Lets find a parameter for the V(S(t),t):

$$V(S(t),\, t) = (T - t)\, \frac{\sigma^2}{2\, r} \left(\frac{K}{1 + \frac{\sigma^2}{2\, r}} \right)^{\frac{2\, r + \sigma^2}{\sigma^2}} (S - a\, t)^{\frac{-2\, r}{\sigma^2}}$$

The first derivative of V by S :

$$\partial_S\, V(S(t),\, t) = (t - T) \left(\frac{K}{1 + \frac{\sigma^2}{2\, r}} \right)^{\frac{2\, r + \sigma^2}{\sigma^2}} (S - a\, t)^{\frac{-2\, r}{\sigma^2} - 1}$$

We can do it by using the conditions of free boundary problem of american put option :

$$\begin{cases} (T - t)\, \frac{\sigma^2}{2\, r} \left(\frac{K}{1 + \frac{\sigma^2}{2\, r}} \right)^{\frac{2\, r + \sigma^2}{\sigma^2}} (S - a\, t)^{\frac{-2\, r}{\sigma^2}} = K - S & \square \\ (t - T) \left(\frac{K}{1 + \frac{\sigma^2}{2\, r}} \right)^{\frac{2\, r + \sigma^2}{\sigma^2}} (S - a\, t)^{\frac{-2\, r}{\sigma^2} - 1} = -1 & \square \end{cases}$$

Then a is equal to :

$$a = \frac{S - (K - S)\, \frac{2\, r}{\sigma^2}}{t}$$

Substitute a into $V(S(t),\, t)$:

$$V(S(t),\, t) = (T - t)\, \frac{\sigma^2}{2\, r} \left(\frac{K}{1 + \frac{\sigma^2}{2\, r}} \right)^{\frac{2\, r + \sigma^2}{\sigma^2}} \left(\frac{2\, r}{\sigma^2}\, (K - S) \right)^{\frac{-2\, r}{\sigma^2}}$$

! Is the derivative equal to -1 in S_0 ?

$$\frac{\partial}{\partial S} \left(\frac{\sigma^2\, (T - t) \left(\frac{K}{\frac{\sigma^2}{2\, r} + 1} \right)^{(2\, r + \sigma^2)/\sigma^2} \left(\frac{2\, r\, (K - S)}{\sigma^2} \right)^{-(2\, r)/\sigma^2}}{2\, r} \right) = -\frac{2\, r\, (t - T) \left(\frac{K\, r}{2\, r + \sigma^2} \right)^{(2\, r)/\sigma^2 + 1} \left(\frac{r\, (K - S)}{\sigma^2} \right)^{-(2\, r)/\sigma^2 - 1}}{\sigma^2}$$

! Ommited conditions :

1. $S \to \infty\; :\; V \to 0$

2. $V(S,\, T) \;=\; K - S$ **but out** $V(S,\, T) = 0\; (T - T)\; -$ **can be easily get around by taking of a piecewise function ?**