**Lets find *a* parameter for the exact solution $V(S,t)$:**

$$V(S,\,t)=(T-t)\,\frac{\sigma^2}{2\,r}\left(\frac{K}{1+\frac{\sigma^2}{2\,r}}\right)^{\frac{2\,r+\sigma^2}{\sigma^2}}(S-a\,t)^{\frac{-2\,r}{\sigma^2}}$$

**The first derivative of $V$ by $S$:**

$$\partial_S V(S,\,t)=(t-T)\left(\frac{K}{1+\frac{\sigma^2}{2\,r}}\right)^{\frac{2\,r+\sigma^2}{\sigma^2}}(S-a\,t)^{\frac{-2\,r}{\sigma^2}-1}$$

**We can do it by using the conditions of free boundary problem of american put option. On each time step :**

$$\begin{cases}(T-t)\,\frac{\sigma^2}{2\,r}\left(\frac{K}{1+\frac{\sigma^2}{2\,r}}\right)^{\frac{2\,r+\sigma^2}{\sigma^2}}(S_0(t)-a\,t)^{\frac{-2\,r}{\sigma^2}}=K-S_0(t) & \square\\ (t-T)\left(\frac{K}{1+\frac{\sigma^2}{2\,r}}\right)^{\frac{2\,r+\sigma^2}{\sigma^2}}(S_0(t)-a\,t)^{\frac{-2\,r}{\sigma^2}-1}=-1 & \square\end{cases}$$

**Then a is equal to :**

$$a=\frac{S_0(t)-(K-S_0(t))\,\frac{2\,r}{\sigma^2}}{t}$$

**Substitute a into $V(S,\,t)$ :**

$$V(S,\,t)=(T-t)\,\frac{\sigma^2}{2\,r}\left(\frac{K}{1+\frac{\sigma^2}{2\,r}}\right)^{\frac{2\,r+\sigma^2}{\sigma^2}}\left(S-S_0(t)+(K-S_0(t))\,\frac{2\,r}{\sigma^2}\right)^{\frac{-2\,r}{\sigma^2}}$$

In[17]:=

**(T-t) (\frac{\sigma^2}{2r})(\frac{K}{1+\frac{\sigma^2}{2r}})^{\frac{2r+\sigma^2}{\sigma^2}}(S-A+ (K-A)\frac{2r}{\sigma^2})^{\frac{-2r}{\sigma^2}}**

Assuming multiplication | Use a list instead

Input:

$$(T-t)\times\frac{\sigma^2}{2\,r}\left(\frac{K}{1+\frac{\sigma^2}{2\,r}}\right)^{(2\,r+\sigma^2)/\sigma^2}\left(S-A+(K-A)\times\frac{2\,r}{\sigma^2}\right)^{(-2\,r)/\sigma^2}$$

Result:

$$\frac{\sigma^2\,(T-t)\left(\frac{K}{\frac{\sigma^2}{2\,r}+1}\right)^{(2\,r+\sigma^2)/\sigma^2}\left(\frac{2\,r\,(K-A)}{\sigma^2}-A+S\right)^{-(2\,r)/\sigma^2}}{2\,r}$$

Alternate form:

$$-\frac{1}{r}\sigma^2\,2^{(2\,r)/\sigma^2}\,(t-T)\left(\frac{K\,r}{2\,r+\sigma^2}\right)^{(2\,r)/\sigma^2+1}\left(-\frac{A\left(2\,r+\sigma^2\right)-2\,K\,r+\sigma^2\,(-S)}{\sigma^2}\right)^{-(2\,r)/\sigma^2}$$

Expanded form:

Step–by–step solution

$$\frac{\sigma^2\, T \left(\frac{K}{\frac{\sigma^2}{2\,r}+1}\right)^{(2\,r+\sigma^2)/\sigma^2} \left(\frac{2\,r\,(K-A)}{\sigma^2} - A + S\right)^{-(2\,r)/\sigma^2}}{2\,r} - \frac{\sigma^2\, t \left(\frac{K}{\frac{\sigma^2}{2\,r}+1}\right)^{(2\,r+\sigma^2)/\sigma^2} \left(\frac{2\,r\,(K-A)}{\sigma^2} - A + S\right)^{-(2\,r)/\sigma^2}}{2\,r}$$

Series expansion at r = 0:

$$K\,(T-t) - \frac{K\,r\,(t-T)\left(-2\log(S-A) + 2\log\left(\frac{K\,r}{\sigma^2}\right) - 2 + \log(4)\right)}{\sigma^2} + O\left(r^2\right)$$

(Puiseux series)

log(x) is the natural logarithm »

Big-O notation »

Derivative:

Step–by–step solution

$$\frac{\partial}{\partial r}\left(\frac{1}{2\,r}(T-t)\,\sigma^2 \left(\frac{K}{1+\frac{\sigma^2}{2\,r}}\right)^{(2\,r+\sigma^2)/\sigma^2} \left(S - A + \frac{(K-A)\,(2\,r)}{\sigma^2}\right)^{-(2\,r)/\sigma^2}\right) =$$

$$-\frac{1}{2\,r^2}\sigma^2\,(T-t)\left(\frac{K}{\frac{\sigma^2}{2\,r}+1}\right)^{(2\,r+\sigma^2)/\sigma^2}\left(\frac{2\,r\,(K-A)}{\sigma^2} - A + S\right)^{-(2\,r)/\sigma^2} + \frac{1}{2\,r}$$

$$\sigma^2\,(T-t)\left(\frac{K}{\frac{\sigma^2}{2\,r}+1}\right)^{(2\,r+\sigma^2)/\sigma^2}\left(\frac{2\log\left(\frac{K}{\frac{\sigma^2}{2\,r}+1}\right)}{\sigma^2} + \frac{2\,r+\sigma^2}{2\,r^2\left(\frac{\sigma^2}{2\,r}+1\right)}\right)\left(\frac{2\,r\,(K-A)}{\sigma^2} - A + S\right)^{-(2\,r)/\sigma^2} +$$

$$\frac{1}{2\,r}\sigma^2\,(T-t)\left(\frac{K}{\frac{\sigma^2}{2\,r}+1}\right)^{(2\,r+\sigma^2)/\sigma^2}\left(\frac{2\,r\,(K-A)}{\sigma^2} - A + S\right)^{-(2\,r)/\sigma^2}$$

$$\left(-\frac{2\log\left(\frac{2\,r\,(K-A)}{\sigma^2} - A + S\right)}{\sigma^2} - \frac{4\,r\,(K-A)}{\sigma^4\left(\frac{2\,r\,(K-A)}{\sigma^2} - A + S\right)}\right)$$

WolframAlpha

In[22]:= **d((T-t) (\frac{\sigma^2}{2r})(\frac{K}{1+\frac{\sigma^2}{2r}})^{\frac{2r+\sigma^2}{\sigma^2}}(S-A+ (K-A)\frac{2r}{\sigma^2})^{\frac{-2r}{\sigma^2}})/dt**

Derivative:

Step–by–step solution

$$\frac{\partial}{\partial t}\left(\frac{1}{2\,r}(T-t)\,\sigma^2\left(\frac{K}{1+\frac{\sigma^2}{2\,r}}\right)^{(2\,r+\sigma^2)/\sigma^2}\left(S-A+\frac{(K-A)\,(2\,r)}{\sigma^2}\right)^{-(2\,r)/\sigma^2}\right)=$$

$$-\frac{K\,\sigma^2\left(\frac{K}{\frac{\sigma^2}{2\,r}+1}\right)^{(2\,r)/\sigma^2}\left(\frac{2\,r\,(K-A)}{\sigma^2}-A+S\right)^{-(2\,r)/\sigma^2}}{2\,r+\sigma^2}$$

Alternate form:

$$-\frac{1}{2\,r+\sigma^2}K\,\sigma^2\,2^{(2\,r)/\sigma^2}\left(\frac{K\,r}{2\,r+\sigma^2}\right)^{(2\,r)/\sigma^2}\left(-\frac{A\left(2\,r+\sigma^2\right)-2\,K\,r+\sigma^2\,(-S)}{\sigma^2}\right)^{-(2\,r)/\sigma^2}$$

Series expansion at r = 0:

$$-K+\frac{2\,K\,r\left(\log(S-A)-\log\left(\frac{2\,K\,r}{\sigma^2}\right)+1\right)}{\sigma^2}+O\left(r^2\right)$$

(Puiseux series)

log(x) is the natural logarithm »

Big-O notation »

WolframAlpha

In[23]:= d((T-t) (\frac{\sigma^2}{2r})(\frac{K}{1+\frac{\sigma^2}{2r}})^{\frac{2r+\sigma^2}{\sigma^2}}(S-A+ (K-A)\frac{2r}{\sigma^2})^{\frac{-2r}{\sigma^2}})/dS

Derivative:

Step-by-step solution

$$\frac{\partial}{\partial S}\left(\frac{1}{2\,r}(T-t)\,\sigma^2\left(\frac{K}{1+\frac{\sigma^2}{2\,r}}\right)^{\left(2\,r+\sigma^2\right)/\sigma^2}\left(S-A+\frac{(K-A)\,(2\,r)}{\sigma^2}\right)^{-(2\,r)/\sigma^2}\right)=$$

$$(t-T)\left(\frac{K}{\frac{\sigma^2}{2\,r}+1}\right)^{\left(2\,r+\sigma^2\right)/\sigma^2}\left(\frac{2\,r\,(K-A)}{\sigma^2}-A+S\right)^{-(2\,r)/\sigma^2-1}$$

Alternate form:

$$-\left(\left(2\,K\,r\,\sigma^2\,(t-T)\left(\frac{K}{\frac{\sigma^2}{2\,r}+1}\right)^{(2\,r)/\sigma^2}\left(\frac{2\,r\,(K-A)}{\sigma^2}-A+S\right)^{-(2\,r)/\sigma^2}\right)\Big/\left(\left(2\,r+\sigma^2\right)\left(2\,A\,r+A\,\sigma^2-2\,K\,r-\sigma^2\,S\right)\right)\right)$$

Expanded form:

$$t\left(\frac{K}{\frac{\sigma^2}{2\,r}+1}\right)^{\left(2\,r+\sigma^2\right)/\sigma^2}\left(\frac{2\,r\,(K-A)}{\sigma^2}-A+S\right)^{-(2\,r)/\sigma^2-1}-T\left(\frac{K}{\frac{\sigma^2}{2\,r}+1}\right)^{\left(2\,r+\sigma^2\right)/\sigma^2}\left(\frac{2\,r\,(K-A)}{\sigma^2}-A+S\right)^{-(2\,r)/\sigma^2-1}$$

WolframAlpha

In[24]:= d^2((T-t) (\frac{\sigma^2}{2r})(\frac{K}{1+\frac{\sigma^2}{2r}})^{\frac{2r+\sigma^2}{\sigma^2}}(S-A+ (K-A)\frac{2r}{\sigma^2})^{\frac{-2r}{\sigma^2}})/dS^2

Derivative:

$$\frac{\partial^2}{\partial S^2}\left(\frac{(T-t)\,\sigma^2\left(\frac{K}{1+\frac{\sigma^2}{2r}}\right)^{(2r+\sigma^2)/\sigma^2}}{(2r)\left(S-A+\frac{(K-A)(2r)}{\sigma^2}\right)^{(2r)/\sigma^2}}\right)=$$

$$\left(-\frac{2r}{\sigma^2}-1\right)(-(T-t))\left(\frac{K}{\frac{\sigma^2}{2r}+1}\right)^{(2r+\sigma^2)/\sigma^2}\left(\frac{2r(K-A)}{\sigma^2}-A+S\right)^{-(2r)/\sigma^2-2}$$

Alternate form:

$$-\left(\left(2Kr\sigma^2(t-T)\left(\frac{K}{\frac{\sigma^2}{2r}+1}\right)^{(2r)/\sigma^2}\left(\frac{2r(K-A)}{\sigma^2}-A+S\right)^{-(2r)/\sigma^2}\right)\Big/\left(2Ar+A\sigma^2-2Kr-\sigma^2S\right)^2\right)$$

Expanded form:

$$-t\left(\frac{K}{\frac{\sigma^2}{2r}+1}\right)^{(2r+\sigma^2)/\sigma^2}\left(\frac{2r(K-A)}{\sigma^2}-A+S\right)^{-(2r)/\sigma^2-2}-\frac{2rt\left(\frac{K}{\frac{\sigma^2}{2r}+1}\right)^{(2r+\sigma^2)/\sigma^2}\left(\frac{2r(K-A)}{\sigma^2}-A+S\right)^{-(2r)/\sigma^2-2}}{\sigma^2}+$$

$$T\left(\frac{K}{\frac{\sigma^2}{2r}+1}\right)^{(2r+\sigma^2)/\sigma^2}\left(\frac{2r(K-A)}{\sigma^2}-A+S\right)^{-(2r)/\sigma^2-2}+\frac{2rT\left(\frac{K}{\frac{\sigma^2}{2r}+1}\right)^{(2r+\sigma^2)/\sigma^2}\left(\frac{2r(K-A)}{\sigma^2}-A+S\right)^{-(2r)/\sigma^2-2}}{\sigma^2}$$

WolframAlpha

**Then the $f$ can be calculated from**

$$\begin{aligned}
&\frac{\partial V}{\partial t}+\frac{\sigma^2}{2}S^2\frac{\partial^2 V}{\partial S^2}+\mathrm{rS}\frac{\partial V}{\partial S}-\mathrm{rV}=-\frac{1}{2r+\sigma^2}K\sigma^2\left(\frac{K}{\frac{\sigma^2}{2r}+1}\right)^{(2r)/\sigma^2}\left(\frac{2r(K-A)}{\sigma^2}-A+S\right)^{-(2r)/\sigma^2}+\\
&\quad\frac{\sigma^2}{2}S^2\left(\left(-\frac{2r}{\sigma^2}-1\right)(t-T)\left(\frac{K}{\frac{\sigma^2}{2r}+1}\right)^{\left(\frac{2r+\sigma^2}{\sigma^2}\right)}\left(\frac{2r(K-A)}{\sigma^2}-A+S\right)^{-(2r)/\sigma^2-2}\right)+\\
&\quad\mathrm{rS}\left((t-T)\left(\frac{K}{\frac{\sigma^2}{2r}+1}\right)^{(2r+\sigma^2)/\sigma^2}\left(\frac{2r(K-A)}{\sigma^2}-A+S\right)^{-(2r)/\sigma^2-1}\right)-\\
&\quad -r\left((T-t)\frac{\sigma^2}{2r}\left(\frac{K}{1+\frac{\sigma^2}{2r}}\right)^{(2r+\sigma^2)/\sigma^2}\left((K-A)\frac{2r}{\sigma^2}-A+S\right)^{(-2r)/\sigma^2}\right)=f
\end{aligned}$$