Distorting lower probabilities using common distortion models

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Abstract

Distortion or neighbourhood models are useful tools in the imprecise probability theory allowing to robustify a probabilistic model by considering a neighbourhood around a given probability measure. In this work, we tackle the more general problem of distorting a lower probability. This problem can be interesting when we believe that a given lower probability is too precise, or in coalitional game theory when the set of solutions is empty. Our main purpose is to investigate how the linear vacuous and pari mutuel models can be defined for the distortion of lower probabilities, and for this aim we address the problem in a more general manner: we extend the vertical barrier models, which includes the linear vacuous and pari mutuel models, and investigate the properties they satisfy.

Keywords. Distortion models, Vertical barrier model, Linear vacuous model, Pari mutuel model, Total variation model

1. Introduction

While probability theory is the most widespread tool used to model uncertainty, a common approach in robust statistics [10] is to consider a neighborhood around a given probability measure. This allows to make the model more robust and also to adopt a more conservative approach when making decisions or inferences, given that the probability measure may not always accurately represent the uncertainty in the experiment due to noise or estimation errors. The classical reference [10] extensively discusses various ways in which a probability measure can be robustified, considering, among others, the linear vacuous or total variation models.

These models were given a common formalisation in [17, 18] as closed topological balls centered at a given probability measure, with a specified radius and a distorting function that compares probability measures. This directly connects distortion models with the imprecise probability theory [1, 30], since taking lower envelopes on the set of probability measures in the ball results in a coherent lower prevision. Indeed, many common distortion models such as the linear vacuous [10, 30] or the pari mutuel model [16, 24, 30] align with this definition. The distorting function may be in particular a distance between probability measures, as is the case with the total variation model [9]. Interestingly, the three aforementioned models can be embedded in the more general framework of vertical barrier models, introduced in [4] and extensively studied in [14, 22, 23]. Essentially, vertical barrier models apply a quasi-linear transformation to a given probability measure. Finally, distortion models can also be defined by directly applying an increasing transformation to the probability measure [2]. Depending on the additional properties satisfied by the transformation, this approach may yield a coherent lower probability or even a 2-monotone one. Alternatively, if the transformation is quasi-linear, it leads to a vertical barrier model.

In this contribution we go a step beyond and instead of distorting a probability measure, we consider the distortion of a lower probability. This idea is not new: for example, in Evidence Theory [25], it is common to discount a belief function by a factor α representing the degree of unreliability of the source. Within imprecise probabilities, Moral [19] proposed distorting, or discounting in his terminology, a credal set or a set of almost desirable gambles using divergences. The problem has also been considered in the context of coalitional games [26, 27], if we take into account that a normalised game is formally equivalent to a lower probability, and its credal set is commonly referred to as the core of the game, containing the solutions that satisfy the requirements of the coalitions [8, 12, 13]. In this context, when the core is empty, several procedures exist to modify the game (i.e., the lower probability) to ensure non-emptiness of the core, such as the strong δ-core [26].

Recently [20, 21], we addressed the distortion of a lower probability by providing a formal definition of a distortion procedure, enumerating several desirable properties that such a procedure should ideally satisfy, and analysing as an example the case when distortion procedure is determined by the total variation distance, thus generalising the total variation model. By doing this, we established a connection between this imprecise total variation model and the strong δ-core from coalitional games. Here, we take our analysis a step further by studying how to extend the linear vacuous and pari mutuel models to the distortion of lower probabilities. We tackle this problem by considering the more general framework of vertical barrier models and then particularising the results.

The rest of the contribution is organised as follows. In Section 2 we recall the tools and notation from imprecise probabilities that we shall use in the remainder of the paper. The basics of distortion models and some examples are introduced in Section 3. In Section 4 we recall the definition given in [21] of a distortion procedure for lower probabilities and enumerate some of the desirable properties such a procedure may satisfy. The main bulk of our work is carried out in Sections 5, 6 and 7, where we define and investigate the distortion procedures based on vertical barrier, linear vacuous and pari mutuel models, respectively.

2. Preliminaries

In this section we make a quick overview about the main notions from imprecise probabilities we shall employ in this paper; we refer to [1, 28, 30] for a detailed introduction.

We denote by $\mathcal{X}$ a finite possibility space, by $\mathbb{P}(\mathcal{X})$ the set of probability measures on $\mathcal{X}$, by $\mathbb{P}^*(\mathcal{X})$ the set of those probability measures satisfying $P(A) > 0$ for any $\emptyset \neq A \subseteq \mathcal{X}$, and by $\mathcal{L}(\mathcal{X})$ the set of gambles or real-valued functions defined on $\mathcal{X}$. A lower prevision $\underline{P} : \mathcal{K} \subseteq \mathcal{L}(\mathcal{X}) \to \mathbb{R}$ is a function assigning a real value to each gamble in $\mathcal{K}$ and, when $\mathcal{K}$ is a linear space, it is called *coherent* whenever it satisfies the following properties for any $f, g \in \mathcal{K}$ and $\lambda > 0$:

- Accepting sure gain: $\underline{P}(f) \geq \min f$.
- Positive homogeneity: $\underline{P}(\lambda f) = \lambda \underline{P}(f)$.
- Superlinearity: $\underline{P}(f+g) \geq \underline{P}(f) + \underline{P}(g)$.

Equivalently, coherence can be defined using the credal set associated with the lower prevision, given by:

$$\mathcal{M}(\underline{P}) = \{P \in \mathbb{P}(\mathcal{X}) \mid P(f) \geq \underline{P}(f)\ \forall f \in \mathcal{K}\}.$$

Then, $\underline{P}$ is coherent if and only if $\underline{P}(f) = \min_{P\in\mathcal{M}(\underline{P})} P(f)$ for any $f \in \mathcal{K}$. Additionally, it satisfies the weaker property of *avoiding sure loss* (ASL) when $\mathcal{M}(\underline{P}) \neq \emptyset$. $\mathcal{M}(\underline{P})$ is a closed and convex set of probability measures, hence it is characterised by the set of extreme points, that we shall denote $\text{ext}\mathcal{M}(\underline{P})$.

We denote by I_A the indicator function of the event A, that is the gamble that takes the value $I_A(x) = 1$ if $x \in A$ and zero otherwise. When $\mathcal{K} = \{I_A \mid A \subseteq \mathcal{X}\}$, we refer to $\underline{P}$ as a lower probability, and use $\underline{P}(A)$ to denote $\underline{P}(I_A)$ for $A \subseteq \mathcal{X}$. Hereon, $\underline{\mathbb{P}}(\mathcal{X})$ shall denote the set of lower probabilities on $\mathcal{X}$ and $\underline{\mathbb{P}}^*(\mathcal{X})$ those lower probabilities satisfying $\underline{P}(A) > 0$ for any $\emptyset \neq A \subseteq \mathcal{X}$.

Any coherent lower prevision (or probability) defined on $\mathcal{K}$ can be extended to $\mathcal{L}(\mathcal{X})$ using the natural extension:

$$\underline{E}(f) = \min\{P(f) \mid P \in \mathcal{M}(\underline{P})\}, \quad \forall f \in \mathcal{L}(\mathcal{X}) \qquad (1)$$

which is again a coherent lower prevision and satisfies $\mathcal{M}(\underline{E}) = \mathcal{M}(\underline{P})$. For this reason, we may always assume w.l.o.g. that $\underline{P}$ is defined on $\mathcal{L}(\mathcal{X})$.

A coherent lower prevision defined on $\mathcal{L}(\mathcal{X})$ is called *k-monotone* [5] when for any $p \leq k$, $f_1, \dots, f_p \in \mathcal{L}(\mathcal{X})$ it holds:

$$\underline{P}\left(\bigvee_{i=1}^{p} f_i\right) \geq \sum_{\emptyset\neq I\subseteq\{1,\dots,p\}} (-1)^{|I|+1} \underline{P}\left(\bigwedge_{i\in I} f_i\right). \qquad (2)$$

Similarly, a coherent lower probability is k-monotone when Equation (2) holds replacing the gambles f_i by events A_i, and the pointwise minimum ($\wedge$) and maximum ($\vee$) by the intersection and union, respectively.

While a coherent lower probability has in general more than one coherent extension to gambles, when the lower probability satisfies k-monotonicity for some $k \geq 2$, the natural extension in Equation (1) is the unique extension to gambles preserving k-monotonicity; moreover, it can be computed as the Choquet integral [3] with respect to its restriction to events.

For conditioning a coherent lower prevision on $\mathcal{L}(\mathcal{X})$ we consider here the procedure of *regular extension* [11, 30]. Given $B \subseteq \mathcal{X}$ satisfying $\underline{P}(B^c) < 1$, it is defined by:

$$\underline{P}(f|B) = \inf\{P(f|B) \mid P \in \mathcal{M}(\underline{P}), P(B) > 0\} \qquad (3)$$

for any $f \in \mathcal{L}(\mathcal{X})$; in events and whenever $\underline{P}$ is 2-monotone this simplifies to:

$$\underline{P}(A \mid B) = \begin{cases} \frac{\underline{P}(A\cap B)}{\underline{P}(A\cap B)+1-\underline{P}(A\cup B^c)} & \text{if } \underline{P}(A\cup B^c) < 1; \\ 1 & \text{otherwise.} \end{cases} \qquad (4)$$

3. Distortion models

Given a probability measure $P_0 \in \mathbb{P}(\mathcal{X})$, a *distortion* or *neighbourhood* model is built by considering the closed ball, with respect to the topology induced by the Euclidean distance, centered at P_0 with a given radius δ and with respect to a function $d : \mathbb{P}(\mathcal{X}) \times \mathbb{P}(\mathcal{X}) \to \mathbb{R}$ used to compare probability measures. Formally [17, 18]:

$$B_d^{\delta}(P_0) = \{P \in \mathbb{P}(\mathcal{X}) \mid d(P, P_0) \leq \delta\}.$$

Whenever the function d is continuous and quasi-convex with respect to the first argument, $B_d^{\delta}(P_0)$ is a closed and convex set of probabilities [17, Prop.3.1], i.e., a *credal set*. This means that its lower envelope on gambles, given by:

$$\underline{P}_d(f) = \min\{P(f) \mid P \in B_d^{\delta}(P_0)\} \quad \forall f \in \mathcal{L}(\mathcal{X})$$

is a coherent lower prevision, and $B_d^\delta(P_0)$ can be equivalently represented as:

$$B_d^\delta(P_0) = \{P \in \mathbb{P}(\mathcal{X}) \mid P(f) \geq \underline{P}_d(f)\ \forall f \in \mathcal{L}(\mathcal{X})\}.$$

Some well-known imprecise probability models, such as the *linear vacuous* (LV) and the *pari mutuel* (PMM) models, can be embedded into the above definition of distortion model. On the one hand, the LV model [10, 30] is defined as the coherent lower probability:

$$\underline{P}_{\text{LV}}(A) = (1-\delta)P_0(A) \quad \forall A \subset \mathcal{X},$$

for some $\delta \in [0,1)$[1] and $\underline{P}_{\text{LV}}(\mathcal{X}) = 1$. Its credal set can be expressed as:

$$\mathcal{M}(\underline{P}_{\text{LV}}) = \{(1-\delta)P_0 + \delta Q \mid Q \in \mathbb{P}(\mathcal{X})\}.$$

$\underline{P}_{\text{LV}}$ is not only coherent but also completely monotone, i.e., k-monotone for every $k \geq 1$.

On the other hand, the PMM [16, 24, 30] is defined as the coherent lower probability:

$$\underline{P}_{\text{PMM}}(A) = \max\{(1+\delta)P_0(A) - \delta, 0\} \quad \forall A \subseteq \mathcal{X}$$

for some $\delta \geq 0$. $\underline{P}_{\text{PMM}}$ is a 2-monotone lower probability, hence its natural extension is the unique extension to gambles preserving 2-monotonicity.

Whenever $P_0 \in \mathbb{P}^*(\mathcal{X})$, both the LV and PMM can be expressed as distortion models with respect to the respective functions d_{LV} and d_{PMM} given by:

$$d_{\text{LV}}(P, P_0) = \max_{\emptyset \neq A \subseteq \mathcal{X}} \frac{P_0(A) - P(A)}{P_0(A)}.$$

$$d_{\text{PMM}}(P, P_0) = \max_{A \subset \mathcal{X}} \frac{P_0(A) - P(A)}{1 - P_0(A)}.$$

We may also obtain a distortion model using a distance between probability measures as a distorting function. If we consider the total variation distance:

$$d_{\text{TV}}(P, Q) = \max_{A \subseteq \mathcal{X}} |P(A) - Q(A)|,$$

the distortion model it induces, $B_{\text{TV}}^\delta(P_0)$, is a credal set whose associated coherent lower prevision is the natural extension to gambles of the lower probability given, for some $\delta \in [0,1)$[2], by:

$$\underline{P}_{\text{TV}}(A) = \max\{P_0(A) - \delta, 0\} \quad \forall A \subset \mathcal{X} \tag{5}$$

and $\underline{P}_{\text{TV}}(\mathcal{X}) = 1$. This lower probability is not only coherent but also 2-monotone and is called the *total variation* (TV) model.

These three particular models, LV, PMM and TV, can be embedded in a broader family of distortion models: *vertical barrier* (VBM) models, introduced in [4] and thoroughly investigated in [14, 22, 23]. Given $a \leq 0$ and $b \geq 0$ such that $a + b \leq 1$, the VBM is defined as the coherent lower probability:

$$\underline{P}_{\text{VBM}}(A) = \max\{bP_0(A) + a, 0\} \quad \forall A \subset \mathcal{X}$$

and $\underline{P}_{\text{VBM}}(\mathcal{X}) = 1$. Besides being 2-monotone, this lower probability satisfies the following properties:

(i) If $a \leq 0$ and $a + b = 1$, it coincides with the pari mutuel model.

(ii) If $a = 0$ and $b \in (0,1]$, it coincides with the LV model.

(iii) If $b = 1$ and $a \in (-1, 0]$, it coincides with the TV model.

(iv) If $a < 0 < b$ and $0 \leq a+b < 1$, $\mathcal{M}(\underline{P}_{\text{VBM}})$ coincides with $B_{d_{\text{VBM}}}^\delta(P_0)$, where [14, Thm.5]:

$$d_{\text{VBM}}(P, P_0) = \max_{A \subseteq \mathcal{X}} \frac{P_0(A) - P(A)}{(1-b)P_0(A) - a}.$$

We conclude this section recalling that distortion models may also be obtained through increasing transformations of probability measures [2]. Given a probability measure P_0, an increasing function $\phi : [0,1] \to [0,1]$ satisfying $\phi(t) \leq t$ for any $t \in [0,1]$, $\phi(0) = 0$ and $\phi(1) = 1$, determines a lower probability via:

$$\underline{P}(A) = \phi(P_0(A)) \quad \forall A \neq \emptyset, \mathcal{X},$$

and $\underline{P}(\emptyset) = 0$, $\underline{P}(\mathcal{X}) = 1$. This lower probability is coherent, and whenever ϕ is convex, it is 2-monotone as well. Obviously, if $\phi(t) = \max\{bt + a, 0\}$ for any $t \in [0,1]$, where $a \leq 0$,$b \geq 0$ and $a + b \leq 1$, it coincides with the VBM. By [17, Prop.3.2], any such transformation of probability measures can also be expressed as a distortion model for a suitable distorting function d_ϕ.

4. Distorting lower probabilities

In this section we consider the problem of creating a neighbourhood around a lower probability. For this aim, we first introduce the general definition of a distortion procedure in Section 4.1, enumerate some desirable properties the procedure may satisfy in Section 4.2 and finally introduce the distortion procedures of interest for this contribution in Section 4.3.

4.1. Distortion procedure. Let us give our definition of distortion procedure, that is slightly more general than our recent proposal in [21]:

Definition 4.1. Let $\{\phi_\lambda\}_{\lambda \in \Lambda}$ be a family of non-decreasing functions $\phi_\lambda : [0,1] \to [0,1]$ satisfying $\phi_\lambda(t) \leq t$ for every $t \in [0,1]$ and $\lambda \in \Lambda$. Given a lower

[1] We exclude here the case $\delta = 1$ since it gives rise to a vacuous model: $\mathcal{M}(\underline{P}_{\text{LV}}) = \mathbb{P}(\mathcal{X})$.

[2] Again, we exclude the case $\delta = 1$ since it gives rise to a vacuous model.

probability $\underline{P}$ and $\lambda \in \Lambda$, we define the lower probability $\underline{Q}_\lambda[\underline{P}] : \mathcal{P}(\mathcal{X}) \to [0,1]$ as

$$\underline{Q}_\lambda[\underline{P}](A) = \begin{cases} 0, & \text{if } A = \emptyset. \\ \phi_\lambda(\underline{P}(A)), & \text{if } A \neq \emptyset, \mathcal{X}. \\ 1, & \text{if } A = \mathcal{X}. \end{cases}$$

The lower probability $\underline{Q}_\lambda[\underline{P}]$ will be called *distortion of $\underline{P}$ with parameter λ through ϕ_λ*, and the family of functions $\{\underline{Q}_\lambda[\cdot]\}_{\lambda\in\Lambda}$, will be called transformation or *distortion procedure of lower probabilities indexed by* Λ. Each function ϕ_λ shall be referred to as *transforming function*.

The spirit of this definition goes in line with that of the transformations of probability measures described in Section 3. The main differences are that (i) the transformation is now applied to the lower probability, and that (ii) the transforming function depends on the distortion parameter.

Definition 4.1 extends our initial proposal in [21], where we considered the index set $[0, \infty)$. Here, we use a generic index set so as to be able to encompasses more general models, such as the VBM.

4.2. Desirable properties. Following the ideas from [21], we next enumerate some desirable properties that the distortion procedure $\{\underline{Q}_\lambda[\cdot]\}_{\lambda\in\Lambda}$ may satisfy. We consider first some basic properties ((P1)–(P4)), then move to invariance properties with respect to some operations ((P5)–(P7)) and finally analyse the connection between the distortion of a lower probability and the distortion of its credal set ((P8)–(P10)).

- (P1) (Expansion) If $\preceq$ is a total order over Λ, $\underline{Q}_{\lambda_1}[\underline{P}] \leq \underline{Q}_{\lambda_2}[\underline{P}]$ for any $\lambda_1, \lambda_2 \in \Lambda$ such that $\lambda_1 \succeq \lambda_2$.
- (P2) (Semigroup) If $\preceq$ is a total order over Λ, then:
  - (a) there exists $\lambda_0 \in \Lambda$ such that $\underline{Q}_{\lambda_0}[\underline{P}] = \underline{P}$ and $\lambda_0 \preceq \lambda$ for each $\lambda \in \Lambda$; and
  - (b) for such $\lambda_0 \in \Lambda$, and any $\lambda_1, \lambda_2 \succeq \lambda_0$ such that $\lambda_1 + \lambda_2 \in \Lambda$ it holds $\underline{Q}_{\lambda_2+\lambda_1}[\underline{P}] = \underline{Q}_{\lambda_2}[\underline{Q}_{\lambda_1}[\underline{P}]]$, where $+$ is an internal binary operation.
- (P3) (Structure preservation) If $\underline{P}$ satisfies one of the properties of avoiding sure loss, coherence, 2-monotonicity or k-monotonicity, then so does $\underline{Q}_\lambda[\underline{P}]$ for every $\lambda \in \Lambda$.
- (P4) (Reversibility) Given λ,$\underline{P}$ there exists $\varphi_\lambda : [0,1] \to [0,1]$ such that $\underline{P}(A) = \varphi_\lambda(\underline{Q}_\lambda[\underline{P}](A))$, for every $A \neq \emptyset, \mathcal{X}$. As a consequence, for every $\lambda \in \Lambda$, the initial model $\underline{P}$ can be recovered from $\underline{Q}_\lambda[\underline{P}]$.

Structure preservation and reversibility were already considered by Destercke in [7]. The next two properties, that deal with invariance with respect to some operations, were considered by Moral in [19].

- (P5) (Invariance under permutations) For every permutation σ of $\{1, \dots, n\}$ and $\lambda \in \Lambda$ it holds $\underline{Q}_\lambda[\underline{P}^\sigma] = \left(\underline{Q}_\lambda[\underline{P}]\right)^\sigma$, where for any lower probability $\underline{P}$ its permuted version is defined by $\underline{P}^\sigma(A) := \underline{P}(A^\sigma)$ for any $A = \{x_{i_1}, \dots, x_{i_m} \mid i_j \neq i_k \ \forall j \neq k\} \subseteq \mathcal{X}$, where A^σ is defined by $A^\sigma := \{x_{\sigma(i_1)}, \dots, x_{\sigma(i_m)}\}$ and the permuted version of $\underline{Q}_\lambda[\underline{P}]$ is defined by $\left(\underline{Q}_\lambda[\underline{P}]\right)^\sigma(A) := \underline{Q}_\lambda[\underline{P}](A^\sigma)$.
- (P6) (Invariance under marginalisation) For each partition of the possibility space, $\Pi = \{A_1, \dots, A_m\}$, and $\lambda \in \Lambda$ it holds $\underline{Q}_\lambda\left[\underline{P}^\Pi\right] = \left(\underline{Q}_\lambda[\underline{P}]\right)^\Pi$, where $\underline{P}^\Pi$ is the lower probability on $\mathcal{P}(\Pi)$ given by $\underline{P}^\Pi(\cup_{j\in J} A_j) := \underline{P}(\cup_{j\in J} A_j)$ for each $J \subseteq \{1, \dots, m\}$, and $\left(\underline{Q}_\lambda[\underline{P}]\right)^\Pi$ is defined analogously.
- (P7) (Invariance under conditioning) For each $B \subset \mathcal{X}$ such that $\underline{Q}_\lambda[\underline{P}](B) > 0$ and $\lambda \in \Lambda$, it holds $\left(\underline{Q}_\lambda[\underline{P}]\right)_B = \underline{Q}_{\lambda^*}[\underline{P}_B]$ for certain $\lambda^* \in \Lambda$, where $\underline{P}_B$ is the lower probability obtained by conditioning $\underline{P}$ on B using Equation (3), and $\left(\underline{Q}_\lambda[\underline{P}]\right)_B$ is defined analogously.

The last three properties establish a connection between the distortion procedure applied to the lower probability and the distortion of the credal set.

- (P8) (Expression as an imprecise neighbourhood model) There exists a function $d : \mathbb{P}(\mathcal{X}) \times \underline{\mathbb{P}}(\mathcal{X}) \to \mathbb{R}$ such that, given $\underline{P} \in \underline{\mathbb{P}}(\mathcal{X})$, for every $\lambda \in \Lambda$, $Q \in \mathbb{P}(\mathcal{X})$ it holds $d(Q, \underline{P}) \leq \mu$ for certain $\mu \in \mathbb{R}$ depending on λ if and only if $Q \in \mathcal{M}\left(\underline{Q}_\lambda[\underline{P}]\right)$. Equivalently:
$$\mathcal{M}\left(\underline{Q}_\lambda[\underline{P}]\right) = \{Q \in \mathbb{P}(\mathcal{X}) \mid d(Q, \underline{P}) \leq \mu\} =: B_d^\mu(\underline{P}).$$
- (P9) (Extreme points commutativity) If $\underline{P}$ is coherent, then for every $\lambda \in \Lambda$ and $A \subseteq \mathcal{X}$ it holds:
$$\underline{Q}_\lambda[\underline{P}](A) = \inf\left\{Q(A) \mid Q \in \bigcup_{P \in \mathrm{ext}\mathcal{M}(\underline{P})} \mathcal{M}\left(\underline{Q}_\lambda[P]\right)\right\}.$$
- (P10) (Strong commutativity) If $\underline{P}$ is coherent, then for every $\lambda \in \Lambda$ it holds:
$$\mathcal{M}\left(\underline{Q}_\lambda[\underline{P}]\right) = \bigcup_{P \in \mathcal{M}(\underline{P})} \mathcal{M}\left(\underline{Q}_\lambda[P]\right).$$

4.3. Particular distortion procedures. In [21] we considered a distortion procedure that extends the TV model to the distortion of coherent lower probabilities.

Definition 4.2 ([21])**.** Given a lower probability $\underline{P}$ and a distortion parameter $\delta \in [0,1)$, the *imprecise total* *variation* (for short, ITV) model is determined by the lower probability $\underline{Q}_\delta[\underline{P}]$ given by:

$$\underline{Q}_\delta[\underline{P}](A) = \begin{cases} \max\{\underline{P}(A) - \delta, 0\}, & \text{if } A \neq \mathcal{X}. \\ 1, & \text{if } A = \mathcal{X}. \end{cases} \tag{ITV}$$

This model corresponds to the transforming functions $\{\phi_\delta\}_{\delta\in[0,1)}$ given by $\phi_\delta(t) = \max\{t - \delta, 0\}$ for any $t \in [0,1]$. Also, the ITV model extends the TV model to coherent lower probabilities in the sense that applying Equation (ITV) to a probability measure P_0 we retrieve the TV model given in Equation (5). In [21] we explored the properties satisfied by this distortion procedure; a summary can be seen in Table 1 given in Section 8.

Our aim in this contribution is to define distortion procedures for lower probabilities similar to the ITV but extending the PMM and LV models. For this aim, instead of focusing in these two particular models, we tackle the problem using a more general approach: directly extending a VBM.

Definition 4.3. Given a lower probability $\underline{P}$, $a \leq 0$ and $b \geq 0$ such that $a + b \leq 1$, the *imprecise vertical barrier* (for short, IVBM) model with parameter (a,b) is defined as:

$$\underline{Q}_{(a,b)}[\underline{P}](A) = \begin{cases} \max\{b\underline{P}(A) + a, 0\}, & \text{if } A \neq \mathcal{X}. \\ 1, & \text{if } A = \mathcal{X}. \end{cases} \tag{IVBM}$$

In this case, the transforming functions $\{\phi_{(a,b)}\}_{(a,b)\in\Lambda}$ are given by $\phi_{(a,b)}(t) = \max\{bt + a, 0\}$ for any $t \in [0,1]$, where $\Lambda = \{(a,b) \mid a \leq 0, b \geq 0, a + b \leq 1\}$.

Moreover, the IVBM generalises the ITV simply considering $b = 1$ and $\delta = -a \geq 0$. We may similarly define the imprecise LV and PMM models:

Definition 4.4. Given a lower probability $\underline{P}$, we consider the following models:

- The *imprecise linear vacuous* (ILV, for short) model corresponds to the IVBM where $a = 0$ and $1 - \delta = b \in (0,1]$:
$$\underline{Q}^{\mathrm{LV}}_\delta[\underline{P}](A) = \begin{cases} (1-\delta)\underline{P}(A), & \text{if } A \neq \mathcal{X}. \\ 1, & \text{if } A = \mathcal{X}. \end{cases} \tag{ILV}$$
- The *imprecise pari mutuel* (IPMM, for short) model, corresponds to the IVBM where $-\delta = a \leq 0$ and $a + b = 1$:

$$\underline{Q}^{\mathrm{PMM}}_\delta[\underline{P}](A) = \begin{cases} \max\{(1+\delta)\underline{P}(A) - \delta, 0\}, & \text{if } A \neq \mathcal{X}. \\ 1, & \text{if } A = \mathcal{X}. \end{cases} \tag{IPMM}$$

The remainder of this contribution explores the properties satisfied by the ILV and IPMM models. For this purpose, we investigate in Section 5 the properties satisfied by the IVBM, and in Sections 6 and 7 we focus on the particular cases of ILV and IPMM, respectively.

5. Imprecise Vertical Barrier Models

Next we investigate in detail which of the properties enumerated in Section 4.2 are satisfied by the IVBM.

5.1. Basic and invariance properties. Due to the structure of the index set Λ, we shall not investigate properties (P1) and (P2) for the IVBM. Regarding structure preservation (P3), we show in the following result that the IVBM preserves ASL, coherence and 2-monotonicity.

Proposition 5.1. *The IVBM is closed for the subfamilies (P3) of avoiding sure loss, coherent and 2-monotone lower probabilities.*

On the other hand, the IVBM does not always preserve k-monotonicity; it suffices to observe that, as shown in [21], the ITV model violates this property.

Regarding reversibility, our next result shows that it holds under an additional condition.

Proposition 5.2. *Given a lower probability $\underline{P}$, the IVBM satisfies reversibility (P4) if $b \neq 0$ and $\min_{x\in\mathcal{X}} \underline{P}(\{x\}) \geq -a/b$. In that case, the inverse transforming function is given by $\varphi_{(a,b)}(t) := \min\{1, (t+a)/b\}$ for any $t \in [0,1]$.*

Note that conditions in the above proposition are always satisfied by the ILV because $a = 0$ and $b \neq 0$.

Conversely, if the previous sufficient condition were not satisfied and $a, b \neq 0$, then the reversibility property does not hold as we show in the following example.

Example 5.1. Consider $a, b \neq 0$ satisfying $a < 0 < b$ and $a + b \leq 1$. Since $-a/b > 0$, we can define two lower probabilities $\underline{P}_1$ and $\underline{P}_2$ on a possibility space $\mathcal{X} = \{x_1, x_2, x_3\}$ as follows. For $j \in \{1,2\}$, let $\underline{P}_1(\{x_j\}) = -a/b$, $\underline{P}_2(\{x_j\}) < -a/b$ and let $\underline{P}_1(A) = \underline{P}_2(A) > -a/b$ for every non-trivial event A such that $A \neq \{x_1\}, \{x_2\}$. Then $\underline{P}_1$ and $\underline{P}_2$ are monotone and normalised (i.e., lower probabilities) even if they may not be coherent nor avoid sure loss. Moreover,

$$\underline{Q}_{(a,b)}[\underline{P}_1](A) = \underline{Q}_{(a,b)}[\underline{P}_2](A) \quad \forall A \subseteq \mathcal{X},$$

hence $\underline{Q}_{(a,b)}[\cdot]$ is not injective. ♦

Since the value of the distorted lower probability of an event does not depend on the event itself, but rather on its original lower probability, our next result states that these transformations are invariant under permutations (P5) and marginalisation (P6).

Proposition 5.3. *The IVBM is invariant under permutations (P5) and marginalisations (P6) of the possibility space.*

Regarding invariance under conditioning, it is known that the VBM are closed under conditioning (see e.g. [22, Prop.5] or [23, Prop.4.3]), and in particular the same property is satisfied by the LV [30, Sec. 6.6.2], PMM [17] and TV [18] models. However, when we start with a lower probability and apply the IVBM, invariance under conditioning will not be satisfied in general. We refer to [21, Ex.2] for an example with the ITV model, and to the forthcoming Examples 6.1 and 7.1 for the ILV and IPMM, respectively, where in the first case we show moreover that conditioning is not closed within IVBMs. Note that in all these examples we use a 2-monotone original model.

5.2. Expression as an imprecise neighbourhood. Let us next express any IVBM as an imprecise neighbourhood model. Inspired by [14, Sec. 3], we restrict ourselves to the values $a < 0 < b$ and $a + b \neq 1$, and we define

$$d_{\text{IVBM}}(Q, \underline{P}) := \max_{A \subseteq \mathcal{X}} d_{\text{IVBM}}(Q, \underline{P}; A), \tag{6}$$

where for each $A \subseteq \mathcal{X}$:

$$d_{\text{IVBM}}(Q, \underline{P}; A) := \frac{\underline{P}(A) - Q(A)}{(1-b)\underline{P}(A) - a}. \tag{7}$$

Reasoning as in [14, Lemma 3], we observe that this is well defined, since $d_{\text{IVBM}}(Q, \underline{P}; \emptyset) = 0$ in any case, hence $d_{\text{IVBM}}(Q, \underline{P}) \geq 0$, and the denominator in the previous equation is different from zero. Indeed, the excluded parameters are those for which $a = 0$ (ILV) or $a + b = 1$ (IPMM), including the identity transformation, since for these cases the denominator in Equation (7) may be null. A similar version of the cited lemma is given next.

Lemma 5.1. *For every $(a, b) \in \Lambda$ and $t \in [0, 1]$ it holds $(1 - b)t - a \geq 0$. Moreover, the equality holds if and only if either (i) $a + b = 1$ and $t = 1$; (ii) $a = 0$ and $t = 0$; or (iii) $b = 1$ and $a = 0$.*

We thus exclude from the analysis of the forthcoming properties in this section the parameters $(a, b) \in \Lambda$ such that $a + b = 1$ or $a = 0$. That is, we exclude the ILV and the IPMM, that shall be treated separately in the next sections. By not having into account these parameters, Equation (6) allows us to express any IVBM as an imprecise neighbourhood model.

Proposition 5.4. *Let $\underline{P} \in \underline{\mathbb{P}}(\mathcal{X})$ and consider a, b such that $a < 0 < b$ and $a + b < 1$. For every $Q \in \mathbb{P}(\mathcal{X})$ it holds $d_{\text{IVBM}}(Q, \underline{P}) \leq 1 \Leftrightarrow Q \in \mathcal{M}\left(\underline{Q}_{(a,b)}[\underline{P}]\right)$. In other words, (P8) is satisfied:*

$$\mathcal{M}\left(\underline{Q}_{(a,b)}[\underline{P}]\right) = B^1_{d_{\text{IVBM}}}(\underline{P}).$$

We remark that the radius of the neighbourhood equals 1 uniformly over (a, b) and the dependence on this parameter is implicit in the credal set (see [14, Rem. 1] for a detailed explanation of the analogous result in the precise case). Also, we should emphasise that this result does not require $\underline{P}$ to be coherent, nor even to avoid sure loss. Indeed, when $\underline{P}$ does not avoid sure loss, $\underline{Q}_{(a,b)}[\underline{P}]$ may not avoid sure loss either, and in that case, both $\mathcal{M}\left(\underline{Q}_{(a,b)}[\underline{P}]\right)$ and $B^1_{d_{\text{IVBM}}}(\underline{P})$ would be empty.

Proposition 5.5. *Let $\underline{P}$ be a coherent lower probability and $a < 0 < b$ such that $a + b < 1$. For every $Q \in \mathbb{P}(\mathcal{X})$ and $A \subseteq \mathcal{X}$, it holds:*

$$d_{\text{IVBM}}(Q, \underline{P}; A) = \min_{P \in \mathcal{M}(\underline{P})} d_{\text{IVBM}}(Q, P; A).$$

From the previous result and Equation (6) it is straightforward to obtain the following corollary.

Corollary 5.1. *Let $\underline{P}$ be a coherent lower probability and $a < 0 < b$ such that $0 \leq a + b < 1$. For every $Q \in \mathbb{P}(\mathcal{X})$ it holds:*

$$d_{\text{IVBM}}(Q, \underline{P}) = \max_{A \subseteq \mathcal{X}} \min_{P \in \mathcal{M}(\underline{P})} d_{\text{IVBM}}(Q, P; A). \tag{8}$$

Regarding commutativity, we first apply [21, Prop.4] to prove that, since the IVBM preserves coherence (Prop. 5.1), the IVBM satisfies extreme point commutativity.

Corollary 5.2. *The IVBM satisfies extreme points commutativity (P9).*

It is worth mentioning that the previous result does not make any additional assumption on the values of a and b, meaning that extreme points commutativity also holds for the IPMM and ILV models.

With respect to strong commutativity, whenever $\underline{P}$ is coherent we deduce from Proposition 5.4 (used in the second equivalence) that:

$$\begin{aligned}
&Q \in \bigcup_{P \in \mathcal{M}(\underline{P})} \mathcal{M}\left(\underline{Q}_{(a,b)}[P]\right) \\
&\quad \Leftrightarrow \exists P \in \mathcal{M}(\underline{P}) \text{ such that } Q \in \mathcal{M}\left(\underline{Q}_{(a,b)}[P]\right) \\
&\quad \Leftrightarrow \exists P \in \mathcal{M}(\underline{P}) \text{ such that } d_{\text{IVBM}}(Q, P) \leq 1 \\
&\quad \Leftrightarrow d'_{\text{IVBM}}(Q, \underline{P}) := \min_{P \in \mathcal{M}(\underline{P})} \max_{A \subseteq \mathcal{X}} d_{\text{IVBM}}(Q, P; A) \leq 1,
\end{aligned}$$

where in the last equation we write a minimum instead of an infimum because $\mathcal{M}(\underline{P})$ is closed. We next prove that d'_{IVBM} and d_{IVBM} from Equation (8) coincide whenever $\underline{P}$ is 2-monotone. As a consequence, strong commutativity (P10) holds as well under this assumption.

Theorem 5.1. *Let $\underline{P}$ be a 2-monotone lower probability and $a < 0 < b$ such that $0 \leq a + b < 1$. For every $Q \in \mathbb{P}(\mathcal{X})$ it holds $d_{\text{IVBM}}(Q, \underline{P}) = d'_{\text{IVBM}}(Q, \underline{P})$. Consequently, the IVBM verifies strong commutativity (P10).*

In general, as proved in [21, Ex.3], strong commutativity may fail for the ITV (hence for the IVBM) when the initial model is only coherent rather than 2-monotone. Thus, a similar result for the IVBM, but relying on coherence alone, is not possible.

6. Imprecise Linear Vacuous Models

Next we turn our attention to the ILV model. Trivially, all the properties satisfied in general by all the IVBM will be satisfied in particular for the ILV model.

6.1. Basic and invariance properties. In this case, we shall investigate the expansion (P1) and semigroup (P2) properties, considering that the ILV model is defined for $\delta \in [0,1)$.

Proposition 6.1. *The ILV model satisfies (P1) and (P2)(a), but it violates (P2)(b) in general. Moreover, it preserves the subfamilies (P3) of k-monotone lower probabilities for every $k \geq 2$.*

The invariance under permutations (P5) and marginalisation (P6) are guaranteed by Proposition 5.3. Also, the following example shows that conditioning an ILV do not necessarily gives rise to another ILV based on the conditioned initial lower probability.

Example 6.1. Consider the possibility space $\mathcal{X} = \{x_1, x_2, x_3\}$ and the coherent lower probability $\underline{P}$ given in the following table, where we also show the ILV model with parameter $\delta = 0.5$, that corresponds with an IVBM with $a = 0$ and $b = 1 - \delta = 0.5$.

	$\{x_1\}$	$\{x_2\}$	$\{x_3\}$	$\{x_1,x_2\}$	$\{x_1,x_3\}$	$\{x_2,x_3\}$
$\underline{P}$	0.1	0.1	0.1	0.2	0.2	0.9
$\underline{Q}_{(a,b)}[\underline{P}]$	0.05	0.05	0.05	0.1	0.1	0.45

$\underline{P}$ is not only coherent but also 2-monotone, because $\mathcal{X}$ only has three elements [29, p.58]. Hence, from Proposition 5.1 the transformed model preserves (P3) 2-monotonicity, hence $\underline{Q}_{(a,b)}[\underline{P}]$ is 2-monotone as well.

Now, let $B = \{x_1, x_2\}$, which satisfies $\underline{P}(B) \geq \underline{Q}_{(a,b)}[\underline{P}](B) > 0$. Since both $\underline{P}$ and the transformed model are 2-monotone, we can apply Equation (4) to obtain the conditional models. The following table shows the conditional initial lower probability, the conditional transformed model for (a,b) and the transformation of the former, $\underline{Q}_{(a^*,b^*)}[\underline{P}_B]$, for some $(a^*,b^*) \in \Lambda$:

	$\{x_1\} \mid \{x_1,x_2\}$	$\{x_2\} \mid \{x_1,x_2\}$
$\underline{P}_B$	$1/9$	$1/2$
$\left(\underline{Q}_{(a,b)}[\underline{P}]\right)_B$	$1/19$	$1/12$
$\underline{Q}_{(a^*,b^*)}[\underline{P}_B]$	$\max\{b^*/9 + a^*, 0\}$	$\max\{b^*/2 + a^*, 0\}$

If there exists some $(a^*,b^*) \in \Lambda$ such that $\underline{Q}_{(a^*,b^*)}[\underline{P}_B] = \left(\underline{Q}_{(a,b)}[\underline{P}]\right)_B$ is satisfied, it must be:

$$\frac{1}{19} - \frac{b^*}{9} = a^* = \frac{1}{12} - \frac{b^*}{2} \Rightarrow b^* = \frac{3}{2 \cdot 19} \text{ and } a^* = \frac{5}{6 \cdot 19},$$

but $a^* > 0$ hence $(a^*,b^*) \notin \Lambda$, a contradiction. ♦

6.2. Expression as an imprecise neighbourhood. For a straightforward generalisation of the distorting function giving rise to the LV in [17, Thm.5.1], we let $\underline{P} \in \underline{\mathbb{P}}^*(\mathcal{X})$ and consider, for each $Q \in \mathbb{P}^*(\mathcal{X})$:

$$d_{\mathrm{ILV}}(Q, \underline{P}; A) = \frac{\underline{P}(A) - Q(A)}{\underline{P}(A)} \quad \forall A \neq \emptyset,$$

defining thus $d_{\mathrm{ILVM}}(Q, \underline{P})$ as:

$$\begin{aligned} d_{\mathrm{ILV}}(Q, \underline{P}) &= \max_{\emptyset \neq A \subseteq \mathcal{X}} d_{\mathrm{ILV}}(Q, \underline{P}; A) \\ &= \max_{\substack{\emptyset \neq A \subseteq \mathcal{X} \\ \underline{P}(A) \geq Q(A)}} d_{\mathrm{ILV}}(Q, \underline{P}; A), \end{aligned}$$

where the last equality is due to the fact that $d_{\mathrm{ILV}}(Q, \underline{P}; \mathcal{X}) = 0$. With this definition, we can reproduce the results obtained with the IVBM regarding commutativity properties. In fact, note that

$$d_{\mathrm{ILV}}(Q, \underline{P}; A) = \delta d_{\mathrm{IVBM}}(Q, \underline{P}; A),$$

where the parameter of the associated IVBM is $(a,b) = (0, 1-\delta)$ and $d_{\mathrm{IVBM}}(Q, \underline{P}; A)$ is given by Equation (7) for each $A \neq \emptyset$ and extended to include the case of $a = 0$. Hence, the only difference is that we do not consider here the empty event, and the assumptions $\underline{P} \in \underline{\mathbb{P}}^*(\mathcal{X})$ and $Q \in \mathbb{P}^*(\mathcal{X})$.

Remarkably, for a general $\underline{P} \in \underline{\mathbb{P}}(\mathcal{X})$ and $Q \in \mathbb{P}(\mathcal{X})$, we may define $d_{\mathrm{ILV}}(Q, \underline{P})$ by taking the maximum over those sets $A \subseteq \mathcal{X}$ for which $\underline{P}(A) > 0$ (extending thus the observation made in [14, Eq.(6)] to the imprecise case). That is,

$$\begin{aligned} d_{\mathrm{ILV}}(Q, \underline{P}) &= \max_{A \mid \underline{P}(A) > 0} d_{\mathrm{ILV}}(Q, \underline{P}; A) \\ &= \max_{\substack{A \mid \underline{P}(A) > 0, \\ \underline{P}(A) \geq Q(A)}} d_{\mathrm{ILV}}(Q, \underline{P}; A), \end{aligned}$$

This aligns with our purposes of expressing the credal set of any ILV as a neighbourhood model for any given lower probability $\underline{P}$.

Proposition 6.2. *Given $\underline{P} \in \underline{\mathbb{P}}(\mathcal{X})$, for every $Q \in \mathbb{P}(\mathcal{X})$ and $\delta \in [0,1)$ it holds $d_{\mathrm{ILV}}(Q, \underline{P}) \leq \delta \Leftrightarrow Q \in \mathcal{M}(\underline{Q}_{\delta}^{\mathrm{LV}}[\underline{P}])$. In other words, (P8) is satisfied:*

$$\mathcal{M}\left(\underline{Q}_{\delta}^{\mathrm{LV}}[\underline{P}]\right) = B_{d_{\mathrm{ILV}}}^{\delta}(\underline{P}).$$

As in Proposition 5.4, for this result $\underline{P}$ need not be coherent or avoid sure loss. In the latter case, if $\underline{Q}_{\delta}^{\mathrm{LV}}[\underline{P}]$ does not avoids sure loss too, both sets coincide and are empty.

Also, under coherence, we can give an alternative expression for d_{ILV}.

Proposition 6.3. *Let $\underline{P}$ be a coherent lower probability and $\delta \in [0,1)$. For every $Q \in \mathbb{P}(\mathcal{X})$ it holds that:*

$$d_{\mathrm{ILV}}(Q, \underline{P}) = \max_{A \mid \underline{P}(A) > 0} \min_{P \in \mathcal{M}(\underline{P})} d_{\mathrm{ILV}}(Q, P; A).$$

On the other hand, we deduce that:

$$Q \in \bigcup_{P \in \mathcal{M}(\underline{P})} \mathcal{M}\left(\underline{Q}_{\delta}^{\mathrm{LV}}[P]\right)$$

$$\Leftrightarrow \exists P \in \mathcal{M}(\underline{P}) \text{ such that } Q \in \mathcal{M}\left(\underline{Q}_\delta^{\mathrm{LV}}[P]\right)$$
$$\Leftrightarrow \exists P \in \mathcal{M}(\underline{P}) \text{ such that } d_{\mathrm{ILV}}(Q,P) \leq \delta$$
$$\Leftrightarrow d'_{\mathrm{ILV}}(Q,\underline{P}) := \min_{P\in\mathcal{M}(\underline{P})} \max_{A|\underline{P}(A)>0} d_{\mathrm{ILV}}(Q,P;A) \leq \delta,$$

where the previous proposition has been used in the second equivalence.

Regarding (P9), it is satisfied from Corollary 5.2. Finally, as shown in the next theorem, strong commutativity (P10) holds for the ILV whenever the initial model $\underline{P}$ is 2-monotone. For this, it suffices to prove the equality between $d_{\mathrm{ILV}}(\cdot,\underline{P})$ and $d'_{\mathrm{ILV}}(\cdot,\underline{P})$ using a minimax theorem.

Theorem 6.1. *Let $\underline{P}$ be a coherent lower probability. It holds that:*

- *The ILV model does not satisfy strong commutativity (P10) in general.*
- *If $\underline{P} \in \underline{\mathbb{P}}^*(\mathcal{X})$ is 2-monotone, the ILV satisfies strong commutativity (P10).*

7. Imprecise Pari Mutuel Models

Next, we analyse the properties satisfied by the IPMM.

7.1. Basic and invariance properties. Considering that for the IPMM the parameter δ belongs to $[0,\infty)$, we shall investigate the expansion (P1) and semigroup (P2) properties. Also, from Proposition 5.1 any IVBM preserves (P3) the subfamilies of ASL, coherence and 2-monotone lower probabilities. Moreover, with an additional condition the IPMM also preserves k-monotonicity.

Proposition 7.1. *The IPMM satisfies (P1) and (P2)(a), but it violates (P2)(b) in general. Moreover, if $\underline{P}$ and δ are such that $\underline{Q}_\delta^{\mathrm{PMM}}[\underline{P}] \in \underline{\mathbb{P}}^*(\mathcal{X})$, the IPMM preserves the subfamily (P3) of k-monotone lower probabilities for every $k \geq 2$.*

As a consequence of Proposition 5.2, an IPMM with parameter $\delta \geq 0$ satisfies (P4) whenever the initial $\underline{P}$ satisfies:

$$\min_{x\in\mathcal{X}} \underline{P}(\{x\}) \geq \delta/(1+\delta).$$

Assuming that $\min_{x\in\mathcal{X}} \underline{P}(\{x\}) < 1$, this is equivalent to

$$\delta \leq \frac{\min_{x\in\mathcal{X}} \underline{P}(\{x\})}{1-\min_{x\in\mathcal{X}} \underline{P}(\{x\})} = \min_{\substack{x\in\mathcal{X}\\ \underline{P}(\{x\})\neq 1}} \frac{\underline{P}(\{x\})}{1-\underline{P}(\{x\})},$$

since the function $h(t) = t/(1-t)$ is strictly increasing in $[0,1)$. Then, letting $\delta^* > 0$ be the unique parameter that turns the inequality above into an equality, the inverse contraction procedure is given by the family $\{\varphi_\delta\}_{\delta\in[0,\delta^*]}$ where $\varphi_\delta(t) := \min\{(t-\delta)/(1+\delta), 1\}$ for each $t \in [0,1]$.

The invariance under permutations (P5) and marginalisation (P6) are guaranteed by Proposition 5.3. Regarding the invariance under conditioning (P7), the next example shows that it does not hold in general, not even for an initial 2-monotone lower probability.

Example 7.1. Let $\mathcal{X} = \{x_1,x_2,x_3\}$, $\underline{Q}_\delta^{\mathrm{PMM}}[\cdot]$ be the IPMM with parameter $\delta = 0.01$ and $\underline{P}$ be the coherent, and also 2-monotone, lower probability given in the following table, where we also specify the transformed model:

	$\{x_1\}$	$\{x_2\}$	$\{x_3\}$	$\{x_1,x_2\}$	$\{x_1,x_3\}$	$\{x_2,x_3\}$
$\underline{P}$	0.1	0.1	0.1	0.2	0.2	0.9
$\underline{Q}_\delta^{\mathrm{PMM}}[\underline{P}]$	0.091	0.091	0.091	0.192	0.192	0.899

Since the IPMM preserves (P3) 2-monotonicity, as proved in Propositions 5.1 and 7.1, $\underline{Q}_\delta^{\mathrm{PMM}}[\underline{P}]$ is also 2-monotone. Take now $B = \{x_1,x_2\}$, which satisfies $\underline{P}(B) \geq \underline{Q}_\delta^{\mathrm{PMM}}[\underline{P}](B) > 0$, and apply Equation (4) to obtain the conditional lower probability $\underline{P}_B$, the conditional transformed model for δ and a generic IPMM transformation of the former, $\underline{Q}_{\delta^*}^{PMM}[\underline{P}_B]$, for certain $\delta^* \geq 0$:

	$\{x_1\} \mid \{x_1,x_2\}$	$\{x_2\} \mid \{x_1,x_2\}$
$\underline{P}_B$	$1/9$	$1/2$
$\left(\underline{Q}_\delta^{\mathrm{PMM}}[\underline{P}]\right)_B$	$91/899$	$91/192$
$\underline{Q}_{\delta^*}^{\mathrm{PMM}}[\underline{P}_B]$	$\max\left\{\frac{(1+\delta^*)}{9}-\delta^*,0\right\}$	$\max\left\{\frac{(1+\delta^*)}{2}-\delta^*,0\right\}$

If there exists some $\delta^* \geq 0$ such that $\underline{Q}_{\delta^*}^{\mathrm{PMM}}[\underline{P}_B] = \left(\underline{Q}_\delta^{\mathrm{PMM}}[\underline{P}]\right)_B$ is satisfied, it must hold:

$$\frac{1+\delta^*}{9} - \delta^* = \frac{91}{899} \text{ and } \frac{1+\delta^*}{2} - \delta^* = \frac{91}{192},$$

but these equalities give rise to different values of δ^*, a contradiction. ♦

7.2. Expression as an imprecise neighbourhood. We restrict our attention now to initial lower probabilities $\underline{P}$ satisfying $\underline{P} \in \underline{\mathbb{P}}^*(\mathcal{X})$. Under this situation, we define, for every $Q \in \mathbb{P}(\mathcal{X})$, the following expression:

$$d_{\mathrm{IPMM}}(Q,\underline{P};A) = \frac{\underline{P}(A)-Q(A)}{1-\underline{P}(A)} \quad \forall A \subset \mathcal{X}.$$

Note that the previous expression is well defined because the denominator is always strictly positive: if it were $1 = \underline{P}(A) \leq \overline{P}(A)$ then $\underline{P}(A^c) = 0$, and since $\underline{P} \in \underline{\mathbb{P}}^*(\mathcal{X})$, this can only be if $A^c = \emptyset$ and $A = \mathcal{X}$. Based on these functions, we define $d_{\mathrm{IPMM}}(Q,\underline{P})$ as:

$$d_{\mathrm{IPMM}}(Q,P) = \max_{A\subset\mathcal{X}} d_{\mathrm{IPMM}}(Q,\underline{P};A).$$

Given $\underline{P} \in \underline{\mathbb{P}}^*(\mathcal{X})$, we use the imprecise distorting functions just introduced to write the IPMM as imprecise neighbourhood model.

Proposition 7.2. *Let $\underline{P} \in \underline{\mathbb{P}}^*(\mathcal{X})$. Then, for every $Q \in \mathbb{P}(\mathcal{X})$ and $\delta \geq 0$ it holds $d_{\mathrm{IPMM}}(Q, \underline{P}) \leq \delta \Leftrightarrow Q \in \mathcal{M}\big(\underline{Q}_\delta^{\mathrm{PMM}}[\underline{P}]\big)$. In other words, (P8) is satisfied:*

$$\mathcal{M}\Big(\underline{Q}_\delta^{\mathrm{PMM}}[\underline{P}]\Big) = B^\delta_{d_{\mathrm{IPMM}}}(\underline{P}).$$

For this result, $\underline{P}$ is not required to satisfy coherence or to avoid sure loss. If none of $\underline{P}$ and $\underline{Q}_\delta^{\mathrm{PMM}}[\underline{P}]$ avoids sure loss, then the two sets of probabilities are empty.

With respect to (P9), it is satisfied from Corollary 5.2. Moreover, when $\underline{P}$ is coherent, we deduce the following result.

Proposition 7.3. *Let $\underline{P} \in \underline{\mathbb{P}}^*(\mathcal{X})$ be coherent. Then, for every $Q \in \mathbb{P}(\mathcal{X})$ and $\delta \geq 0$ it holds:*

$$d_{\mathrm{IPMM}}(Q, \underline{P}) = \max_{A \subset \mathcal{X}} \min_{P \in \mathcal{M}(\underline{P})} d_{\mathrm{IPMM}}(Q, P; A).$$

On the other hand, we deduce that for every $Q \in \mathbb{P}(\mathcal{X})$ and $\delta \geq 0$:

$$Q \in \bigcup_{P \in \mathcal{M}(\underline{P})} \mathcal{M}\Big(\underline{Q}_\delta^{\mathrm{PMM}}[P]\Big)$$
$$\Leftrightarrow \exists P \in \mathcal{M}(\underline{P}) \text{ such that } Q \in \mathcal{M}\Big(\underline{Q}_\delta^{\mathrm{PMM}}[P]\Big)$$
$$\Leftrightarrow \exists P \in \mathcal{M}(\underline{P}) \text{ such that } d_{\mathrm{IPMM}}(Q, P) \leq \delta$$
$$\Leftrightarrow d'_{\mathrm{IPMM}}(Q, \underline{P}) := \min_{P \in \mathcal{M}(\underline{P})} \max_{A \subset \mathcal{X}} d_{\mathrm{IPMM}}(Q, P; A) \leq \delta,$$

where Proposition 7.2 has been used in the second equivalence. We next prove that, for a 2-monotone initial model, the IPMM satisfies strong commutativity. Again, the key point of the proof is showing the equality between d_{IPMM} and d'_{IPMM} using a minimax theorem.

Theorem 7.1. *Let $\underline{P}$ be a coherent lower probability. It holds that:*

- *The IPMM does not satisfy strong commutativity (P10) in general.*
- *If $\underline{P} \in \underline{\mathbb{P}}^*(\mathcal{X})$ is 2-monotone, the IPMM satisfies strong commutativity (P10).*

8. Conclusions

Our aim with this work was to develop distortion models that allow for the input to be a lower probability instead of a probability measure. Following our initial steps in [21], where we generalised the total variation model, in this work we have extended the linear vacuous and pari mutuel model. In order to achieve this, we have considered a more general approach: that of extending vertical barrier models, that include the linear vacuous and pari mutuel model as particular cases.

After formally defining the imprecise vertical barrier models, as well as the imprecise linear vacuous and imprecise pari mutuel distortion procedures, we investigated the properties that each approach satisfies. A summary of the results is presented in Table 1 Note that all

Table 1. *Summary of the properties satisfied by each distortion procedure, where a double check indicates that the property is always satisfied, a single check means that the property holds under certain additional conditions, and a cross denotes that the property does not generally hold*

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Property	IVBM	ITV	ILV	IPMM
Expansion	N.A.	✓✓	✓✓	✓✓
Semigroup				
(a)	N.A.	✓✓	✓✓	✓✓
(b)	N.A.	✓✓	✗	✗
Structure preservation				
ASL	✓✓	✓✓	✓✓	✓✓
Coherence	✓✓	✓✓	✓✓	✓✓
2-monotonicity	✓✓	✓✓	✓✓	✓✓
k-monotonicity	✗	✗	✓✓	✓
Reversibility	✓	✓	✓✓	✓
Invariance under permutation	✓✓	✓✓	✓✓	✓✓
Invariance under marginalisation	✓✓	✓✓	✓✓	✓✓
Invariance under conditioning	✗	✗	✗	✗
Expression as neighbourhood	✓✓	✓✓	✓✓	✓
Extreme point commutativity	✓✓	✓✓	✓✓	✓✓
Strong Commutativity	✓	✓	✓	✓

the properties satisfied by the general approach of imprecise vertical barrier models are also satisfied by its particular submodels, with the qualifications about the expressions as imprecise neighbourhoods made earlier. However, it is worth noting that some properties that do not hold in general may still be satisfied by specific models. In particular, we highlight that the imprecise linear vacuous model always preserves k-monotonicity, meaning that distorting a belief function results in another belief function. Moreover, it always satisfies reversibility without requiring additional constraints. For completeness, we have also included in the table the properties of the imprecise total variation model from [21]. We see from the table that the three particular cases of IVBMs (ITV,ILV,IPMM) behave somewhat similarly, the imprecise linear vacuous model being the one with the better properties.

Properties (P8)–(P10) establish a connection between the distortion of the lower probability and the distortion of its associated credal set. Theorems 5.1, 6.1, 7.1, and [21, Thm.1] show that distorting a 2-monotone lower probability is equivalent to distorting any probability in the credal set and taking the union of the neighbourhoods.

This opens the door to an interesting future problem: since under 2-monotonicity distorting a lower probability is equivalent to distorting its credal set, is it possible to characterise the distortion in terms of the associated set of almost desirable gambles? Some preliminary results in this direction were given in [19, Thm.4.1] for the total variation model, showing that distorting each probability in the credal set using the total variation model is equivalent to distorting the set of almost desirable gambles. We would like to determine whether, under 2-monotonicity, distorting the lower probability, the credal set, or the set of almost desirable gambles lead to the same model.

Additionally, it would be interesting to investigate the properties satisfied by other distortion procedures that do not align with vertical barrier models, such as the Kolmogorov [18], the Euclidean [15], or the Kullback-Leibler [15, 19] models. Finally, the problem of distorting a lower probability can be connected to coalitional game theory, where the lower probability represents the minimal reward required by each coalition. When the set of possible solutions to the game (i.e., the credal set) is empty, one possibility is to distort the lower probability, relaxing the requirements of the coalitions to ensure that there is a solution that satisfies all the conditions. As we did in [21, Sec.5] for the imprecise total variation, we could apply the imprecise linear vacuous or imprecise pari mutuel models in this setting. Finally, it would be interesting to analyse to which extent our results still hold on infinite spaces. While some of them appear to be satisfied in the general setting, others, such as the expressions as imprecise neighborhoods and those whose proof depend on topological considerations, seem more challenging.

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A. Proofs

Proof of Proposition 5.1

(P3)(ASL) From $\phi_{(a,b)}(t) \leq t$ for each $t \in [0,1]$ we get $\mathcal{M}(\underline{P}) \subseteq \mathcal{M}\left(\underline{Q}_{(a,b)}[\underline{P}]\right)$. If $\underline{P}$ avoids sure loss, then $\mathcal{M}(\underline{P}) \neq \emptyset$ and we conclude that $\mathcal{M}\left(\underline{Q}_{(a,b)}[\underline{P}]\right) \neq \emptyset$ as well.

(P3)(Coherence) In [4, Prop.4.1] it is proved that for every $P \in \mathbb{P}(\mathcal{X})$, $\underline{Q}_{(a,b)}[P]$ is coherent. The preservation of coherence when starting from precise probabilities implies [21, Prop.2] the analogous property for any initial coherent lower probability.

(P3)(2-monotonicity) Since $\phi_{(a,b)}$ is convex for every $(a,b) \in \Lambda$, we obtain (use [6, Ex.2.1] or [31, Thm.1(5)]) that $\underline{Q}_{(a,b)}[\underline{P}]$ is 2-monotone whenever $\underline{P}$ is. $\square$

Proof of Proposition 5.2

Assume $\min_{x\in\mathcal{X}} \underline{P}(\{x\}) \geq -a/b$. Then, from the monotonicity of $\underline{P}$, for each $A \neq \emptyset, \mathcal{X}$ we have $\underline{P}(A) \geq -a/b$, so that $b\underline{P}(A) + a \geq 0$ and:

$$\underline{Q}_{(a,b)}[\underline{P}](A) = \max\{0, b\underline{P}(A) + a\} = b\underline{P}(A) + a$$

is equivalent to:

$$\underline{P}(A) = \frac{\underline{Q}_{(a,b)}[\underline{P}](A) - a}{b}.$$

Defining $\varphi_{(a,b)} : [0,1] \to [0,1]$ by $\varphi_{(a,b)}(t) := \min\{1, (t-a)/b\}$ for any $t \in [0,1]$, it holds:

$$\varphi_{(a,b)}\left(\underline{Q}_{(a,b)}[\underline{P}](A)\right) = \min\{1, \underline{P}(A)\} = \underline{P}(A)$$

for any $A \neq \emptyset, \mathcal{X}$. $\square$

Proof of Proposition 5.3

(P5)(Invariance under permutations) Consider a permutation $\sigma : \{1,\dots,n\} \to \{1,\dots,n\}$. Then, for any $(a,b) \in \Lambda$ and $A \neq \emptyset, \mathcal{X}$ it holds:

$$\begin{aligned}\left(\underline{Q}_{(a,b)}[\underline{P}]\right)^{\sigma}(A) &= \underline{Q}_{(a,b)}[\underline{P}](A^{\sigma}) = \max\{b\underline{P}(A^{\sigma}) + a, 0\} \\ &= \max\{b\underline{P}^{\sigma}(A) + a, 0\} = \underline{Q}_{(a,b)}[\underline{P}^{\sigma}](A).\end{aligned}$$

(P6)(Invariance under marginalisations) Let $\Pi = \{A_1, \dots, A_m\} \subseteq \mathcal{P}(\mathcal{X})$ be a partition of the possibility space $\mathcal{X}$. For every $(a,b) \in \Lambda$ it holds that:

$$\begin{aligned}\left(\underline{Q}_{(a,b)}[\underline{P}]\right)^{\Pi}(A) &= \underline{Q}_{(a,b)}[\underline{P}](A^{\Pi}) = \max\{b\underline{P}(A^{\Pi}) + a, 0\} \\ &= \max\{b\underline{P}^{\Pi}(A) + a, 0\} = \underline{Q}_{(a,b)}[\underline{P}^{\Pi}](A)\end{aligned}$$

for any $\emptyset \neq A \subset \Pi$. □

Proof of Lemma 5.1
Condition $a + b \leq 1$ is equivalent to $1 - b \geq a$. Taking also into account that $t \in [0,1]$ and $a \leq 0$, we get:

$$(1-b)t - a \geq at - a = a(t-1) \geq 0.$$

Moreover, the equality holds if and only if $(1-b)t = a$, hence it must be $a = 0$ and either $b = 1$ or $t = 0$, or $t = 1$ and $a + b = 1$. □

Proof of Proposition 5.4
(P8)(Expression as an imprecise neighbourhood model) Given a lower probability $\underline{P}$ and $a < 0 < b$ with $a + b < 1$, we aim to prove that $d_{\mathrm{IVBM}}(Q, \underline{P}) \leq 1 \Leftrightarrow Q \in \mathcal{M}(\underline{Q}_{(a,b)}[\underline{P}])$, which is equivalent to the equality $\mathcal{M}(\underline{Q}_{(a,b)}[\underline{P}]) = B^1_{d_{\mathrm{IVBM}}}(\underline{P})$. Indeed, $Q \in \mathcal{M}(\underline{Q}_{(a,b)}[\underline{P}])$ if and only if for any $A \subset \mathcal{X}$ it holds that:

$$\begin{aligned}Q(A) &\geq \underline{Q}_{(a,b)}[\underline{P}](A) = \max\{b\underline{P}(A) + a, 0\} \\ &\Leftrightarrow (\forall A \subset \mathcal{X})\; Q(A) \geq b\underline{P}(A) + a \\ &\Leftrightarrow (\forall A \subset \mathcal{X})\; \underline{P}(A) - Q(A) \leq \underline{P}(A) - b\underline{P}(A) - a \\ &\Leftrightarrow (\forall A \subset \mathcal{X})\; \frac{\underline{P}(A) - Q(A)}{(1-b)\underline{P}(A) - a} \leq 1 \\ &\Leftrightarrow \max_{A \subset \mathcal{X}} \left(\frac{\underline{P}(A) - Q(A)}{(1-b)\underline{P}(A) - a}\right) \leq 1 \\ &\Leftrightarrow d_{\mathrm{IVBM}}(Q, \underline{P}) \leq 1 \Leftrightarrow Q \in B^1_{d_{\mathrm{IVBM}}}(\underline{P}),\end{aligned}$$

where Lemma 5.1 has been used in the fourth equivalence. □

Proof of Proposition 5.5
Given $Q \in \mathbb{P}(\mathcal{X})$, $a < 0 < b$ such that $a + b < 1$ and $A \subseteq \mathcal{X}$, we define $h : [0,1] \to \mathbb{R}$ by:

$$h(t) = \frac{t - Q(A)}{(1-b)t - a}$$

for each $t \in [0,1]$. Then:

$$d(Q, \underline{P}; A) = h(\underline{P}(A)) = h\Big(\min_{P \in \mathcal{M}(\underline{P})} P(A)\Big),$$

since $\underline{P}$ is coherent by hypothesis. We also have that

$$\min_{P \in \mathcal{M}(\underline{P})} h\big(P(A)\big) = \min_{P \in \mathcal{M}(\underline{P})} d(Q, P; A),$$

hence it suffices to prove that

$$h\Big(\min_{P \in \mathcal{M}(\underline{P})} P(A)\Big) = \min_{P \in \mathcal{M}(\underline{P})} h\big(P(A)\big).$$

This holds because h is increasing:

$$\begin{aligned}h(t_1) \leq h(t_2) &\Leftrightarrow (t_1 - Q(A))\big[(1-b)t_2 - a\big] \\ &\qquad \leq (t_2 - Q(A))\big[(1-b)t_1 - a\big] \\ &\Leftrightarrow t_1((1-b)Q(A) - a) \\ &\qquad \leq t_2((1-b)Q(A) - a) \\ &\Leftrightarrow t_1 \leq t_2,\end{aligned}$$

where the first equivalence is due to Lemma 5.1. □

Proof of Corollary 5.1
The proof trivially follows from Proposition 5.5 and Equation (6). □

Proof of Corollary 5.2
The proof trivially follows from [21, Prop.4]. □

Proof of Theorem 5.1
(P10)(Strong commutativity) Let $\underline{P} \in \mathbb{P}(\mathcal{X})$ be a 2-monotone lower probability and $a < 0 < b$ such that $a + b < 1$. We aim to prove that $d_{\mathrm{IVBM}}(Q, \underline{P}) = d'_{\mathrm{IVBM}}(Q, \underline{P})$ for each $Q \in \mathbb{P}(\mathcal{X})$. This would imply, from Proposition 5.4 and Equation (**??**), that:

$$\begin{aligned}\mathcal{M}\Big(\underline{Q}_{(a,b)}[\underline{P}]\Big) &= B^1_{d_{\mathrm{IVBM}}}(\underline{P}) \\ &= B^1_{d'_{\mathrm{IVBM}}}(\underline{P}) = \bigcup_{P \in \mathcal{M}(\underline{P})} \mathcal{M}\Big(\underline{Q}_{(a,b)}[P]\Big).\end{aligned}$$

Since $\underline{P}$ is coherent, d_{IVBM} is given by Equation (8). Then, the equality between $d_{\mathrm{IVBM}}(\cdot, \underline{P})$ and $d'_{\mathrm{IVBM}}(\cdot, \underline{P})$ reduces to:

$$\begin{aligned}&\max_{A \subseteq \mathcal{X}} \min_{P \in \mathcal{M}(\underline{P})} \frac{P(A) - Q(A)}{(1-b)P(A) - a} \\ &\quad = \min_{P \in \mathcal{M}(\underline{P})} \max_{A \subseteq \mathcal{X}} \frac{P(A) - Q(A)}{(1-b)P(A) - a} \quad \forall Q \in \mathbb{P}(\mathcal{X}). \quad (9)\end{aligned}$$

Let $\mathcal{H} = \{g \in \mathcal{L}(\mathcal{X}) \mid g(x) \in [0,1]\ \forall x \in \mathcal{X}\}$, which satisfies $\mathcal{I} := \{I_A \mid A \subseteq \mathcal{X}\} \subset \mathcal{H}$, and consider each probability measure in $\mathbb{P}(\mathcal{X})$ as its unique extension to a linear operator on the set of gambles. For each $Q \in \mathbb{P}(\mathcal{X})$, define the map:

$$\begin{aligned}f_Q : \mathcal{M}(\underline{P}) \times \mathcal{H} &\to \mathbb{R} \\ (P, g) &\hookrightarrow \frac{P(g) - Q(g)}{(1-b)P(g) - a}.\end{aligned}$$

We shall prove that the equality:

$$\max_{g \in \mathcal{H}} \min_{P \in \mathcal{M}(\underline{P})} f_Q(P, g) = \min_{P \in \mathcal{M}(\underline{P})} \max_{g \in \mathcal{H}} f_Q(P, g) \qquad (10)$$

for any $Q \in \mathbb{P}(\mathcal{X})$, is satisfied and that it is equivalent to Equation (9).

In what follows, the two first steps together prove this equivalence, while in the three remaining ones the hypothesis of the minimax theorem in [30, App.E6] are shown to be satisfied, implying that Equation (10) holds.

Step 1 Let us prove that

$$\max_{g\in\mathcal{H}} \min_{P\in\mathcal{M}(\underline{P})} f_Q(P,g) = \max_{A\subseteq\mathcal{X}} \min_{P\in\mathcal{M}(\underline{P})} \frac{P(A)-Q(A)}{(1-b)P(A)-a}.$$

On the one hand, the set of indicator functions $\mathcal{I}$ is included in $\mathcal{H}$ so we straightforwardly obtain:

$$\max_{g\in\mathcal{H}} \min_{P\in\mathcal{M}(\underline{P})} f_Q(P,g) \geq \max_{A\subseteq\mathcal{X}} \min_{P\in\mathcal{M}(\underline{P})} \frac{P(A)-Q(A)}{(1-b)P(A)-a}.$$

For the converse inequality, we use that for every $g \in \mathcal{H}$ there exists a permutation $\sigma : \{1,\dots,n\} \to \{1,\dots,n\}$ such that $g(x_{\sigma(1)}) \geq \dots \geq g(x_{\sigma(n)})$ and g can be written as $g = \sum_{i=1}^n \alpha_i I_{A_i}$, where $A_i = \{x_{\sigma(1)},\dots,x_{\sigma(i)}\}$, $\alpha_i = g(x_{\sigma(i)}) - g(x_{\sigma(i+1)})$ for all $i \in \{1,\dots,n-1\}$, $A_n = \mathcal{X}$ and $\alpha_n = g(x_{\sigma(n)})$. Also every 2-monotone lower probability has a unique 2-monotone extension to gambles given by the Choquet integral. Assume that there exists certain $\varepsilon \geq 0$ such that:

$$\max_{A\subseteq\mathcal{X}} \min_{P\in\mathcal{M}(\underline{P})} \frac{P(A)-Q(A)}{(1-b)P(A)-a} \leq \varepsilon \Leftrightarrow \forall A\subseteq\mathcal{X} \quad \underline{P}(A)-Q(A) \leq \varepsilon\big((1-b)\underline{P}(A)-a\big),$$

which implies that for each $g \in \mathcal{L}(\mathcal{X})$:

$$\begin{aligned}\frac{\underline{P}(g)-Q(g)}{(1-b)\underline{P}(g)-a} &= \frac{\underline{P}\big(\sum_{i=1}^n \alpha_i I_{A_i}\big) - Q\big(\sum_{i=1}^n \alpha_i I_{A_i}\big)}{(1-b)\underline{P}\big(\sum_{i=1}^n \alpha_i I_{A_i}\big)-a} \\ &= \frac{\sum_{i=1}^n \alpha_i(\underline{P}(A_i)-Q(A_i))}{(1-b)\sum_{i=1}^n \alpha_i\underline{P}(A_i)-a} \\ &\leq \varepsilon\frac{\sum_{i=1}^n \alpha_i\big((1-b)\underline{P}(A_i)-a\big)}{(1-b)\sum_{i=1}^n \alpha_i\underline{P}(A_i)-a} \\ &\leq \varepsilon\frac{\sum_{i=1}^n \alpha_i\big((1-b)\underline{P}(A_i)-a\big)}{\sum_{i=1}^n \alpha_i\big((1-b)\underline{P}(A_i)-a\big)} = \varepsilon,\end{aligned}$$

where in the second equality we have used the fact that $\underline{P}(g) = \sum_{i=1}^n \alpha_i\underline{P}(A_i)$ and the last inequality is due to $-a\sum_{i=1}^n \alpha_i \leq -a$, since $\sum_{i=1}^n \alpha_i \in [0,1]$. Then, $\max_{g\in\mathcal{H}} \min_{P\in\mathcal{M}(\underline{P})} f_Q(P,g) \leq \varepsilon$ and we deduce the claimed equality.

Step 2 Let us prove that

$$\min_{P\in\mathcal{M}(\underline{P})} \max_{g\in\mathcal{H}} f_Q(P,g) = \min_{P\in\mathcal{M}(\underline{P})} \max_{A\subseteq\mathcal{X}} \frac{P(A)-Q(A)}{(1-b)P(A)-a}.$$

Similarly to the first equation in the step before, from $\mathcal{I} \subseteq \mathcal{H}$ we get:

$$\min_{P\in\mathcal{M}(\underline{P})} \max_{g\in\mathcal{H}} f_Q(P,g) \geq \min_{P\in\mathcal{M}(\underline{P})} \max_{A\subseteq\mathcal{X}} \frac{P(A)-Q(A)}{(1-b)P(A)-a}.$$

We finally note that for any $\varepsilon \geq 0$:

$$\min_{P\in\mathcal{M}(\underline{P})} \max_{A\subseteq\mathcal{X}} \frac{P(A)-Q(A)}{(1-b)P(A)-a} \leq \varepsilon$$

holds if and only if there exists $P \in \mathcal{M}(\underline{P})$ such that for any $A \subseteq \mathcal{X}$

$$\frac{P(A)-Q(A)}{(1-b)P(A)-a} \leq \varepsilon.$$

For this P, writing $g = \sum_{i=1}^n \alpha_i I_{A_i} \in \mathcal{H}$ as in the previous step, and following the same reasoning we obtain:

$$f_Q(P,g) = \frac{P(g)-Q(g)}{(1-b)P(g)-a} \leq \varepsilon \quad \forall g \in \mathcal{H}.$$

Thus, $\min_{P\in\mathcal{M}(\underline{P})} \max_{g\in\mathcal{H}} f_Q(P,g) \leq \varepsilon$, for every $\varepsilon \geq 0$, and we deduce then the desired equality.

Step 3 $\mathcal{M}(\underline{P})$ and $\mathcal{H}$ trivially are convex and compact sets in $\mathbb{R}^n$, where $n = |\mathcal{X}|$.

Step 4 Let us prove that for any $g \in \mathcal{H}$ and $\mu \in \mathbb{R}$ fixed, $C_{g,\mu} := \{P \in \mathcal{M}(\underline{P}) \mid f_Q(P,g) \leq \mu\}$ is convex and closed. Let $\alpha \in [0,1]$, $P_1, P_2 \in C_{g,\mu}$ and assume in first place that $1-\mu(1-b) > 0$, so that

$$P \in C_{g,\mu} \Leftrightarrow f_Q(P,g) \leq \mu \Leftrightarrow P(g) \leq \frac{Q(g)-\mu a}{1-\mu(1-b)}.$$

Then,

$$\begin{aligned}\alpha P_1(g) + (1-\alpha)P_2(g) &\leq \alpha\frac{Q(g)-\mu a}{1-\mu(1-b)} + (1-\alpha)\frac{Q(g)-\mu a}{1-\mu(1-b)} \\ &= \frac{Q(g)-\mu a}{1-\mu(1-b)},\end{aligned}$$

so $\alpha P_1 + (1-\alpha)P_2 \in C_{g,\mu}$. If, on the other hand, $1-\mu(1-b) < 0$, the same equations hold with the reversed inequalities. Finally, if $1 = \mu(1-b)$, we have $b \neq 1$ and:

$$P \in C_{g,\mu} \Leftrightarrow Q(g) \geq \mu a = \frac{a}{1-b}.$$

Whenever $b < 1$ it holds $a/(1-b) \leq 0$, so $C_{g,\mu} = \mathbb{P}(\mathcal{X})$, hence it is convex. Also, $b > 1$ yields $a/(1-b) = -a/(b-1) > 1$, since $a+b < 1$, concluding that $C_{g,\mu} = \emptyset$, which is convex as well.

To prove that the set is closed, consider a sequence $(P_k)_k \subset C_{g,\mu}$ such that $P_k \to P$ for some $P \in \mathbb{P}(\mathcal{X})$ and let us prove that $P \in C_{g,\mu}$. For this, we firstly note that $(P_k)_k \subseteq \mathcal{M}(\underline{P})$, which is closed, hence $P \in \mathcal{M}(\underline{P})$. Also, $(P_k)_k$ converging to P implies that $P_k(g) \to P(g)$ for every $g \in \mathcal{H}$. When $1-\mu(1-b) > 0$, by taking limits:

$$P_k(g) \leq \frac{Q(g) - \mu a}{1 - \mu(1-b)} \quad \forall k \in \mathbb{N}$$

implies that

$$P(g) \leq \frac{Q(g) - \mu a}{1 - \mu(1-b)}.$$

When $1 - \mu(1-b) < 0$, we obtain the same conclusion with the converse inequalities. Then, $P \in C_{g,\mu}$ whenever $1 \neq (1-b)\mu$ and we conclude that $C_{g,\mu}$ is closed in that case. When instead $1 = (1-b)\mu$, this set coincides with either $\mathbb{P}(\mathcal{X})$ or $\emptyset$, as already shown, which are closed.

Step 5 Let us prove that for any $P \in \mathcal{M}(\underline{P})$ and $\mu \in \mathbb{R}$ fixed, $S_{P,\mu} := \{g \in \mathcal{H} \mid f_Q(P,g) \geq \mu\}$ is convex and closed.

Let $\alpha \in [0,1]$ and $g_1, g_2 \in S_{P,\mu}$. Since

$$f_Q(P,g) \geq \mu \Leftrightarrow P(g) - Q(g) \geq \mu\big((1-b)P(g) - a\big),$$

we get

$$\begin{aligned}
&P(\alpha g_1 + (1-\alpha)g_2) - Q(\alpha g_1 + (1-\alpha)g_2)\\
&\quad = \alpha(P(g_1) - Q(g_1)) + (1-\alpha)(P(g_2) - Q(g_2))\\
&\quad \geq \mu\big(\alpha\big((1-b)P(g_1) - a\big)\\
&\qquad + (1-\alpha)\big((1-b)P(g_2) - a\big)\big)\\
&\quad = \mu\big((1-b)P(\alpha g_1 + (1-\alpha)g_2) - a\big),
\end{aligned}$$

so $\alpha g_1 + (1-\alpha)g_2 \in S_{P,\mu}$ and this is a convex set.

Finally, consider a sequence $(g_k)_k \subset S_{P,\mu}$ such that $g_k \to g$ for certain $g \in \mathcal{L}(\mathcal{X})$. As stated in Step 3, $\mathcal{H}$ is closed and we deduce that the limit g must belong to $\mathcal{H}$ as well. We conclude by noting, from P, Q being continuous and taking limits, that:

$$P(g_k) - Q(g_k) \geq \mu\big((1-b)P(g_k) - a\big) \quad \forall k \in \mathbb{N}$$

implying

$$P(g) - Q(g) \geq \mu\big((1-b)P(g) - a\big)$$

so $g \in S_{P,\mu}$ and this is a closed set. $\square$

Proof of Proposition 6.1

(P1)(Expansion) For every $A \subset \mathcal{X}$ such that $\underline{P}(A) = 0$ it clearly holds $\underline{Q}^{\mathrm{LV}}_{\delta_1}[\underline{P}](A) = \underline{Q}^{\mathrm{LV}}_{\delta_2}[\underline{P}](A)$ because $\phi^{\mathrm{LV}}_{\delta}(0) = 0$ for each $\delta \in [0,1)$. On the other hand, whenever $\underline{P}(A) \neq 0$ we have:

$$\begin{aligned}
\underline{Q}^{\mathrm{LV}}_{\delta_1}[\underline{P}](A) \leq \underline{Q}^{\mathrm{LV}}_{\delta_2}[\underline{P}](A) &\Leftrightarrow \phi_{\delta_1}(\underline{P}(A)) \leq \phi_{\delta_2}(\underline{P}(A))\\
&\Leftrightarrow (1-\delta_1)\underline{P}(A) \leq (1-\delta_2)\underline{P}(A) \Leftrightarrow \delta_1 \geq \delta_2.
\end{aligned}$$

We conclude that $\delta_1 \geq \delta_2$ is equivalent to $\underline{Q}^{\mathrm{LV}}_{\delta_1}[\underline{P}] \leq \underline{Q}^{\mathrm{LV}}_{\delta_2}[\underline{P}]$.

(P2)(a)(Semigroup) For $\delta = 0$, $\phi^{\mathrm{LV}}_{\delta} = \phi_0$ is the identity function, hence $\underline{Q}^{\mathrm{LV}}_{0}[\underline{P}](A) = (\phi_0 \circ \underline{P})(A) = \underline{P}(A)$ for each $A \subseteq \mathcal{X}$.

(P2)(b)(Semigroup) does not hold in general, since $\delta_1, \delta_2 \in [0,1)$ and $A \subset \mathcal{X}$ yield:

$$\begin{aligned}
\underline{Q}^{\mathrm{LV}}_{\delta_1}\big[\underline{Q}^{\mathrm{LV}}_{\delta_2}[\underline{P}]\big](A) &= (1-\delta_1)(1-\delta_2)\underline{P}(A)\\
&= (1-\delta_1-\delta_2+\delta_1\delta_2)\underline{P}(A),
\end{aligned}$$

while, if $\delta_1 + \delta_2 \in [0,1)$, we obtain

$$\underline{Q}^{\mathrm{LV}}_{\delta_1+\delta_2}[\underline{P}](A) = (1-\delta_1-\delta_2)\underline{P}(A).$$

Thus, (P2)(b) holds iff either $\delta_1 = 0$ or $\delta_2 = 0$.

(P3)(k-monotonicity) Let $\underline{P}$ be a k-monotone lower probability, for some natural $k \geq 2$, and let $A_1, \dots, A_p \subset \mathcal{X}$ be a collection of p events, with $1 \leq p \leq k$. Assume that there exists some $i^* \in \{1, \dots, p\}$ such that $A_{i^*} = \mathcal{X}$, hence $\cup_{i=1}^{p} A_i = \mathcal{X}$. Then,

$$\underline{Q}^{\mathrm{LV}}_{\delta}[\underline{P}]\big(\cup_{i=1}^{p} A_i\big) = \underline{Q}^{\mathrm{LV}}_{\delta}[\underline{P}](\mathcal{X}) = 1$$

and

$$\begin{aligned}
&\sum_{\emptyset \neq I \subseteq \{1,\dots,p\}} (-1)^{|I|+1} \underline{Q}^{\mathrm{LV}}_{\delta}[\underline{P}]\big(\cap_{i \in I} A_i\big)\\
&= \underline{Q}^{\mathrm{LV}}_{\delta}[\underline{P}](A_{i*})\\
&\quad + \sum_{\substack{\emptyset \neq I \subseteq \{1,\dots,p\}\\ i^* \notin I}} (-1)^{|I|+1} \underline{Q}^{\mathrm{LV}}_{\delta}[\underline{P}]\big(\cap_{i \in I} A_i\big)\\
&\quad + \sum_{\substack{\emptyset \neq I' \subseteq \{1,\dots,p\}\\ i^* \in I', I' \neq \{i^*\}}} (-1)^{|I'|+1} \underline{Q}^{\mathrm{LV}}_{\delta}[\underline{P}]\big(\cap_{i \in I'} A_i\big)\\
&= 1 + \sum_{\substack{\emptyset \neq I \subseteq \{1,\dots,p\}\\ i^* \notin I}} (-1)^{|I|+1} \underline{Q}^{\mathrm{LV}}_{\delta}[\underline{P}]\big(\cap_{i \in I} A_i\big)\\
&\quad + \sum_{\substack{\emptyset \neq I \subseteq \{1,\dots,p\}\\ i^* \notin I}} (-1)^{|I|+2} \underline{Q}^{\mathrm{LV}}_{\delta}[\underline{P}]\big(\cap_{i \in I} A_i\big)\\
&= 1+
\end{aligned}$$

$$\sum_{\substack{\emptyset\neq I\subseteq\{1,\dots,p\}\\ i^*\notin I}} ((-1)^{|I|+1}-(-1)^{|I|+1})\underline{Q}^{\mathrm{LV}}_{\delta}[\underline{P}](\cap_{i\in I}A_i)=1,$$

so they coincide.

On the other hand, if $\cup_{i=1}^{p}A_i=\mathcal{X}$ and $A_i\neq\mathcal{X}$ for any $i\in\{1,\dots,p\}$, then $\cap_{i\in I}A_i\neq\mathcal{X}$ for any $\emptyset\neq I\subseteq\{1,\dots,p\}$ and we deduce

$$\begin{aligned}
&\underline{Q}^{\mathrm{LV}}_{\delta}[\underline{P}](\cup_{i=1}^{p}A_i)\\
&\quad=\underline{Q}^{\mathrm{LV}}_{\delta}[\underline{P}](\mathcal{X})=1\geq(1-\delta)=(1-\delta)\underline{P}(\cup_{i=1}^{p}A_i)\\
&\quad\geq(1-\delta)\sum_{\emptyset\neq I\subseteq\{1,\dots,p\}}(-1)^{|I|+1}\underline{P}(\cap_{i\in I}A_i)\\
&\quad=\sum_{\emptyset\neq I\subseteq\{1,\dots,p\}}(-1)^{|I|+1}\underline{Q}^{\mathrm{LV}}_{\delta}[\underline{P}](\cap_{i\in I}A_i).
\end{aligned}$$

Finally, if $\cup_{i=1}^{p}A_i\neq\mathcal{X}$, hence $\cap_{i\in I}A_i\neq\mathcal{X}$ for any $\emptyset\neq I\subseteq\{1,\dots,p\}$, and we have that

$$\begin{aligned}
\underline{Q}^{\mathrm{LV}}_{\delta}[\underline{P}](\cup_{i=1}^{p}A_i)&=(1-\delta)\underline{P}(\cup_{i=1}^{p}A_i)\\
&\geq(1-\delta)\sum_{\emptyset\neq I\subseteq\{1,\dots,p\}}(-1)^{|I|+1}\underline{P}(\cap_{i\in I}A_i)\\
&=\sum_{\emptyset\neq I\subseteq\{1,\dots,p\}}(-1)^{|I|+1}\underline{Q}^{\mathrm{LV}}_{\delta}[\underline{P}](\cap_{i\in I}A_i),
\end{aligned}$$

hence we conclude that $\underline{Q}^{\mathrm{LV}}_{\delta}[\underline{P}]$ is k-monotone. □

Proof of Proposition 6.2

(P8)(Expression as an imprecise neighbourhood model) Given $\underline{P}\in\underline{\mathbb{P}}(\mathcal{X})$, for each $Q\in\mathbb{P}(\mathcal{X})$ and $\delta\in[0,1)$, we aim to prove that $d_{\mathrm{ILV}}(Q,\underline{P})\leq\delta$ if and only if $Q\in\mathcal{M}(\underline{Q}^{\mathrm{LV}}_{\delta}[\underline{P}])$, which is equivalent to the equality $B^{\delta}_{d_{\mathrm{ILV}}}(\underline{P})=\mathcal{M}(\underline{Q}^{\mathrm{LV}}_{\delta}[\underline{P}])$. From Equation (**??**) we have that:

$$\begin{aligned}
Q\in&B^{\delta}_{d_{\mathrm{ILV}}}(\underline{P})\Leftrightarrow d_{\mathrm{ILV}}(Q,\underline{P})\leq\delta\\
&\Leftrightarrow(\forall A\neq\emptyset\mid\underline{P}(A)>0,\ \underline{P}(A)\geq Q(A))\\
&\qquad\frac{\underline{P}(A)-Q(A)}{\underline{P}(A)}\leq\delta\\
&\Leftrightarrow(\forall A\subset\mathcal{X})\\
&\qquad Q(A)\geq\underline{Q}^{\mathrm{LV}}_{\delta}[\underline{P}](A)=(1-\delta)\underline{P}(A)\\
&\Leftrightarrow Q\in\mathcal{M}\Big(\underline{Q}^{\mathrm{LV}}_{\delta}[\underline{P}]\Big),
\end{aligned}$$

where the last but one equivalence is owe to the fact that if $Q(A)>\underline{P}(A)$ or $\underline{P}(A)=0$ for some $A\subseteq\mathcal{X}$, then it holds $Q(A)\geq\underline{P}(A)\geq\underline{Q}^{\mathrm{LV}}_{\delta}[\underline{P}](A)$, using the expansion property (P1) of the ILV. □

Proof of Proposition 6.3

Let $\underline{P}\in\underline{\mathbb{P}}(\mathcal{X})$ be a coherent lower probability. Given $Q\in\mathbb{P}(\mathcal{X})$ and $\delta\in[0,1]$, for each $A\subseteq\mathcal{X}$ such that $\underline{P}(A)>0$, define $h:(0,1]\to\mathbb{R}$ by:

$$h(t)=\frac{t-Q(A)}{t},$$

which is non-decreasing for each $t\in(0,1]$. Thus, for any $A\subseteq\mathcal{X}$ for which $\underline{P}(A)>0$, the coherence of $\underline{P}$ yields:

$$\begin{aligned}
h(\underline{P}(A))&=h\Big(\min_{P\in\mathcal{M}(\underline{P})}P(A)\Big)=\min_{P\in\mathcal{M}(\underline{P})}h(P(A))\\
&=\min_{P\in\mathcal{M}(\underline{P})}d_{\mathrm{ILV}}(Q,P;A),
\end{aligned}$$

where the second equality is a consequence of monotonicity of h. Finally, from Equation (**??**) we get:

$$\begin{aligned}
d_{\mathrm{ILV}}(Q,\underline{P})&=\max_{A\mid\underline{P}(A)>0}h(\underline{P}(A))\\
&=\max_{A\mid\underline{P}(A)>0}\min_{P\in\mathcal{M}(\underline{P})}d_{\mathrm{ILV}}(Q,P;A),
\end{aligned}$$

and we obtain the desired equality. □

Proof of Theorem 6.1

Let $\underline{P}\in\underline{\mathbb{P}}(\mathcal{X})$ be 2-monotone, and let us prove that the ILV satisfies strong commutativity. For every given $Q\in\mathbb{P}(\mathcal{X})$, we aim to prove that $d_{\mathrm{ILV}}(Q,\underline{P})=d'_{\mathrm{ILV}}(Q,\underline{P})$. This will imply that:

$$\begin{aligned}
\mathcal{M}\Big(\underline{Q}^{\mathrm{LV}}_{\delta}[\underline{P}]\Big)&=B^{\delta}_{d_{\mathrm{ILV}}}(\underline{P})=B^{\delta}_{d'_{\mathrm{ILV}}}(\underline{P})\\
&=\bigcup_{P\in\mathcal{M}(\underline{P})}\mathcal{M}\Big(\underline{Q}^{\mathrm{LV}}_{\delta}[P]\Big),
\end{aligned}$$

for each $\delta\in[0,1]$, so ILVs satisfy strong commutativity (P10).

Since $\underline{P}$ is coherent, d_{ILV} is given by Equation (**??**). Then, the equality between $d_{\mathrm{ILV}}(\cdot,\underline{P})$ and $d'_{\mathrm{ILV}}(\cdot,\underline{P})$ reduces to verifying:

$$\begin{aligned}
&\max_{A\mid\underline{P}(A)>0}\min_{P\in\mathcal{M}(\underline{P})}\frac{P(A)-Q(A)}{P(A)}\\
&\qquad=\min_{P\in\mathcal{M}(\underline{P})}\max_{A\mid\underline{P}(A)>0}\frac{P(A)-Q(A)}{P(A)},
\end{aligned}\tag{11}$$

for each $Q\in\mathbb{P}(\mathcal{X})$. Let

$$\begin{aligned}
\varepsilon&=\min\{\underline{P}(A)\mid A\subseteq\mathcal{X},\ \underline{P}(A)>0\},\\
\mathcal{H}&=\{g\in\mathcal{L}(\mathcal{X})\mid 0\leq g(x)\leq 1\ \forall x\in\mathcal{X}\},\text{ and}\\
\mathcal{H}_{\varepsilon}(\underline{P})&=\{g\in\mathcal{H}\mid\underline{P}(g)\geq\varepsilon\},
\end{aligned}$$

which satisfies $\mathcal{I}^*=\{I_A\mid A\subseteq\mathcal{X},\ \underline{P}(A)>0\}\subset\mathcal{H}_{\varepsilon}(\underline{P})$. For each $Q\in\mathbb{P}(\mathcal{X})$, define the map:

$$\begin{aligned}
f_Q:\mathcal{M}(\underline{P})\times\mathcal{H}_{\varepsilon}(\underline{P})&\to\mathbb{R}\\
(P,g)&\hookrightarrow\frac{P(g)-Q(g)}{P(g)},
\end{aligned}$$

where $P(g), Q(g)$ stand for the expectation operators associated with P, Q. We note that the previous map is well defined, since $P(g) \geq \underline{P}(g) > 0$ for each $P \in \mathcal{M}(\underline{P})$ and $g \in \mathcal{H}_\varepsilon(\underline{P})$. We shall prove that the equality:

$$\max_{g\in\mathcal{H}_\varepsilon(\underline{P})} \min_{P\in\mathcal{M}(\underline{P})} f_Q(P,g) = \min_{P\in\mathcal{M}(\underline{P})} \max_{g\in\mathcal{H}_\varepsilon(\underline{P})} f_Q(P,g) \quad \forall Q \in \mathbb{P}(\mathcal{X}) \quad (12)$$

is equivalent to Equation (11) and that they are satisfied. We note that the minima and maxima in the previous equation are indeed attained, since f_Q is continuous (the denominator does not vanish and the function is hence the composition of continuous functions) and $\mathcal{M}(\underline{P}), \mathcal{H}_\varepsilon(\underline{P})$ are compact sets. The latter set being closed follows from [30, Sec.2.6.1(l)], which allows us to deduce, for any $(g_k)_k \subset \mathcal{H}_\varepsilon(\underline{P})$ with $(g_k)_k \to g \in \mathcal{L}(\mathcal{X})$ and by taking limits, that:

$$\underline{P}(g_k) \geq \varepsilon \quad \forall k \in \mathbb{N} \Rightarrow \underline{P}(g) \geq \varepsilon, \quad (13)$$

since pointwise convergence is equivalent to uniform convergence in the finite dimensional setting that we are considering, and the lower prevision $\underline{P}$ is coherent. Also, $(g_k)_k \subset \mathcal{H}$ implies $g \in \mathcal{H}$, since $\mathcal{H}$ is compact. Then, $g \in \mathcal{H}_\varepsilon(\underline{P})$.

The following two first steps together prove the claimed equivalence, while the three remaining ones check the verification of the hypothesis of the minimax theorem in [30, App.E6], implying that Equation (12) holds.

Step 1 Let us prove that

$$\max_{g\in\mathcal{H}_\varepsilon(\underline{P})} \min_{P\in\mathcal{M}(\underline{P})} f_Q(P,g) = \max_{A|\underline{P}(A)>0} \min_{P\in\mathcal{M}(\underline{P})} \frac{P(A)-Q(A)}{P(A)}.$$

On the one hand, the set of indicator functions $\mathcal{I}^* = \{I_A \mid A \subseteq \mathcal{X},\ \underline{P}(A) > 0\}$ is included in $\mathcal{H}_\varepsilon(\underline{P})$ so we straightforwardly obtain:

$$\max_{g\in\mathcal{H}_\varepsilon(\underline{P})} \min_{P\in\mathcal{M}(\underline{P})} f_Q(P,g) \geq \max_{A|\underline{P}(A)>0} \min_{P\in\mathcal{M}(\underline{P})} \frac{P(A)-Q(A)}{P(A)}.$$

For the converse inequality, we use that for every $g \in \mathcal{H}_\varepsilon(\underline{P})$ there exists a permutation $\sigma : \{1,\dots,n\} \to \{1,\dots,n\}$ such that $g(x_{\sigma(1)}) \geq \dots \geq g(x_{\sigma(n)})$ and g can be written as $g = \sum_{i=1}^n \alpha_i I_{A_i}$, where $A_i = \{x_{\sigma(1)},\dots,x_{\sigma(i)}\}$, $\alpha_i = g(x_{\sigma(i)}) - g(x_{\sigma(i+1)})$ for all $i \in \{1,\dots,n-1\}$, $A_n = \mathcal{X}$ and $\alpha_n = g(x_{\sigma(n)})$. Also every 2-monotone lower probability has a unique 2-monotone extension to gambles given by the Choquet integral. Denoting

$$\delta = \max_{A|\underline{P}(A)>0} \min_{P\in\mathcal{M}(\underline{P})} \frac{P(A)-Q(A)}{P(A)},$$

the coherence of $\underline{P}$ implies that:

$$\underline{P}(A) - Q(A) \leq \delta \underline{P}(A) \quad \forall A \subseteq \mathcal{X},$$

since the inequality is trivially satisfied for any event with null lower probability. This implies that for each $g \in \mathcal{H}_\varepsilon(\underline{P})$:

$$\begin{aligned}
\min_{P\in\mathcal{M}(\underline{P})} \frac{P(g)-Q(g)}{P(g)} &= \frac{\underline{P}(g)-Q(g)}{\underline{P}(g)} \\
&= \frac{\underline{P}\big(\sum_{i=1}^n \alpha_i I_{A_i}\big) - Q\big(\sum_{i=1}^n \alpha_i I_{A_i}\big)}{\underline{P}\big(\sum_{i=1}^n \alpha_i I_{A_i}\big)} \\
&= \frac{\sum_{i=1}^n \alpha_i(\underline{P}(A_i) - Q(A_i))}{\sum_{i=1}^n \alpha_i \underline{P}(A_i)} \\
&\leq \delta \frac{\sum_{i=1}^n \alpha_i \underline{P}(A_i)}{\sum_{i=1}^n \alpha_i \underline{P}(A_i)} = \delta \\
&= \max_{A|\underline{P}(A)>0} \min_{P\in\mathcal{M}(\underline{P})} \frac{P(A)-Q(A)}{P(A)},
\end{aligned}$$

where in the second equality we use the fact that $\underline{P}(g) = \sum_{i=1}^n \alpha_i \underline{P}(A_i)$, which holds because $\underline{P}$ is 2-monotone, and the linearity of Q. Then, we have obtained the converse inequality and deduce thus the claimed equality.

Step 2 Let us prove that

$$\min_{P\in\mathcal{M}(\underline{P})} \max_{g\in\mathcal{H}_\varepsilon(\underline{P})} f_Q(P,g) = \min_{P\in\mathcal{M}(\underline{P})} \max_{A|\underline{P}(A)>0} \frac{P(A)-Q(A)}{P(A)}.$$

Similarly to the first inequality in the step before, from $\mathcal{I}^* \subseteq \mathcal{H}_\varepsilon(\underline{P})$ we get:

$$\min_{P\in\mathcal{M}(\underline{P})} \max_{g\in\mathcal{H}_\varepsilon(\underline{P})} f_Q(P,g) \geq \min_{P\in\mathcal{M}(\underline{P})} \max_{A|\underline{P}(A)>0} \frac{P(A)-Q(A)}{P(A)}.$$

Reciprocally, denote

$$\delta = \min_{P\in\mathcal{M}(\underline{P})} \max_{A|\underline{P}(A)>0} \frac{P(A)-Q(A)}{P(A)},$$

so there exists some $P \in \mathcal{M}(\underline{P})$ satisfying

$$P(A) - Q(A) \leq \delta P(A) \quad \forall A \subseteq \mathcal{X},$$

since this is trivially verified for any subset with null probability. For that probability measure P, writing $g = \sum_{i=1}^n \alpha_i I_{A_i} \in \mathcal{H}_\varepsilon(\underline{P})$ as in the previous step and following the same reasoning, we obtain:

$$f_Q(P,g) = \frac{P(g)-Q(g)}{P(g)} \leq \delta$$
$$= \min_{P\in\mathcal{M}(\underline{P})} \max_{A|\underline{P}(A)>0} \frac{P(A)-Q(A)}{P(A)} \quad \forall g \in \mathcal{H}_\varepsilon(\underline{P}),$$

from where we obtain the converse inequality and deduce then the desired equality.

Step 3 Let us prove that $\mathcal{M}(\underline{P})$ and $\mathcal{H}_\varepsilon(\underline{P})$ are convex and compact sets in $\mathbb{R}^n$, where $n = |\mathcal{X}|$. We already know that the credal set $\mathcal{M}(\underline{P})$ is so and, from Equation (13) $\mathcal{H}_\varepsilon(\underline{P})$ is closed. Letting $g_1, g_2 \in \mathcal{H}_\varepsilon(\underline{P})$ and $\alpha \in [0,1]$, it is clear that $\alpha g_1 + (1-\alpha)g_2 \in \mathcal{H}$, because this is a convex set. Also, a coherent lower prevision $\underline{P}$ is concave [30, Sec.2.6.1(g)], hence:

$$\underline{P}\big(\alpha g_1 + (1-\alpha)g_2\big) \geq \alpha\underline{P}(g_1) + (1-\alpha)\underline{P}(g_2) \geq \varepsilon$$

and we deduce that $\alpha g_1 + (1-\alpha)g_2 \in \mathcal{H}_\varepsilon(\underline{P})$ and therefore that this is a convex set.

Step 4 Let us prove that for any $g \in \mathcal{H}_\varepsilon(\underline{P})$ and $\mu \in \mathbb{R}$ fixed, $C_{g,\mu} := \{P \in \mathcal{M}(\underline{P}) \mid f_Q(P,g) \leq \mu\}$ is convex and closed. Let $\alpha \in [0,1]$, $P_1, P_2 \in C_{g,\mu}$. If $\mu < 1$, we have that

$$P \in C_{g,\mu} \Leftrightarrow f_Q(P,g) \leq \mu \Leftrightarrow P(g) \leq \frac{Q(g)}{1-\mu}.$$

Then,

$$\alpha P_1(g) + (1-\alpha)P_2(g)$$
$$\leq \alpha\frac{Q(g)}{1-\mu} + (1-\alpha)\frac{Q(g)}{1-\mu} = \frac{Q(g)}{1-\mu},$$

so $\alpha P_1 + (1-\alpha)P_2 \in C_{g,\mu}$. When, instead, $\mu \geq 1$, we have:

$$P \in C_{g,\mu} \Leftrightarrow f_Q(P,g) \leq \mu \Leftrightarrow (1-\mu)P(g) \leq Q(g),$$

since $(1-\mu)P(g) \leq 0$, the latter inequality is trivially satisfied, hence $C_{g,\mu} = \mathbb{P}(\mathcal{X})$, which is convex.

To prove that the set is closed, consider a sequence $(P_k)_k \subset C_{g,\mu}$ such that $P_k \to P$ for some $P \in \mathbb{P}(\mathcal{X})$ and let us prove that $P \in C_{g,\mu}$. For this, we firstly note that $(P_k)_k \subseteq \mathcal{M}(\underline{P})$, which is closed, hence $P \in \mathcal{M}(\underline{P})$. When $\mu < 1$, by taking limits:

$$P_k(g) \leq \frac{Q(g)}{1-\mu} \quad \forall k \in \mathbb{N} \Rightarrow P(g) \leq \frac{Q(g)}{1-\mu}.$$

Then, $P \in C_{g,\mu}$ and we conclude that $C_{g,\mu}$ is closed. In the case $\mu \geq 1$, the set is $\mathbb{P}(\mathcal{X})$, which is closed as well.

Step 5 Let us prove that for any $P \in \mathcal{M}(\underline{P})$ and $\mu \in \mathbb{R}$ fixed, $S_{P,\mu} := \{g \in \mathcal{H}_\varepsilon(\underline{P}) \mid f_Q(P,g) \geq \mu\}$ is convex and closed.

Let $\alpha \in [0,1]$ and $g_1, g_2 \in S_{P,\mu}$. Since

$$f_Q(P,g) \geq \mu \Leftrightarrow P(g) - Q(g) \geq \mu P(g),$$

we get

$$P(\alpha g_1 + (1-\alpha)g_2) - Q(\alpha g_1 + (1-\alpha)g_2)$$
$$= \alpha(P(g_1) - Q(g_1)) + (1-\alpha)(P(g_2) - Q(g_2))$$
$$\geq \mu\big(\alpha\delta P(g_1) + (1-\alpha)\delta P(g_2)\big)$$
$$= \mu P(\alpha g_1 + (1-\alpha)g_2),$$

so $\alpha g_1 + (1-\alpha)g_2 \in S_{P,\mu}$ and this is a convex set.

Finally, consider a sequence $(g_k)_k \subset S_{P,\mu}$ such that $g_k \to g$ for some $g \in \mathcal{L}(\mathcal{X})$. As stated in Step 3, $\mathcal{H}_\varepsilon(\underline{P})$ is closed and we deduce that the limit g must belong to $\mathcal{H}_\varepsilon(\underline{P})$ as well. We conclude by noting, from P, Q being continuous and taking limits, that:

$$P(g_k) - Q(g_k) \geq \mu P(g_k) \quad \forall k \in \mathbb{N}$$
$$\Rightarrow P(g) - Q(g) \geq \mu P(g)$$

so $g \in S_{P,\mu}$ and this is a closed set.

Without the hypothesis of 2-monotonicity, it follows from Example A.1 below that strong commutativity does not hold. $\square$

Example A.1. Consider the lower prevision $\underline{P}$ on $\mathcal{X} = \{x_1, x_2, x_3, x_4, x_5\}$:

A	$\underline{P}(A)$	$\underline{Q}(A)$	$P(A)$
$\{x_1\}$	$2/9.9$	0.2	0.2
$\{x_2\}$	$1/9.9$	0.1	0.2
$\{x_3\}$	$1/9.9$	0.2	0.2
$\{x_4\}$	$1/9.9$	0.2	0.2
$\{x_5\}$	$0.5/9.9$	0.05	0.2
$\{x_1,x_2\}$	$3.5/9.9$	0.35	0.4
$\{x_1,x_3\}$	$3.6/9.9$	0.36	0.4
$\{x_1,x_4\}$	$3/9.9$	0.3	0.4
$\{x_1,x_5\}$	$4/9.9$	0.4	0.4
$\{x_2,x_3\}$	$3/9.9$	0.3	0.4
$\{x_2,x_4\}$	$1.5/9.9$	0.15	0.4
$\{x_2,x_5\}$	$2/9.9$	0.2	0.4
$\{x_3,x_4\}$	$3/9.9$	0.3	0.4
$\{x_3,x_5\}$	$2/9.9$	0.2	0.4
$\{x_4,x_5\}$	$2/9.9$	0.2	0.4
$\{x_1,x_2,x_3\}$	$6/9.9$	0.6	0.6
$\{x_1,x_2,x_4\}$	$4/9.9$	0.4	0.6
$\{x_1,x_2,x_5\}$	$5/9.9$	0.5	0.6
$\{x_1,x_3,x_4\}$	$6/9.9$	0.6	0.6
$\{x_1,x_3,x_5\}$	$5/9.9$	0.5	0.6
$\{x_1,x_4,x_5\}$	$4.5/9.9$	0.45	0.6
$\{x_2,x_3,x_4\}$	$4.5/9.9$	0.45	0.6
$\{x_2,x_3,x_5\}$	$4.1/9.9$	0.41	0.6
$\{x_2,x_4,x_5\}$	$3.5/9.9$	0.35	0.6
$\{x_3,x_4,x_5\}$	$4/9.9$	0.4	0.6
$\{x_1,x_2,x_3,x_4\}$	$7/9.9$	0.7	0.8
$\{x_1,x_2,x_3,x_5\}$	$7.5/9.9$	0.75	0.8
$\{x_1,x_2,x_4,x_5\}$	$35/36\cdot 9.9$	$35/36$	0.8
$\{x_1,x_3,x_4,x_5\}$	$7/9.9$	0.7	0.8
$\{x_2,x_3,x_4,x_5\}$	$6.5/9.9$	0.65	0.8
$\mathcal{X}$	1	1	1

It can be checked that $\underline{P}$ is coherent and that $\underline{Q} = \underline{Q}^{\text{LV}}_{0.01}[\underline{P}]$. Moreover, $P \geq \underline{Q}$ but there is no $\underline{P}' \geq \underline{P}$ such that $P \in \mathcal{M}(\underline{Q}^{\text{LV}}_{0.01}[\underline{P}'])$: if there were, it should be $\underline{P}'(A) = \underline{P}(A)$ for $A = \{x_1\},\{x_1,x_5\},\{x_1,x_2,x_3\},\{x_1,x_3,x_4\},\mathcal{X}$, from which it follows that $\underline{P}'$ would be given by the mass function $(2/9.9, 1.9/9.9, 2.1/9.9, 1.9/9.9, 2/9.9)$. However, $P(\{x_3\}) < 0.99P'(\{x_3\})$. Therefore, strong commutativity does not hold. ♦

Proof of Proposition 7.1

(P1)(Expansion) Let $\delta_1 \geq \delta_2 \geq 0$. For each $t \in [0,1]$, we have:

$$(1+\delta_2)t - \delta_2 \geq (1+\delta_1)t - \delta_1 \Leftrightarrow \delta_2(t-1) \geq \delta_1(t-1),$$

which holds if and only if $t = 1$ or $\delta_2 \leq \delta_1$, since $t < 1$ otherwise. In particular, for every $A \subseteq \mathcal{X}$ it holds:

$$\begin{aligned}\underline{Q}^{\text{PMM}}_{\delta_1}[\underline{P}](A) &= \phi^{\text{PMM}}_{\delta_1}(\underline{P}(A)) \\ &= \max\{(1+\delta_1)\underline{P}(A) - \delta_1, 0\} \\ &\leq \max\{(1+\delta_2)\underline{P}(A) - \delta_2, 0\} \\ &= \phi^{\text{PMM}}_{\delta_2}(\underline{P}(A)) = \underline{Q}^{\text{PMM}}_{\delta_2}[\underline{P}](A).\end{aligned}$$

(P2)(a)(Semigroup) For $\delta = 0$, $\phi^{\text{PMM}}_{\delta} = \phi_0$ is the identity function, hence $\underline{Q}^{\text{PMM}}_{0}[\underline{P}](A) = (\phi_0 \circ \underline{P})(A) = \underline{P}(A)$ for each $A \subseteq \mathcal{X}$.

(P2)(b)(Semigroup) It does not hold in general. Indeed, given $\underline{P} \in \underline{\mathbb{P}}^*(\mathcal{X})$, for every $\delta_1, \delta_2 \geq 0$ such that $\underline{Q}^{\text{PM}}_{\delta_1+\delta_2}[\underline{P}](A) > 0$ for every $\emptyset \neq A \subset \mathcal{X}$ (whence from expansion (P1) $\underline{Q}^{\text{PM}}_{\delta_1}[\underline{P}](A), \underline{Q}^{\text{PM}}_{\delta_2}[\underline{P}](A) > 0$), we have:

$$\begin{aligned}\underline{Q}^{\text{PM}}_{\delta_1}[\underline{Q}^{\text{PM}}_{\delta_2}[\underline{P}]](A) &= (1+\delta_1)((1+\delta_2)\underline{P}(A) - \delta_2) - \delta_1 \\ &= (1+\delta_1+\delta_2+\delta_1\delta_2)\underline{P}(A) - \delta_1 - \delta_2 - \delta_1\delta_2.\end{aligned}$$

On the other hand,

$$\underline{Q}^{\text{PM}}_{\delta_1+\delta_2}[\underline{P}](A) = (1+\delta_1+\delta_2)\underline{P}(A) - \delta_1 - \delta_2,$$

so (P2)(b) is not verified unless $\delta_1 = 0$ or $\delta_2 = 0$.

(P3)(k-monotonicity) Let $\underline{P}$ be a k-monotone lower probability (equivalently, $\overline{P}$ is k-alternating), for some natural $k \geq 2$, and such that $(1+\delta)\overline{P}(A) \leq 1$ for every $A \subset \mathcal{X}$. Let us prove that $\overline{Q}^{\text{PMM}}_{\delta}[\underline{P}]$ is k-alternating as well. Letting $A_1, \dots, A_p \subseteq \mathcal{X}$ be a collection of p events, with $1 \leq p \leq k$, we have that:

$$\begin{aligned}\overline{Q}^{\text{PMM}}_{\delta}[\underline{P}](\cap_{i=1}^p A_i) &= \min\{(1+\delta)\overline{P}(\cap_{i=1}^p A_i), 1\} \\ &= (1+\delta)\overline{P}(\cap_{i=1}^p A_i) \\ &\leq (1+\delta)\sum_{\emptyset \neq I \subseteq \{1,\dots,p\}} (-1)^{|I|+1}\overline{P}(\cup_{i\in I} A_i) \\ &= \sum_{\emptyset \neq I \subseteq \{1,\dots,p\}} (-1)^{|I|+1}\overline{Q}^{\text{PMM}}_{\delta}[\underline{P}](\cup_{i\in I} A_i),\end{aligned}$$

where the inequality holds because $\overline{P}$ is k-alternating and the last equality is satisfied by hypothesis. We conclude that $\overline{Q}^{\text{PMM}}_{\delta}[\underline{P}]$ is k-alternating, and therefore $\underline{Q}^{\text{PMM}}_{\delta}[\underline{P}]$ is k-monotone. □

Proof of Proposition 7.2

(P8)(Expression as an imprecise neighbourhood model) Given $\underline{P} \in \underline{\mathbb{P}}(\mathcal{X})$, for each $Q \in \mathbb{P}(\mathcal{X})$ and $\delta \geq 0$, we aim to prove that $d_{\text{IPMM}}(Q,\underline{P}) \leq \delta \Leftrightarrow Q \in \mathcal{M}(\underline{Q}^{\text{PMM}}_{\delta}[\underline{P}])$, which is equivalent to the equality $B^{\delta}_{d_{\text{IPMM}}}(\underline{P}) = \mathcal{M}(\underline{Q}^{\text{PMM}}_{\delta}[\underline{P}])$. We have, from Equation (**??**), that:

$$\begin{aligned}&Q \in B^{\delta}_{d_{\text{IPMM}}}(\underline{P}) \Leftrightarrow d_{\text{IPMM}}(Q,\underline{P}) \leq \delta \\ &\Leftrightarrow (\forall A \subseteq \mathcal{X} \mid \underline{P}(A) < 1) \quad \frac{\underline{P}(A) - Q(A)}{1 - \underline{P}(A)} \leq \delta \\ &\Leftrightarrow (\forall A \subseteq \mathcal{X})\ Q(A) \geq \underline{Q}^{\text{PMM}}_{\delta}[\underline{P}](A) = (1+\delta)\underline{P}(A) - \delta\end{aligned}$$

$$\Leftrightarrow Q \in \mathcal{M}\Big(\underline{Q}_{\delta}^{\text{PMM}}[\underline{P}]\Big),$$

hence we conclude that property (P8) is satisfied. □

Proof of Proposition 7.3
Let $\underline{P} \in \underline{\mathbb{P}}^*(\mathcal{X})$ be a coherent lower probability. Given $Q \in \mathbb{P}(\mathcal{X})$ and $\delta \geq 0$, for each $A \neq \mathcal{X}$ we define $h : [0,1) \to \mathbb{R}$ by:

$$h(t) = \frac{t - Q(A)}{1-t},$$

which is non-decreasing for each $t \in (0,1]$. Thus, for any $A \neq \mathcal{X}$, the coherence of $\underline{P}$ yields:

$$\begin{aligned} h\big(\underline{P}(A)\big) &= h\Big(\min_{P\in\mathcal{M}(\underline{P})} P(A)\Big) = \min_{P\in\mathcal{M}(\underline{P})} h\big(P(A)\big) \\ &= \min_{P\in\mathcal{M}(\underline{P})} d_{\text{IPMM}}(Q,P;A), \end{aligned}$$

where the second equality is a consequence of the monotonicity of h. Finally, from Equation (**??**) we get:

$$\begin{aligned} d_{\text{IPMM}}(Q,\underline{P}) &= \max_{A\subset\mathcal{X}} h\big(\underline{P}(A)\big) \\ &= \max_{A|\underline{P}(A)<1} \min_{P\in\mathcal{M}(\underline{P})} d_{\text{IPMM}}(Q,P;A), \end{aligned}$$

and we obtained the desired equality. □

Proof of Theorem 7.1
Let $\underline{P} \in \underline{\mathbb{P}}^*(\mathcal{X})$ be 2-monotone. For every given $Q \in \mathbb{P}(\mathcal{X})$ we aim to prove that $d_{\text{IPMM}}(Q,\underline{P}) = d'_{\text{IPMM}}(Q,\underline{P})$. This will imply, from Proposition 7.2 and Equation (**??**), that:

$$\begin{aligned} \mathcal{M}\Big(\underline{Q}_{\delta}^{\text{PMM}}[\underline{P}]\Big) &= B^{\delta}_{d_{\text{IPMM}}}(\underline{P}) = B^{\delta}_{d'_{\text{IPMM}}}(\underline{P}) \\ &= \bigcup_{P\in\mathcal{M}(\underline{P})} \mathcal{M}\Big(\underline{Q}_{\delta}^{\text{PMM}}[P]\Big), \end{aligned}$$

for each $\delta \geq 0$, so IPMMs satisfy strong commutativity (P10).

Since $\underline{P}$ is coherent, d_{IPMM} is given by Equation (**??**). Then, the equality between $d_{\text{IPMM}}(\cdot,\underline{P})$ and $d'_{\text{IPMM}}(\cdot,\underline{P})$ reduces to verifying, since $\underline{P} \in \underline{\mathbb{P}}^*(\mathcal{X})$ and hence $\overline{P}(A) < 1$ for each $A \subset \mathcal{X}$:

$$\begin{aligned} &\max_{A\subset\mathcal{X}} \min_{P\in\mathcal{M}(\underline{P})} \frac{P(A)-Q(A)}{1-P(A)} \\ &\quad = \min_{P\in\mathcal{M}(\underline{P})} \max_{A\subset\mathcal{X}} \frac{P(A)-Q(A)}{1-P(A)} \quad \forall Q \in \mathbb{P}(\mathcal{X}). \quad (14) \end{aligned}$$

Let

$$\begin{aligned} \varepsilon &= \max\big\{\overline{P}(A) \mid A \subseteq \mathcal{X},\ \overline{P}(A) < 1\big\}, \\ \mathcal{H} &= \{g \in \mathcal{L}(\mathcal{X}) \mid 0 \leq g(x) \leq 1\ \forall x \in \mathcal{X}\}, \text{ and} \\ \mathcal{H}^{\varepsilon}(\underline{P}) &= \{g \in \mathcal{H} \mid \overline{P}(g) \leq \varepsilon\}, \end{aligned}$$

which is such that $\mathcal{I}^* = \{I_A \mid A \subset \mathcal{X}\} \subset \mathcal{H}^{\varepsilon}(\underline{P})$. For each $Q \in \mathbb{P}(\mathcal{X})$, define the map:

$$\begin{aligned} f_Q : \mathcal{M}(\underline{P}) \times \mathcal{H}^{\varepsilon}(\underline{P}) &\to \mathbb{R} \\ (P,g) &\hookrightarrow \frac{P(g)-Q(g)}{1-P(g)}, \end{aligned}$$

where $P(g), Q(g)$ stand for the expectation operators associated with the probability measures P, Q. We note that the previous map is well defined, since $P(g) \leq \overline{P}(g) < 1$ for each $P \in \mathcal{M}(\underline{P})$ and $g \in \mathcal{H}^{\varepsilon}(\underline{P})$. We shall prove that the equality:

$$\begin{aligned} &\max_{g\in\mathcal{H}^{\varepsilon}(\underline{P})} \min_{P\in\mathcal{M}(\underline{P})} f_Q(P,g) \\ &\quad = \min_{P\in\mathcal{M}(\underline{P})} \max_{g\in\mathcal{H}^{\varepsilon}(\underline{P})} f_Q(P,g) \quad \forall Q \in \mathbb{P}(\mathcal{X}) \quad (15) \end{aligned}$$

is equivalent to Equation (14) and that it is satisfied. We note that the minima and maxima in the previous equation are indeed attained, since f_Q is continuous (the denominator does not vanish and the function is hence the composition of continuous functions) and $\mathcal{M}(\underline{P}), \mathcal{H}^{\varepsilon}(\underline{P})$ are compact sets. The latter set being closed follows from [30, Sec.2.6.1(l)], which allows us to deduce, for any $(g_k)_k \subset \mathcal{H}_{\varepsilon}(\underline{P})$ with $(g_k)_k \to g \in \mathcal{L}(\mathcal{X})$ and by taking limits, that:

$$\overline{P}(g_k) \leq \varepsilon \quad \forall k \in \mathbb{N} \Rightarrow \overline{P}(g) \leq \varepsilon, \quad (16)$$

since pointwise convergence is equivalent to uniform convergence in the finite dimensional setting that we are considering, and the lower prevision $\underline{P}$ is coherent. Also, $(g_k)_k \subset \mathcal{H}$ implies $g \in \mathcal{H}$, since $\mathcal{H}$ is compact. Then, $g \in \mathcal{H}^{\varepsilon}(\underline{P})$.

The following two first steps together prove the claimed equivalence, while the three remaining ones check the verification of the hypothesis of the minimax theorem in [30, App.E6], implying that Equation (15) holds.

Step 1 Let us prove that

$$\max_{g\in\mathcal{H}^{\varepsilon}(\underline{P})} \min_{P\in\mathcal{M}(\underline{P})} f_Q(P,g) = \max_{A\subset\mathcal{X}} \min_{P\in\mathcal{M}(\underline{P})} \frac{P(A)-Q(A)}{1-P(A)}.$$

On the one hand, the set of indicator functions $\mathcal{I}^* = \{I_A \mid A \subset \mathcal{X}\}$ is included in $\mathcal{H}^{\varepsilon}(\underline{P})$ so we straightforwardly obtain:

$$\max_{g\in\mathcal{H}^{\varepsilon}(\underline{P})} \min_{P\in\mathcal{M}(\underline{P})} f_Q(P,g) \geq \max_{A\subset\mathcal{X}} \min_{P\in\mathcal{M}(\underline{P})} \frac{P(A)-Q(A)}{1-P(A)}.$$

For the converse inequality, we use that for every $g \in \mathcal{H}^{\varepsilon}(\underline{P})$ there exists a permutation $\sigma : \{1,\ldots,n\} \to \{1,\ldots,n\}$ such that $g(x_{\sigma(1)}) \geq \ldots \geq g(x_{\sigma(n)})$ and g can be written as $g = \sum_{i=1}^n \alpha_i I_{A_i}$, where $A_i = \{x_{\sigma(1)},\ldots,x_{\sigma(i)}\}$, $\alpha_i = g(x_{\sigma(i)}) -$

$g(x_{\sigma(i+1)})$ for all $i \in \{1,\dots,n-1\}$, $A_n = \mathcal{X}$ and $\alpha_n = g(x_{\sigma(n)})$. Also every 2-monotone lower probability has a unique 2-monotone extension to gambles given by the Choquet integral. Denoting

$$\delta = \max_{A\subset\mathcal{X}} \min_{P\in\mathcal{M}(\underline{P})} \frac{P(A)-Q(A)}{1-P(A)},$$

from the coherence of $\underline{P}$ the latter is equivalent to:

$$\underline{P}(A) - Q(A) \le \delta\big(1-\underline{P}(A)\big) \quad \forall A \subset \mathcal{X}.$$

This implies that for each $g \in \mathcal{H}^\varepsilon(\underline{P})$ it holds:

$$\begin{aligned}
\min_{P\in\mathcal{M}(\underline{P})} \frac{P(g)-Q(g)}{1-P(g)} &= \frac{\underline{P}(g)-Q(g)}{1-\underline{P}(g)} \\
&= \frac{\underline{P}\big(\sum_{i=1}^n \alpha_i I_{A_i}\big) - Q\big(\sum_{i=1}^n \alpha_i I_{A_i}\big)}{1-\underline{P}\big(\sum_{i=1}^n \alpha_i I_{A_i}\big)} \\
&\le \frac{\sum_{i=1}^n \alpha_i(\underline{P}(A_i)-Q(A_i))}{\sum_{i=1}^n \alpha_i(1-\underline{P}(A_i))} \\
&= \frac{\sum_{i=1}^{n-1} \alpha_i(\underline{P}(A_i)-Q(A_i))}{\sum_{i=1}^{n-1} \alpha_i(1-\underline{P}(A_i))} \\
&\le \frac{\delta\sum_{i=1}^{n-1} \alpha_i(1-\underline{P}(A_i))}{\sum_{i=1}^{n-1} \alpha_i(1-\underline{P}(A_i))} = \delta \\
&= \max_{A\subset\mathcal{X}} \min_{P\in\mathcal{M}(\underline{P})} \frac{P(A)-Q(A)}{1-P(A)},
\end{aligned}$$

where in the first inequality we used that Q is linear, $\underline{P}(g)$ equals $\sum_{i=1}^n \alpha_i \underline{P}(A_i)$, since $\underline{P}$ is 2-monotone, and $\sum_{i=1}^n \alpha_i = \sup_{x\in\mathcal{X}} g(x) \le 1$, which follows from $g \in \mathcal{H}^\varepsilon(\underline{P}) \subset \mathcal{H}$. In the third equality we used the fact $\underline{P}(A_n) = \underline{P}(\mathcal{X}) = 1 = Q(\mathcal{X})$. Then, we have obtained the converse inequality and deduce thus the claimed equality.

Step 2 Let us prove that

$$\min_{P\in\mathcal{M}(\underline{P})} \max_{g\in\mathcal{H}^\varepsilon(\underline{P})} f_Q(P,g) = \min_{P\in\mathcal{M}(\underline{P})} \max_{A\subset\mathcal{X}} \frac{P(A)-Q(A)}{1-P(A)}.$$

Similarly to the first inequality in the step before, from $\mathcal{I}^* \subseteq \mathcal{H}^\varepsilon(\underline{P})$ we get:

$$\min_{P\in\mathcal{M}(\underline{P})} \max_{g\in\mathcal{H}^\varepsilon(\underline{P})} f_Q(P,g) \ge \min_{P\in\mathcal{M}(\underline{P})} \max_{A\subset\mathcal{X}} \frac{P(A)-Q(A)}{1-P(A)}.$$

Reciprocally, denote

$$\delta = \min_{P\in\mathcal{M}(\underline{P})} \max_{A\subset\mathcal{X}} \frac{P(A)-Q(A)}{1-P(A)}.$$

This is equivalent to the existence of certain $P \in \mathcal{M}(\underline{P})$ for which

$$P(A) - Q(A) = \delta(1-P(A)) \quad \forall A \subseteq \mathcal{X},$$

since this is trivially verified for $A = \mathcal{X}$. For that probability measure P, writing $g = \sum_{i=1}^n \alpha_i I_{A_i} \in \mathcal{H}^\varepsilon(\underline{P})$ as in the previous step and following the same reasoning, we obtain:

$$\begin{aligned}
f_Q(P,g) &= \frac{P(g)-Q(g)}{1-P(g)} \le \delta \\
&= \min_{P\in\mathcal{M}(\underline{P})} \max_{A\subset\mathcal{X}} \frac{P(A)-Q(A)}{1-P(A)} \quad \forall g \in \mathcal{H}^\varepsilon(\underline{P}). \quad (17)
\end{aligned}$$

Thus, $\min_{P\in\mathcal{M}(\underline{P})} \max_{g\in\mathcal{H}^\varepsilon(\underline{P})} f_Q(P,g) \le \delta$ and we deduce then the desired equality.

Step 3 Let us prove that $\mathcal{M}(\underline{P})$ and $\mathcal{H}^\varepsilon(\underline{P})$ are convex and compact sets in $\mathbb{R}^n$, where $n = |\mathcal{X}|$. We already know that the credal set $\mathcal{M}(\underline{P})$ is so and, from Equation (16) $\mathcal{H}^\varepsilon(\underline{P})$ is closed. Letting $g_1, g_2 \in \mathcal{H}^\varepsilon(\underline{P})$ and $\alpha \in [0,1]$, it is clear that $\alpha g_1 + (1-\alpha)g_2 \in \mathcal{H}$, because this is a convex set. Also, the upper conjugate Choquet integral of a coherent lower prevision $\underline{P}$ is convex [30, Sec.2.6.1(g)], hence:

$$\overline{P}\big(\alpha g_1 + (1-\alpha)g_2\big) \le \alpha\overline{P}(g_1) + (1-\alpha)\overline{P}(g_2) \le \varepsilon$$

and we deduce that $\alpha g_1 + (1-\alpha)g_2 \in \mathcal{H}^\varepsilon(\underline{P})$ and this is a convex set.

Step 4 Let us prove that for any $g \in \mathcal{H}^\varepsilon(\underline{P})$ and $\mu \in \mathbb{R}$ fixed, $C_{g,\mu} := \{P \in \mathcal{M}(\underline{P}) \mid f_Q(P,g) \le \mu\}$ is convex and closed. Let $\alpha \in [0,1]$, $P_1, P_2 \in C_{g,\mu}$ and assume $\mu > -1$, which yields

$$P \in C_{g,\mu} \Leftrightarrow f_Q(P,g) \le \mu \Leftrightarrow P(g) \le \frac{Q(g)+\mu}{1+\mu}.$$

Then,

$$\begin{aligned}
&\alpha P_1(g) + (1-\alpha)P_2(g) \\
&\le \alpha\frac{Q(g)+\mu}{1+\mu} + (1-\alpha)\frac{Q(g)+\mu}{1+\mu} = \frac{Q(g)+\mu}{1+\mu},
\end{aligned} \quad (18)$$

so $\alpha P_1 + (1-\alpha)P_2 \in C_{g,\mu}$. When, instead, $\mu \le -1$, we obtain that:

$$P \in C_{g,\mu} \Leftrightarrow f_Q(P,g) \le \mu \Leftrightarrow Q(g) \ge (1+\mu)P(g) - \mu.$$

Since $(1+\mu)P(g) - \mu \ge 1$ because $P(g) \ge 0$, and the identity gamble does not belongs to $\mathcal{H}^\varepsilon(\underline{P})$, the latter inequality is not satisfied for any P. Hence, in this case it holds $C_{g,\mu} = \emptyset$, which is a convex set.

To prove that the set is closed, consider a sequence $(P_k)_k \subset C_{g,\mu}$ such that $P_k \to P$ for some $P \in \mathbb{P}(\mathcal{X})$ and let us prove that $P \in C_{g,\mu}$. For this, we firstly note that $(P_k)_k \subseteq \mathcal{M}(\underline{P})$, which is closed, hence $P \in \mathcal{M}(\underline{P})$. Also, $(P_k)_k$ converging to P implies that

$P_k(g) \to P(g)$ for every $g \in \mathcal{H}^\varepsilon(\underline{P})$. When $\mu > -1$, by taking limits:

$$P_k(g) \leq \frac{Q(g)+\mu}{1+\mu} \quad \forall k \in \mathbb{N} \Rightarrow P(g) \leq \frac{Q(g)+\mu}{1+\mu}.$$

Then, $P \in C_{g,\mu}$ and we conclude that $C_{g,\mu}$ is closed. On the other hand, whenever $\mu \leq -1$ this is the empty set, which is closed as well.

Step 5 Let us prove that for any $P \in \mathcal{M}(\underline{P})$ and $\mu \in \mathbb{R}$ fixed, $S_{P,\mu} := \{g \in \mathcal{H}^\varepsilon(\underline{P}) \mid f_Q(P,g) \geq \mu\}$ is convex and closed.

Let $\alpha \in [0,1]$ and $g_1, g_2 \in S_{P,\mu}$. Since

$$f_Q(P,g) \geq \mu \Leftrightarrow P(g) - Q(g) \geq \mu(1-P(g)),$$

we get

$$\begin{aligned} &P(\alpha g_1 + (1-\alpha)g_2) - Q(\alpha g_1 + (1-\alpha)g_2) \\ &\quad = \alpha(P(g_1) - Q(g_1)) + (1-\alpha)(P(g_2) - Q(g_2)) \\ &\quad \geq \mu\big(\alpha(1-P(g_1)) + (1-\alpha)(1-P(g_2))\big) \\ &\quad = \mu\big(1 - P(\alpha g_1 + (1-\alpha)g_2)\big), \end{aligned}$$

so $\alpha g_1 + (1-\alpha)g_2 \in S_{P,\mu}$ and this is a convex set.

Finally, consider a sequence $(g_k)_k \subset S_{P,\mu}$ such that $g_k \to g$ for certain $g \in \mathcal{L}(\mathcal{X})$. As stated in Step 3, $\mathcal{H}^\varepsilon(\underline{P})$ is closed and we deduce that the limit g must belong to $\mathcal{H}^\varepsilon(\underline{P})$ as well. We conclude by noting, from P, Q being continuous and taking limits, that:

$$\begin{aligned} &P(g_k) - Q(g_k) \geq \mu(1-P(g_k)) \quad \forall k \in \mathbb{N} \\ &\Rightarrow P(g) - Q(g) \geq \mu(1-P(g)), \end{aligned}$$

so $g \in S_{P,\mu}$ and this is a closed set.

Without the hypothesis of 2-monotonicity, it follows from Example A.2 below that strong commutativity does not hold. □

Example A.2. Consider the upper prevision $\overline{P}$ on $\mathcal{X} = \{x_1, x_2, x_3, x_4, x_5\}$:

A	$\overline{P}(A)$	$\overline{Q}(A)$	$P(A)$
$\{x_1\}$	$2/10.1$	0.2	0.2
$\{x_2\}$	$2.5/10.1$	0.25	0.2
$\{x_3\}$	$2.5/10.1$	0.25	0.2
$\{x_4\}$	$2.5/10.1$	0.25	0.2
$\{x_5\}$	$2.5/10.1$	0.25	0.2
$\{x_1, x_2\}$	$4.5/10.1$	0.45	0.4
$\{x_1, x_3\}$	$4.2/10.1$	0.42	0.4
$\{x_1, x_4\}$	$4.5/10.1$	0.45	0.4
$\{x_1, x_5\}$	$4/10.1$	0.4	0.4
$\{x_2, x_3\}$	$4.9/10.1$	0.49	0.4
$\{x_2, x_4\}$	$5/10.1$	0.5	0.4
$\{x_2, x_5\}$	$5/10.1$	0.5	0.4
$\{x_3, x_4\}$	$4.9/10.1$	0.49	0.4
$\{x_3, x_5\}$	$5/10.1$	0.5	0.4
$\{x_4, x_5\}$	$5/10.1$	0.5	0.4
$\{x_1, x_2, x_3\}$	$6/10.1$	0.6	0.6
$\{x_1, x_2, x_4\}$	$7/10.1$	0.7	0.6
$\{x_1, x_2, x_5\}$	$6.5/10.1$	0.65	0.6
$\{x_1, x_3, x_4\}$	$6/10.1$	0.6	0.6
$\{x_1, x_3, x_5\}$	$6.5/10.1$	0.65	0.6
$\{x_1, x_4, x_5\}$	$6.5/10.1$	0.65	0.6
$\{x_2, x_3, x_4\}$	$7/10.1$	0.7	0.6
$\{x_2, x_3, x_5\}$	$7.4/10.1$	0.74	0.6
$\{x_2, x_4, x_5\}$	$7.5/10.1$	0.75	0.6
$\{x_3, x_4, x_5\}$	$7.4/10.1$	0.74	0.6
$\{x_1, x_2, x_3, x_4\}$	$8.5/10.1$	0.85	0.8
$\{x_1, x_2, x_3, x_5\}$	$8.5/10.1$	0.85	0.8
$\{x_1, x_2, x_4, x_5\}$	$9/10.1$	0.9	0.8
$\{x_1, x_3, x_4, x_5\}$	$8.5/10.1$	0.85	0.8
$\{x_2, x_3, x_4, x_5\}$	$9/10.1$	0.9	0.8
$\mathcal{X}$	1	1	1

It can be checked that $\overline{P}$ is coherent and that $\overline{P} = \overline{Q}_{0.01}^{\mathrm{PMM}}[\underline{P}]$. Moreover, $P \leq \overline{Q}$ but there is no $P' \leq \overline{P}$ such that $P \in \mathcal{M}(\overline{Q}_{0.01}^{\mathrm{PMM}}[P'])$: if there were, it should be $P'(A) = \overline{P}(A)$ for $A = \{x_1\}$, $\{x_1, x_5\}$, $\{x_1, x_2, x_3\}$, $\{x_1, x_3, x_4\}$, $\mathcal{X}$, from which it follows that P' would be given by the mass function $(2/10.1, 2.1/1.01, 1.9/1.01, 2.1/10.1, 2/10.1)$. However, $P(\{x_3\}) > 1.01P'(\{x_3\})$. Therefore, strong commutativity does not hold. ♦