mathrm{marg}_S\, \mathcal{D}_K = \mathcal{D}_{\mathrm{Marg}_S K}$.*

Proof. Proposition 4.1 tells us that $\mathrm{Marg}_S\, K$ is represented by $\mathrm{marg}_S\, \mathcal{D}_K$, which then is necessarily a subset of $\mathcal{D}_{\mathrm{Marg}_S K}$ by Theorem 3.1. It therefore suffices to show that $\mathcal{D}_{\mathrm{Marg}_S K} \subseteq \mathrm{marg}_S\, \mathcal{D}_K$. To this end, we consider any D_S in $\overline{\mathcal{D}}(\mathcal{X}_S)$ such that $D_S \notin \mathrm{marg}_S\, \mathcal{D}_K$, and will show that then $D_S \notin \mathcal{D}_{\mathrm{Marg}_S K}$. Use Lemma 4.2 to infer that $\mathrm{marg}_S\, D \nsubseteq D_S$ for all D in $\mathcal{D}_K$, and therefore in particular $(\forall D \in \mathcal{D})\, \mathrm{marg}_S\, D \nsubseteq D_S$, where we let $\mathcal{D} \subseteq \mathcal{D}_K$ be a finite representation of K. This implies that, for every D in $\mathcal{D}$, there is some $f_D \in \mathrm{marg}_S\, D$ such that $f_D \notin D_S$. Collect all these gambles f_D in $F := \{f_D : D \in \mathcal{D}\}$, which is finite because $\mathcal{D}$ is, and therefore a valid gamble set on $\mathcal{X}_S$. Then $f_D \in D$ – and therefore $F \cap D \neq \emptyset$ – for every D in $\mathcal{D}$, so $F \in K_{\mathcal{D}} = K$, where we took into account that $\mathcal{D}$ is a representation of K. Moreover, since $F \in \mathcal{Q}(\mathcal{X}_S)$, we find that $F \in K \cap \mathcal{Q}(\mathcal{X}_S) = \mathrm{Marg}_S\, K$. Since $f_D \notin D_S$ for every D in $\mathcal{D}$, we find also that $F \cap D_S = \emptyset$, so $F \notin K_{D_S}$ and therefore $\mathrm{Marg}_S\, K \nsubseteq K_{D_S}$, whence, indeed, $D_S \notin \mathcal{D}_{\mathrm{Marg}_S K}$. □

5. The marginal problem

Let us now turn to the main topic of this paper: the marginal problem. Suppose that we are given $m \in \mathbb{N}$ coherent SoDGSes $K_\ell \subseteq \mathcal{Q}(\mathcal{X}_{S_\ell})$ for some non-empty index sets $S_\ell \subseteq N$, where these index sets may overlap. The marginal problem is this:

> "When is there a joint SoDGS $K \subseteq \mathcal{Q}(\mathcal{X})$ that marginalises to the given SoDGSes?"

This question has been solved for sets of desirable gambles by Miranda and Zaffalon [23], who also discussed an interesting connection with valuation algebras. This connection has been further developed by Casanova, Kohlas, and Zaffalon [5, 6]. In this work, we build on these results to provide a solution for SoDGSes. At this point, we would like to stress that the marginal problem, as we understand it, is a *satisfiability problem*, which naturally arises in different subfields in artificial intelligence – we refer to Miranda and Zaffalon [23, Section 1] for an overview.

Let us first make this problem more precise. We generalise Miranda and Zaffalon [23, Definitions 9 and 10] from sets of desirable gambles to the current setting.

Definition 5.1 (Pairwise compatibility)**.** Two coherent SoDGSes $K_1 \subseteq \mathcal{Q}(\mathcal{X}_{S_1})$ and $K_2 \subseteq \mathcal{Q}(\mathcal{X}_{S_2})$ are called *pairwise compatible* if

$$\mathrm{Marg}_{S_1 \cap S_2}\, K_1 = \mathrm{Marg}_{S_1 \cap S_2}\, K_2.$$

The m coherent SoDGSes $K_\ell \subseteq \mathcal{Q}(\mathcal{X}_{S_\ell})$, $\ell \in \{1, \dots, m\}$, are called *pairwise compatible* if any two of them are pairwise compatible. Similarly, two coherent sets of desirable gambles $D_1 \subseteq \mathcal{L}(\mathcal{X}_{S_1})$ and $D_2 \subseteq \mathcal{L}(\mathcal{X}_{S_2})$ are *pairwise compatible* if $\mathrm{marg}_{S_1 \cap S_2}\, D_1 = \mathrm{marg}_{S_1 \cap S_2}\, D_2$, and m coherent sets of desirable gambles $D_\ell \subseteq \mathcal{Q}(\mathcal{X}_{S_\ell})$, $\ell \in \{1, \dots, m\}$, are called *pairwise compatible* if any two of them are pairwise compatible.

In order to conclude that K_1 and K_2 are pairwise compatible, it suffices that the marginalisations of their representations coincide.

Proposition 5.1. *Consider two coherent SoDGSes $K_1 \subseteq \mathcal{Q}(\mathcal{X}_{S_1})$ and $K_2 \subseteq \mathcal{Q}(\mathcal{X}_{S_2})$. If*

$$\mathrm{marg}_{S_1 \cap S_2}\, \mathcal{D}_{K_1} = \mathrm{marg}_{S_1 \cap S_2}\, \mathcal{D}_{K_2}$$

then they are pairwise compatible.

Proof. Use Proposition 4.1 to infer that $\mathrm{marg}_{S_1 \cap S_2}\, \mathcal{D}_{K_1}$ represents $\mathrm{Marg}_{S_1 \cap S_2}\, K_1$, and, similarly, that $\mathrm{marg}_{S_1 \cap S_2}\, \mathcal{D}_{K_2}$ represents $\mathrm{Marg}_{S_1 \cap S_2}\, K_2$. Since these two representations coincide, necessarily also $\mathrm{Marg}_{S_1 \cap S_2}\, K_1 = \mathrm{Marg}_{S_1 \cap S_2}\, K_2$, so that K_1 and K_2 are pairwise compatible, indeed. □

Moreover, it turns out that for coherent SoDGSes with finite representations, this sufficient condition is also necessary, making it an equivalent property to pairwise compatibility.

Proposition 5.2. *Consider two coherent SoDGSes $K_1 \subseteq \mathcal{Q}(\mathcal{X}_{S_1})$ and $K_2 \subseteq \mathcal{Q}(\mathcal{X}_{S_2})$ that have finite representations. Then they are pairwise compatible if and only if*

$$\mathrm{marg}_{S_1 \cap S_2}\, \mathcal{D}_{K_1} = \mathrm{marg}_{S_1 \cap S_2}\, \mathcal{D}_{K_2}.$$

Proof. Sufficiency is, in a more general context, established in Proposition 5.1, so we focus on necessity. Use Theorem 3.1 to infer that $\mathrm{Marg}_{S_1\cap S_2} K_1$ is represented by $\mathcal{D}_{\mathrm{Marg}_{S_1\cap S_2} K_1}$ and that $\mathrm{Marg}_{S_1\cap S_2} K_2$ is represented by $\mathcal{D}_{\mathrm{Marg}_{S_1\cap S_2} K_2}$. Since K_1 has a finite representation, it is represented by a finite set $\mathcal{D}$, so Proposition 4.1 tells us that $\mathrm{Marg}_{S_1\cap S_2} K_1$ is represented by the set $\mathrm{marg}_{S_1\cap S_2} \mathcal{D}$ which is finite, implying that $\mathrm{Marg}_{S_1\cap S_2} K_1$ has a finite representation, too. A similar argument shows that $\mathrm{Marg}_{S_1\cap S_2} K_2$ has a finite representation.

Now use Proposition 3.2 to infer that

$$\mathrm{Marg}_{S_1\cap S_2} K_1 = \mathrm{Marg}_{S_1\cap S_2} K_2 \Leftrightarrow \min \mathcal{D}_{\mathrm{Marg}_{S_1\cap S_2} K_1} = \min \mathcal{D}_{\mathrm{Marg}_{S_1\cap S_2} K_2}.$$

The proof follows by observing that $\mathcal{D}_{\mathrm{Marg}_{S_1\cap S_2} K_1} = \uparrow \min \mathcal{D}_{\mathrm{Marg}_{S_1\cap S_2} K_1} = \uparrow \min \mathrm{marg}_{S_1\cap S_2} \mathcal{D}_{K_1}$, using Theorem 3.3 in the first equality and Proposition 4.2 in the second one, and similarly $\mathcal{D}_{\mathrm{Marg}_{S_1\cap S_2} K_2} = \uparrow \min \mathrm{marg}_{S_1\cap S_2} \mathcal{D}_{K_2}$. □

Another way to interpret Proposition 5.2 is that K_1 and K_2 are pairwise compatible if and only if every D_1 in $\mathcal{D}_{K_1}$ has a pairwise compatible D_2 in $\mathcal{D}_{K_2}$, and every D_2 in $\mathcal{D}_{K_2}$ has a pairwise compatible D_1 in $\mathcal{D}_{K_1}$. We will also need a global version of compatibility, which is the requirement in the marginal problem.

Definition 5.2 (Compatibility)**.** The m coherent SoDGSes $K_\ell \subseteq \mathcal{Q}(\mathcal{X}_{S_\ell})$, $\ell \in \{1,\dots,m\}$, are called *compatible* if there is a coherent SoDGS $K \subseteq \mathcal{Q}(\mathcal{X})$ that is pairwise compatible with each of them, in the sense that $\mathrm{Marg}_{S_\ell} K = K_\ell$ for every ℓ in $\{1,\dots,m\}$. We then also call K *compatible* with $K_1, \dots, K_m$. Similarly, the m coherent sets of desirable gambles $D_\ell \subseteq \mathcal{L}(\mathcal{X}_{S_\ell})$, $\ell \in \{1,\dots,m\}$, are called *compatible* if there is a coherent set of desirable gambles $D \subseteq \mathcal{L}(\mathcal{X})$ that is pairwise compatible with each of them, in the sense that $\mathrm{marg}_{S_\ell} D = D_\ell$ for every ℓ in $\{1,\dots,m\}$. We then also call D *compatible* with $D_1, \dots, D_m$.

The following result is directly inspired by Miranda and Zaffalon [23, Proposition 1], and generalises it to SoDGSes. It provides an equivalent condition to compatibility.

Proposition 5.3. *Consider any coherent SoDGSes $K_1 \subseteq \mathcal{Q}(\mathcal{X}_{S_1})$, $K_2 \subseteq \mathcal{Q}(\mathcal{X}_{S_2})$, ..., $K_m \subseteq \mathcal{Q}(\mathcal{X}_{S_m})$. Then they are compatible if and only if the natural extension $K := \mathrm{cl}_{\overline{\mathcal{K}}}(\bigcup_{\ell\leq m} K_\ell)$ is compatible with them, or, in other words, if and only if $\mathrm{Marg}_{S_i} K = K_i$ for every i in $\{1,\dots,m\}$.*

Proof. For necessity, consider any i in $\{1,\dots,m\}$. Since K_1, ..., K_m are compatible, there is a coherent SoDGS $K^\star \subseteq \mathcal{Q}(\mathcal{X})$ such that, for all $\ell \in \{1,\dots,m\}$, $\mathrm{Marg}_{S_\ell} K^\star = K_\ell$, or equivalently, $K^\star \cap \mathcal{Q}(\mathcal{X}_{S_\ell}) = K_\ell$. Hence $\bigcup_{\ell\leq m} K_\ell \subseteq K^\star$, whence also $K_i \subseteq \mathrm{cl}_{\overline{\mathcal{K}}}(\bigcup_{\ell\leq m} K_\ell) = K \subseteq \mathrm{cl}_{\overline{\mathcal{K}}}(K^\star) = K^\star$, where we used the facts that $\mathrm{cl}_{\overline{\mathcal{K}}}$ is a closure operator and that $K^\star$ is coherent. Taking into account that Marg_{S_i} preserves the order, we obtain that $K_i = \mathrm{Marg}_{S_i} K_i \subseteq \mathrm{Marg}_{S_i} K \subseteq \mathrm{Marg}_{S_i} K^\star = K_i$, whence, indeed, $\mathrm{Marg}_{S_i} K = K_i$.

For sufficiency, it suffices to show that K is coherent, because then it serves as a coherent SoDGS on $\mathcal{X}$ that is compatible with $K_1, \dots, K_m$. To this end, taking into account Theorem 2.2 with $\mathcal{A} = K$, it suffices to show that $\emptyset \notin K$ and $\{0\} \notin K$. For every i in $\{1,\dots,m\}$, note that $\emptyset \notin K_i$ and $\{0\} \notin K_i$ by K_i's coherence. Since $\emptyset$ and $\{0\}$ belong to $\mathcal{Q}(\mathcal{X}_{S_i})$ for every i in $\{1,\dots,m\}$, we find by the compatibility of K with $K_1, \dots, K_m$ that, indeed, $\emptyset \notin K$ and $\{0\} \notin K$. □

We are now in a position to establish the main result of this paper. When we are given coherent SoDGSes that are pairwise compatible, under what condition are they compatible? In the special case of probability measures, Beeri, Fagin, Maier, and Yannakakis [2] showed that a sufficient condition is that the index sets satisfy the *running intersection property* [cf. also 21]. Miranda and Zaffalon [23, Theorem 2] have established that this is also sufficient for the compatibility of sets of desirable gambles, and it will play an important role in the present setting, too.

Definition 5.3 (Running intersection property [23, Definition 11])**.** The index sets $S_1, \dots, S_m$ satisfy the *running intersection property* when

$$(\forall \ell \in \{2,\dots,m\})(\exists i^\star < \ell) S_\ell \cap S_{i^\star} = S_\ell \cap \textstyle\bigcup_{i<\ell} S_i. \tag{RIP}$$

Note that it may happen that $S_1, \dots, S_m$ does not satisfy the running intersection property, while a reordering $S_{\sigma(1)}, \dots, S_{\sigma(m)}$, with σ a permutation of $\{1,\dots,m\}$, does. To keep our exposition simple, we will assume from here on that the index sets $S_1, \dots, S_m$ are ordered in such a way that they satisfy the running intersection property, whenever this is possible. This is not a substantial constraint: if they are not in this order, then simply reorder them, and the following results will continue to hold.

Theorem 5.1. *Consider any coherent SoDGSes $K_1 \subseteq \mathcal{Q}(\mathcal{X}_{S_1})$, $K_2 \subseteq \mathcal{Q}(\mathcal{X}_{S_2})$, ..., $K_m \subseteq \mathcal{Q}(\mathcal{X}_{S_m})$ that have finite representations. If $S_1, \dots, S_m$ satisfy (RIP) and $K_1, \dots, K_m$ are pairwise compatible, then $K_1, \dots, K_m$ are compatible.*

Proof. For any $\ell \in \{1,\dots,m\}$ let $\hat{K}_\ell := \mathrm{cl}_{\overline{\mathcal{K}}}(\bigcup_{j\leq\ell} K_j)$. We will show, using induction on ℓ, that $\hat{K}_\ell$ is compatible with $K_1, \dots, K_\ell$. Proposition 5.3 then guarantees that K_1, ..., K_m are compatible.

For the base case, start with $\ell = 1$. Note that $\hat{K}_1 = K_1$, so it is trivially compatible with K_1.

For the induction step, consider any $\ell \in \{2,\dots,m\}$ and assume that $\hat{K}_{\ell-1}$ is compatible with $K_1, \dots, K_{\ell-1}$. We will show that $\hat{K}_\ell = \mathrm{cl}_{\overline{\mathcal{K}}}(\bigcup_{i\leq\ell} K_i) = \mathrm{cl}_{\overline{\mathcal{K}}}(\hat{K}_{\ell-1} \cup K_\ell)$ is compatible with $\hat{K}_{\ell-1}$ and K_ℓ. Using the compatibility of $\hat{K}_{\ell-1}$ with $K_1, \ldots, K_{\ell-1}$, the desired result follows. Let

$$\begin{aligned}\hat{\mathcal{D}}_\ell := \{\mathrm{cl}_{\overline{\mathcal{D}}}(D_{\ell-1} \cup D_\ell) \colon D_{\ell-1} \in \mathcal{D}_{\hat{K}_{\ell-1}}, D_\ell \in \mathcal{D}_{K_\ell},\\ D_{\ell-1}, D_\ell \text{ pairwise compatible}\},\end{aligned}$$

and we will show that $K_{\hat{\mathcal{D}}_\ell}$ is compatible with $\hat{K}_{\ell-1}$ and K_ℓ, and that $\hat{K}_\ell = K_{\hat{\mathcal{D}}_\ell}$.

We will first show that $\mathrm{marg}_{S_\ell} \hat{\mathcal{D}}_\ell = \mathcal{D}_{K_\ell}$. To this end, consider any D_ℓ in $\mathcal{D}_{K_\ell}$. By (RIP) there is some $i^\star < \ell$ such that $S_\ell \cap S_{i^\star} = S_\ell \cap \bigcup_{i<\ell} S_i$. By the pairwise compatibility of K_ℓ and $K_{i^\star}$, there is some $D_{i^\star}$ in $\mathcal{D}_{K_{i^\star}}$ that is pairwise compatible with D_ℓ, taking into account Proposition 5.2. But by the pairwise compatibility of $K_{i^\star}$ with any K_i (with $i \in \{1, \ldots, \ell-1\} \setminus \{i^\star\}$) there is some $D_i \in \mathcal{D}_{K_i}$ that is pairwise compatible with $D_{i^\star}$. Since $S_\ell \cap S_{i^\star} \supseteq S_\ell \cap S_i$, this implies that

$$\begin{aligned}\mathrm{marg}_{S_\ell \cap S_i} D_\ell &= \mathrm{marg}_{S_\ell \cap S_i \cap S_{i^\star}} D_\ell\\ &= \mathrm{marg}_{S_\ell \cap S_i \cap S_{i^\star}} D_{i^\star}\\ &= \mathrm{marg}_{S_\ell \cap S_i \cap S_{i^\star}} D_i = \mathrm{marg}_{S_\ell \cap S_i} D_i\end{aligned}$$

so we infer that D_ℓ and D_i are pairwise compatible, and therefore, so is D_ℓ with $\hat{D}_{\ell-1} := \mathrm{cl}_{\overline{\mathcal{D}}}(\bigcup_{i<\ell} D_i)$, which is coherent and compatible with $D_1, \ldots, D_{\ell-1}$ due to [23, Proposition 1 and Theorem 2].

Infer that $K_i \subseteq K_{D_i} \subseteq K_{\hat{D}_{\ell-1}}$ for any i in $\{1, \ldots, \ell-1\}$, and therefore $\bigcup_{i<\ell} K_i \subseteq K_{\hat{D}_{\ell-1}}$. Taking $K_{\hat{D}_{\ell-1}}$'s coherence into account, we infer that then also $\hat{K}_{\ell-1} \subseteq K_{\hat{D}_{\ell-1}}$, so $\hat{D}_{\ell-1}$ belongs to $\mathcal{D}_{\hat{K}_{\ell-1}}$. Consider now $\mathrm{cl}_{\overline{\mathcal{D}}}(\hat{D}_{\ell-1} \cup D_\ell)$, which belongs to $\hat{\mathcal{D}}_\ell$ because $\hat{D}_{\ell-1}$ and D_ℓ are pairwise compatible, $\hat{D}_{\ell-1}$ belongs to $\mathcal{D}_{\hat{K}_{\ell-1}}$, and D_ℓ to $\mathcal{D}_{K_\ell}$. Then, again invoking the pairwise compatibility of $\hat{D}_{\ell-1}$ and D_ℓ, we find that $\mathrm{marg}_{S_{\ell-1}} \mathrm{cl}_{\overline{\mathcal{D}}}(\hat{D}_{\ell-1} \cup D_\ell) = D_\ell$, so D_ℓ belongs to $\mathrm{marg}_{S_{\ell-1}} \hat{\mathcal{D}}_\ell$, indeed.

The converse set inclusion follows since $\mathrm{marg}_{S_\ell} \mathrm{cl}_{\overline{\mathcal{D}}}(D_{\ell-1} \cup D_\ell) \supseteq D_\ell$ for any D_ℓ, and hence the former belongs to $\mathcal{D}_{K_\ell}$, too. Using Proposition 4.1, we infer that $\mathrm{Marg}_{S_\ell} K_{\hat{\mathcal{D}}_\ell} = K_\ell$.

Next, we will show that $\mathrm{marg}_{\hat{S}_{\ell-1}} \hat{\mathcal{D}}_\ell = \mathcal{D}_{\hat{K}_{\ell-1}}$, where $\hat{S}_{\ell-1} := \bigcup_{i<\ell} S_i$. To this end, consider any $\hat{D}_{\ell-1}$ in $\mathcal{D}_{\hat{K}_{\ell-1}}$. Then $\mathrm{marg}_{S_1} \hat{D}_{\ell-1}, \ldots, \mathrm{marg}_{S_{\ell-1}} \hat{D}_{\ell-1}$ are pairwise compatible, and belong to $\mathcal{D}_{K_1}, \ldots, \mathcal{D}_{K_{\ell-1}}$, respectively. By (RIP) there is some $i^\star$ such that $S_\ell \cap S_{i^\star} = S_\ell \cap \hat{S}_{\ell-1}$. Taking into account the pairwise compatibility of $K_{i^\star}$ and K_ℓ, we find using that Proposition 5.2 that there is a D_ℓ in $\mathcal{D}_{K_\ell}$ pairwise compatible with $\mathrm{marg}_{S_{i^\star}} \hat{D}_{\ell-1}$. But then

$$\begin{aligned}\mathrm{marg}_{\hat{S}_{\ell-1} \cap S_\ell} \hat{D}_{\ell-1} &= \mathrm{marg}_{S_{i^\star} \cap S_\ell} \hat{D}_{\ell-1}\\ &= \mathrm{marg}_{S_{i^\star} \cap S_\ell} \mathrm{marg}_{S_{i^\star}} \hat{D}_{\ell-1}\\ &= \mathrm{marg}_{S_{i^\star} \cap S_\ell} D_\ell = \mathrm{marg}_{\hat{S}_{\ell-1} \cap S_\ell} D_\ell,\end{aligned}$$

so $\hat{D}_{\ell-1}$ and D_ℓ are pairwise compatible. We find that $\mathrm{cl}_{\overline{\mathcal{D}}}(\hat{D}_{\ell-1} \cup D_\ell)$ belongs to $\hat{\mathcal{D}}_\ell$, and, taking into account [23, Theorem 2], we find that $\mathrm{marg}_{\hat{S}_{\ell-1}} \mathrm{cl}_{\overline{\mathcal{D}}}(\hat{D}_{\ell-1} \cup D_\ell) = \hat{D}_{\ell-1}$, so that $\hat{D}_{\ell-1} \in \mathrm{marg}_{\hat{S}_{\ell-1}} \hat{\mathcal{D}}_\ell$. Since the choice of $\hat{D}_{\ell-1}$ in $\mathcal{D}_{\hat{K}_{\ell-1}}$ was arbitrary, this implies that $\mathcal{D}_{\hat{K}_{\ell-1}} \subseteq \mathrm{marg}_{\hat{S}_{\ell-1}} \hat{\mathcal{D}}_\ell$.

The converse set inclusion follows since $D_{\ell-1} \subseteq \mathrm{marg}_{S_{\ell-1}} \mathrm{cl}_{\overline{\mathcal{D}}}(D_{\ell-1} \cup D_\ell)$ for any $D_{\ell-1}$, and taking into account that $\mathcal{D}(K_{\ell-1})$ is an upset, the latter belongs to it as well. Using Proposition 4.1 again, we infer that $\mathrm{Marg}_{\hat{S}_{\ell-1}} K_{\hat{\mathcal{D}}_\ell} = \hat{K}_{\ell-1}$, so that $K_{\hat{\mathcal{D}}_\ell}$ is compatible with $\hat{K}_{\ell-1}$ and K_ℓ, as desired.

To finish the proof, we show that $\hat{K}_\ell = K_{\hat{\mathcal{D}}_\ell}$. Since $K_{\hat{\mathcal{D}}_\ell}$ is compatible with $\hat{K}_{\ell-1}$ and K_ℓ, we know that $\hat{K}_{\ell-1} \cup K_\ell \subseteq K_{\hat{\mathcal{D}}_\ell}$, and therefore $\hat{K}_\ell = \mathrm{cl}_{\overline{\mathcal{K}}}(\hat{K}_{\ell-1} \cup K_\ell) \subseteq K_{\hat{\mathcal{D}}_\ell}$ since $K_{\hat{\mathcal{D}}_\ell}$ is coherent. So it suffices to prove that $K_{\hat{\mathcal{D}}_\ell} \subseteq \hat{K}_\ell$, with $\hat{K}_\ell = \bigcap\{K_D : D \in \overline{\mathcal{D}}, \hat{K}_{\ell-1} \cup K_\ell \subseteq K_D\}$ taking into account Theorem 3.2, and $K_{\hat{\mathcal{D}}_\ell} = \bigcap\{K_D : D \in \hat{\mathcal{D}}_\ell\}$. In turn, it suffices to show that $\{D \in \overline{\mathcal{D}} : \hat{K}_{\ell-1} \cup K_\ell \subseteq K_D\} \subseteq \uparrow\hat{\mathcal{D}}_\ell$, using [31, Proposition 4]. To this end, consider any $D \in \overline{\mathcal{D}}$ such that $\hat{K}_{\ell-1} \subseteq K_D$ and $K_\ell \subseteq K_D$, implying that $\hat{K}_{\ell-1} \subseteq K_{\mathrm{marg}_{\hat{S}_{\ell-1}} D}$ and $K_\ell \subseteq K_{\mathrm{marg}_{S_\ell} D}$ taking into account Proposition 4.1. But $\mathrm{marg}_{\hat{S}_{\ell-1}} D$ and $\mathrm{marg}_{S_\ell} D$ are pairwise compatible because they are derived from the same joint, and $D \supseteq \mathrm{marg}_{\hat{S}_{\ell-1}} D \cup \mathrm{marg}_{S_\ell} D$, so $D \in \uparrow\hat{\mathcal{D}}_\ell$, indeed. This establishes that $\hat{K}_\ell = K_{\hat{\mathcal{D}}_\ell}$, which is therefore compatible with $\hat{K}_{\ell-1}$ and K_ℓ. □

Theorem 5.1 does not only guarantee that (RIP) and pairwise compatibility of coherent SoDGSes with finite representations imply compatibility, but its proof also provides a way to construct a representation of the smallest compatible joint $\mathrm{cl}_{\overline{\mathcal{K}}}(\bigcup_{\ell \leq m} K_\ell)$, which we spell out next.

Proposition 5.4. [4] *Consider coherent and compatible SoDGSes $K_1 \subseteq \mathcal{Q}(\mathcal{X}_{S_1})$, $K_2 \subseteq \mathcal{Q}(\mathcal{X}_{S_2})$, …, $K_m \subseteq \mathcal{Q}(\mathcal{X}_{S_m})$. Then the natural extension $K := \mathrm{cl}_{\overline{\mathcal{K}}}(\bigcup_{\ell \leq m} K_\ell)$ is coherent and represented by*

$$\begin{aligned}\mathcal{D} := \{&\mathrm{cl}_{\overline{\mathcal{D}}}(D_1 \cup \cdots \cup D_m) :\\ &D_1 \in \mathcal{D}_{K_1}, \ldots, D_m \in \mathcal{D}_{K_m}, D_1, \ldots, D_m \text{ compatible}\}.\end{aligned}$$

Proof. Since $K_1, \ldots, K_m$ are compatible, they are derived from a joint coherent $\hat{K}$, which then necessarily is a superset of K. This implies that $\hat{K} := \bigcup_{\ell \leq m} K_\ell$ is consistent, guaranteeing K's coherence by Theorem 2.2.

We show that $K_{\mathcal{D}_K} \subseteq K_{\mathcal{D}}$. To this end, consider any D in $\mathcal{D}$. Then $D = \mathrm{cl}_{\overline{\mathcal{D}}}(D_1 \cup \cdots \cup D_m)$ for some $D_1 \in \mathcal{D}_{K_1}$, …, $D_m \in \mathcal{D}_{K_m}$, whence $K_\ell \subseteq K_{D_\ell} \subseteq K_D$ for every ℓ

[4] We are indebted to Reviewer 4 for providing us with the current version of the result, and its proof.

in $\{1, \ldots, m\}$, so indeed $\hat{K} \subseteq K_D$ and hence $D \in \mathcal{D}_{\hat{K}}$. Since the choice of D in $\mathcal{D}$ was arbitrary, this implies that $\mathcal{D} \subseteq \mathcal{D}_{\hat{K}}$, which in turn implies that $K_{\mathcal{D}_{\hat{K}}} \subseteq K_{\mathcal{D}}$.

Next, we show that $K_{\mathcal{D}} \subseteq K_{\mathcal{D}_{\hat{K}}}$. To this end, consider any F in $\mathcal{Q}$ such that $F \notin K_{\mathcal{D}_{\hat{K}}}$. This implies that there is some D in $\mathcal{D}_{\hat{K}}$ such that $F \cap D = \emptyset$. Consider any ℓ in $\{1, \ldots, m\}$. That $D \in \mathcal{D}_{\hat{K}}$ implies that $K_\ell \subseteq K_D$, and therefore $K_\ell = \mathrm{Marg}_{S_\ell} K_\ell \subseteq \mathrm{Marg}_{S_\ell} K_D = K_{\mathrm{margs}_{S_\ell} D}$, so $\mathrm{margs}_{S_\ell} D \in \mathcal{D}_{K_\ell}$. Since $D \supseteq \mathrm{cl}_{\overline{\mathcal{D}}}(\mathrm{margs}_{S_1} D \cup \cdots \cup \mathrm{margs}_{S_m} D)$, we infer from $F \cap D = \emptyset$ that $F \cap \mathrm{cl}_{\overline{\mathcal{D}}}(\mathrm{margs}_{S_1} D \cup \cdots \cup \mathrm{margs}_{S_m} D) = \emptyset$, But $\mathrm{margs}_{S_1} D, \ldots, \mathrm{margs}_{S_m} D$ are compatible because they are derived from the common coherent joint D, so $F \notin K_{\mathcal{D}}$. Since the choice of F was arbitrary, together with the result established in the paragraph above, this implies that $K_{\mathcal{D}} = K_{\mathcal{D}_{\hat{K}}}$.

Taking into account Theorem 3.2, we have that $K_{\mathcal{D}_{\hat{K}}} = \mathrm{cl}_{\overline{\mathcal{K}}}(\hat{K}) = K$, and therefore $K_{\mathcal{D}} = K$, so $\mathcal{D}$ represents K, indeed. □

It is noteworthy that Proposition 5.4 does not require the coherent SoDGSes to admit a finite representation. If the coherent SoDGSes $K_1 \subseteq \mathcal{Q}(\mathcal{X}_{S_1})$, $K_2 \subseteq \mathcal{Q}(\mathcal{X}_{S_2})$, …, $K_m \subseteq \mathcal{Q}(\mathcal{X}_{S_m})$ do admit finite representations, then Proposition 5.4 provides a way to find a representation of the smallest compatible joint $\mathrm{cl}_{\overline{\mathcal{K}}}(\bigcup_{\ell \leq m} K_\ell)$.

6. Conclusions

We have established in Theorem 5.1 that coherent SoDGSes with a finite representation that are pairwise compatible, and whose index sets satisfy the running intersection property, are compatible. In order to do so, we had to study the representation of coherent SoDGSes in more detail. We have established in Theorem 3.3 that the representation $\mathcal{D}_K$ of a coherent SoDGS K is determined by its minimal elements $\min \mathcal{D}_K$, which by Proposition 3.1 is a finite set whenever K admits a finite representation. Moreover, Proposition 3.2 establishes that the coherent SoDGSes that admit a finite representation are *uniquely* determined by finite antichains of coherent sets of desirable gambles.

Our solution to the marginal problem in Theorem 5.1 generalises Miranda and Zaffalon [23, Theorem 2], the result for sets of desirable gambles. Importantly, they note [23, Appendix A.4] that coherent sets of desirable gambles, with the standard definition of marginalisation, can be interpreted as a *valuation algebra* [19]. As made explicit by Casanova, Kohlas, and Zaffalon [5, 6], the structure of a valuation algebra is rich enough to prove Theorem 5.1 directly.

Preliminary work indicates that the coherent SoDGSes, with our definition of marginalisation in Definition 4.2, also form a valuation algebra. As a consequence, Theorem 5.1 will come for free, *even for coherent SoDGSes that do not necessarily admit a finite representation*. However, a valuation algebra does not allow us to derive a representation in the sense of Proposition 5.4, at least not as far as we see. It is an interesting open question whether the representation Theorem 3.1 can be incorporated in the framework of valuation algebras, and we intend to pursue this in the future. Answering this would be useful for the study of compatibility. This paper – specifically Propositions 4.2 and 5.2 – establishes foundations for this project.

Another direction for future work, is to study the compatibility of conditional SoDGSes. Miranda and Zaffalon have investigated this problem, which they call the 'compatibility problem', for sets of desirable gambles in quite some detail [23, Section 3]. This can be done best using valuation algebras, so incorporating representation in the framework of valuation algebras would be useful for this project, too.

The new interesting subclass of coherent SoDGSes that we have introduced in this paper, namely SoDGSes with a finite representation, served its main purpose in the proofs of Lemma 3.1 and Proposition 3.2. We have indicated in Example 3.1 that this subclass is sufficiently broad: it contains all the SoDGSes based on Sen–Walley maximality, and also based on E-admissibility with a finite set of probabilities. However, preliminary work indicates that mixing Archimedean SoDGSes [15] also have the property described in Proposition 3.2, *even when they do not admit a finite representation*; we intend to report on this in the future. An interesting open and foundational question is what property on SoDGSes is *necessary* for the conclusion of Proposition 3.2 to hold.

Additional Author Information

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