Programming Assignment 3

CS 474

https://github.com/alexander-novo/CS474-PA3

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1 Experiment 1

1.1 Theory

The Discrete Fourier Transform (DFT) is defined as a transformation that converts a finite sequence of complex values from the time or spatial domain into the frequency domain. Mathematically, the DFT is defined as

$$F(u) = \frac{1}{N} \sum_{x=0}^{N-1} f(x) \exp\left(\frac{-j2\pi ux}{N}\right)$$

The DFT is a generalization of the continuous Fourier Transform, which transforms a continuous function from the time domain to the frequency domain. In order to return the values into the time time domain, the inverse DFT can be used, which is defined as

$$f(x) = \sum_{u=0}^{N-1} F(u) \exp\left(\frac{j2\pi ux}{N}\right)$$

The main benefit of the DFT stems from the ability to view the frequency information of a sequence of discrete values. This is particularly useful in fields such as spectral analysis and signal processing, as it allows for the removal of undesired or noisy frequencies from the data, which produces a smoother signal.

In order to perform the DFT on a continuous function, the function must first be sampled into a finite sequence of discrete, equally spaced values. This is equivalent to multiplying the continuous function by a series of equally spaced impulse functions, which can be defined as

$$f(x)s(x) = \sum_{k=-\infty}^{\infty} f(x)\delta(x - k\Delta x)$$

where

$$\Delta x$$

is the interval between each sample.

The DFT has the property of being periodic with period N, where N is the number of samples. In order to view the full period, it is necessary to shift the DFT by N/2. This can be achieved using the property

$$f(x)(-1)^x \leftrightarrow F(u - N/2)$$

1.2 Implementation

In order to implement the DFT, the Fast Fourier Transform (FFT) is used due to its more efficient time complexity of O(N log N) compared to the naive implementation which runs at O(

$$N^2$$

). The main data structure used to store the sample points is an array of complex values. The data points that are to be plotted, including the magnitude and phase information, are stored in output files. These files are then read by Gnuplot in order to generate the necessary plots and graphs.

In order to sample the cosine function, the interval between each sample was calculated by subtracting the endpoints of one period and dividing that by the number of samples. The ith cosine function sample was then evaluated at x = (i * interval) for 0 ¡= i ¡= 128.

1.3 Results and Discussion

Figure 1 showcases the results of performing the DFT on the function f sample data. According to the results, the real part of the first sample remains positive and the last three samples become negative. The DFT also introduces imaginary values to two of the samples. The magnitudes all remain positive as expected, and according to the phase plot the phase value flips sign for the second value of 4 in f.

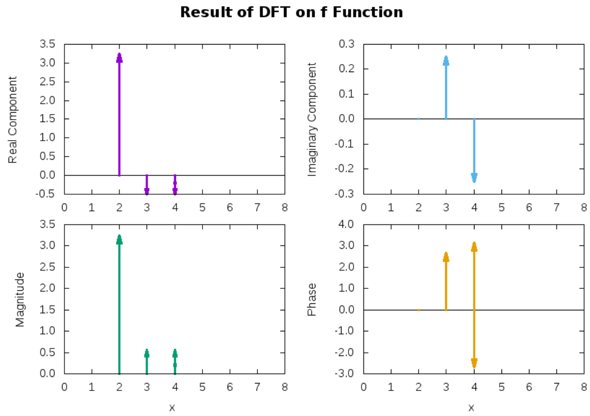

Figure 1: The result of performing the DFT on the sample data of the function f.

The cosine function along with its samples are shown in figure 2. After performing the DFT on the cosine sample data, the results were plotted in figure 3. According to the real component, the two non-zero values offset by 8 samples from the shifted center of the period represent the frequency of the cosine function, which is also 8. The imaginary values are all essentially zero as expected, and hence the magnitude corresponds to the absolute value of the real values. The phase also properly corresponds to the cyclical nature of the cosine function.

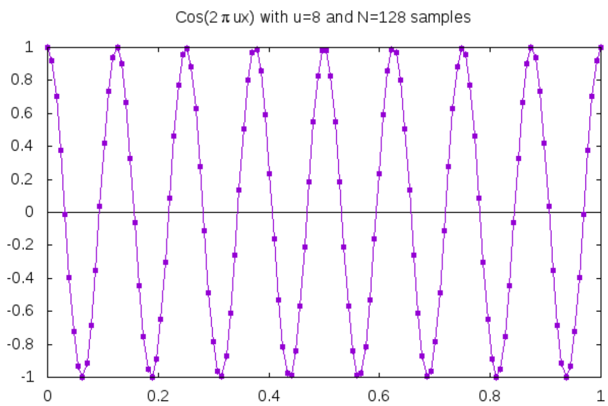

Figure 2: The given cosine was sampled along 128 points within its first period.

Lastly figure 4 shows similar DFT data for the Rect function. Similarly to the cosine function, the imaginary values of the DFT are zero. However, the real values seem to follow the general shape of a

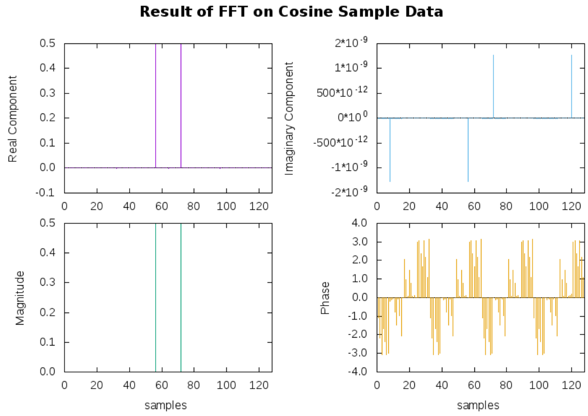

Figure 3: The result of performing the DFT on the cosine sample data.

discrete sinc function as expected. Since the imaginary parts of the frequencies are zero, the magnitude of the transformed Rect function corresponds to taking the absolute value of the real values.

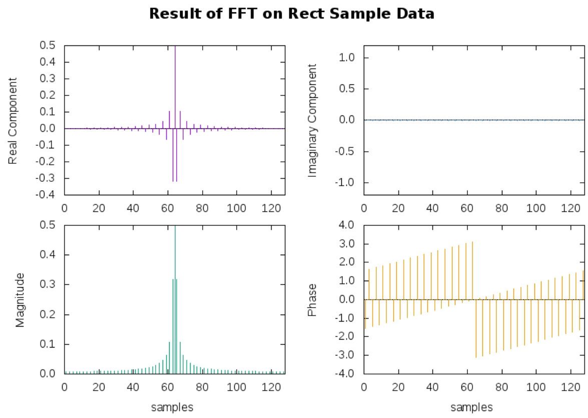

Figure 4: The result of performing the DFT on the Rect sample data.

2 Experiment 2

2.1 Theory

The 2-Dimensional Fourier Transform has a property called "separability", which shows that it can be computed as nested 1-Dimensional Fourier Transforms. A proof of this fact is given below:

$$\begin{aligned}
\mathcal{F}\{f(x,y)\}(u,v) &= \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)\exp\left(-i2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)\right) \\
&= \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)\exp\left(-i2\pi\frac{ux}{M}\right)\exp\left(-i2\pi\frac{vy}{N}\right) \\
&= \sum_{x=0}^{M-1}\left[\exp\left(-i2\pi\frac{ux}{M}\right)\sum_{y=0}^{N-1} f(x,y)\exp\left(-i2\pi\frac{vy}{N}\right)\right] \\
&= \sum_{x=0}^{M-1}\exp\left(-i2\pi\frac{ux}{M}\right)\mathcal{F}_y\{f(x,y)\}(x,v) \\
&= \mathcal{F}_x\{\mathcal{F}_y\{f(x,y)\}(x,v)\}(u,v),
\end{aligned} \tag{1}$$