Logistic Difference Equation Calculator

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In [ ]:
```python
# Logistic Difference Equation Sequence Calculator, written in Python 3.10.6
# by Alexander Malfregeot
#
# BE WARNED: it does not handle integer overflow, so if you put in a large r
# will throw exception and crash.

import matplotlib
import matplotlib.pyplot as plt
from decimal import Decimal as dec

p0 = ''
k = ''
n_terms = 0
terms = []
sequence = {}

# start main function
def main():
    while True: # get input for p_0
        try:
            p0 = str(input('enter p_0 as a decimal number between [0, 1]\nor
            p0 = dec(p0)
            break
        except:
            print('invalid input')

    if p0 < 0 or p0 > 1:
        return

    terms.append(p0) # add p_0 to the list of terms
    sequence['p0'] = p0 # add p_0 to the dictionary of indicies

    while True: # get input for k
        try:
            k = str(input('enter k as a decimal number between [0, 4]\nor er
            k = dec(k)
            break
        except:
            print('invalid input')

    if k < 0:
        return

    while True: # get desired n terms of the sequence
        try:
            n_terms = int(input('enter desired number of terms for the graph
            break
        except:
            print('invalid input')
```

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    if n_terms <= 0:
        return

    i = 1
    while i < n_terms:
        term = (k * terms[i-1]) * (1 - terms[i-1]) # here is the equation
        terms.append(term)
        key = f'p{i}'
        sequence[key] = term
        i += 1

    plt.figure(figsize=(17,7))
    plt.plot(terms, 'ro')
    plt.plot(terms, 'b-')
    plt.title(r'$p_{n+1} = kp_n(1 - p_n)$')
    plt.xlabel(r'$n$')
    plt.ylabel(r'$p_n$')
    plt.show()

    print('exact terms of the sequence:')
    print(sequence)
# end main function

if __name__ == '__main__': # run main function
    main()
```

What is the Logistic Difference Equation?

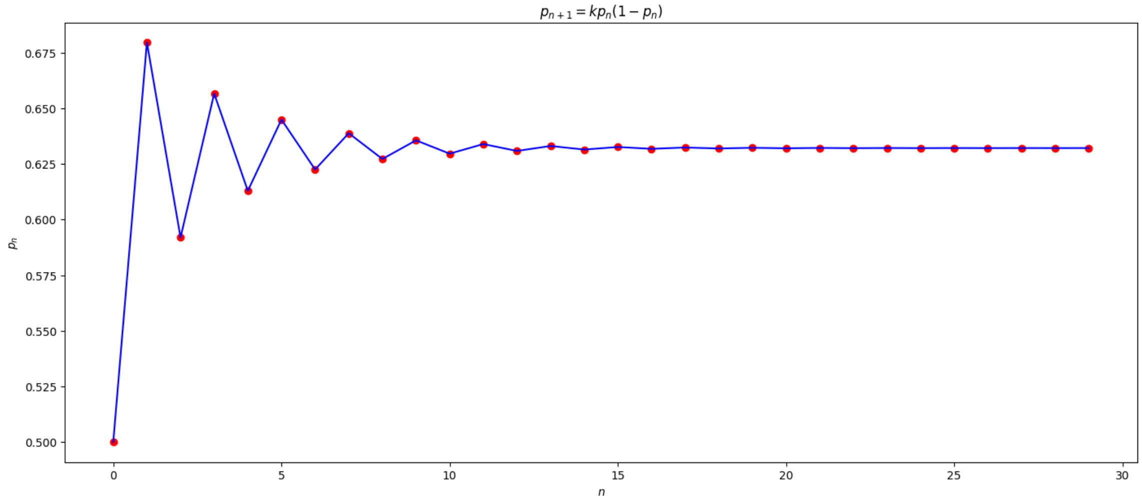
The **logistic difference equation** is the explicit form of a sequence that arises in ecology which describes a population size p_n of the n^{th} generation. p_n is a proportion where 1 is the maximum size of the population and 0 is the minimum; therefore, $0 \leq p_n \leq 1$. The constant k is obtained from other statistics. For our purposes, $0 \leq k \leq 4$.

The logistic difference equation is used in place of the logistic differential equation for modeling the population of species with periodic mating and death cycles, such as species of insects.

Part 1

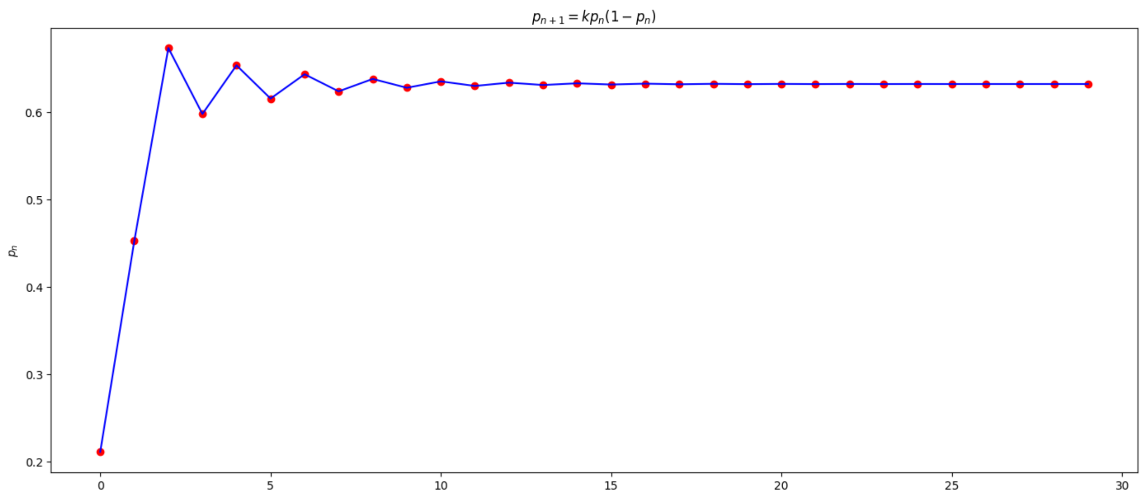
Calculate 20 or 30 terms of the sequence for $p_0 = \frac{1}{2}$ and for two values of $k \mid 1 < k < 3$. Graph the sequences. Do they appear to converge? Repeat for a different value of $p_0 \mid 0 < p_0 < 1$. Does the limit depend on the choice of p_0? Does it depend on the choice of k?

Figure 1. $p_0 = \frac{1}{2},\ k \approx e\ (2.7182)$ first 30 terms



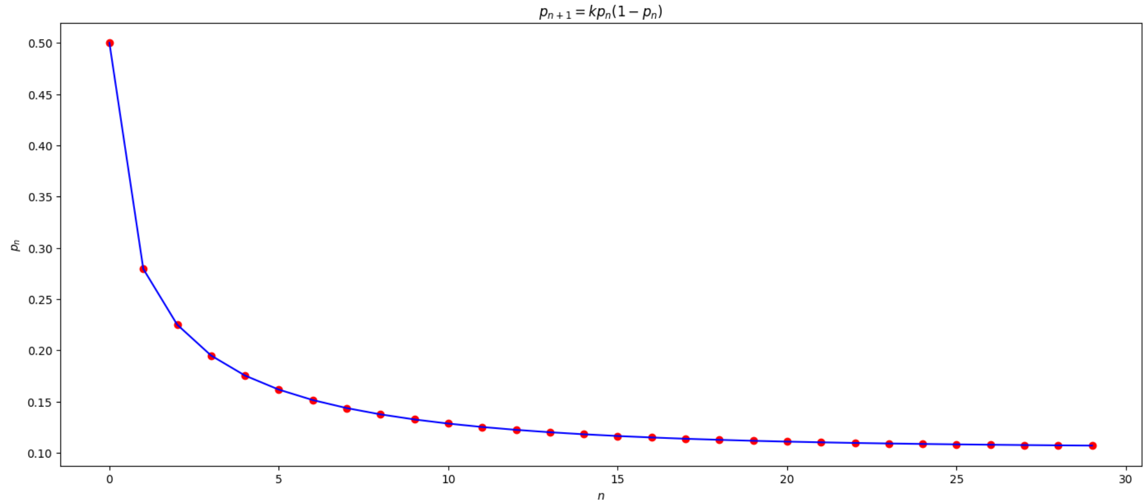
In this figure, the population starts out at half of maximum capacity. Given a constant of approximately e, the population stabilizes and converges to about 0.63 of maximum capacity around generation 14.

Figure 1a. $p_0 = 0.2109, \ k \approx e \ (2.7182)$ first 30 terms



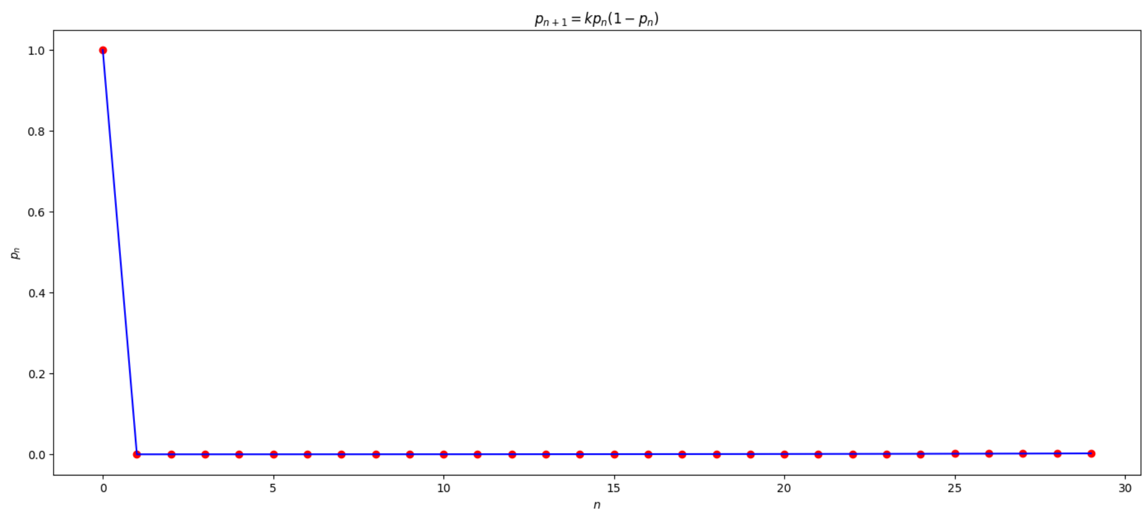
Keeping k at approximately e, we change p_0 to an arbitrary number. Despite changing p_0 as we have, the population once again converges to 0.63 at roughly generation 14.

Figure 2. $p_0 = \frac{1}{2}, \ k = 1.1177$ first 30 terms



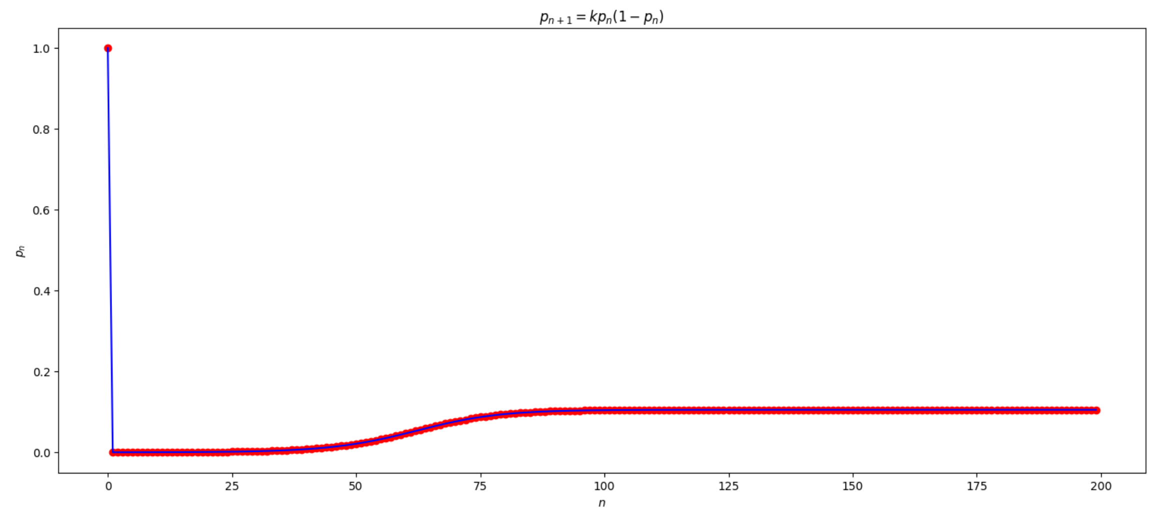
The population again starts out at half of max capacity, but this time declines with each successive generation and eventually converges to about 0.11 of maximum capacity.

Figure 2a. $p_0 = 0.9999,\ k = 1.1177$ first 30 terms



Keeping k at the same arbitrary value, we set p_0 very near to 1. This results in a dramatic decline in population between the first and second generations, and then a similar convergence, but it looks lower. In reality, this sequence still converges to about 0.11 of maximum capacity, it just takes more generations to reach the limit. See figure 2b.

Figure 2b. $p_0 = 0.9999,\ k = 1.1177$ first 200 terms



The sequence converges to roughly 0.11.

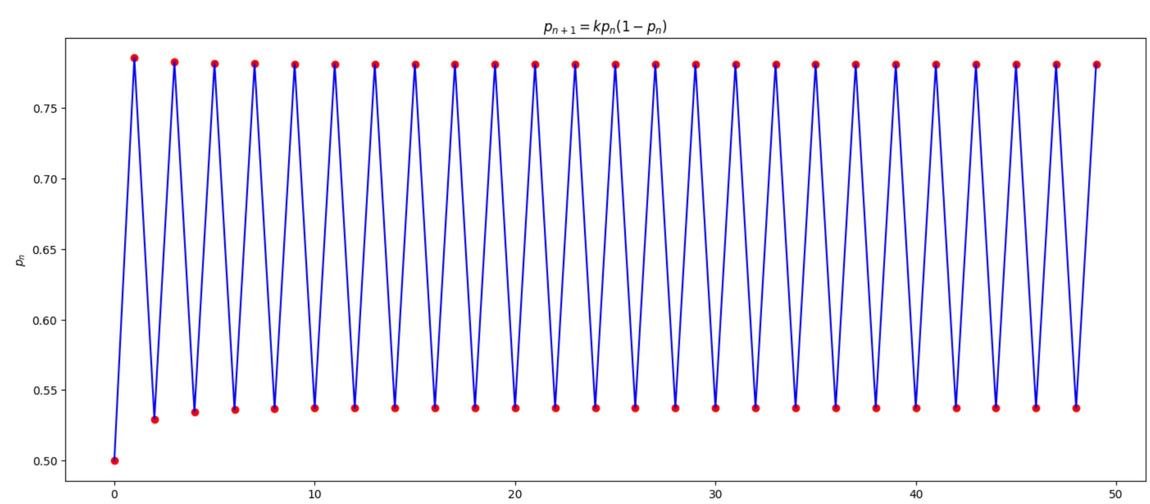
Part 1 Conclusion

Studying these figures, we can assume that the constant k has a much greater impact on the limit of the sequence than the initial population p_0 does. The intuition here is that no matter how many individuals make up the initial population, the given ecosystem can only support so many at a time. While the value of p_0 may affect the first few terms, it does not have much bearing on the entire sequence. As $n \to \infty$, the limit of the sequence is dictated by k.

Part 2

Calculate terms of the sequence for a value of $k \mid 3 < k < 3.4$. What do you notice about the behavior of the terms?

Figure 3. $p_0 = \frac{1}{2},\ k \approx \pi\ (3.1415)$ first 50 terms



In figure 3, the population oscillates between a high value, about 0.78, and low value, about 0.54, for every successive generation. This sequence diverges. Maybe, for each bug generation, natural selection dictates only a fraction survives to pass on genes to the next generation. Maybe other factors, like the life and mating cycles, sync up to the seasons. Anecdotally, this reminds me of the spotted lantern fly. They disappear in late fall, are absent in the winter, come back as nymphs in the spring, and are everywhere and full grown in the summer and early-mid fall.

Part 2 Conclusion

In part 1, we saw examples where the sequence converges at a limit depending on k. In part 2, we see an example of a divergent sequence that changes in a *cyclical* fashion. This means the population fluctuates between a maximum and minimum every successive generation.

Part 3

Experiment with values of $k \mid 3.4 < k < 3.5$. *What happens to the terms?*

Figure 4. $p_0 = \frac{1}{2},\ k = 3.4001$ first 50 terms

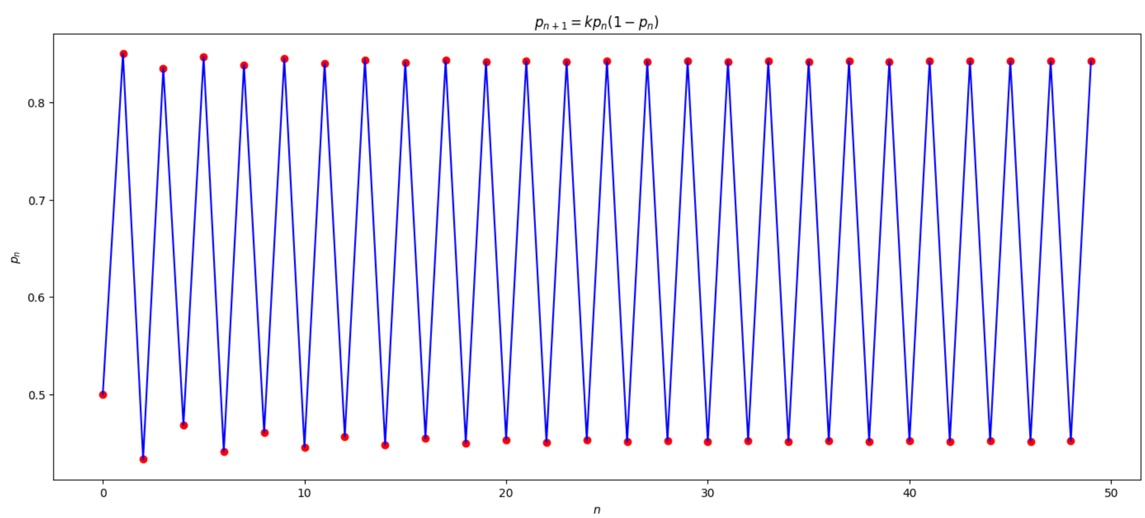


Figure 4a. $p_0 = \frac{1}{2},\ k = 3.4555$ first 50 terms

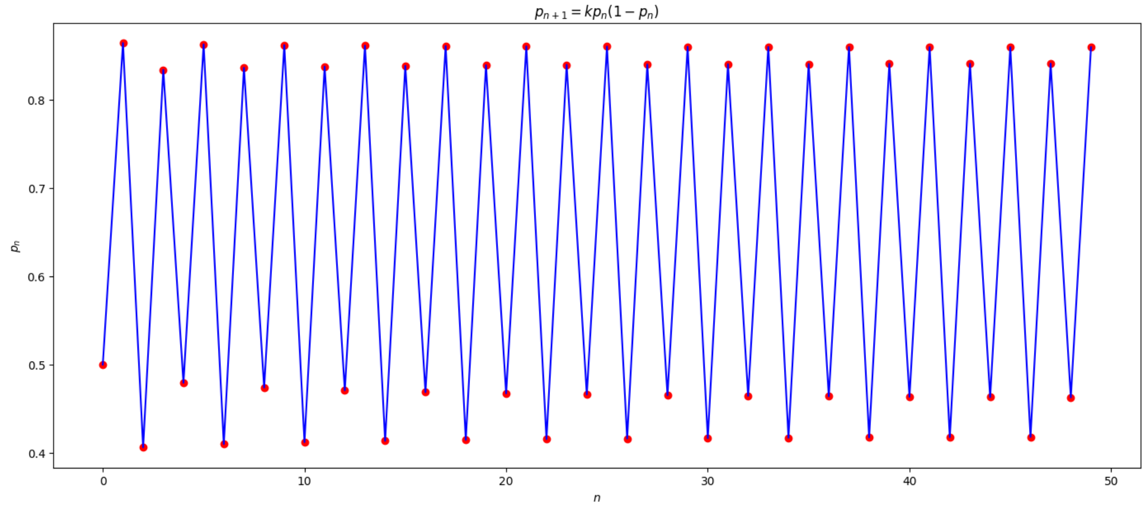
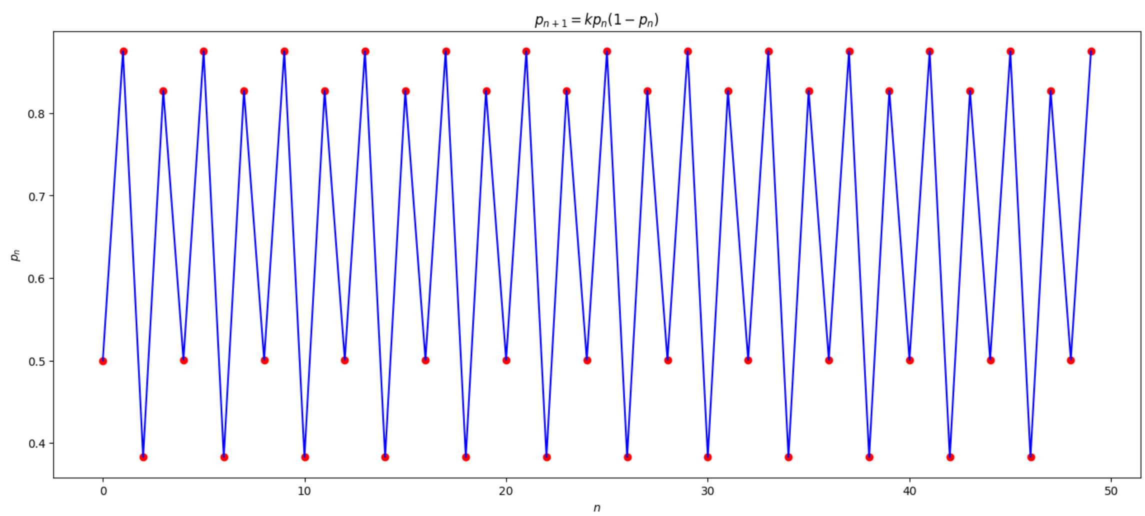


Figure 4b. $p_0 = \frac{1}{2},\ k = 3.4999$ first 50 terms



Part 3 Conclusion

As k approaches 3.5, a pattern appears in the figures. This pattern is a repeated 'staggering' of the regular 'oscillations' seen in figure 3. As seen in the difference from figure 4a to 4b, the closer k is to 3.5, the more pronounced and predictable the staggered pattern becomes. It is almost as if the main cycle of generations contains two more cycles, a high cycle and low cycle, and it switches between them every 3 generations.

Part 4

For values of $k \mid 3.6 < k < 4$, compute at least 100 terms and comment on the behavior of the sequence. What happens if you change p_0 by 0.001? This type of behavior is called chaotic and is exhiited by insect populations under certain conditions.

Figure 5. $p_0 = \frac{1}{2},\ k = 3.6001$ first 200 terms

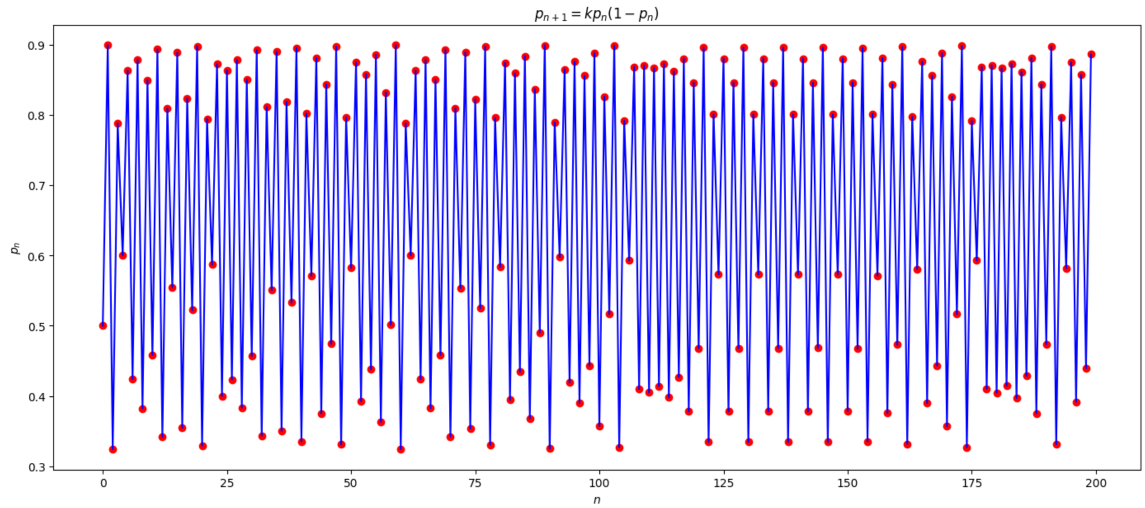


Figure 5a. $p_0 = 0.4989,\ k = 3.6001$ first 200 terms

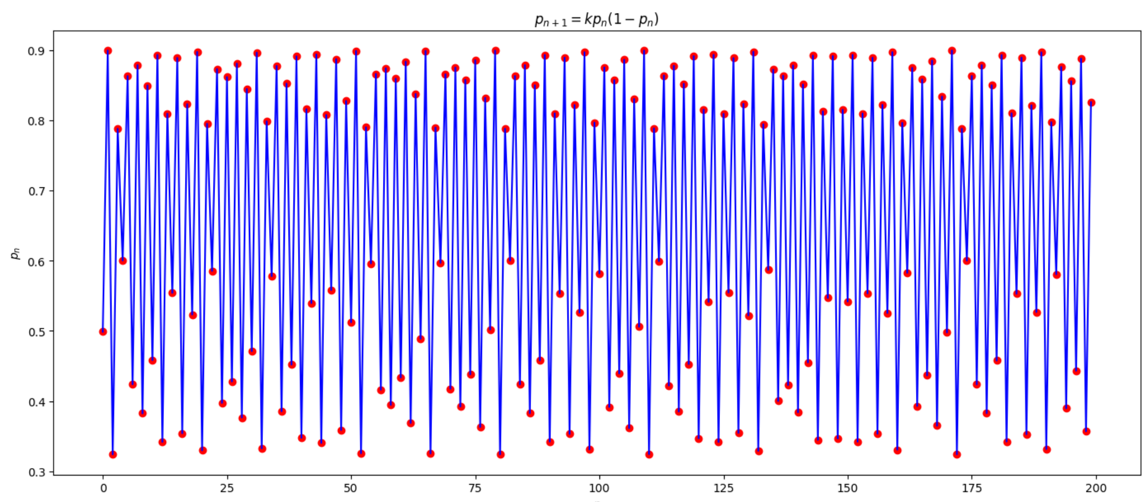


Figure 6. $p_0 = 0.3333,\ k = 3.7555$ first 200 terms

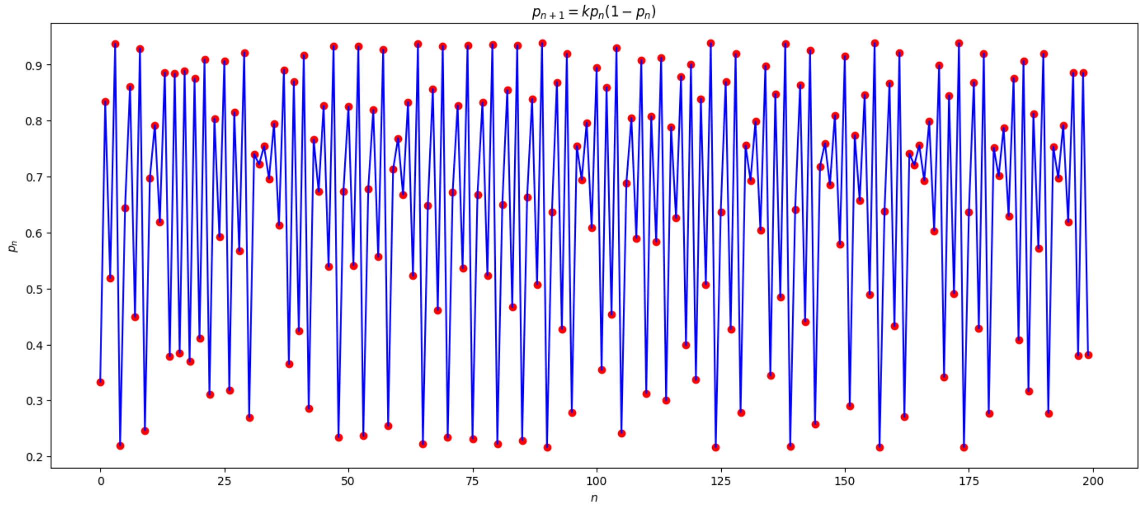


Figure 6a. $p_0 = 0.3323,\ k = 3.7555$ **first 200 terms**

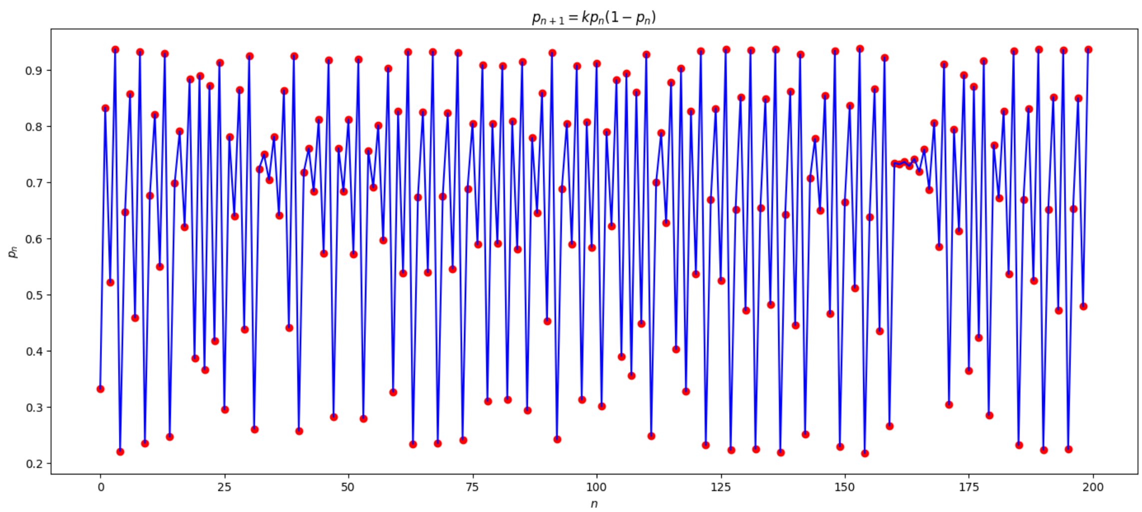


Figure 7. $p_0 = 0.9999,\ k = 3.9999$ **first 200 terms**

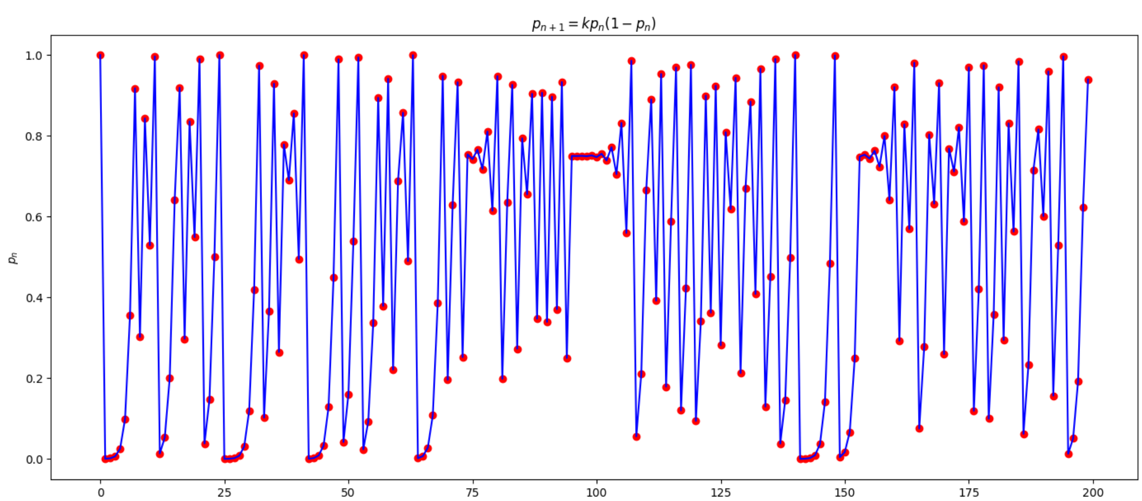
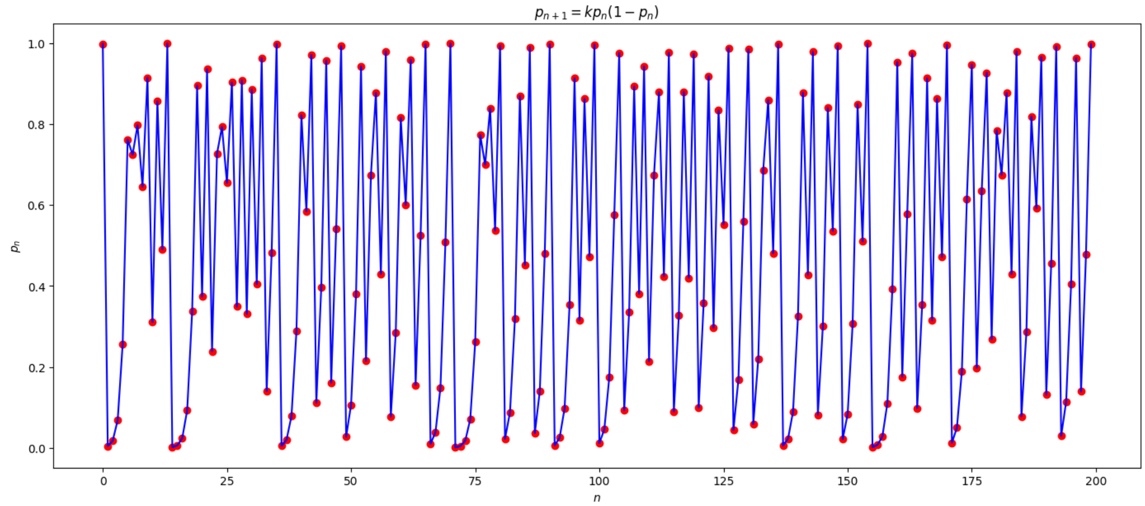


Figure 7a. $p_0 = 0.9989,\ k = 3.9999$ first 200 terms



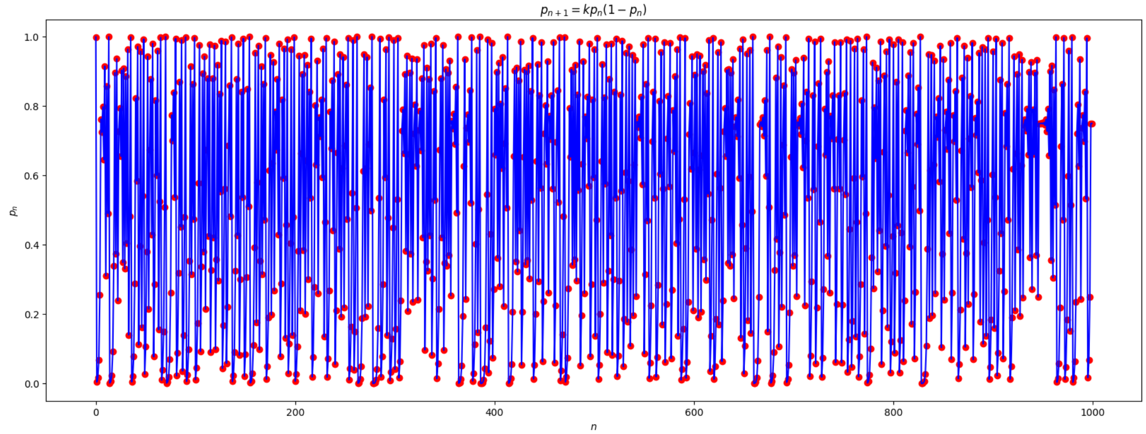
Part 4 Conclusion

The *chaotic* quality mentioned in part 4's header is plain to see in figures 5 through 7. Even when k is close to 3.6, notice that we already see the chaotic behavior. For figures 5 through 7, each has a part *a* in which 0.001 is subtracted from p_0. Despite the very small change, the *a* figures are noticably very different from their original counterparts. This is most obvious in figure 7, which has k very close to 4. Despite the chaotic behavior, we can still discern some small patterns, such as the clusters of points in figures 7 and 7a; which form a sort of 'knife edge' (if you ask me, it sort of looks like a chef's knife) shape when connected with a line.

Part 5: Project Conclusion

Despite the almost identical name, the logistic *difference* equation $p_{n+1} = kp_n(1 - p_n)$ differs from the logistic *differential* equation $\frac{dP}{dt} = kP(1 - \frac{P}{K})$. The *difference* equation is a discrete model better suited for populations with periodic cycles of growth and decline such as insects, while the *differential* is a continuous model better suited for populations which level out at a certain limit, such as humans.

Just for fun, here is one last figure. Its the same as 7a, but with the first 1000 terms.



At no point in the first 1000 do we see a repeating pattern emerge on a large scale, but when we looked at the first couple hundred, we could pick out some, such as the small 'knife edge' clusters. This reminds me of a Mandelbrot Set, which can be commonly described as the following sequence: $Z_{n+1} = {Z_n}^2 + C$. The visual metaphor between the chaotic behavior of the sequence and the Mandelbrot Set is made more apparent by these wikimedia commons sourced images: