1 Iterated THH - Loday Functor

We proceed to introduce topological Hochschild homology based on a space X and the structure it carries for $X = T^n$ the n-dimensional torus, following [BCD10] as well as [CDD11]. For details on bicategories, confer [Bén67].

[Should have two subsections here: spans + bicategories?]

Spans of finite sets

Definition 1.1. The category of spans V has as objects the class of finite sets, while the morphism set $V(Y,X)$ for two finite sets X,Y is given by the set of equivalence classes of diagrams of the form $Y \longleftarrow A \longrightarrow X$, called spans. Two such diagrams $Y \longleftarrow A \longrightarrow X$, $Y \longleftarrow A' \longrightarrow X$ are said to be equivalent if there is a bijection $A \longrightarrow A'$ making the resulting triangles commutative. To compose two maps represented by $[Y \longleftarrow A \longrightarrow X]$ and $[X \longleftarrow B \longrightarrow W]$, we take the pullback $A \longleftarrow C \longrightarrow B$ of $A \longrightarrow X \longleftarrow B$ and compose with the maps $A \longrightarrow Y$, $B \longrightarrow W$ to obtain $[Y \longleftarrow C \longrightarrow W]$.

Lemma 1.2. *The product and coproduct in V are given by disjoint union.*

Definition 1.3. Given a finite set X with a left action of a group G, we define the action on X through automorphisms in V by mapping $g \longmapsto [X \xleftarrow{\mathrm{id}_X} X \xrightarrow{g} X]$, which can easily be seen to be a left action. Thus G also acts (from the left) on $V(Y,X)$ for any finite set Y, by functoriality. We define a function

$$\begin{aligned} \phi : V(Y, X/G) &\longrightarrow V(Y,X)^G \\ [Y \longleftarrow A \longrightarrow X/G] &\longmapsto [Y \longleftarrow A \times_{X/G} X \longrightarrow X]. \end{aligned}$$

This function is bijective.

Definition 1.4. A bicategory $\mathcal{C}$ consists of the following data:

Example 1.5. The strict bicategory of small categories

Definition 1.6. A functor $F : \mathcal{C}$

Definition 1.7. The bicategory of spans W ...

Definition 1.8. subcategories of epimorphisms and isomorphisms of spans;

Definition 1.9. the lax functor from bicategory of spans to subcategory of epimorphisms

Definition 1.10. functors from finite sets and finite sets op to iW: composition of inclusion into V and L, actually lands in iW.

Definition 1.11. functor from iW to Cat Cat ...

Definition 1.12. pointed category, functor, smash, adjoining basepoint to category,

Definition 1.13. weak functor of bicategories, ...

Definition 1.14. We define J and the two functors...

Left lax transformations, bimodules and homotopy colimits

Definition 1.15. A left lax transformation ...

Remark 1.16. remark about notation of 2-arrows in diagrams, axioms of a left lax 2-transformation, ...

Remark 1.17. Given two functors $E, F : J \longrightarrow Cat_*$ what does it mean to be a left lax transformation $E \Rightarrow F$?

Definition 1.18. category Bimod^J/E of bimodules (of J over E?) consists of pairs (F, G) ...

Definition 1.19. The homotopy colimit ...

Remark 1.20. Given a pointed small category K with all small coproducts, we let $E = K$ be the constant functor $J \longrightarrow Cat_*$ ...

The left lax transformation G^A

Definition 1.21. Let Σ be the category of finite sets with bijections. We choose a strong symmetric monoidal functor $S : \Sigma \longrightarrow Top$

NB We have to start working here!

Definition 1.22. Given a connective commutative ring spectrum$A$, we define a left lax transformation

Definition 1.23. We combine the two above definitions to obtain a left lax transformation G^A ...

Rectification

Definition 1.24. Street's first construction looks as follows; We take a lax functor $F : J \longrightarrow Cat_*$ ...

The Loday functor for finite sets

Definition 1.25. Applying Street's first construction to ...

Definition 1.26. Given a finite set S and a connective commutative ring spectrum$A$, we define the Loday functor of A at S to be ...

Lemma 1.27. *Can drop r for fixed points under finite group...*

Proof. should be formal? □

Lemma 1.28. *The Loday functor (at a finite set) preserves connectivity of commutative ring spectra, sends stable equivalences to point-wise equivalences (check this statement!) and [comparison to smash product].*

Proof. depends on connectivity of a map $S^1 \wedge A(S^n) \longrightarrow A(S^{n+1})$ - corresponding map in orthogonal spectra context should be structure map. check definition of transformation G^A or rather transformation A! Connection of connectivity to "functors of simplicial sets"? □

Corollary 1.29. *Let A be a cofibrant? flat? connective commutative ring spectrum, then Loday of A at S is stably equivalent as a spectrum to $S \otimes A$, and the morphisms are natural in S.*

A bicategory $\mathcal{C}$ is in part made up of a class of 0-cells, and for any two zero-cells A, B a category $\mathcal{C}(A, B)$, whose objects form the 1-cells from A to B and whose morphisms form the 2-cells between two given 1-cells. The bicategory of spans W has 0-cells all finite cells. Given finite sets X, Y the 1-cells are given as spans $X \leftarrow A \rightarrow Y$ for some finite set A, and a 2-cell between two spans from X to Y is given as the vertical map in the following commutative diagram:

$$\begin{tikzcd} & A \arrow[dl] \arrow[dr] \arrow[dd] & \\ X & & Y \\ & A' \arrow[ul] \arrow[ur] & \end{tikzcd}$$

Horizontal composition is given by a functorial and conrete choice of pullback applied to the 1-cells and taking the map induced by the 2-cells between pullbacks, [make clearer or scratch - this should explain horizontal while vertical composition is composition of maps.

The bicategory Cat of small categories has small categories as 0-cells, functors as 1-cells and natural transformations as 2-cells. add all the technical things you need from covering homology: spans, functor $\mathcal{J}$, nat traf G_S^A (gamma spaces, hom space (fibrant replacement), $(\Lambda_X A)^G$ functor of conn. comm S-algebras that preserves conn. and has values in very special gamma spaces (Cor. 5.1.5 in Covering homology), how diagonal is constructed (Street rectification necessary! H-set, and so on...); adapt to orthogonal spectra!?

Definition 1.30. We define the Loday functor for a finite set S and a commutative $\mathbb{S}$-algebra A as hocolim category functor ...

Definition 1.31. Let $(\mathcal{C}, ,\!\nVdash)$ be a symmetric monoidal category.
TODO complete this

Lemma 1.32. *The Loday functor is a simplicial functor.*

[loday functor simplicial! add remark? add proof? add remark? source (Γ-spaces)?]

References

[BCD10] Morten Brun, Gunnar Carlsson, and Bjørn Ian Dundas. Covering homology. *Advances in Mathematics*, 225(6):3166–3213, 2010.

[Bén67] Jean Bénabou. Introduction to bicategories. In *Reports of the Midwest Category Seminar*, pages 1–77. Springer, 1967.

[CDD11] Gunnar Carlsson, Christopher L Douglas, and Bjørn Ian Dundas. Higher topological cyclic homology and the Segal conjecture for tori. *Advances in Mathematics*, 226(2):1823–1874, 2011.