

Dynamique des breathers

Alexander Semenov

Remerciements

Table des matières

1	Introduction	9
1.1	Historique	10
1.1.1	Découverte de l'onde solitaire	10
1.1.2	Equation pour l'onde solitaire : Korteweg-de Vries	11
1.1.3	Naissance du soliton, découverte de l'intégrabilité de (KdV)	12
1.1.4	Equation de Korteweg-de Vries modifiée	14
1.2	Dispersion et concentration	15
1.2.1	Equations dispersives, équation de Korteweg-de Vries généralisée	15
1.2.2	Symétries	16
1.2.3	Lois de conservation	16
1.2.4	Dispersion de l'équation d'Airy et conséquences	17
1.2.5	Concurrence entre dispersion et concentration	19
1.2.6	Problème de Cauchy	20
1.2.7	Scattering linéaire	21
1.3	Intégrabilité et scattering inverse	21
1.3.1	Remarque sur l'utilisation de l'intégrabilité dans cette thèse	21
1.3.2	Lois de conservation supplémentaires pour (KdV) et (mKdV)	21
1.3.3	Paires de Lax pour (KdV) et (mKdV) et méthode de scattering inverse	22
1.4	Solutions particulières : solitons, breathers	23
1.4.1	Solitons	23
1.4.2	Multi-soliton pour (KdV) ou (mKdV)	25
1.4.3	Breather pour (mKdV)	26
1.4.4	Résolution en solitons-breathers comme conséquence du scattering inverse	27
1.4.5	Multi-breather pour (mKdV)	29
1.4.6	Dipôles et multi-pôles	30
1.4.7	Stabilité des solitons, des breathers et construction des multi-solitons dans le cas non intégrable	31
1.5	Résultats démontrés dans cette thèse	32
1.5.1	Existence, régularité, unicité	34
1.5.2	Stabilité orbitale	35
1.5.3	Conséquences de la formule	36
1.5.4	Idées de démonstration	37
1.5.5	Perspectives	43
1.5.6	Organisation des chapitres	46

2	Existence et unicité des multi-breathers de (mKdV)	47
2.1	Introduction	47
2.1.1	Setting of the problem	47
2.1.2	Main results	50
2.1.3	Outline of the proof	52
2.1.4	Acknowledgments	55
2.2	Construction of a multi-breather in $H^2(\mathbb{R})$	55
2.2.1	Elementary results	55
2.2.2	Almost-conservation of localized conservation laws	56
2.2.3	Modulation	60
2.2.4	Study of coercivity	67
2.2.5	Proof of Proposition 2.9 (Bootstrap)	74
2.3	p is a smooth multi-breather	78
2.3.1	Estimates in higher order Sobolev norms	78
2.3.2	Uniformity of constants	83
2.4	Uniqueness	84
2.4.1	A solution converging super polynomially to a multi-breather is this multi-breather	84
2.4.2	A solution converging to a multi-breather converges exponentially to this multi-breather, if all the velocities (except possibly one) are positive	93
2.4.3	Proof of Theorem 2.4	114
2.5	Appendix	114
2.5.1	Elliptic equation satisfied by a soliton	114
2.5.2	Study of coercivity of the quadratic form associated to a soliton	115
2.5.3	Coercivity of the quadratic form associated to a soliton	116
2.5.4	Coercivity with almost orthogonality conditions (to be used for the uniqueness)	117
2.5.5	Computations for the third localized integral (to be used for the uniqueness)	119
3	Stabilité orbitale d'une somme de solitons et de breathers de (mKdV)	121
3.1	Introduction	121
3.1.1	Setting of the problem	121
3.1.2	Main results	124
3.1.3	Organisation of the proof	126
3.1.4	Acknowledgments	126
3.2	Reduction of the proof to an induction	126
3.2.1	Some useful notations	127
3.2.2	Modulation lemma	127
3.2.3	Bootstrap	129
3.2.4	Proof by induction	132
3.3	Orbital stability of a sum of solitons and breathers in $H^2(\mathbb{R})$	136
3.3.1	Almost decay of conservation laws at the right	136
3.3.2	Quadratic approximation for conservation laws at the right	139
3.3.3	Lyapunov functional and simplifications	142
3.3.4	Coercivity	144
3.3.5	Proof of \mathcal{P}_j	145
3.3.6	Choice of suitable A and Z_j	150
3.4	A consequence of Theorem 3.1 : orbital stability of a multi-breather	150

4	Calculs sur les breathers et les multi-breathers	153
4.1	Mass, energy and second energy of a soliton and of a breather	153
4.2	Consequences of the formula for multi-breathers	155
4.2.1	Formula for a 2-soliton	158
4.2.2	Pointwise convergence	159
4.2.3	Shift after a collision between two objects	170
4.2.4	Uniform convergence of p independent from translation parameters	171
4.2.5	Proof of Proposition 4.5	172
4.3	Liouville theorem in 0 for (mKdV)	175
4.3.1	Study of a virial	175
4.3.2	A bound for the \dot{H}^1 norm	176
4.3.3	Conclusion in the case when $\omega > 0$	176
	Bibliographie	185

Chapitre 1

Introduction

Cette thèse s'intéresse à certaines propriétés des breathers de l'équation de Korteweg-de Vries modifiée (mKdV), qui est un exemple d'équation aux dérivées partielles non linéaire dispersive et qui est un modèle de système hamiltonien en dimension infinie. (mKdV) admet deux types de solutions intéressantes : les solitons et les breathers. On s'intéresse tout particulièrement aux propriétés asymptotiques d'interaction et de stabilité.

Les solitons ont été beaucoup étudiés, surtout depuis qu'il a été découvert qu'ils se comportent comme des particules qui peuvent interagir entre elles. Les résultats sur leur stabilité ou sur l'existence et l'unicité de multi-solitons ont déjà été établis. L'objectif de cette thèse est de voir dans quelle mesure ces différentes propriétés s'étendent aux breathers. Ce sont des objets dont l'intérêt de l'étude est notamment justifié par la conjecture de résolution en solitons et en breathers.

Les équations dispersives donnent lieu à un domaine de recherche vaste. L'objectif de l'introduction est de situer les résultats de cette thèse au sein de ce domaine de recherche. Nous présentons dans cette introduction un aperçu de l'état des lieux des connaissances en problématiques et en techniques différentes et toutefois inter-dépendantes. Enfin, nous donnons un aperçu des résultats nouveaux qui seront développés dans cette thèse : l'unicité des multi-breathers de (mKdV) et la stabilité orbitale d'une somme de solitons et de breathers de (mKdV). Nous justifierons également les raisonnements choisis dans cette thèse.

Nous allons commencer par une introduction historique à la théorie non linéaire autour des solitons. Il s'agit d'un cadre commun à un grand nombre d'équations dispersives. On peut citer plusieurs grandes familles d'équations dispersives : les équations de Schrödinger non linéaires (NLS), les équations de Korteweg-de Vries généralisées (gKdV) ou l'équation des ondes non linéaire. (mKdV) est un cas particulier de (gKdV).

Puis, nous allons expliquer pourquoi la théorie des solitons et des équations dispersives est devenue un sujet à la mode en parlant notamment de la découverte de l'intégrabilité. L'intégrabilité est une propriété particulière qui n'est pas vérifiée par toutes les équations dispersives. En particulier, dans la famille (gKdV), il n'y a probablement que l'équation de Korteweg-de Vries (KdV), l'équation de Gardner et l'équation de Korteweg-de Vries modifiée (mKdV) qui la vérifient. Cette propriété nous donne la possibilité de trouver des formules pour un grand nombre de solutions, mais aussi nous fournit une famille infinie de lois de conservation. Ce fait donne un mode de raisonnement qu'il est possible de réaliser avec les équations intégrables. Il a le désavantage de ne pas pouvoir se généraliser aux équations non intégrables. Les raisonnements dans cette thèse ne s'appuient pas sur l'intégrabilité, toutefois nous ferons usage des lois de conservation supplémentaires fournies par l'intégrabilité.

Ensuite, nous allons parler de la découverte des breathers par la méthode de scattering inverse et indiquer qu'une formule pour les multi-breathers a été trouvée dans ce cadre. Le fait que la formule correspond bien à la définition naturelle des multi-breathers se fait avec des calculs fastidieux.

Nous poursuivrons en présentant le point de vue consistant à voir les solitons comme un minimum local d'une

fonctionnelle de Lyapunov bien choisie à quelques conditions d'orthogonalité près. C'est le point de départ de la technique permettant d'étudier la dynamique de solutions d'équations dispersives qui ne s'appuie pas sur l'intégrabilité. Cette technique est utilisée pour l'étude de la dynamique des solitons des familles d'équations dispersives différentes. Alejo et Muñoz ont adapté ce point de vue aux breathers de (mKdV).

Enfin, nous allons présenter les résultats de Martel et Merle permettant d'étudier des sommes de solitons de manière asymptotique. Leurs méthodes permettent de montrer l'existence et l'unicité de multi-solitons, ainsi que la stabilité orbitale de sommes de solitons. Cette étude est inspirée de l'étude de la stabilité d'un seul soliton et du fait que les solitons interagissent très peu entre eux quand ils sont loin les uns des autres.

Les nouveaux résultats de cette thèse consistent à étudier les sommes de solitons et de breathers de manière asymptotique. Ils s'inspirent des résultats de Martel et Merle et des résultats d'Alejo et Muñoz. Toutefois, pour rendre cette adaptation possible, il s'agit d'étudier les interactions de deux types d'objets non linéaires, sachant que le soliton a naturellement une structure variationnelle H^1 , alors que le breather a une structure variationnelle H^2 . Les démonstrations effectuées dans cette thèse ne s'appuient pas sur l'intégrabilité de (mKdV), ce qui permet d'éviter des calculs fastidieux, de réaliser des raisonnements généralisables, et de montrer l'unicité des multi-breathers dans un cadre satisfaisant. On précise toutefois lesquels des résultats de cette thèse peuvent aussi être déduits de l'intégrabilité et comment.

On y montre l'existence, l'unicité et la régularité des multi-breathers de (mKdV), et puis un résultat de stabilité orbitale pour une somme de solitons et de breathers, ainsi que pour les multi-breathers.

1.1 Historique

1.1.1 Découverte de l'onde solitaire

L'histoire des solitons commence en 1834 par une observation de John Scott Russell, ingénieur britannique qui travaillait sur la conception de bateaux. Etant donné qu'il concevait des grands bateaux, les questions liées aux vagues sur l'eau l'intéressaient particulièrement car il lui a été important de comprendre les interactions entre le profil d'un bateau et les vagues. Lorsqu'il se promenait le long du canal qui reliait Edimbourg à Glasgow (il s'agissait d'un canal de 6 m de largeur et 1,5 m de profondeur, ce qui du point de vue de l'hydrodynamique correspond à un écoulement d'eau en faible profondeur), une barge s'est arrêtée subitement, ce qui a provoqué une montée locale de l'eau devant elle. Cette montée d'eau a créé une vague en forme de courbe en cloche (une bosse) qui s'est mise à se déplacer, et ce qui est étonnant c'est qu'elle ne s'est pas rapidement dispersée et disparue. Au contraire, elle a existé relativement longtemps en gardant sa forme. Elle avait une hauteur d'environ 40 cm et une longueur d'environ 4-5 m. Intrigué, il l'a alors suivi sur son cheval pendant quelques kilomètres. Il a alors remarqué que cette vague se propageait sans déformation le long du canal avec une vitesse constante (environ 14 km/h). Cette vague semblait donc d'ores et déjà posséder une stabilité remarquable (sa hauteur a légèrement diminué à cause des frottements mais on peut tout de même la considérer comme quasi-constante). Il n'a dû arrêter de la suivre qu'à cause du fait que le canal s'est élargi (au bout de 3 km de parcours), ce qui a fait que la vague a disparu. Cette vague qui se propage, seule, à vitesse constante, sans déformation, a été baptisée *onde solitaire* ou onde de translation. Russell a décrit ses observations à plusieurs reprises dans des rapports pour l'Association britannique pour la Science, par exemple en 1838 ou en 1844 [101, 102].

Russell est parvenu à produire ce genre de vague dans des conditions expérimentales. En effet, il a fait creuser une sorte de piscine dans son domaine pour essayer de créer et d'observer des vagues solitaires. Il a regroupé les résultats de ses observations dans un rapport en 1844 [102] puis dans un mémoire publié en 1849, *The Great Solitary Wave* [103], qui lui a valu d'être élu membre de la Société Royale de Londres. Il a notamment observé que la vitesse d'une onde solitaire dépend de son amplitude et de la profondeur du canal. Il a aussi observé qu'une vague quelconque suffisamment grande se sépare en deux ou plusieurs ondes solitaires de vitesses différentes avec ou sans apparition d'une queue qui reste derrière les ondes solitaires et qui finit par disparaître, ce qui est un premier

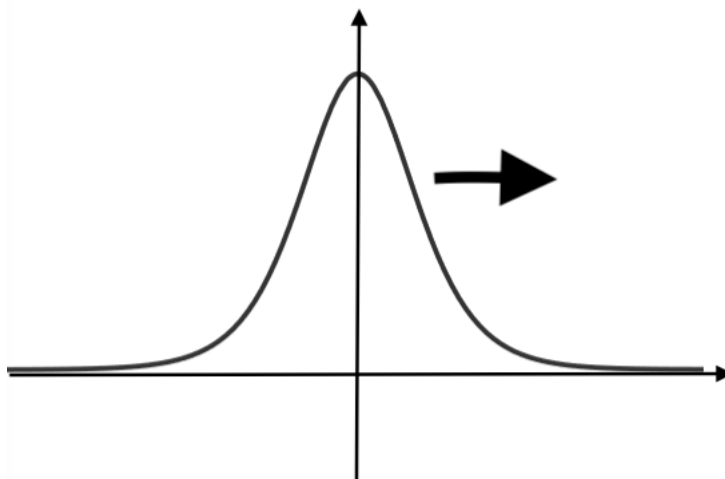


FIGURE 1.1.1 – Une onde solitaire correspond à un profil de ce type là qui se déplace à vitesse constante strictement positive sans déformation.

pas vers la conjecture de résolution en solitons dont on parlera plus loin. Enfin, il a même observé que les ondes solitaires peuvent se traverser les unes les autres et que leur forme ne change pas après cette traversée.¹

Ces observations ont provoqué beaucoup de débats dans la communauté scientifique britannique. Elles ont été accueillies avec beaucoup de scepticisme, notamment par Airy en 1841 et Stokes en 1847 [4, 110]. Ils sont arrivés à une conclusion catégorique : une vague ne peut pas avoir une forme constante même dans des conditions de frottements négligeables. En effet, il était admis à l'époque que le monde peut être modélisé par des équations linéaires. Or, il n'y a effectivement pas d'ondes solitaires pour les équations linéaires, car elles sont en contradiction avec la *dispersion* vérifiée par les équations linéaires. Il n'était non plus pas très clair pourquoi on arrivait à observer des ondes solitaires sous forme de bosses vers le haut mais pas des ondes solitaires sous formes de bosses vers le bas. Il y avait donc une asymétrie qui paraissait suspecte. Or, cette asymétrie peut très bien exister pour des EDP non linéaires, contrairement aux EDP linéaires où la symétrie par rapport à l'axe des abscisses va de soi.

Cette critique issue de mastodontes de la science britannique a fait que l'onde solitaire a été rapidement oubliée par la communauté scientifique. Il n'y a quasiment que Russell qui y pensait jusqu'à la fin de ses jours. Ses réflexions ont été publiées dans un ouvrage posthume [104].

1.1.2 Equation pour l'onde solitaire : Korteweg-de Vries

Dès lors, pour ceux qui croyaient en l'existence de l'onde solitaire, la question de l'équation modélisant l'écoulement d'un canal de faible profondeur qui pouvait admettre une onde solitaire comme solution se posait. Il est apparu clair qu'il fallait trouver une équation non linéaire, d'autant plus que les expériences de Russell ont pu être répétées et ses observations confirmées, notamment par Bazin en 1865 [12]. Les premières descriptions mathématiques ont été données en 1871 par Boussinesq [17], puis par Raileigh en 1876 [99], encore du vivant de Russell. La clarté a été apportée par Korteweg et de Vries, deux néerlandais, qui ont établi *l'équation de Korteweg-de Vries* en 1895 (qui a été pour la première fois écrite par Boussinesq en 1877 [18]) à partir des équations de l'hydrodynamique

1. Cette dernière observation n'a pas suscité beaucoup d'intérêt de sa part, mais elle sera considérée comme extraordinaire par les scientifiques qui la redécouvriront dans les années 60 du XXe siècle (cf. 1.1.3).

[62], en ayant généralisé la méthode de Raileigh, et ils ont trouvé ses solutions périodiques. L'onde solitaire est apparue comme une limite de solutions périodiques lorsque l'amplitude est grande. Cette équation non linéaire est bien appropriée pour décrire l'écoulement dans un canal de faible profondeur, et correspond étonnamment bien aux observations de Russell. Cependant, une fois que l'équation a été trouvée et que l'existence des ondes solitaires comme solutions de cette équation a été établie, il n'y a plus eu beaucoup de travaux sur ces questions jusqu'aux années 60. Notons tout de même qu'il y a eu quelques travaux isolés sur ce sujet pendant cet intervalle temporel : notamment Lavrentiev a donné la première démonstration rigoureuse de l'existence des ondes solitaires sur un écoulement de faible profondeur en 1946 [66], puis Friedrichs et Hyers en ont donné une démonstration plus simple en 1954 [36].

L'équation de Korteweg-de Vries (KdV) est la suivante :

$$\begin{cases} u_t + (u_{xx} + u^2)_x = 0, \\ u(t, x) \in \mathbb{R}. \end{cases} \quad (\text{KdV})$$

On peut vérifier que les ondes solitaires sont solutions de cette équation, données par la formule :

$$R_c(t, x; x_0) := \frac{3c}{2 \cosh^2 \left(\frac{\sqrt{c}}{2} (x - ct - x_0) \right)}, \quad (1.1.1)$$

où $c > 0$ est le paramètre déterminant à la fois sa vitesse et sa taille, et x_0 est sa position initiale.

1.1.3 Naissance du soliton, découverte de l'intégrabilité de (KdV)

Dans les années 60, le sujet des ondes solitaires et de l'équation de Korteweg-de Vries est devenu à la mode. En effet, il a été établi que l'équation de Korteweg-de Vries décrit bien les ondes dans un plasma, ce dernier sujet de recherche étant très en vogue parmi les physiciens de l'époque. Tout d'abord, Adlam et Allen [3] d'un côté et Sagdeev [105] de l'autre ont découvert des ondes solitaires qui se propagent dans un plasma en 1958, puis Gardner et Morikawa ont montré que les équations décrivant un plasma sont analogues à (KdV) [40] en 1960. C'est pourquoi on a commencé à étudier (KdV) numériquement.

Et l'étude numérique de (KdV) a révélé des propriétés étonnantes. D'un côté, Kruskal et Zabusky [125], en 1965, ont observé de façon numérique que si on observe deux ondes solitaires où l'un va plus vite que l'autre, alors les deux conservent leur forme après leur collision (elles ne sont que translatées). Ceci semblait suggérer un comportement *intégrable*. Ainsi, chaque onde solitaire se comporte comme une particule bien identifiée : c'est cela qui justifie l'appellation *soliton* pour une onde solitaire, qu'ils ont introduite. Ce n'est que là que l'intérêt de cette observation, qui a déjà été réalisée par Russell, a été compris. Notons que Kruskal et Zabusky sont les premiers à avoir remarqué les translations que subissent les solitons après une collision entre eux. Ils ont aussi constaté numériquement qu'en regardant suffisamment longtemps une solution quelconque de (KdV), on voit apparaître une superposition de solitons, ce qui semble justifier la *conjecture de résolution en solitons* dont on parlera plus loin.

Ensuite, Gardner, Greene, Kruskal et Miura ont donné l'explication mathématique pour les observations numériques de Kruskal et Zabusky en introduisant la méthode du *scattering inverse* en 1967 [38], qu'ils ont précisée en 1974 [39], et qui permet de trouver des solutions explicites à (KdV). Enfin, en 1968, Peter Lax a généralisé la méthode du scattering inverse et a déterminé l'algèbre qui se cache derrière le travail de Gardner, Greene, Kruskal et Miura en exhibant une *paire de Lax* pour (KdV) [67], autrement dit deux opérateurs différentiels dont la commutation est équivalente à (KdV). Il s'agit d'une découverte fondamentale pour l'étude de (KdV) en tant qu'équation intégrable. Parallèlement, Miura, Gardner et Kruskal ont démontré qu'il existe une infinité de lois de conservation pour (KdV) [92] en partant d'une loi de conservation pour l'équation de Gardner, puis en 1971, Kruskal, Miura, Gardner et Zabusky [63] ont explicité une méthode pour calculer de manière effective ses lois de conservation [63].

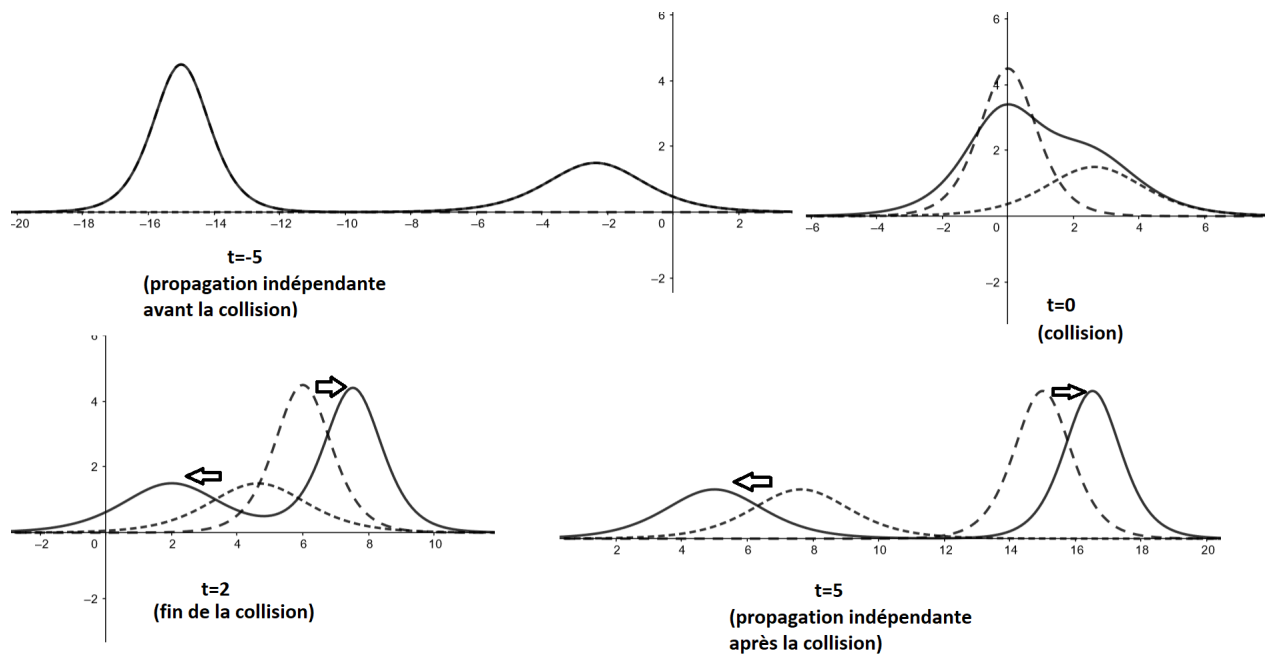


FIGURE 1.1.2 – On représente ici la solution de (KdV) formée par deux solitons à quatre instants successifs : celui qui est initialement à gauche a une vitesse $c = 3$, celui qui est initialement à droite a une vitesse $c = 1$. Le trait plein représente la solution qui tient compte des interactions entre les deux solitons, et les pointillés représentent la propagation des deux solitons sans tenir compte de leurs interactions. On voit donc qu'après collision, les deux solitons conservent leurs formes et sont juste translatés : le plus rapide est translaté un peu vers la droite et le plus lent est translaté un peu vers la gauche. On peut observer que pendant la collision, les deux solitons ont l'air d'échanger leurs places.

Lax a également indiqué une méthode permettant de trouver les lois de conservation de (KdV) en partant de la paire de Lax [67], en attribuant cette méthode à Gardner. Enfin, en 1971, Zakharov et Faddeev [126] d'un côté et Gardner [37] de l'autre ont mis en évidence le caractère *hamiltonien* de (KdV), où le rôle de l'hamiltonien est joué par l'énergie. Ainsi, (KdV) est un système hamiltonien complètement intégrable. Enfin, Wadati et Toda [119] ont appliqué la méthode de scattering inverse pour trouver une formule pour une solution approchant une somme de solitons de (KdV) en 1972. Il s'agit de la première EDP intégrable découverte, d'autres l'ont été par la suite, comme indiqué dans [1] en 1973, et cela a donné lieu à un domaine de recherche riche.

Tout ceci démontre une structure très riche de (KdV) et a été une motivation pour la recherche. En particulier, cela a donné espoir d'arriver à décrire explicitement les solutions génériques de (KdV).

1.1.4 Equation de Korteweg-de Vries modifiée

La modification la plus simple du terme non linéaire de (KdV) donne lieu à l'équation de Korteweg-de Vries *modifiée* :

$$\begin{cases} u_t + (u_{xx} + u^3)_x = 0, \\ u(t, x) \in \mathbb{R}. \end{cases} \quad (\text{mKdV})$$

L'équation considérée ici est l'équation de Korteweg-de Vries modifiée *focalisante*. Elle existe aussi sous une deuxième version où le signe devant u^3 est négatif, dans ce cas il s'agit de l'équation (mKdV) *défocalisante*. Cette terminologie existe aussi pour l'équation de Schrödinger non linéaire. En général, une équation défocalisante a un comportement plus proche du cas linéaire, on y reviendra dans la section suivante. On peut remarquer que cette disjonction n'a pas lieu d'être pour (KdV) parce qu'il s'agit de la même équation quel que soit le signe devant le terme non linéaire (en effet, si u est solution de (KdV), alors $v = -u$ est solution de (KdV) avec un signe négatif devant u^2).

L'équation (mKdV) apparaît pour la première fois, sans être nommée ainsi (elle était désignée comme une généralisation de (KdV) avec une nonlinéarité cubique), en 1967 dans l'étude d'un réseau anharmonique pour décrire une onde acoustique par Zabusky [124]. En 1968, Miura remarque qu'il existe une transformation entre les solutions de (KdV) et les solutions de (mKdV) [90] (qu'il désigne comme une modification de (KdV)), qui porte désormais son nom. Il s'agit d'un cas rare où il existe une transformation entre deux EDP telles que chacune d'entre elles a un intérêt propre. Miura a considéré le cas focalisant, même si le cas défocalisant se traite de manière similaire. Cependant, dans le cas où l'on considère l'équation (mKdV) focalisante, il s'agit d'une transformation qui ne préserve pas le côté réel d'une solution, il n'y a donc pas de correspondance entre les solutions physiques (i.e. réelles) de (KdV) et de (mKdV) focalisante. C'est pourquoi l'étude de (mKdV) ne peut pas se réduire à l'étude de (KdV).

Explicitons le résultat de Miura : si v est une solution de (mKdV) focalisante, alors $u = \frac{3}{2}v^2 \pm 3i\frac{\sqrt{2}}{2}v_x$ est solution de (KdV). On peut énoncer ce résultat dans le cas défocalisant aussi : si w est solution de (mKdV) défocalisante, alors $u = -\frac{3}{2}w^2 \pm \frac{3\sqrt{2}}{2}w_x$ est solution de (KdV). La transformation qui envoie une solution de (mKdV) sur une solution de (KdV) s'appelle la *transformation de Miura*.

Ainsi, l'étude de (mKdV) avait, au départ, surtout un intérêt mathématique : il est en effet intéressant de voir ce qui change quand on change le degré de la nonlinéarité, de voir si on peut ou non prouver des choses similaires et comment. Ceci a été fait avec Miura [90] en 1968, puis avec Kametaka [55] en 1969 qui a été le premier à étudier le problème de Cauchy pour (mKdV). Cependant, l'idée d'étudier cette équation est venue d'un problème physique comme le montre Zabusky [124] en 1967. D'autres applications physiques sont apparues par la suite. Ce sont des applications où il est important de prendre en compte la nonlinéarité à un ordre plus grand que quadratique, plus précisément à un ordre cubique ici. En effet, (mKdV) peut aussi décrire l'onde de Alfvén dans un plasma froid sans collision, comme décrit par Kakutani et Ono en 1969 [54] (il s'agit d'ailleurs de la première fois où (mKdV) est

désignée par son nom définitif). (mKdV) apparaît aussi dans l'étude asymptotique de l'auto-piégeage des impulsions de chaleur dans les solides comme montré par Tappert et Varma en 1970 [114]. Enfin, (mKdV) a aussi des applications en électrohydrodynamique lorsqu'on étudie des ondes sur la surface libre d'un liquide idéalement conducteur avec la présence d'un champ électrique transverse comme établi par Perel'man, Fridman et El'yashevich en 1974 [97]. Cette liste d'exemples, qui est loin d'être exhaustive, montre suffisamment l'intérêt physique de l'équation.

Hormis son lien avec (KdV) par la transformation de Miura, (mKdV) apparaît comme intéressante mathématiquement par ses propriétés d'*intégrabilité* qui sont proches de celles de (KdV). Ce qui fait que, tout comme (KdV), (mKdV) peut être résolue grâce à la méthode de scattering inverse. Ceci a été initié par Miura, Gardner et Kruskal [92] en 1968 qui ont montré que tout comme (KdV), (mKdV) a une infinité de lois de conservation et de constantes du mouvement. Ceci a été prolongé par Tanaka [111], Wadati [116, 117] et Hirota [51] en 1972 et 1973 avec une construction de solutions explicites pour (mKdV) par scattering inverse, et systématisé par Ablowitz, Kaup, Newell et Segur [1] en 1973. Ceci montre que (mKdV) est une équation particulièrement intéressante mathématiquement. De plus, (mKdV) est une des rares équations qui vérifie une version adaptée de la conjecture de résolution en solitons. Enfin, Wadati [117] a découvert en 1973 que (mKdV) a un type de solutions particulières en plus des solitons qui lui est propre, les breathers, dont l'étude est au cœur de cette thèse.

1.2 Dispersion et concentration

1.2.1 Equations dispersives, équation de Korteweg-de Vries généralisée

Les équations dispersives représentent une famille des EDP non linéaires. Les deux principales sous-familles de la famille des EDP dispersives sont l'équation de Schrödinger non linéaire (NLS) et l'équation de Korteweg-de Vries généralisée (gKdV). On peut aussi citer des équations comme sine-Gordon, Benjamin-Ono, Zakharov-Kuznetsov, Kadomtsev-Petviashvili entre autres. Ces équations ont beaucoup d'applications en physique [106, 20, 97, 122, 50].

Ici, on s'intéresse principalement à un cas particulier de (gKdV). L'équation de Korteweg-de Vries généralisée correspond au cas où la nonlinéarité est générale :

$$\begin{cases} u_t + (u_{xx} + u^p)_x = 0, \\ u(t, x) \in \mathbb{R}, \end{cases} \quad (\text{gKdV})$$

où $p \in \mathbb{N}$ avec $p \geq 2$ est le degré de la nonlinéarité. On peut aussi généraliser la situation en remplaçant u^p par $f(u)$, notamment pour des puissances p non entières en posant $f(u) = \pm|u|^p$ ou $f(u) = \pm|u|^{p-1}u$. Ainsi, (KdV) correspond au cas $p = 2$, et (mKdV) correspond au cas $p = 3$. L'équation de Gardner correspond au cas $f(u) = u^2 + \mu u^3$, où $\mu \in \mathbb{R}^*$. Le comportement de l'équation est déterminé par le degré de la nonlinéarité.

Dans le cas où p est pair, les nonlinéarités $f(u) = u^p$ et $f(u) = -u^p$ sont équivalentes. Ce n'est pas le cas lorsque p est impair : dans ce cas, lorsque $f(u) = u^p$, on dit que l'équation est *focalisante* ; lorsque $f(u) = -u^p$, on dit que l'équation est *défocalisante*. Dans le cas où p est pair, on dit que l'équation est focalisante. Ainsi, (mKdV) est l'équation (gKdV) dont la nonlinéarité a le degré le plus petit possible à avoir une version défocalisante.

L'autre grand exemple d'équations dispersives est l'équation de Schrödinger non linéaire (NLS) :

$$\begin{cases} iu_t + \Delta u \pm |u|^{p-1}u = 0, \\ u(t, x) \in \mathbb{C}, \end{cases} \quad (t, x) \in [0, T] \times \mathbb{R}^d. \quad (\text{NLS})$$

La plupart des observations développées dans cette section pour (gKdV) peuvent être adaptées pour (NLS). En particulier, le cas L^2 -sous-critique correspond au cas où $p - 1 < \frac{4}{d}$, le cas \dot{H}^1 -sous-critique correspond au cas où $p - 1 < \frac{4}{d-2}$ pour $d \geq 3$ ou au cas où $d = 1, 2$, le cas focalisant correspond au cas où le signe devant la nonlinéarité

est $+$ et le cas défocalisant correspond au cas où le signe devant la nonlinéarité est $-$. Il existe un cas particulier intégrable de (NLS) : c'est sa version cubique en dimension 1 [2].

Citons aussi la première version de l'équation de Kadomtsev-Petviashvili (KP-I), qui est intégrable et dont la dynamique a notamment été étudiée par Tzvetkov et Rousset [100] :

$$\begin{cases} (u_t + uu_x + u_{xxx})_x - u_{yy} = 0, \\ u(t, x) \in \mathbb{R}, \end{cases} \quad (t, x, y) \in [0, T[\times \mathbb{R} \times \mathbb{R}/(2\pi\mathbb{Z}). \quad (\text{KP-I})$$

Cette équation est un modèle pour la propagation d'ondes longues en x faiblement modulées dans la direction transverse (en y) [53, 98]. Elle apparaît comme une généralisation possible de l'équation (KdV). Elle existe aussi dans une version où le signe devant u_{yy} est positif, c'est l'équation (KP-II).

1.2.2 Symétries

(gKdV) possède les symétries suivantes :

- elle est stable par *translation* en temps et en espace : pour tout $(x_0, t_0) \in \mathbb{R}^2$, si $u(t, x)$ est solution alors $u(t + t_0, x + x_0)$ aussi,
- elle est stable par *symétrie centrale* : si $u(t, x)$ est solution, alors $u(-t, -x)$ aussi,
- dans le cas où p est impair (par exemple pour (mKdV)), elle est stable par *réflexion par rapport à l'axe des abscisses* : si $u(t, x)$ est solution alors $-u(t, x)$ aussi,
- elle est stable par *changement d'échelle* : pour tout $\lambda > 0$, si $u(t, x)$ est solution alors

$$\frac{1}{\lambda^{\frac{2}{p-1}}} u\left(\frac{t}{\lambda^3}, \frac{x}{\lambda}\right) \quad (1.2.1)$$

aussi.

En $t = 0$, le changement d'échelle laisse invariante la norme \dot{H}^{s_c} de $u(0, x)$, où

$$s_c = \frac{1}{2} - \frac{2}{p-1} \quad (1.2.2)$$

est l'*exposant d'échelle critique*. L'équation est dite \dot{H}^{s_c} -critique. La position de l'espace L^2 par rapport à \dot{H}^{s_c} détermine le mode dans lequel se trouve l'équation :

- si $p < 5$ (i.e. $s_c < 0$), l'équation (gKdV) est dite L^2 -sous-critique,
- si $p = 5$ (i.e. $s_c = 0$), l'équation (gKdV) est dite L^2 -critique,
- si $p > 5$ (i.e. $s_c > 0$), l'équation (gKdV) est dite L^2 -sur-critique.

(KdV) et (mKdV) correspondent donc au cas L^2 -sous-critique. On peut aussi remarquer qu'en $t = 0$, le changement d'échelle laisse invariante la norme L^1 pour (mKdV) : on dit donc que l'équation (mKdV) est L^1 -critique.

1.2.3 Lois de conservation

L'équation (gKdV) possède formellement les trois lois de conservation suivantes (autrement dit, des quantités conservées en temps pour une solution u quelconque) :

- la norme L^1 :

$$I_1(t) := \int u(t, x) dx = I_1(0), \quad (1.2.3)$$

- la masse :

$$I_2(t) := \frac{1}{2} \int u(t, x)^2 dx = I_2(0), \quad (1.2.4)$$

— l'énergie (découverte par Whitham [123]) :

$$I_3(t) := \frac{1}{2} \int u_x^2 dx - \frac{1}{p+1} \int u^{p+1} dx = I_3(0), \quad (1.2.5)$$

qui se généralise pour des nonlinéarités autres que $f(u) = u^p$ de la manière suivante :

$$I_3(t) := \frac{1}{2} \int u_x^2 dx - \int F(u) dx = I_3(0), \quad (1.2.6)$$

où F désigne une primitive de f .

Dans la mesure où, dans la suite, on travaillera principalement dans les espaces à carré intégrable, ce sont surtout les deux dernières lois de conservation qui nous intéresseront (la masse et l'énergie). De plus ; I_1 a le désavantage de ne pas avoir de signe préférentiel. La masse est souvent notée $M[u](t)$ et l'énergie est souvent notée $E[u](t)$.

1.2.4 Dispersion de l'équation d'Airy et conséquences

La *dispersion*, au sens physique, correspond à un phénomène où des fréquences différentes se propagent à des vitesses différentes. Cela se vérifie bien sur la partie linéaire de (gKdV), appelée *équation d'Airy* [112],

$$u_t + u_{xxx} = 0. \quad (\text{Airy})$$

En effet, une fréquence est bien représentée par l'*onde plane progressive monochromatique* qui lui est associée. Ainsi, l'onde plane progressive monochromatique de pulsation ω est

$$u_\omega(t, x) = \exp(i(kx + \omega t)), \quad (1.2.7)$$

où le nombre d'onde k dépend de ω . La vitesse de propagation d'une telle onde est donnée par $-\frac{\omega}{k}$. Pour la trouver, il suffit de déterminer comment k dépend de ω .

Si l'on suppose que u_ω est solution de (Airy), on a que

$$i\omega + (ik)^3 = 0, \quad (1.2.8)$$

donc $k = \omega^{1/3}$, qui est la *relation de dispersion* de (Airy). Donc, la vitesse de u_ω est $-(\omega^{1/3})^2 = -k^2 < 0$. Donc, la vitesse de propagation dépend de la fréquence. Il est une propriété importante de (gKdV) que la dispersion se fait toujours vers la gauche.

On peut noter W la fonction qui à un nombre d'onde k associe la pulsation $\omega = W(k)$ telle que la relation de dispersion est vérifiée. Pour (Airy), $W(k) = k^3$. Ceci nous permet de calculer la *vitesse de groupe* $v_g(k) = -W'(k) = -3k^2$ et la *vitesse de phase* $v_\varphi(k) = -W(k)/k = -k^2$. Comme la vitesse de groupe est différente de la vitesse de phase, (Airy) est dispersive.

Il est possible de résoudre (Airy) dont l'état initial est u_0 explicitement (grâce à un passage en Fourier), pour tout $t \neq 0$:

$$u(t) = u_0 * K_t, \quad (1.2.9)$$

où $*$ désigne le produit de convolution et

$$K_t(x) := \frac{1}{(3t)^{1/3}} \text{Ai} \left(\frac{x}{(3t)^{1/3}} \right), \quad (1.2.10)$$

où Ai désigne la *fonction d'Airy* qui est une fonction spéciale donnée par

$$Ai(x) := \frac{1}{\pi} \int_0^{+\infty} \cos(\xi x + \frac{1}{3}\xi^3) d\xi, \quad (1.2.11)$$

dont le comportement asymptotique quand $x \rightarrow +\infty$ et quand $x \rightarrow -\infty$ détermine les propriétés de scattering linéaire de (gKdV).

Comme la fonction d'Airy est bornée, on en déduit que pour tout $t \neq 0$,

$$\|K_t\|_{L^\infty} \leq \frac{C}{|t|^{1/3}}. \quad (1.2.12)$$

En combinant cette dernière observation avec le fait que (Airy) préserve la norme L^2 d'une solution, et en utilisant le théorème d'interpolation complexe de Riesz-Thorin, on obtient une propriété mathématique qui traduit la dispersion de (Airy).

Proposition 1.1 (Dispersion ponctuelle). *Soit $t \neq 0$ et $p \geq 2$, alors pour toute solution $u(t)$ de (Airy) telle que $u(0) = u_0$, on a*

$$\|u(t)\|_{L^p} \leq \frac{C}{|t|^{\frac{1}{3}(\frac{1}{p'} - \frac{1}{p})}} \|u_0\|_{L^{p'}}, \quad (1.2.13)$$

où p' est défini par $\frac{1}{p} + \frac{1}{p'} = 1$.

En particulier, pour tout $p > 2$, $\|u(t)\|_{L^p}$ tend vers 0 lorsque $|t| \rightarrow +\infty$, ce qui traduit bien le fait qu'une solution a tendance à s'aplatir.

On peut également obtenir des énoncés plus sophistiqués pour la dispersion :

Théorème 1.2 (Kenig-Ponce-Vega [59]). *Soit u une solution de (Airy) telle que $u(0) = u_0$.*

Alors

$$\|D_x^\alpha u(t)\|_{L_x^p L_t^q} \leq C \|u_0\|_{L^2}, \quad (1.2.14)$$

où D_x^α est défini par

$$\widehat{D_x^\alpha u}(\xi) = |\xi|^\alpha \widehat{u}(\xi), \quad (1.2.15)$$

et (p, q, α) est un triplet admissible, dans le sens où

$$\frac{1}{p} + \frac{1}{2q} = \frac{1}{4}, \quad \alpha = \frac{2}{q} - \frac{1}{p}, \quad 1 \leq p, q \leq \infty, \quad -1/4 \leq \alpha \leq 1. \quad (1.2.16)$$

Remarquons que la condition définissant l'admissibilité de (p, q, α) peut partiellement être justifiée par les symétries de l'équation. En effet, pour $\lambda > 0$, si $u(t, x)$ est solution de (Airy), alors $v(t, x) = u(\frac{t}{\lambda}, \frac{x}{\lambda^3})$ aussi. L'inégalité démontrée par le théorème doit être homogène par rapport à ce changement d'échelle. On a que $\|v(0)\|_{L^2} = \lambda^{1/2} \|u(0)\|_{L^2}$, donc on devrait nécessairement avoir que $\|D_x^\alpha v(t)\|_{L_x^p L_t^q} = \lambda^{1/2} \|D_x^\alpha u(t)\|_{L_x^p L_t^q}$. On voit que ceci équivaut à $\frac{1}{p} - \alpha + \frac{3}{q} = \frac{1}{2}$, ce qui est une conséquence de l'admissibilité de (p, q, α) .

Cela traduit bien le fait que les différentes normes de u ont tendance à tendre vers 0, ce qui est en accord avec l'intuition physique. En effet, si les différentes fréquences se propagent à des vitesses différentes, une solution (constituée par un ensemble de telles fréquences) a tendance à s'aplatir, parce que les paquets d'ondes se séparent.

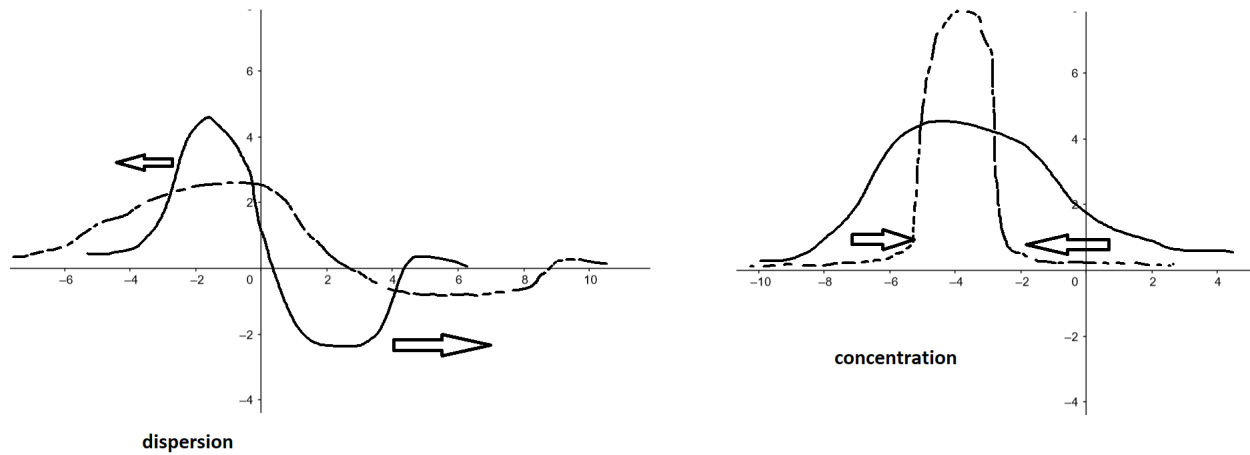


FIGURE 1.2.1 – On représente en traits pleins l’allure d’une solution pour des temps petits et en pointillés l’allure d’une solution pour des temps plus grands. La dispersion a tendance à uniformiser la répartition de la masse de la solution, alors que la concentration a tendance à la concentrer en un seul point. Le soliton, dont le profil ne change pas, est un équilibre entre les deux phénomènes.

1.2.5 Concurrence entre dispersion et concentration

Il est important de souligner que la dispersion est un phénomène qui est causé par la partie linéaire de l’équation. Lorsque l’on considère (gKdV) sous sa forme focalisante (non linéaire), la partie non linéaire de l’équation est à l’origine d’un autre phénomène qui est la *concentration*. Ce phénomène entre en concurrence avec la dispersion. En effet, la dispersion a tendance à disperser la masse de la solution, donc de la répartir sur des portions de plus en plus grandes de l’espace et vers la gauche dans le cas de (gKdV) ; alors que la concentration a tendance au contraire à concentrer la masse de la solution autour d’un certain point. Ainsi, dans le cas où (gKdV) est focalisante, la dispersion ne suffit pas pour expliquer le comportement de ses solutions et on peut observer des phénomènes intéressants s’expliquant par un équilibre formé entre la dispersion et la concentration.

Notamment, l’existence des solitons dans le cas focalisant est une illustration de l’équilibre entre ces deux phénomènes. S’agissant d’une bosse qui se propage à une vitesse constante dans le sens contraire à la dispersion sans déformation, cet équilibre s’exprime dans le fait que sa forme ne change pas ; elle n’est ni aplatie avec le temps (ce qui aurait témoigné d’une action de la dispersion), ni concentrée autour de son maximum (ce qui aurait témoigné d’une action de la concentration), l’équilibre est donc parfait.

Le phénomène de concentration peut donner lieu à une explosion en temps fini d’une solution. Cependant, pour (gKdV), ceci n’arrive pas dans le cas L^2 -sous-critique pour des espaces de Sobolev suffisamment réguliers (intuitivement, on peut le comprendre en se disant qu’une nonlinéarité de degré petit engendre un phénomène de concentration trop faible qui est insuffisant pour faire exploser une solution), on a existence globale pour le problème de Cauchy (cf. paragraphe suivant). Cependant, dans le cas L^2 -critique et L^2 -sur-critique, on peut s’attendre à ce qu’une solution puisse exploser dans les espaces de Sobolev usuels (la nonlinéarité est de degré plus grand et donne lieu donc à un phénomène de concentration plus fort) dans le cas où elle n’est pas assez petite. Pour le moment, on ne dispose de résultats d’explosion que pour le cas L^2 -critique grâce aux travaux de Martel et Merle [77, 75, 76] et de Martel, Merle et Raphaël [83, 84, 85, 86].

1.2.6 Problème de Cauchy

Le problème de Cauchy est localement bien posé pour (gKdV) lorsque la donnée initiale est dans un espace de Sobolev suffisamment régulier. La régularité requise de l'espace de Sobolev requise a été étudiée par Kenig, Ponce et Vega [58]. Grâce aux lois de conservation de (gKdV) (la masse et l'énergie), ils ont été en mesure d'établir que le problème de Cauchy est globalement bien posé dans H^1 dans le cas L^2 -sous-critique. Il ne pouvaient pas obtenir l'existence globale dans un espace de Sobolev moins régulier car l'utilisation de l'énergie requiert une régularité au moins H^1 . Ce résultat est largement suffisant pour les besoins de cette thèse : on aura juste besoin du fait que le problème de Cauchy est globalement bien posé dans H^2 pour (mKdV).

Ensuite, d'autres méthodes ont été trouvées pour établir le problème de Cauchy global, ce qui a permis d'obtenir l'existence globale dans des espaces ayant une régularité plus faible que H^1 . Pour (KdV) et (mKdV) cela a été fait par Colliander, Keel, Staffilani, Takaoka et Tao [23]. Citons le résultat ainsi obtenu dans le cadre de (mKdV) :

Théorème 1.3 (Kenig-Ponce-Vega [58] et Colliander-Keel-Staffilani-Takaoka-Tao [23]). *Soit $s > \frac{1}{4}$. Alors le problème de Cauchy pour (mKdV) est globalement bien posé pour une donnée initiale dans H^s . Plus précisément, pour $u_0 \in H^s(\mathbb{R})$, il existe une unique solution de (mKdV) $u \in \mathcal{C}(\mathbb{R}, H^s(\mathbb{R}))$ telle que $u(0) = u_0$. De plus, pour tout $t \in \mathbb{R}$, $\|u(t)\|_{H^s} \leq C\|u_0\|_{H^s}$ pour une certaine constante $C > 0$. L'application $u_0 \in H^s(\mathbb{R}) \mapsto u(t) \in H^s(\mathbb{R})$ qui à une donnée initiale u_0 associe la solution $u(t)$ en un certain temps t de (mKdV) est lipschitzienne.*

Pour $s = \frac{1}{4}$, d'après [58], la continuité uniforme en la donnée initiale énoncée dans le théorème précédent est toujours vraie. [58] donne également une liste d'estimées vérifiées par une solution de (mKdV).

On ne peut pas espérer obtenir le même résultat dans un cadre d'espaces de Sobolev moins réguliers. En effet, on a :

Théorème 1.4 (Kenig-Ponce-Vega [60]). *Soit $s < \frac{1}{4}$. Alors, l'application $u_0 \in H^s(\mathbb{R}) \mapsto u(t) \in H^s(\mathbb{R})$ qui à une donnée initiale u_0 associe la solution $u(t)$ en un certain temps t de (mKdV) n'est pas uniformément continue.*

On peut toutefois avoir un résultat d'existence globale plus faible dans des espaces de Sobolev moins réguliers. Un tel résultat a été trouvé par Harrop-Griffiths, Killip et Visan [46]. Il garantit l'existence globale pour le problème de Cauchy dans H^s pour $s \geq -\frac{1}{2}$, cependant il n'y a plus d'uniforme continuité en la donnée initiale. Notons également que dans ce cadre, le sens de la nonlinéarité et donc de l'EDP n'est plus très clair. En effet, si $s < 1/6$ et $u \in H^s$, d'après les injections de Sobolev, on trouve que $u \in L^q$ avec $q < 3$, mais dans ce cas u^3 n'est a priori pas dans L^1_{loc} . C'est pour cela que la notion de solution est définie par l'unique prolongement de l'application continue définissant la notion de solution dans un cadre où le sens de celle-ci est clair.

Théorème 1.5 (Harrop-Griffiths-Killip-Visan [46]). *Soit $s > -\frac{1}{2}$. Alors, (mKdV) est globalement bien posée pour toute donnée initiale dans $H^s(\mathbb{R})$ dans le sens où le flot Φ s'étend de façon unique de l'espace de Schwartz à une application continue $\Phi : \mathbb{R} \times H^s(\mathbb{R}) \rightarrow H^s(\mathbb{R})$.*

Harrop-Griffiths, Killip et Visan [46] montrent également que $s = -\frac{1}{2}$ est nécessairement exclu d'un tel énoncé d'existence globale. De plus, Birnir, Ponce et Svanstedt [15] ont montré que le problème de Cauchy pour (mKdV) est mal posé dans $H^s(\mathbb{R})$ pour $s < -\frac{1}{2}$.

Signalons également que dans un cadre similaire, le problème de Cauchy est globalement bien posé dans H^s pour $s > -\frac{3}{4}$ pour (KdV) [58, 23], et même pour $s \geq -1$ [56, 61] si l'on accepte de perdre l'uniforme continuité en la donnée initiale [60]. Le problème de Cauchy est localement bien posé dans H^s pour $s \geq \frac{1}{12}$ et globalement bien posé dans H^s pour $s \geq 1$ lorsque $p = 4$ [58]. Pour $p \geq 5$, il existe des résultats d'existence locale dans H^s (où s est l'exposant critique donné par (1.2.2)) et des résultats d'existence globale pour des données initiales suffisamment petites [58]. En particulier, pour $p = 5$ (dans le cas L^2 -critique), (gKdV) est localement bien posée dans L^2 .

1.2.7 Scattering linéaire

On dispose de résultats permettant de dire que sous certaines conditions le comportement d'une solution de (gKdV) en temps long est celui d'une solution de (Airy), autrement dit le comportement d'une solution en temps long est essentiellement linéaire. Il s'agit de résultats de *scattering linéaire*. Le phénomène de scattering linéaire est favorisé par des nonlinéarités qui sont petites pour des solutions petites et comme indiqué chez Hayashi et Naumkin [49, 47] l'exposant $p = 3$ joue un rôle critique en termes de scattering linéaire. Ainsi, pour $p = 2$, on ne connaît pas de résultat de scattering linéaire, alors que pour $p > 3$, on dispose d'un résultat de scattering linéaire pour des solutions petites dans un espace de Sobolev à poids d'après Hayashi et Naumkin [48] (les solutions pour lesquelles le scattering linéaire s'applique doivent être plus petites que tout soliton dans l'espace de Sobolev à poids considéré). Tao [113] a étudié un résultat de scattering linéaire au voisinage des solitons pour $p = 4$. Pour $p = 3$, Hayashi et Naumkin [49, 47] ont étudié à quel point peut-on espérer avoir un résultat de scattering modifié. Il s'agit d'un résultat limité par les solitons dans le cas focalisant et par les solutions auto-similaires dans le cas défocalisant (ce sont deux types de solutions qui ne vérifient pas de résultat de scattering modifié).

Le fait que $p = 3$ joue un rôle critique dans les résultats de scattering linéaire s'explique par le comportement asymptotique de la fonction d'Airy. Plus précisément, (1.2.12) nous permet de déduire que

$$\|Ai^{p-2}Ai'\|_{L^\infty} \leq \frac{C}{t^{p/3}}. \quad (1.2.17)$$

Le terme non linéaire de (gKdV) peut être vu comme Vu où $V = pu^{p-2}u_x$. Et il se trouve qu'une condition qui permet d'espérer un résultat de scattering linéaire est $V \in L_t^1 L_x^\infty$. Or, dans le cas de la recherche d'un résultat de scattering linéaire, le comportement de la solution u peut être résumé par le comportement de la fonction d'Airy, donc d'après (1.2.17), il est raisonnable de s'attendre à un résultat de scattering linéaire pour $p > 3$. Le cas $p = 3$ joue un rôle de cas tampon pour lequel on peut obtenir un résultat de scattering modifié car Ai' croît avec beaucoup d'oscillations.

1.3 Intégrabilité et scattering inverse

1.3.1 Remarque sur l'utilisation de l'intégrabilité dans cette thèse

(KdV), (mKdV) et l'équation de Gardner sont des EDP intégrables. Cela signifie que l'on peut calculer explicitement les solutions génériques de ces équations. Toutefois, pour démontrer les résultats de cette thèse, on n'utilise pas l'intégrabilité de (mKdV), mis à part le fait qu'on utilise l'énergie seconde (une loi de conservation supplémentaire) dont l'existence est spécifique au cas intégrable, bien que le fait qu'on l'utilise est justifié par le fait que l'on s'intéresse à des objets ayant une structure H^2 . Ainsi, on peut dire que les raisonnements utilisés dans cette thèse ne sont pas spécifiques au cas intégrable mais sont réalisés à un niveau H^2 (alors qu'en général, dans les cas non intégrables, on ne peut que raisonner à un niveau H^1).

1.3.2 Lois de conservation supplémentaires pour (KdV) et (mKdV)

Pour (KdV), (mKdV) et l'équation de Gardner, la liste des lois de conservation ne s'arrête pas à la norme L^1 (1.2.3), la masse (1.2.4) et l'énergie (1.2.5). Elles ont en fait une infinité de lois de conservation, ce qui est dû à leur *intégrabilité*. Pour les besoins de cette thèse, nous avons besoin d'énoncer une loi de conservation supplémentaire pour (mKdV) car on travaillera dans l'espace H^2 (les fonctions de carré intégrable avec deux dérivées spatiales), donc l'utilité d'utiliser une loi de conservation faisant intervenir une dérivée seconde est justifié. On l'appelle parfois *énergie seconde*.

Pour (KdV), l'énergie seconde (découverte par Kruskal et Zabusky [124]) s'exprime comme suit :

$$I_{4,KdV}(t) := \frac{1}{5} \int u_{xx}^2 - \frac{2}{3} \int uu_x^2 + \frac{1}{9} \int u^4 = I_4(0), \quad \text{lorsque } p = 2. \quad (1.3.1)$$

Pour (mKdV) (et c'est dans ce cas qu'on s'en servira), l'énergie seconde (qui se déduit de celle de (KdV) grâce à la transformée de Miura [90]) a l'expression suivante :

$$I_4(t) := \frac{1}{2} \int u_{xx}^2 - \frac{5}{2} \int u^2 u_x^2 + \frac{1}{4} \int u^6 = I_4(0), \quad \text{lorsque } p = 3. \quad (1.3.2)$$

L'énergie seconde, qui est souvent notée $F[u](t)$, ne se généralise pas à (gKdV), ce qui fait qu'on a moins d'outils lorsqu'on sort du cas intégrable.

On peut également énoncer la loi de conservation avec une dérivation d'ordre 3 qui a elle aussi été trouvée par Kruskal et Zabusky pour (KdV) et dont la version pour (mKdV) se trouve grâce à la transformation de Miura. On a :

$$I_{5,KdV}(t) := \frac{3}{7} \int u_{xxx}^2 - 2 \int uu_{xx}^2 + \frac{10}{3} \int u^2 u_x^2 - \frac{2}{9} \int u^5, \quad \text{lorsque } p = 2, \quad (1.3.3)$$

et

$$I_5(t) := \frac{1}{5} \int u_{xxx}^2 + \frac{7}{10} \int u_x^4 - \frac{7}{5} \int u^2 u_{xx}^2 + \frac{7}{2} \int u^4 u_x^2 - \frac{1}{8} \int u^8, \quad \text{lorsque } p = 3. \quad (1.3.4)$$

Une liste de lois de conservation pour (KdV) et (mKdV) a été donnée par Miura, Gardner et Kruskal dans [92]. Ils y ont par ailleurs montré qu'il existe une suite infinie de lois de conservation pour (KdV) et (mKdV).

1.3.3 Paires de Lax pour (KdV) et (mKdV) et méthode de scattering inverse

C'est Lax [67] qui a provoqué un tournant dans la compréhension de l'intégrabilité de (KdV) en trouvant une paire de Lax (qui est une paire d'opérateurs différentiels) pour (KdV) qui décrit l'évolution de l'équation et qui permet d'en déduire une infinité de lois de conservation pour celle-ci.

Une *paire de Lax* correspond à deux opérateurs différentiels (en x) $L := L(t)$ auto-adjoint et $M := M(t)$ antisymétrique qui sont définis à partir de u tels que le fait que u vérifie l'équation considérée est équivalent à : $[\partial_t + M, L] = 0$, ce qui est équivalent à $L_t = [L, M]$, autrement dit l'équation est équivalente à la commutation de deux opérateurs différentiels. Le travail de Lax [67] a consisté à trouver ces deux opérateurs pour (KdV) et à montrer leur intérêt pour l'étude de la conjecture de résolution en solitons et pour l'étude de l'interaction de deux solitons.

Pour (KdV), la paire de Lax est donnée par les opérateurs suivants :

$$L_{KdV} = \partial_x^2 + \frac{u}{3}, \quad M_{KdV} = 4 \left(\partial_x^3 + \frac{u}{2} \partial_x + \frac{u_x}{4} \right), \quad (1.3.5)$$

où u désigne une solution de (KdV). L'opérateur $\partial_x^2 + \frac{u}{3}$ porte le nom d'*opérateur de Schrödinger*.

Pour (mKdV) focalisant, d'après Wadati [117], la paire de Lax est donnée par les opérateurs :

$$L = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \partial_x + \begin{pmatrix} 0 & \frac{u}{\sqrt{2}} \\ \frac{u}{\sqrt{2}} & 0 \end{pmatrix}, \quad (1.3.6)$$

et

$$M = 4 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \partial_x^3 + 2 \begin{pmatrix} 3 \left(\frac{u^2}{2} - i \frac{u_x}{\sqrt{2}} \right) & 0 \\ 0 & 3 \left(\frac{u^2}{2} + i \frac{u_x}{\sqrt{2}} \right) \end{pmatrix} \partial_x + \begin{pmatrix} 3 \left(uu_x - i \frac{u_{xx}}{\sqrt{2}} \right) & 0 \\ 0 & 3 \left(uu_x + i \frac{u_{xx}}{\sqrt{2}} \right) \end{pmatrix}, \quad (1.3.7)$$

où u désigne une solution de (mKdV).

Parmi l'infinité de lois de conservation garanties par la paire de Lax, on a énoncé les quelques premières dans la sous-section précédente. L'existence de n lois de conservation nous donne la possibilité d'étudier la stabilité d'un n -soliton dans H^n , comme cela a été par Maddocks et Sachs [70] pour (KdV), puis par Le Coz et Wang [69] pour (mKdV).

L'équation $L_t = [L, M]$ implique que l'ensemble des valeurs propres de l'opérateur auto-adjoint L ne dépend pas du temps. Les valeurs propres de L sont des quantités conservées en temps associées à une solution u de l'équation. L'étude de ces valeurs propres et des fonctions propres associées est à la base de la méthode de scattering inverse. Elle donne la possibilité de décrire le comportement asymptotique de u à partir de l'étude de l'ensemble des valeurs propres de L , et on arrive à produire la décomposition asymptotique en solitons de u à partir d'eux.

Les articles de Tanaka [111], de Wadati [117] ou d'Ablowitz, Kaup, Newell et Segur [1] explicitent la méthode de scattering inverse qui permet de trouver des formules pour les solutions des équations intégrables en passant par un problème à valeurs propres dans laquelle la solution de l'équation apparaît comme un potentiel.

On indiquera plus loin que la méthode de scattering inverse permet aussi de traiter la conjecture de résolution en solitons.

Il semble que (KdV), (mKdV) et Gardner sont les seules équations de la famille (gKdV) qui sont intégrables comme nous l'indiquent les résultats de non-élasticité de la collision des solitons développés dans la section suivante.

1.4 Solutions particulières : solitons, breathers

1.4.1 Solitons

Le soliton est la motivation première pour l'étude des équations dispersives, notamment (KdV). On peut dire que ces équations (construites à partir de considérations physiques pour les plus simples d'entre elles) ont été construites pour que les solitons puissent en être solutions. Dans le cas de (gKdV), il est important de remarquer qu'un soliton ne peut se déplacer que vers la droite, c'est-à-dire dans le sens contraire à la dispersion. L'expression mathématique de ce fait heuristique est le découplage entre les solitons et la dispersion, appelé *propriété de monotonie*.

Définition 1.6. On appelle *soliton* de vitesse c une solution R de (gKdV) telle qu'il existe une fonction $Q_c \in H^1(\mathbb{R})$ non nulle qui vérifie

$$R(t, x) = Q_c(x - ct). \quad (1.4.1)$$

A partir de (gKdV), on peut établir que Q_c satisfait l'équation elliptique suivante :

$$(Q_c)_{xx} - cQ_c + Q_c^p = 0. \quad (\text{GS})$$

Il vérifie également

$$(Q_c)_x^2 - cQ_c^2 + \frac{2}{p+1}Q_c^{p+1} = 0. \quad (1.4.2)$$

Q_c porte le nom d'*état fondamental* ou *ground state* en anglais. En particulier, $R(t, x) = Q_c(x - ct)$ vérifie (gKdV) si et seulement si Q_c vérifie (GS).

On peut établir que les solitons n'existent que dans le cas focalisant et que pour des vitesses $c > 0$ strictement positives. De plus, pour tout $c > 0$, il existe un unique soliton de (gKdV) de vitesse c à translation près (quelque soit p), et à réflexion près par rapport à l'axe des abscisses dans le cas où p est impair. Il existe un unique état fondamental de vitesse $c > 0$ qui soit strictement positif et pair dans le cas focalisant. Par convention, c'est cette solution là qu'on désignera par Q_c par la suite.

Avec cette convention, on a :

$$\partial_c Q_c = \frac{1}{2c} (Q_c + x(Q_c)_x). \quad (1.4.3)$$

On peut alors désigner un soliton par ses paramètres. Un soliton de paramètre de forme $c > 0$ (c'est aussi sa vitesse) et de paramètre de translation x_0 se définit par :

$$R_c(t, x; x_0) := Q_c(x - ct - x_0). \quad (1.4.4)$$

Dans le cas où p est impair, ce qui est notamment le cas de (mKdV), on peut aussi rajouter un paramètre $\kappa \in \{-1, 1\}$ qui désigne le signe du soliton (chose qu'on fera dans cette thèse) :

$$R_{c,\kappa}(t, x; x_0) := \kappa Q_c(x - ct - x_0). \quad (1.4.5)$$

Un soliton négatif ($\kappa = -1$) est parfois appelé *antisoliton*.

On peut trouver la formule explicite pour Q_c grâce à des techniques standard sur les équations différentielles (et aussi par la méthode de scattering inverse) :

$$Q_c(x) = \left(\frac{c(p+1)}{2 \cosh^2\left(\frac{p-1}{2}\sqrt{cx}\right)} \right)^{\frac{1}{p-1}}. \quad (1.4.6)$$

Notamment, pour (KdV), l'état fondamental s'exprime comme suit :

$$Q_{c,KdV}(x) = \frac{3c}{2 \cosh^2\left(\frac{\sqrt{c}}{2}x\right)}, \quad \text{lorsque } p = 2, \quad (1.4.7)$$

pour (mKdV), l'état fondamental s'exprime ainsi :

$$Q_c(x) = \frac{\sqrt{2c}}{\cosh(\sqrt{c}x)}, \quad \text{lorsque } p = 3. \quad (1.4.8)$$

Notamment, on peut remarquer que plus c est grand, plus un soliton est grand et étroit. L'amplitude d'un soliton est proportionnelle à $c^{\frac{1}{p-1}}$, la largeur est d'ordre $c^{-1/2}$. La formule implique immédiatement qu'un soliton est exponentiellement localisé :

$$|Q_c(x)| \leq \frac{M}{\exp(\sqrt{c}|x|)}, \quad (1.4.9)$$

où $M > 0$ est une certaine constante proportionnelle à $c^{\frac{1}{p-1}}(p+1)^{\frac{1}{p-1}}$. Cette estimée justifie notamment l'appellation *paramètre de forme* pour c .

Un changement d'échelle permet de se ramener de Q_c à $Q = Q_1$ et permet de trouver les valeurs prises par les lois de conservation en Q_c pour (mKdV). On renvoie au Chapitre 4 pour les valeurs.

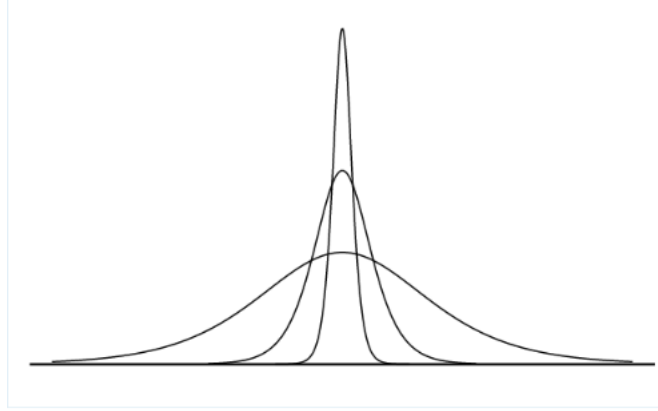


FIGURE 1.4.1 – On représente trois solitons de vitesses c distinctes. Plus c est grand, plus le soliton est grand et étroit (et rapide).

1.4.2 Multi-soliton pour (KdV) ou (mKdV)

On peut s'intéresser aux objets consistant en une superposition de plusieurs solitons. Soient $0 < c_1 < \dots < c_K$ et $x_1, \dots, x_K \in \mathbb{R}$ les paramètres de forme et de translation des solitons R_1, \dots, R_K . Autrement dit, pour $1 \leq k \leq K$, on pose

$$R_k(t, x) := R_{c_k}(t, x; x_k). \quad (1.4.10)$$

Définition 1.7. Un *multi-soliton* est une solution $r \in \mathcal{C}([T^*, +\infty[, H^1(\mathbb{R}))$ de (gKdV) telle qu'il existe des solitons R_1, \dots, R_K de (gKdV) tels que

$$\left\| r(t) - \sum_{k=1}^K R_k(t) \right\|_{H^1} \xrightarrow{t \rightarrow +\infty} 0. \quad (1.4.11)$$

La méthode de scattering inverse donne la possibilité de trouver des formules pour les multi-solitons pour (KdV) et (mKdV) [119, 117]. On renvoie au chapitre 4 pour la formule d'un multi-soliton de (mKdV). Donnons la formule d'un multi-soliton de (KdV).

Soient des paramètres $\delta_1, \dots, \delta_K \in \mathbb{R}$. On définit $V(t, x)$ une matrice $K \times K$ dont les coefficients sont donnés de la manière suivante :

$$V_{i,j}(t, x) := \frac{2(c_i c_j)^{1/4}}{\sqrt{c_i} + \sqrt{c_j}} \exp\left(\frac{\sqrt{c_i}}{2}(x - c_i t - \delta_i)\right) \exp\left(\frac{\sqrt{c_j}}{2}(x - c_j t - \delta_j)\right). \quad (1.4.12)$$

Un K -soliton de (KdV), constitué de solitons R_1, \dots, R_K dont les paramètres de translation sont donnés par

$$x_i = \delta_i - \frac{2}{\sqrt{c_i}} \sum_{j=1}^{i-1} \ln\left(\frac{\sqrt{c_i} - \sqrt{c_j}}{\sqrt{c_i} + \sqrt{c_j}}\right), \quad (1.4.13)$$

pour tout $1 \leq i \leq K$. Ce K -soliton est alors donné par la formule suivante :

$$r(t, x) := 6 \frac{\partial^2}{\partial x^2} \ln(\det(I + V(t, x))), \quad (1.4.14)$$

où I désigne la matrice identité $K \times K$.

En particulier, on peut simplement énoncer la formule d'un 2-soliton de (KdV) :

$$r(t, x) := 6 \frac{\partial^2}{\partial x^2} \ln \left(1 + e^{\sqrt{c_1}(x-c_1t-\delta_1)} + e^{\sqrt{c_2}(x-c_2t-\delta_2)} + \left(\frac{\sqrt{c_1} - \sqrt{c_2}}{\sqrt{c_1} + \sqrt{c_2}} \right)^2 e^{\sqrt{c_1}(x-c_1t-\delta_1)} e^{\sqrt{c_2}(x-c_2t-\delta_2)} \right). \quad (1.4.15)$$

Les formules impliquent en particulier que lorsque r est un multi-soliton associé aux solitons R_1, \dots, R_K , alors

$$\left\| r(t) - \sum_{k=1}^K \overline{R_k}(t) \right\|_{H^1} \xrightarrow{t \rightarrow -\infty} 0, \quad (1.4.16)$$

où $\overline{R_k}$ est une translation de R_k . Plus précisément, dans le cas de (KdV) (le cas de (mKdV) sera considéré dans la proposition 1.14),

$$\overline{R_k}(t, x) := R_{c_k}(t, x; x_k^{-\infty}), \quad (1.4.17)$$

où

$$x_k^{-\infty} := x_k - \frac{2}{\sqrt{c_k}} \sum_{j=k+1}^K \ln \left(\frac{\sqrt{c_j} - \sqrt{c_k}}{\sqrt{c_j} + \sqrt{c_k}} \right) + \frac{2}{\sqrt{c_k}} \sum_{j=1}^{k-1} \ln \left(\frac{\sqrt{c_k} - \sqrt{c_j}}{\sqrt{c_k} + \sqrt{c_j}} \right). \quad (1.4.18)$$

Ainsi, dans le cas intégrable, les solitons préservent leurs paramètres de forme et subissent seulement des translations après collisions les uns avec les autres. On dit que la collision entre deux solitons est *élastique* dans le cas intégrable. Il s'agit d'une confirmation de l'observation numérique réalisée par Kruskal et Zabusky [125].

Cette dernière propriété est spécifique au cas intégrable. En effet, pour $p = 4$, d'après Martel et Merle [81, 82], la collision entre deux solitons n'est pas élastique : il est établi que les paramètres de forme des solitons sont modifiés après collision (ceci dit, cette modification est limitée, donc on est quand même très proche de l'élasticité dans les cas non intégrables). Muñoz [93] a généralisé ce résultat pour une classe plus large de nonlinéarités non intégrables. En particulier, on peut affirmer que les équations pour lesquelles les collisions entre solitons ne sont pas élastiques sont non intégrables.

Les méthodes permettant de trouver les formules pour les solutions de (KdV) par scattering inverse ont engendré la *conjecture de résolution en solitons*. Celle-ci dit que toute solution générique de (KdV) est un multi-soliton modulo un terme de dispersion. Pour (KdV), celle-ci a été établie par Schuur [107] en utilisant la méthode de scattering inverse. Dans le cas de (mKdV), il s'agit d'une résolution en solitons et en breathers, qui sera considérée plus loin.

1.4.3 Breather pour (mKdV)

En 1973, grâce à la méthode de scattering inverse, un nouveau type de solution de (mKdV) est découvert par Wadati [117] : le *breather*. Il s'agit d'une fonction périodique en temps qui se propage à vitesse constante. Tout comme le soliton, un breather est exponentiellement localisé, mais la propagation contient un phénomène périodique. Un breather de paramètres de forme $\alpha, \beta > 0$ et de paramètres de translation $x_1, x_2 \in \mathbb{R}$ a pour expression :

$$B_{\alpha, \beta}(t, x; x_1, x_2) := 2\sqrt{2} \partial_x \left[\arctan \left(\frac{\beta \sin(\alpha y_1)}{\alpha \cosh(\beta y_2)} \right) \right], \quad (1.4.19)$$

où

$$y_1 := x + \delta t + x_1 \quad (1.4.20)$$

désigne la phase,

$$y_2 := x + \gamma t + x_2 \quad (1.4.21)$$

désigne l'éloignement du centre,

$$\delta := \alpha^2 - 3\beta^2 \quad (1.4.22)$$

désigne l'opposé de la pulsation et

$$\gamma := 3\alpha^2 - \beta^2 \quad (1.4.23)$$

désigne l'opposé de la vitesse. On peut observer que le paramètre α détermine la fréquence spatiale du phénomène périodique et que le paramètre β détermine la décroissance exponentielle du breather. Ainsi, on peut en déduire la localisation exponentielle d'un breather :

$$|B_{\alpha,\beta}(t, x; x_1, x_2)| \leq K(\alpha, \beta) \exp(-\beta|x + x_2 + \gamma t|), \quad (1.4.24)$$

où $K(\alpha, \beta) > 0$ est une constante qui dépend de α et de β .

Contrairement aux solitons, le signe de la vitesse $-\gamma$ d'un breather peut être quelconque. Ainsi, les arguments de monotonie ne s'appliquent pas pour les breathers en général. Toutefois, ils s'appliquent pour des breathers à vitesses strictement positives (qui vont dans le sens contraire à la dispersion). Notons qu'il n'y a pas lieu de parler d'« antibreathers » car quand on remplace le paramètre x_1 en $x_1 + \frac{\pi}{\alpha}$, on transforme un breather en son opposé.

Les valeurs des lois de conservation de (mKdV) prises en $B_{\alpha,\beta}$ sont énoncées dans le **Chapitre 4**. Notons que la formule de l'énergie seconde d'un breather semble être énoncée pour la première fois ici.

Les breathers ont été observés expérimentalement dans les fluides stratifiés (voir l'article de Lamb, Polukhina, Talipova, Pelinovsky, Xiao et Kurkin [65]). Il s'agit donc d'une solution qui a un intérêt physique.

Il est aussi intéressant de noter l'intérêt mathématique des breathers pour l'étude de (mKdV). En effet, Kenig, Ponce et Vega [60] les ont utilisés pour montrer que (mKdV) est mal posé pour des données insuffisamment régulières. Il s'agit du théorème 1.4. Pour montrer que le flot de (mKdV) n'est pas uniformément continu en la donnée initiale dans H^s pour $s < \frac{1}{4}$, ils ont établi que deux breathers proches dans H^s pour $t = 0$ peuvent s'éloigner aussi vite qu'on le veut dans H^s en choisissant des paramètres α suffisamment grands pour cela, lorsque $s < \frac{1}{4}$ (heuristiquement, plus les oscillations sont fortes, plus le comportement des breathers est chaotique).

Il est important de noter que les breathers (on définit un breather comme une solution spatialement localisée et périodique en temps, sans être constante, aux symétries de l'équation près) sont spécifiques au cas de (mKdV) et de l'équation de Gardner, on renvoie à l'article de Ponce et Muñoz [94] pour plus de détails. En particulier, il n'y en a pas pour (KdV) ou pour $p = 4$.

Il existe une auto-transformation définie sur les solutions de (mKdV), appelée *transformation de Bäcklund* (voir Lamb [64]). Les transformations de Bäcklund existent aussi pour (KdV), pour sine-Gordon, pour (NLS) unidimensionnel cubique et peuvent exister pour d'autres équations intégrables. Dans le cas de (mKdV), elle a pour propriété remarquable de transformer un breather en un soliton complexe et un soliton complexe en 0 (voir Alejo et Muñoz [9]), ce qui peut permettre de ramener un problème au voisinage d'un breather en un problème au voisinage de 0.

1.4.4 Résolution en solitons-breathers comme conséquence du scattering inverse

Quand on décrit la dynamique d'une solution de (mKdV) en temps grand, on peut séparer l'espace en trois régions : la région de la dispersion correspondant à $x < 0$, avec la dispersion qui se propage avec une vitesse négative par rapport au temps ; la région *auto-similaire* correspondant à $|x/t^{1/3}| = \mathcal{O}(1)$, avec des solutions auto-similaires dont la position est de l'ordre de $t^{1/3}$; et enfin la région des solitons correspondant à $x > 0$, où les

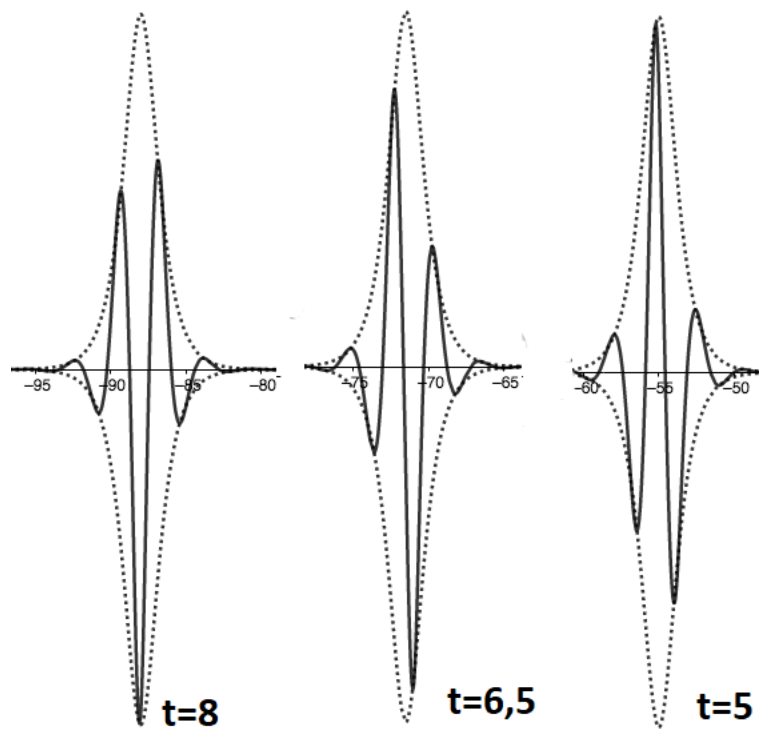


FIGURE 1.4.2 – On représente la propagation d'un breather de paramètres $\alpha = 2$ et $\beta = 1$ à trois instants différents, représenté en traits pleins. Les pointillés représentent l'enveloppe exponentielle du breather. On a là un exemple de breather qui se propage vers la gauche.

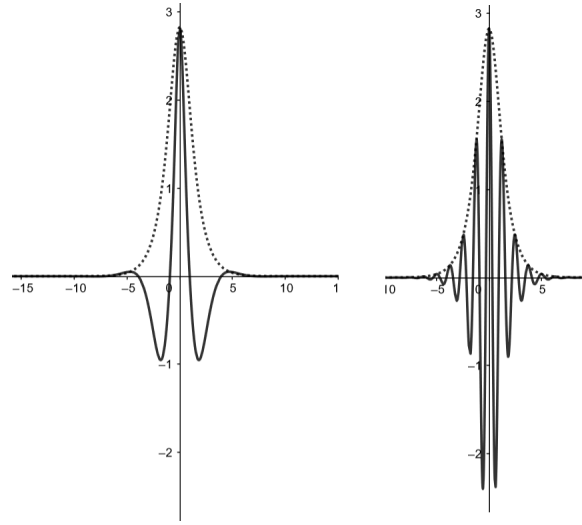


FIGURE 1.4.3 – Influence du paramètre α d'un breather. À gauche, $\alpha = 1$. À droite, $\alpha = 5$. Dans les deux cas, $\beta = 1$ (l'influence du paramètre β est semblable à l'influence du paramètre c pour les solitons).

solitons se propagent avec une vitesse positive par rapport au temps. Les breathers, quant à eux, peuvent exister dans la région de la dispersion et dans la région des solitons et se propagent avec une vitesse de signe quelconque par rapport au temps. Cette séparation de l'espace permet de décrire de façon satisfaisante le comportement d'une solution générique de (mKdV). En effet, une solution générique de (mKdV) se présente comme un certain nombre de solitons situés dans la région des solitons, un certain nombre de breathers situés dans la région des solitons et dans la région de la dispersion (avec éventuellement des breathers de vitesse nulle qui restent à l'intérieur de la région auto-similaire) et un terme de radiation qui tend vers 0 dans la région des solitons, se comporte comme une solution de l'équation de Painlevé II dans la région auto-similaire et se comporte quasiment comme une solution linéaire dans la région de la dispersion sauf au voisinage des breathers où l'influence du breather modifie le comportement du terme de radiation. C'est la *conjecture de résolution en solitons-breathers pour* (mKdV), décrite par Chen et Liu [21]. Dans le cas défocalisant, un résultat similaire a été obtenu par Deift et Zhou [33]. On renvoie aussi à Harrop-Griffiths [45] et Germain, Pusateri et Rousset [41] qui sont des articles qui utilisent cette description.

Les solutions auto-similaires de (mKdV) sont délicates à étudier. Elles apparaissent dans l'étude du scattering modifié. Il s'agit de solutions invariantes par changement d'échelle. Deift et Zhou [33] ont étudié l'asymptotique des solutions auto-similaires. Notamment, Correia, Côte et Vega [27, 26] ont étudié le flot de (mKdV) au voisinage d'une solution auto-similaire, ainsi que le comportement asymptotique des solutions auto-similaires, et Correia et Côte [25] ont étudié les solutions auto-similaires pour montrer un résultat de stabilité sur l'explosion auto-similaire de (mKdV).

1.4.5 Multi-breather pour (mKdV)

En 1973, Wadati [117] a montré que la méthode de scattering inverse s'applique bien pour la résolution de (mKdV). La méthode de scattering inverse consiste à associer à l'équation considérée un problème à valeurs propres où la solution dont on cherche la formule joue le rôle de potentiel. Le raisonnement mené dans son article donnant la formule pour les multi-solitons s'était restreint à la recherche de valeurs propres imaginaires pures, ce qui simplifie les raisonnements et donne les formules pour tous les multi-solitons possibles. Toutefois, il a remarqué qu'on obtient

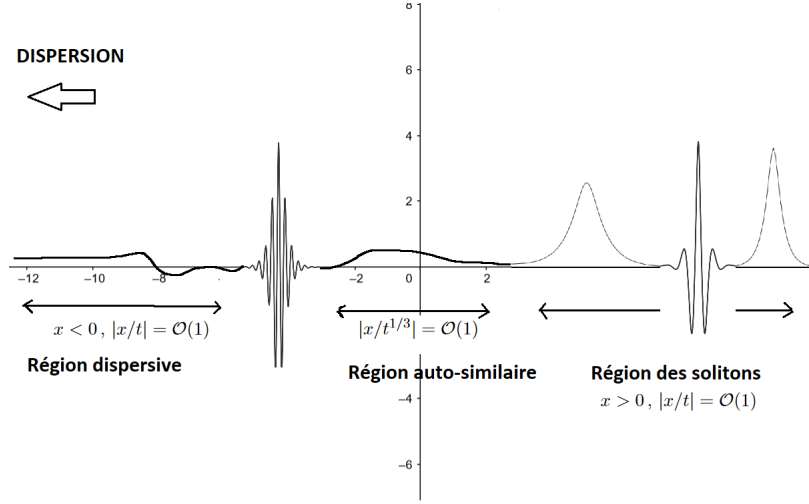


FIGURE 1.4.4 – Comportement asymptotique d’une solution générique de (mKdV) focalisante. Il apparaît trois régions avec les breathers qui peuvent apparaître dans n’importe quelle région. Aux termes dispersifs et auto-similaires près, on voit apparaître un multi-breather.

un nouveau type de solutions lorsqu’on ne restreint pas le raisonnement aux valeurs propres imaginaires pures : les breathers.

Le problème de scattering inverse pour (mKdV) a été poussé plus loin en 1982 par Wadati et Okhuma [118]. Notamment, en s’autorisant à considérer des ensembles de valeurs propres simples regroupées par paires symétriques par rapport à l’axe des imaginaires purs ou de valeurs propres doubles imaginaires pures, il est parvenu à établir une formule pour les multi-breathers.

Définition 1.8. Un *multi-breather* est une solution $p \in \mathcal{C}([T^*, +\infty[, H^2(\mathbb{R}))$ de (mKdV) telle qu’il existe P_1, \dots, P_J des solitons ou des breathers de (mKdV) tels que

$$\left\| p(t) - \sum_{j=1}^J P_j(t) \right\|_{H^2} \rightarrow_{t \rightarrow +\infty} 0. \quad (1.4.25)$$

La conjecture de résolution en solitons-breathers dit qu’à peu de choses près, toute solution générique de (mKdV) se présente comme un multi-breather. Il s’agit là d’une des motivations pour l’étude des multi-breathers.

Toutefois, le fait que la formule correspond bien à la définition naturelle d’un multi-breather demande des calculs conséquents. C’est pourquoi il est intéressant d’avoir une démonstration de l’existence des multi-breathers qui n’utilise pas cette formule.

1.4.6 Dipôles et multi-pôles

Un *dipôle*, découvert par Wadati et Okhuma [118], consiste en un couple soliton-antisoliton (où chacun a pour paramètre de forme $c = \beta^2$) dont le centre de masse se propage à une vitesse constante mais la distance entre le soliton et l’antisoliton est logarithmique en temps. Un dipôle de paramètre de forme $\beta > 0$ et de paramètres de

translation $x_1, x_2 \in \mathbb{R}$ est donné par la formule suivante (ou par son opposée) :

$$B_{0,\beta}(t, x; x_1, x_2) := 2\sqrt{2}\partial_x \left[\arctan \left(\beta \frac{y_1}{\cosh(\beta y_2)} \right) \right], \quad (1.4.26)$$

où $y_1 := x + \delta t + x_1$, $y_2 := x + \gamma t + x_2$, $\delta := -3\beta^2$ et $\gamma := -\beta^2$. Un dipôle peut aussi être vu comme la limite d'un breather lorsque $\alpha \rightarrow 0$, ce qui a été mis en évidence par Alejo [5].

Nguyen [95] a établi l'existence des dipôles dans le cas non intégrable. Plus précisément, il existe une solution de (gKdV) L^2 -sous-critique ou L^2 -sur-critique qui peut être approchée en norme H^1 lorsque $t \rightarrow +\infty$ par une somme donnée de deux solitons de même forme dont le centre de masse se propage à la vitesse correspondant à leur forme et qui s'éloignent l'un de l'autre logarithmiquement. Notons que dans le cas L^2 -sous-critique, il s'agit d'un couple soliton-antisoliton, en accord avec (mKdV) ; alors que dans le cas L^2 -sur-critique, il s'agit d'un couple soliton-soliton. On ne connaît pas de résultat sur les dipôles dans le cas L^2 -critique pour le moment.

Plus généralement, Wadati et Okhuma [118] ont montré l'existence de *multi-pôles* pour (mKdV).

1.4.7 Stabilité des solitons, des breathers et construction des multi-solitons dans le cas non intégrable

Les propriétés de stabilité d'un soliton de (gKdV) L^2 -sous-critique ont été étudiées par Weinstein [121], Cazenave et Lions [19], Benjamin [13], Bona, Souganidis et Strauss [16] ou Grillakis, Shatah et Strauss [44] notamment.

L'équation elliptique vérifiée par un soliton R de vitesse $c > 0$ pour (gKdV) L^2 -sous-critique implique qu'il est minimiseur local (modulo deux directions négatives) pour une *fonctionnelle de Lyapunov* :

$$\mathcal{F}[u] := I_3[u] + cI_2[u]. \quad (1.4.27)$$

Il est important de noter qu'une telle fonctionnelle de Lyapunov est préservée par toute solution.

Notons que le fait qu'il est point critique de cette fonctionnelle de Lyapunov est une conséquence directe de l'équation elliptique (GS). Le fait que c'est un minimiseur local modulo deux directions négatives est une conséquence de l'étude de la *coercivité* de la hessienne de \mathcal{F} au voisinage de R :

$$\mathcal{Q}[z] := \int z_x^2 - 3 \int R^2 z^2 + \frac{c}{2} \int z^2. \quad (1.4.28)$$

En partant d'une solution u qui est proche du soliton R au temps initial, grâce à un argument de *modulation*, en rajoutant une translation appropriée au soliton pour tout temps, on arrive à éviter les directions négatives en faisant en sorte à ce que la différence ϵ entre u et le translaté de R soit orthogonale à celles-ci. En réalité, cette technique nous permet de nous débarrasser de l'une des deux directions négatives ; pour la deuxième, un argument de majoration permet de s'en sortir. On peut ainsi utiliser la coercivité de la forme quadratique et en déduire que $\|\epsilon\|_{H^1}$ est bornée par $\mathcal{Q}[\epsilon]$, qui elle est presque constante en temps, car $\mathcal{F}[u]$ l'est.

On en déduit qu'un soliton de (gKdV) L^2 -sous-critique est stable dans H^1 à translation près : on dit qu'il est *orbitalement stable* dans H^1 .

Grâce aux travaux de Alejo et Muñoz [7, 8], on sait que les breathers de (mKdV) aussi jouissent de propriétés de stabilité remarquables, mais il faut introduire une fonction de Lyapunov plus sophistiquée. Ceci est dû au fait qu'un breather B de paramètres de forme α, β vérifie une équation elliptique d'ordre 4 (et on ne connaît pas d'équation elliptique d'ordre plus petit qui soit vérifiée par un breather) :

$$B_{xxxx} + 5BB_x^2 + 5B^2B_{xx} + \frac{3}{2}B^5 - 2(\beta^2 - \alpha^2)(B_{xx} + B^3) + (\alpha^2 + \beta^2)^2 B = 0. \quad (1.4.29)$$

Ceci implique qu'il est un minimum local (modulo trois directions négatives) de la fonctionnelle de Lyapunov suivante (elle aussi préservée par toute solution) :

$$\mathcal{H}[u] := I_4[u] + 2(\beta^2 - \alpha^2)I_3[u] + (\alpha^2 + \beta^2)^2 I_2[u]. \quad (1.4.30)$$

Plus précisément, le fait que B est un point critique de \mathcal{H} est une conséquence de l'équation elliptique vérifiée par B (1.4.29). Le fait qu'il est minimum local modulo trois directions négatives vient de l'étude de la coercivité de la hessienne de \mathcal{H} au voisinage de B :

$$\begin{aligned} H[z] := & \frac{1}{2} \int z_{xx}^2 - \frac{5}{2} \int B^2 z_x^2 + \frac{5}{2} \int B_x^2 z^2 + 5 \int B B_{xx} z^2 + \frac{15}{4} \int B^4 z^2 \\ & + (\beta^2 - \alpha^2) \left(\int z_x^2 - 3 \int B^2 z^2 \right) + (\alpha^2 + \beta^2)^2 \frac{1}{2} \int z^2. \end{aligned} \quad (1.4.31)$$

Alejo et Muñoz en ont déduit qu'un breather est orbitalement stable dans H^2 , en raisonnant essentiellement de la même manière que dans le cas du soliton mais avec la fonctionnelle de Lyapunov et la forme quadratique adaptées au breather, où la stabilité orbitale est la stabilité modulo des translations des paramètres x_1 et x_2 du breather. Notons que la stabilité orbitale est prouvée dans H^2 et non dans H^1 , parce que la forme quadratique H contient une dérivée seconde de z . Comme il y a trois directions négatives pour la forme quadratique et non pas deux, il faut traduire selon deux paramètres pour la modulation associée au breather (et non plus un seul comme pour un soliton). Ce résultat peut être amélioré en un résultat de stabilité orbitale H^1 grâce à l'utilisation de la transformée de Bäcklund (voir Alejo et Muñoz [9]). De même, les breathers de l'équation de Gardner sont eux aussi orbitalement stables (voir Alejo [6]).

Il est intéressant de noter que la preuve de la stabilité orbitale d'un breather ne passe pas à la limite $\alpha \rightarrow 0$. On peut même remarquer que la notion de stabilité développée ici n'est pas adaptée pour les dipôles. En effet, il y a davantage de directions d'instabilité au niveau d'un dipôle, car un dipôle peut à la fois être approché par des breathers et par des 2-solitons.

En utilisant les propriétés de stabilité d'un soliton, Martel [71] a démontré que pour (gKdV) L^2 -sous-critique ou L^2 -critique, il existe un multi-soliton dans H^1 associé à un ensemble de solitons R_1, \dots, R_K de vitesses deux à deux distinctes. De plus, la convergence du multi-soliton vers la somme des solitons correspondante est exponentielle dans tout espace H^s . De plus, il montre qu'un tel multi-soliton est unique. Il s'agit d'un raisonnement qui n'utilise aucun argument d'intégrabilité. Il est basé sur les propriétés de stabilité des solitons, sur des arguments de localisation, sur des propriétés de monotonie et un *bootstrap*. L'hypothèse disant que les vitesses des objets doivent être deux à deux distinctes est essentielle pour ce raisonnement, car les arguments de localisation ne fonctionnent que lorsque les solitons sont suffisamment découplés (et donc interagissent très peu entre eux et de moins en moins). Heuristiquement, grâce à leur découplage, une somme de solitons de vitesses deux à deux distinctes est quasiment une solution de (gKdV).

Le même genre d'arguments a permis à Martel, Merle et Tsai [87] d'étendre le résultat de stabilité orbitale (déjà connu pour un seul soliton) à une somme de solitons de vitesses deux à deux distinctes pour (gKdV) L^2 -sous-critique.

1.5 Résultats démontrés dans cette thèse

On s'intéresse à un ensemble de $J = K + L$ objets constitué de K breathers et L solitons.

Soient $\alpha_k, \beta_k > 0$ et $x_{1,k}^0, x_{2,k}^0 \in \mathbb{R}$ définis pour tout $k = 1, \dots, K$, et $c_l > 0$, $\kappa_l \in \{-1, 1\}$, et $x_{0,l}^0 \in \mathbb{R}$ définis pour $l = 1, \dots, L$, les paramètres des breathers et des solitons considérés. Définissons maintenant les objets considérés.

Pour $k = 1, \dots, K$, soit

$$B_k(t, x) := B_{\alpha_k, \beta_k}(t, x; x_{1,k}^0, x_{2,k}^0). \quad (1.5.1)$$

On définit aussi les paramètres de vitesses de phase et de groupe associés au breather B_k . Pour $k = 1, \dots, K$, soit

$$\delta_k := \alpha_k^2 - 3\beta_k^2, \quad (1.5.2)$$

et

$$\gamma_k := 3\alpha_k^2 - \beta_k^2. \quad (1.5.3)$$

Pour $l = 1, \dots, L$, soit

$$R_l(t, x) := \kappa_l R_{c_l}(t, x; x_{0,l}^0). \quad (1.5.4)$$

Pour $k = 1, \dots, K$, la vitesse de B_k vaut

$$v_k^b := -\gamma_k. \quad (1.5.5)$$

Pour $l = 1, \dots, L$, la vitesse de R_l vaut

$$v_l^s := c_l. \quad (1.5.6)$$

On suppose que les vitesses sont deux à deux distinctes, autrement dit que

$$\forall k \neq k', v_k^b \neq v_{k'}^b, \quad \forall l \neq l', v_l^s \neq v_{l'}^s, \quad \forall k, l, v_k^b \neq v_l^s. \quad (1.5.7)$$

On peut donc, sans perte de généralité, supposer que

$$v_1^b < \dots < v_K^b, \quad (1.5.8)$$

et que

$$v_1^s < \dots < v_L^s. \quad (1.5.9)$$

On peut donc définir une application strictement croissante

$$\begin{aligned} \underline{v} : \llbracket 1, J \rrbracket &\rightarrow \{v_k^b, 1 \leq k \leq K\} \cup \{v_l^s, 1 \leq l \leq L\} \\ j &\mapsto v_j. \end{aligned} \quad (1.5.10)$$

Et, pour $j = 1, \dots, J$, on désigne par P_j l'objet B_k ou R_l dont la vitesse est v_j .

On a en particulier,

$$v_1 < \dots < v_J. \quad (1.5.11)$$

Notre étude aura pour objet la somme

$$P(t, x) = \sum_{j=1}^J P_j(t, x) \quad (1.5.12)$$

des solitons et breathers.

Pour $j = 1, \dots, J$, on note $x_j(t)$ la position de P_j au temps $t \in \mathbb{R}$. Plus précisément,

$$\forall t \in \mathbb{R}, \quad x_j(t) := x_j(0) + v_j t, \quad (1.5.13)$$

où

$$x_j(0) := -x_{2,k}^0, \quad \text{si } P_j = B_k, \quad (1.5.14)$$

ou

$$x_j(0) := x_{0,l}^0, \quad \text{si } P_j = R_l. \quad (1.5.15)$$

Ces définitions seront reprises dans les chapitres suivants de cette thèse. Nous utiliserons ces notations pour énoncer les théorèmes dans cette section. Le terme "multi-breather" sera utilisé selon la Définition 1.8.

1.5.1 Existence, régularité, unicité

Le chapitre 1.5.6 est consacré à la démonstration des résultats d'existence, d'unicité, de régularité et de vitesse de convergence. Plus précisément, on y démontre que :

Théorème 1.9 (S.). *Sous l'hypothèse (1.5.7), il existe $\theta > 0$, $T^* > 0$ et $A_s > 0$ pour tout $s \geq 0$, tels qu'il existe un multi-breather p associé à P_1, \dots, P_J qui vérifie $p \in C^\infty(\mathbb{R} \times \mathbb{R}) \cap \mathcal{C}(\mathbb{R}, H^s(\mathbb{R}))$ pour tout $s \geq 0$ et*

$$\forall t \geq T^*, \quad \|p(t) - P(t)\|_{H^s} \leq A_s e^{-\theta t}. \quad (1.5.16)$$

De plus, θ ne dépend que des paramètres de forme de P_1, \dots, P_J . S'il existe $D > 0$ tel que $\forall j \geq 2$, $x_j(0) \geq x_{j-1}(0) + D$, alors A_s et T^ ne dépendent que des paramètres de forme de P_1, \dots, P_J et de D . Enfin, il est possible de choisir $D > 0$ suffisamment grand pour pouvoir avoir $T^* = 0$.*

En d'autres mots, on construit p un multi-breather associé aux objets P_1, \dots, P_J et on démontre qu'il a la régularité H^∞ et qu'il converge exponentiellement vers P dans tout H^s . Notons que la démonstration de ce théorème n'utilise pas d'argument de monotonie. D'après le chapitre 4, ce résultat peut aussi être déduit de la formule pour les multi-breathers obtenue par scattering inverse moyennant des calculs fastidieux. L'intérêt de l'approche choisie dans le chapitre 1.5.6 est d'éviter tous ces calculs et de donner un raisonnement qui peut potentiellement être généralisé à des situations non intégrables. De plus, cela permet d'avoir un raisonnement qui utilise la structure variationnelle du breather et les techniques utilisées sont utiles pour les théorème 1.10 et proposition 1.11 d'unicité pour lesquels les méthodes issues de l'intégrabilité ne semblent pas être exploitables.

Notons que l'hypothèse (1.5.7) est essentielle : les vitesses des objets doivent être deux à deux distinctes. Cela est nécessaire pour que les différents objets puissent se découpler lorsque $t \rightarrow +\infty$ et ainsi que leurs interactions tendent vers 0. Si on avait deux objets de même vitesse, il nous serait impossible de négliger leurs interactions, et donc leur somme ne serait pas approchée par une solution.

Observons également que le taux de convergence exponentielle θ peut être donné explicitement en fonction des paramètres des objets P_1, \dots, P_J en jeu. Soit

$$\beta := \min(\{\beta_k, 1 \leq k \leq K\} \cup \{\sqrt{c_l}, 1 \leq l \leq L\}), \quad (1.5.17)$$

le paramètre de localisation exponentielle adapté à chaque P_j . En d'autres mots, on a pour tout $j = 1, \dots, J$,

$$|P_j(t, x)| \leq C \exp(-\beta|x - x_j(t)|), \quad (1.5.18)$$

où C est une constante qui se déduit des paramètres de forme et de translation des P_j . Soit

$$\tau := \min_{1 \leq j \leq J-1} (v_{j+1} - v_j), \quad (1.5.19)$$

la distance minimale entre deux vitesses v_j . Alors, le taux de décroissance exponentielle qui convient pour le théorème ci-dessus est :

$$\theta := \frac{\beta\tau}{32}. \quad (1.5.20)$$

On démontre aussi un résultat d'unicité sur les multi-breathers, dans le cas où toutes les vitesses, sauf possiblement une, sont strictement positives.

Théorème 1.10 (S.). *Sous l'hypothèse (1.5.7) et si $v_2 > 0$, alors il existe un unique multi-breather associé à P_1, \dots, P_J .*

L'unicité du multi-breather est établie avec un argument de monotonie, c'est pour cette raison que l'on a besoin d'avoir des breathers qui vont vers la droite. Néanmoins, en réduisant la classe dans laquelle on cherche à établir l'unicité de p , on peut établir un résultat d'unicité indépendant de la vitesse des breathers :

Proposition 1.11 (S.). *Sous l'hypothèse (1.5.7), il existe $N > 0$ suffisamment grand tel qu'il existe une unique solution*

$$p \in \mathcal{C}([T_0, +\infty[, H^2(\mathbb{R})) \quad (1.5.21)$$

de (mKdV) telle que

$$\|p(t) - P(t)\|_{H^2} = O\left(\frac{1}{t^N}\right), \quad \text{lorsque } t \rightarrow +\infty. \quad (1.5.22)$$

La propriété de monotonie peut être utilisée dans des raisonnements réalisés sur (gKdV) où les solitons vont tous vers la droite, mais ne peut pas être utilisée pour (NLS). En effet, pour (NLS), il n'y a pas de direction privilégiée pour la propagation des solitons. C'est la raison pour laquelle les raisonnements de cette thèse qui n'utilisent pas la propriété de monotonie (Théorème 1.9 et Proposition 1.11) sont adaptés des raisonnements similaires réalisés pour (NLS). C'est aussi la raison pour laquelle on a moins d'outils pour démontrer des résultats sur la dynamique des multi-solitons pour (NLS) que pour (gKdV). En particulier, le théorème 1.9 s'inspire du théorème de l'existence des multi-solitons pour (NLS) établi par Martel et Merle [79]; la proposition 1.11 s'inspire du théorème sur l'unicité des multi-solitons pour (NLS) établi par Côte et Friederich [28].

Bien que l'existence peut être vue comme une conséquence de la formule obtenue par scattering inverse, l'unicité est un résultat nouveau qui ne peut être déduit du scattering inverse. En effet, le scattering inverse ne permet de décrire que les solutions génériques appartenant à un espace à poids. Or, d'un côté, on s'intéresse aux solutions appartenant à une classe plus large qu'un espace à poids, et d'un autre côté, la restriction aux solutions génériques ne peut être satisfaisante pour un résultat d'unicité. En effet, des solutions exceptionnelles (non génériques) pourraient mettre à mal un résultat d'unicité obtenu par scattering inverse. Cela est aussi une justification de pourquoi on n'utilise pas les techniques issues de l'intégrabilité dans cette thèse, sauf dans le fait qu'on utilise une loi de conservation supplémentaire donnée par l'intégrabilité et qui est nécessaire pour l'étude des breathers.

1.5.2 Stabilité orbitale

Le chapitre 3 est consacré à la démonstration des résultats de stabilité. Plus précisément, on y démontre un énoncé de stabilité orbitale pour une somme de solitons et de breathers de vitesses deux à deux distinctes et dont toutes les vitesses sauf possiblement une sont positives :

Théorème 1.12 (S.). *Sous l'hypothèse (1.5.7) et si $v_2 > 0$, alors il existe $A_0, \theta_0, D_0, a_0 > 0$ tels qu'on a ce qui suit. Soit $u \in \mathcal{C}(\mathbb{R}, H^2(\mathbb{R}))$ une solution de (mKdV), $D \geq D_0$ et $a \in [0, a_0]$ tels que*

$$\|u(0) - P(0)\|_{H^2} \leq a, \quad \text{et } \forall j = 1, \dots, J, \quad x_j(0) > x_{j-1}(0) + D. \quad (1.5.23)$$

Alors, il existe $x_{0,l}(t)$, $x_{1,k}(t)$ et $x_{2,k}(t)$ des paramètres de translation définis pour tout $t \geq 0$ tels que,

$$\forall t \geq 0, \quad \|u(t) - \bar{P}(t)\|_{H^2} \leq A_0 (a + e^{-\theta_0 D}), \quad (1.5.24)$$

où

$$\bar{P} = \sum_{j=1}^J \bar{P}_j, \quad (1.5.25)$$

où

$$\overline{P}_j(t) := B_{\alpha_k, \beta_k}(t, x; x_{1,k}(t), x_{2,k}(t)), \quad \text{si } P_j = B_k, \quad (1.5.26)$$

ou

$$\overline{P}_j(t) := R_{c_l}(t, x; x_{0,l}(t)), \quad \text{si } P_j = R_l. \quad (1.5.27)$$

Notons que la deuxième partie de la condition (1.5.23) est nécessaire parce que le *bootstrap* qu'on réalise pour prouver le Théorème 1.12 demande à ce que les différents solitons et breathers soient suffisamment découplés, et le restent au cours du temps. On en déduit la stabilité orbitale des multi-breathers :

Proposition 1.13 (S.). *Supposons (1.5.7). Soit p un multi-breather associé à P_1, \dots, P_J par le théorème 1.9. Si $v_2 > 0$, alors il existe $\eta_0 > 0$ suffisamment petit et $C_0 > 0$ suffisamment grand tels que ce qui suit est vrai pour tout $0 < \eta < \eta_0$.*

Soit $u \in \mathcal{C}(\mathbb{R}, H^2(\mathbb{R}))$ une solution de (mKdV) telle que

$$\|u(0) - p(0)\|_{H^2} \leq \eta. \quad (1.5.28)$$

Alors, il existe des paramètres de translation définis pour tout $t \geq 0$, $x_{0,l}(t)$, $x_{1,k}(t)$ et $x_{2,k}(t)$, tels que

$$\forall t \geq 0, \quad \|u(t) - \overline{p}(t)\|_{H^2} \leq C_0 \eta, \quad (1.5.29)$$

où \overline{p} correspond au multi-breather associé à $\overline{P}_1, \dots, \overline{P}_J$ qui sont définis pour tout $t \geq 0$ par (1.5.26) et (1.5.27).

On peut remarquer que la deuxième partie de la condition (1.5.23) n'est plus nécessaire dans cette proposition. En effet, étant donné que l'on s'intéresse ici à p qui est une solution de (mKdV) et non plus à P qui n'en est pas une, on peut utiliser la continuité du flot de (mKdV) donnée par le Théorème 1.3. En effet, même si au temps initial, les différents solitons et breathers ne sont pas dans le bon ordre et suffisamment éloignés les uns des autres, il suffit toujours d'attendre un certain temps T avant qu'ils le soient. Le fait que $u(T)$ est suffisamment proche de $p(T)$ est assuré par une proximité plus grande de $u(0)$ et $p(0)$ et la continuité du flot dans H^2 .

Notons que le temps T dépend du multi-breather p choisi (notamment, de ses paramètres de translation). Il en est donc de même pour les constantes η_0 et C_0 qui apparaissent dans la Proposition 1.13.

1.5.3 Conséquences de la formule

Dans le chapitre 4, nous menons des calculs, relativement lourds, manipulant la formule pour les multi-breathers trouvée par scattering inverse, établie par Wadati [118]. Nous y vérifions que le théorème 1.9 peut être démontré en faisant des calculs en partant de cette formule. L'avantage de cette méthode est qu'on a la possibilité de démontrer des résultats spécifiques au cas intégrable. Notamment, on démontre que le multi-breather ainsi construit qui tend vers la somme de solitons et de breathers correspondante dans H^2 lorsque $t \rightarrow +\infty$ est aussi un multi-breather lorsque $t \rightarrow -\infty$ pour des objets ayant les mêmes paramètres de forme mais d'autres paramètres de translation. En d'autres mots, on prouve l'élasticité d'une collision entre deux breathers, entre deux solitons ou entre un soliton et un breather de (mKdV). On ne peut pas s'attendre à déduire une chose pareille d'un raisonnement généralisable aux cas non intégrables comme celui qui a été fait pour démontrer le théorème 1.9. On renvoie à la section 1.4.2 pour des exemples de résultats de non-élasticité dans des cas non intégrables.

Le résultat de l'élasticité des interactions entre les solitons ou breathers de (mKdV) s'énonce comme suit :

Proposition 1.14 (S.). *Supposons (1.5.7) pour un ensemble P_1, \dots, P_J de solitons et de breathers de (mKdV), et utilisons l'ensemble des notations données dans le préambule à la Section 1.5. Il existe $\theta > 0$ et un multi-breather p associé à P_1, \dots, P_J tel que pour tout $s \geq 0$, il existe $A_s > 0$ tel que*

$$\forall t \geq 0, \quad \|p(t) - P(t)\|_{H^s} \leq A_s \exp(-\theta t), \quad (1.5.30)$$

et

$$\forall t \leq 0, \quad \left\| p(t) - \sum_{j=1}^J P_j^{-\infty}(t) \right\|_{H^s} \leq A_s \exp(-\theta|t|), \quad (1.5.31)$$

où, pour $j = 1, \dots, J$, si $P_j = B_k$ est un breather, en notant $l := \max(\{0\} \cup \{n, v_n^s < v_j\})$,

$$P_j^{-\infty}(t, x) := B_{\alpha_k, \beta_k}(t, x; x_{1,k}^{-\infty}, x_{2,k}^{-\infty}), \quad (1.5.32)$$

où les formules pour $x_{1,k}^{-\infty}$ et $x_{2,k}^{-\infty}$ sont données dans le chapitre 4,

et si $P_j = R_l$ est un soliton, en notant $k := \max(\{0\} \cup \{n, v_n^b < v_j\})$,

$$P_j^{-\infty} := \kappa_l R_{c_l}(t, x; x_{0,l}^{-\infty}), \quad (1.5.33)$$

où la formule pour $x_{0,l}^{-\infty}$ est donnée dans le chapitre 4.

Ainsi, pour P_1, \dots, P_J donnés, il existe un multi-breather lorsque $t \rightarrow +\infty$ qui est aussi un multi-breather lorsque $t \rightarrow -\infty$ associé à $\bar{P}_1, \dots, \bar{P}_J$ où \bar{P}_j est une translation de P_j (en espace si P_j est un soliton, ou en espace et en phase si P_j est un breather). Il n'y a pas de condition sur le signe des vitesses pour ce résultat.

Ce résultat permet de déterminer l'effet de la collision d'un soliton ou d'un breather avec un ensemble d'objets. Tout d'abord, on constate que l'effet d'une collision avec plusieurs objets correspond à la somme des effets des collisions avec chaque objet. Ainsi, pour connaître les décalages subis par une suite de collisions quelconque, il suffit de connaître les décalages subis après une collision entre deux objets. Les formules exactes des décalages subis après une collision sont données dans le chapitre 4. Notons ici qu'après une collision, l'objet le plus rapide subit un décalage vers la droite, alors que l'objet le plus lent subit un décalage vers la gauche. Le rapport des valeurs absolues des décalages correspond à l'inverse du rapport entre les paramètres β des deux objets considérés (plus précisément, β pour un breather et $\sqrt{c}/2$ pour un soliton). La raison pour laquelle le paramètre associé à un soliton doit être divisé par 2 est qu'on peut voir un soliton comme la moitié d'un breather, car un breather est une généralisation d'un couple soliton-antisoliton (un dipôle) qui peut être naturellement vu comme deux solitons. Un breather subit également un décalage au niveau de sa phase. Après une collision entre deux breathers, ces décalages sont de signes opposés, et le rapport des valeurs absolues des décalages de phase correspond à l'inverse du rapport des paramètres α des breathers.

1.5.4 Idées de démonstration

Cette thèse se consacre à la généralisation des résultats de Martel et Merle (obtenus pour les solitons) au cadre des solitons et des breathers. L'un des défis pour y arriver est d'arriver à considérer les solitons et les breathers à un niveau de régularité H^2 . Alors que le cadre H^2 est idéal pour les breathers (ce ne sont pas des objets que l'on peut facilement considérer au niveau de régularité H^1 étant donné que l'équation (1.4.29) est d'ordre 4), les solitons sont généralement considérés à un niveau de régularité H^1 . Dans cette thèse, on s'aperçoit que les solitons vérifient les propriétés des breathers passés à la limite $\alpha \rightarrow 0$, un peu comme si un soliton vérifiait les propriétés vérifiées par le dipôle qui lui correspond.

Ainsi, un soliton R de vitesse c vérifie l'équation elliptique d'ordre 4 suivante :

$$R_{xxxx} + 5RR_x^2 + 5R^2R_{xx} + \frac{3}{2}R^5 - 2c(R_{xx} + R^3) + c^2R = 0, \quad (1.5.34)$$

la démonstration de cette égalité se fait à partir de (GS) et (1.4.2). Cette équation est analogue à l'équation elliptique vérifiée par un breather (1.4.29). Ceci implique qu'il est un minimum local (modulo deux directions négatives) de la fonctionnelle de Lyapunov :

$$\mathcal{H}[u] := I_4[u] + 2cI_3[u] + c^2I_2[u], \quad (1.5.35)$$

qui est analogue à la fonctionnelle de Lyapunov associée à un breather (1.4.30). On constate que l'équation elliptique et la fonctionnelle de Lyapunov associées à un soliton de vitesse c correspondent à celles associées à un breather de paramètres $\alpha = 0$ et $\beta = \sqrt{c}$. Un soliton a donc quasiment la structure variationnelle H^2 d'un breather dégénéré. Cette observation sera un argument clé pour réaliser les preuves de cette thèse.

Les arguments utilisés dans cette thèse n'utilisent pas plus l'intégrabilité. Par contre, on utilise des arguments de localisation, de monotonie et de bootstrap, analogues à ceux qu'on peut trouver dans les preuves de Martel et Merle.

Soulignons le fait que l'utilisation des arguments de monotonie a posé des limites dans nos possibilités de démonstration sur les breathers. En effet, ces arguments ne fonctionnent pas pour des breathers de vitesse négative. Ainsi, à chaque fois qu'on a utilisé un argument de monotonie, on a été contraint de restreindre le résultat démontré aux breathers qui vont à droite. Néanmoins, le fait que certains des raisonnements faits dans cette thèse n'utilisent pas d'argument de monotonie a rendu possible d'énoncer quelques résultats qui ne dépendent pas du signe de la vitesse des breathers en jeu. Les raisonnements qui évitent les arguments de monotonie sont en général adaptés de raisonnements réalisés pour (NLS), car pour cette équation il n'y a pas de sens privilégié pour la propagation des solitons (cf. Martel et Merle [79], Côte et Friederich [28]).

Existence d'un multi-breather associé à P

Donnons les idées principales de la preuve du Théorème 1.9. Il s'agit d'une preuve qui n'utilise pas d'argument de monotonie, c'est pourquoi le théorème démontré est valable pour des breathers de vitesse quelconque.

Tout d'abord, observons que lorsque le temps t est suffisamment grand, en raison du fait que les vitesses v_j sont deux à deux distinctes et du fait que les objets P_j sont exponentiellement localisés (cf. (1.4.24) et (1.4.9)), les endroits où P_j est grand font partie des endroits où tous les autres objets P_k , pour $k \neq j$, sont petits. Comme P_j est une solution de (mKdV), on en déduit que P est, à peu de choses près, une solution de (mKdV) pour tout temps t suffisamment grand. Ceci justifie le théorème et donne une piste pour sa démonstration. En particulier, avec ces observations, il est naturel que le paramètre de la convergence exponentielle θ doit dépendre de β et τ , ce qui justifie la définition (1.5.20) qui ressortira de la preuve.

Dans un premier temps, nous allons chercher une solution p de (mKdV) qui converge exponentiellement vers P dans H^2 . Ensuite, on améliorera le résultat démontré en établissant le fait que cette convergence exponentielle a lieu dans tout H^s .

La première observation nous suggère de construire p à partir de P . En effet, pour une suite de temps (T_n) telle que $T_n \rightarrow +\infty$, nous allons définir une suite de solutions (p_n) de (mKdV) telles que $p_n(T_n) = P(T_n)$. Heuristiquement, p_n correspond à ce qu'on attend pour p avec $t \rightarrow T_n$ à la place de $t \rightarrow +\infty$. Autrement dit, on simplifie le problème en remplaçant $+\infty$ par T_n .

Compte tenu du fait que l'on cherche une convergence exponentielle de p vers P dans H^2 lorsque $t \rightarrow +\infty$, il est raisonnable de vouloir démontrer une convergence exponentielle de p_n vers P dans H^2 lorsque t tend vers T_n par la gauche. Autrement dit, on cherchera à démontrer qu'il existe $T^* \in \mathbb{R}$ et $C > 0$ tels que

$$\forall n, \quad \forall t \in [T^*, T_n], \quad \|p_n(t) - P(t)\|_{H^2} \leq C \exp(-\theta t), \quad (1.5.36)$$

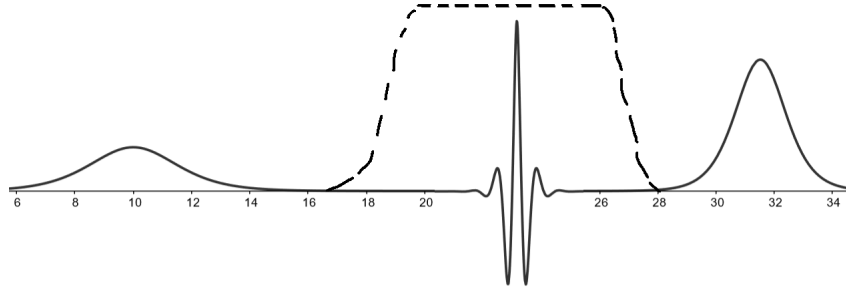


FIGURE 1.5.1 – Localisation effectuée pour la preuve de l'existence (on représente la fonction réalisant la filtration en pointillés : ici, elle est à support compact et permet d'isoler le breather au milieu).

ce qui sera le coeur de cette démonstration.

Comme $(\|p_n(T^*)\|_{H^2})$ est une suite bornée, $(p_n(T^*))$ a une valeur d'adhérence faible p^* dans H^2 . La solution p recherchée est la solution de (mKdV) avec la donnée initiale p^* à $t = T^*$. En effet, grâce à la continuité faible du flot de (mKdV) dans H^2 , l'inégalité (1.5.36) passe à la limite.

La démonstration de (1.5.36) se fait à n fixé. C'est pourquoi, pour simplifier les notations, on pourra omettre l'indice n dès à présent : on écrira donc T pour T_n et p pour p_n . Il est clair que (1.5.36) est vérifié pour tout t suffisamment proche de T . Pour étendre l'inégalité pour tout temps t qui nous intéresse, on tirera profit d'un argument de *bootstrap*. On démontrera que la connaissance d'une estimée de type (1.5.36) permet d'en déduire une estimée meilleure, ce qui permet d'étendre l'intervalle de temps sur lequel (1.5.36) est vérifiée. La preuve est donc réduite à la preuve du bootstrap : pour $t \in [T^*, T]$, si

$$\forall t \in [t^*, T], \quad \|p(t) - P(t)\|_{H^2} \leq A \exp(-\theta t), \quad (1.5.37)$$

alors

$$\forall t \in [t^*, T], \quad \|p(t) - P(t)\|_{H^2} \leq \frac{A}{2} \exp(-\theta t), \quad (1.5.38)$$

où A est une constante appropriée qui ne dépend pas de T .

Pour y parvenir, nous allons adapter les fonctionnelles de Lyapunov (1.5.35) et (1.4.30) qu'on connaît pour les solitons et les breathers à la somme P considérée. Cette fonctionnelle de Lyapunov doit être telle qu'au voisinage de $x_j(t)$, elle se comporte comme la fonctionnelle de Lyapunov associée à P_j . Pour y arriver, nous allons avoir recours à des masses (1.2.4), des énergies (1.2.5) et des énergies secondes (1.3.2) localisées au voisinage de x_j , notées M_j , E_j et F_j . La localisation se fera grâce à des fonctions dont les dérivées sont à support compact (que l'on peut utiliser dans une preuve qui ne repose pas sur un argument de monotonie), qui sont égales à 1 pour x près de $x_j(t)$, qui sont égales à 0 pour x près de x_k pour $k \neq j$ et qui varient à mi-distance entre $x_j(t)$ et les positions de ses voisins. Comme ce sont des intégrales conservées localisées, on arrive à démontrer une presque-conservation pour M_j et E_j pour toute solution de (mKdV) qui est petite dans des régions où la fonction assurant la localisation varie, ce qui est le cas pour p (on n'a pas besoin de considérer la presque-conservation de F_j pour cette preuve).

La fonctionnelle de Lyapunov localisée autour de x_j et adaptée à P_j se définit de la manière suivante :

$$\mathcal{H}_j[p](t) := F_j[p](t) + 2(b_j^2 - a_j^2)E_j[p](t) + (a_j^2 + b_j^2)^2 M_j[p](t), \quad (1.5.39)$$

où $(a_j, b_j) = (\alpha, \beta)$ si P_j est un breather de paramètres α, β , et $(a_j, b_j) = (0, \sqrt{c})$ si P_j est un soliton de paramètre c .

La fonctionnelle de Lyapunov adaptée à P est naturellement :

$$\mathcal{H}[p](t) := \sum_{j=1}^J \mathcal{H}_j[p](t). \quad (1.5.40)$$

De la presque-conservation des M_j et E_j , on en déduit que $\mathcal{H}[p](t)$ est presque-conservée.

Le point est que P est, à peu de choses près, un minimum local pour \mathcal{H} modulo quelques directions négatives. Le fait que P est un point critique pour \mathcal{H} est une conséquence des équations elliptiques (1.5.34) et (1.4.29) vérifiées par les P_j . Le fait que c'est, en plus, un minimum local (modulo quelques directions négatives) vient de l'étude de la hessienne de \mathcal{H} considérée autour de P , qui est une conséquence de l'étude de la coercivité des formes quadratiques (1.4.28) et (1.4.31).

Comme dans le cas d'un seul objet, on arrive à éviter les directions négatives par une modulation, en réalisant des translations pour chaque objet P_j . Le point est que (1.5.37) implique que les paramètres de modulation tendent exponentiellement vers 0. C'est pour cela que la majoration qu'on obtient pour $p - \tilde{P}$ (où \tilde{P} est la somme modulée) est aussi valable pour $p - P$. Enfin, la majoration pour $p - \tilde{P}$ vient de la coercivité de la hessienne de \mathcal{H} et de la presque-conservation de \mathcal{H} . En choisissant bien les constantes intermédiaires, on arrive à obtenir le bootstrap.

Notons que la modulation laisse en place une direction négative pour les breathers (on a deux paramètres de modulation alors qu'il y a trois directions négatives pour chaque breather) qu'on arrive à gérer en rajoutant un terme coercif pour la direction négative en question à la hessienne. Il s'agit donc aussi de majorer ce terme en plus, ce qui se fait en majorant sa dérivée en temps. Les solitons ne sont pas concernés par ce problème, mais ils en causent un autre car contrairement aux breathers, leur modulation modifie légèrement leur forme, il faut donc en prendre compte dans les calculs (notamment pour borner le terme linéaire apparaissant dans la fonctionnelle de Lyapunov, ce qui se fait bien grâce aux conditions d'orthogonalité choisies pour chaque soliton).

Régularité d'un multi-breather associé à P

La convergence exponentielle vers 0 de $p - P$ est établie dans H^2 dans la preuve de l'existence, on l'étend ensuite à tout H^s . Pour y arriver on prouve (1.5.36) dans H^s à la place de H^2 . On n'utilise pas le bootstrap ici. Le raisonnement adopté est un raisonnement par récurrence sur s , initialisé en $s = 2$.

Preuve de la Proposition 1.11 : une solution convergeant super-polynomialement vers un multi-breather lui est égale

Soit p le multi-breather associé à P par le théorème 1.9. Soit u une solution de (mKdV) telle que pour tout t suffisamment grand,

$$\|u(t) - p(t)\|_{H^2} \leq \frac{C}{t^N}, \quad (1.5.41)$$

où N est suffisamment grand.

On démontre qu'alors $z := u - p = 0$, sans aucune condition sur le signe des vitesses. On étudie z , qui est une différence de deux solutions de (mKdV), ce qui fait que l'on peut écrire une équation pour z .

On s'intéresse à la coercivité de la hessienne de (1.5.40) au voisinage de p . Pour l'avoir, il s'agit de modifier z pour être orthogonal aux directions négatives. Cela se fait en rajoutant à z une combinaison linéaire de $\partial_x R$, de $\partial_{x_1} B$ et de $\partial_{x_2} B$ où R et B sont des solitons et des breathers qui interviennent dans la somme P . On note \tilde{z} le résultat. En le faisant, on borne soigneusement $\tilde{z} - z$ et sa variation en temps.

On en déduit que $\|\tilde{z}\|_{H^2}$ est majorée par la hessienne de (1.5.40), et on montre que cette dernière est presque-constante en majorant sa dérivée en temps par un majorant qui dépend de z et de \tilde{z} . On obtient au final le fait que

la norme H^2 de z peut être bornée en fonction de \tilde{z} et l'inégalité suivante pour tout temps t suffisamment grand et une constante $C > 0$ qui ne dépend ni de z ni de N :

$$\|\tilde{z}(t)\|_{H^2} \leq C \int_t^{+\infty} \frac{1}{s} \|\tilde{z}(s)\|_{H^2} ds + C \frac{1}{t^{N-2}} \int_t^{+\infty} \|\tilde{z}(s)\|_{H^2} ds. \quad (1.5.42)$$

Ainsi, en choisissant N suffisamment grand, on trouve que le seul cas où cette inégalité peut être vérifiée est le cas $\tilde{z} = 0$. On en déduit que $z = 0$, ce qui termine la démonstration.

Preuve du théorème 1.10 : une solution convergeant vers un multi-breather converge exponentiellement vers celui-ci

Ici, si u est une solution de (mKdV) telle que

$$\|u - p\|_{H^2} \rightarrow_{t \rightarrow +\infty} 0, \quad (1.5.43)$$

on montre que $u - p = 0$, à condition que toutes les vitesses des P_j (sauf possiblement une) sont strictement positives, autrement dit, $v_2 > 0$.

Pour y arriver, on montre que (1.5.43) implique que $\|u - p\|_{H^2}$ est exponentiellement décroissante, ce qui implique par Proposition 1.11 que $u - p = 0$.

Ici, z désigne $u - P$ et on montrera que $\|z(t)\|_{H^2}$ est exponentiellement décroissante. Comme, $\|p(t) - P(t)\|_{H^2}$ est exponentiellement décroissante par le théorème 1.9, la décroissance exponentielle de $\|u - p\|_{H^2}$ s'ensuit.

L'hypothèse (1.5.43) est bien moins forte que celles qu'on a connues jusqu'à présent. C'est pourquoi on ne peut plus faire certaines majorations qu'on pouvait faire avant, ce qui fait qu'on ne peut pas établir la presque-conservation de M_j et E_j de manière aussi satisfaisante qu'avant (en particulier, il est impossible de borner leurs variations exponentiellement en temps, ou même par une borne intégrable en temps). C'est la raison pour laquelle on va avoir recours à un argument de monotonie et on démontre une propriété légèrement différente pour M_j , E_j et F_j en les définissant différemment aussi.

Plus précisément, on fait en sorte de ne pas avoir à borner certains termes qui apparaissent dans l'étude des variations de M_j , E_j et F_j en utilisant le fait qu'il s'agit de termes ayant un signe. La conséquence est que désormais on montre que M_j , E_j et F_j sont, à peu de choses près, *presque-croissantes* et non plus presque-conservées. Il en suivra que les fonctionnelles de Lyapunov qu'on construira seront elles aussi presque-croissantes, ce qui est en fait suffisant pour les inégalités qu'on veut établir pour mener à bien notre démonstration (la presque-conservation n'est en fait pas nécessaire). Cependant, pour obtenir que les termes "embêtants" soient signés, on doit aussi changer notre façon de localiser les lois de conservation : on localise avec une fonction *monotone* et dont les dérivées supérieures doivent pouvoir être bornées par sa dérivée première. L'autre contre-partie de cette méthode est qu'elle dépend du *signe* des vitesses v_j , en effet $m'_j \in (v_{j-1}, v_j)$ apparaît dans le coefficient des termes qu'on veut éliminer grâce à leur signe, il est donc nécessaire que *tous les m'_j soient strictement positifs* pour que le raisonnement fonctionne. Cela est possible dès que toutes les vitesses sauf possiblement une sont positives, i.e. quand $v_2 > 0$.

Précisons la manière dont on modifie la fonction Φ_j qui réalise la localisation. Premièrement, pour qu'elle soit monotone, on ne peut plus ici localiser les lois de conservation autour de P_j (dans ce cas, à gauche de P_j , Φ_j devrait croître et à droite de P_j , Φ_j devrait décroître). La solution est que les lois de conservations localisées M_j , E_j et F_j sont localisées autour des $j - 1$ premiers objets : Φ_j n'a alors à varier qu'entre P_{j-1} et P_j (ainsi, sa vitesse est m'_j). C'est la raison pour laquelle, on effectue un raisonnement par *réurrence* (finie) : on montre que la norme H^2 de $z(t)$ localisée autour des $j - 1$ premiers objets est exponentiellement décroissante pour $j = 1, \dots, J + 1$. Dans les faits, on effectue l'étape du raisonnement par récurrence permettant de montrer la propriété pour les $j - 1$ premiers objets en utilisant la fonctionnelle de Lyapunov adaptée à P_{j-1} : elle est bien adaptée aux termes associés à P_{j-1} , sachant que les termes associés aux objets précédents ne posent pas de problèmes grâce à l'hypothèse de récurrence.

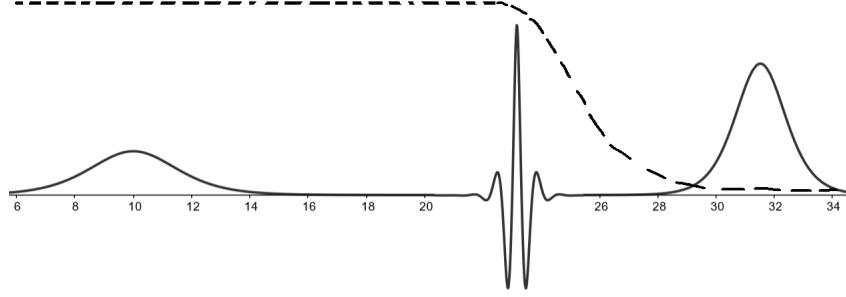


FIGURE 1.5.2 – Filtration réalisée pour la preuve de l'unicité par monotonie.

Deuxièmement, pour que les dérivées supérieures de Φ_j puissent être bornées par sa dérivée première, il apparaît que la dérivée de Φ_j ne peut plus être à support compact : on opte pour une fonction qui tend exponentiellement vers 1 lorsque x tend vers $-\infty$ et qui tend exponentiellement vers 0 lorsque x tend vers $+\infty$.

Ainsi, Φ_j peut se présenter comme une translation de

$$\Psi(x) := \frac{2}{\pi} \arctan \left(\exp \left(-\sqrt{\sigma} x / 2 \right) \right), \quad (1.5.44)$$

où $\sigma > 0$ est une constante suffisamment petite à choisir judicieusement.

En tenant compte des modifications décrites ci-dessus, à la récurrence finie près, dans le reste de ses aspects, le principe du raisonnement effectué pour démontrer le théorème 1.10 est similaire à celui qu'on a l'habitude de faire. La fonctionnelle de Lyapunov localisée autour des $j - 1$ premiers objets \mathcal{H}_j est presque-croissante. Une fois les bonnes modulations réalisées sur P_{j-1} (il s'agit essentiellement du même type de modulations que d'habitude), en notant \tilde{P} la somme modulée, on trouve que \tilde{P} est un minimum local de la fonctionnelle de Lyapunov. Ainsi, la norme H^2 de $u - \tilde{P}$ localisée en les $j - 1$ premiers objets est majorée par la hessienne de \mathcal{H}_j , qui est à peu près égale à la différence entre $\mathcal{H}_j[u]$ et $\mathcal{H}_j[\tilde{P}]$ (la dernière étant à peu de choses près une constante). On utilise ensuite la presque-croissance de $\mathcal{H}_j[u]$ pour la majorer par sa limite en $+\infty$, la décroissance exponentielle s'ensuit. Pour en déduire la décroissance exponentielle de la norme H^2 de $z(t)$ localisée autour des $j - 1$ premiers objets, on montre que $\|P - \tilde{P}\|_{H^2}$ est exponentiellement décroissante.

Stabilité orbitale d'une somme de solitons et de breathers

Le théorème 1.12 démontre la stabilité orbitale d'une somme de solitons et de breathers.

P désigne une somme de solitons et de breathers où les objets sont rangés dans l'ordre de vitesses croissant et qui sont suffisamment éloignés les uns des autres pour tout $t \geq 0$. Le but est de montrer que si une solution u est proche dans H^2 de P pour $t = 0$, alors elle est proche de \bar{P} pour tout $t \geq 0$, où \bar{P} désigne la somme P modulée où chaque soliton peut recevoir une translation en espace et chaque breather peut recevoir une translation en espace et en phase (*attention* : on n'autorise pas de modification pour le paramètre de forme pour les solitons dans \bar{P} qui apparaissent dans le théorème 1.12, contrairement à ce qu'il en est d'habitude pour \tilde{P}).

Néanmoins, dans la preuve, on réalisera une modulation au niveau de deux paramètres pour les solitons et de deux paramètres pour les breathers, de sorte à ce que les quatre conditions d'orthogonalité usuelles soient vérifiées pour tout temps, et on notera \tilde{P} la somme modulée. Autrement dit, on autorise un changement dans les paramètres de forme des solitons pour la modulation. La contre-partie est que pour finir la preuve, il faut arriver à borner la distance entre les paramètres de forme des solitons dans \tilde{P} et les paramètres de forme des solitons dans P . Ce qui

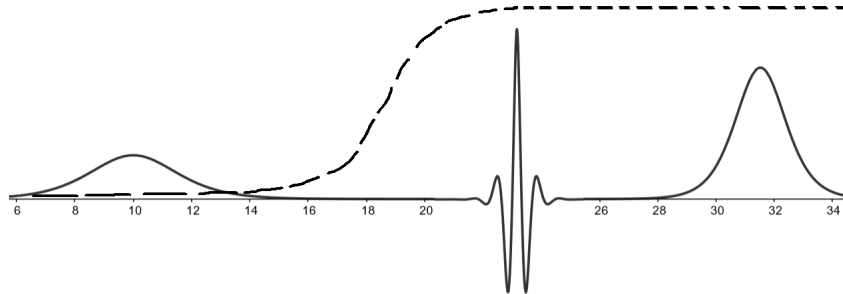


FIGURE 1.5.3 – Filtration réalisée pour la localisation dans la preuve de la stabilité orbitale.

fait qu'il y a au final deux choses à montrer : le fait que $\varepsilon := u - \tilde{P}$ est bornée en norme H^2 pour tout temps et le fait que la distance entre les paramètres de formes $c_l^0 := c_l$ des solitons et leurs versions modulées $c_l(t)$ sont bornées pour tout temps. Il s'agira aussi de décrire cette bornitude en fonction de la distance initiale entre u et P dans H^2 et de l'éloignement initial entre les objets de P .

La preuve s'appuie essentiellement sur un bootstrap : on montre que si, sur un intervalle de temps, $\varepsilon(t)$ et $c_l^0 - c_l(t)$ sont bornées, alors on peut améliorer la constante par laquelle ils sont bornés.

La stratégie de preuve est similaire à celle qu'on a utilisée pour le théorème 1.10 (l'unicité). On ne peut à nouveau pas reproduire le raisonnement qu'on a utilisé pour l'existence où l'on pouvait localiser avec des fonctions de dérivées à support compact. En effet, l'hypothèse dans le bootstrap est le fait que $\varepsilon(t)$ est bornée sur un intervalle du temps, donc il n'y a pas d'hypothèse de décroissance exponentielle en temps, ce qui nous empêche de prouver un résultat de presque-conservation des lois de conservation localisées.

Nous utilisons ainsi des arguments de monotonie, comme pour la preuve du Théorème 1.10. C'est la raison pour laquelle, on établira que les lois de conservation localisées sont presque monotones et qu'on raisonnera par récurrence. Ainsi, la localisation se fait quasiment de la même manière que dans la preuve du Théorème 1.10, et le principe du raisonnement est similaire, avec quelques différences. La différence principale réside dans le fait qu'ici, l'information de référence se trouve en $t = 0$, alors que dans l'énoncé sur l'unicité, l'information de référence se trouvait pour $t \rightarrow +\infty$. C'est pourquoi, pour l'unicité, il convenait de travailler avec une fonctionnelle de Lyapunov presque-croissante pour pouvoir majorer ce qui se passe en un temps donné par ce qui se passe pour $t \rightarrow +\infty$, alors qu'ici il convient de travailler avec une fonctionnelle de Lyapunov presque-décroissante pour pouvoir majorer ce qui se passe en un temps donné par ce qui se passe en $t = 0$. Ainsi, la localisation se fait avec une fonction strictement croissante ici (et non plus strictement décroissante) : on localise autour des $j - 1$ derniers objets ici (les objets les plus rapides).

Une conséquence du fait que la localisation se fait avec des fonctions croissantes est que la récurrence se fait de droite à gauche (et non plus de gauche à droite) : on établit la borne au voisinage d'un objet en supposant que la borne est déjà établie au voisinage des objets plus rapides. L'argument de monotonie repose sur la positivité de la vitesse de l'objet considéré par l'étape du raisonnement par récurrence pour marcher, à l'exception de la dernière étape de la récurrence où il n'est pas nécessaire de faire d'hypothèses sur les vitesses des objets.

1.5.5 Perspectives

Il existe d'autres problèmes sur les breathers de (mKdV) sur lesquels il est possible de poursuivre la recherche :

Théorème de Liouville pour les breathers

Le théorème de Liouville pour les solitons (gKdV) L^2 -sous-critique a été établi par Martel et Merle [73, 72]. Il s'agit d'un résultat surprenant de rigidité des solitons qui dit que toute solution non dispersive proche d'un soliton est un soliton :

Théorème 1.15 (Martel-Merle [73]). *Soit $p = 2, 3, 4$, et soit $c_0 > 0$. Soit $u_0 \in H^1(\mathbb{R})$, et soit $u(t)$ une solution de (gKdV) pour tout $t \in \mathbb{R}$ telle que $u(0) = u_0$. Il existe $\alpha_0 > 0$ tel que si*

$$\|u_0 - Q_{c_0}\|_{H^1} < \alpha_0, \quad (1.5.45)$$

et s'il existe $x(t)$ tel que $v(t, x) = u(t, x + x(t))$ vérifie une condition de compacité L^2 :

$$\forall \delta_0 > 0, \exists A_0 > 0, \forall t \in \mathbb{R}, \quad \int_{|x| > A_0} v^2(t, x) \leq \delta_0, \quad (1.5.46)$$

alors il existe $c_1 > 0$, $x_1 \in \mathbb{R}$ tels que

$$\forall t \in \mathbb{R}, \forall x \in \mathbb{R}, \quad u(t, x) = Q_{c_1}(x - x_1 - c_1 t). \quad (1.5.47)$$

La démonstration de ce théorème passe par la démonstration d'un *théorème de Liouville* linéaire. L'opérateur linéaire associé (correspondant à l'approximation linéaire de la perturbation de (gKdV) autour d'un soliton) est

$$\mathcal{L}u = -\partial_x^2 u + u - pQ^{p-1}u. \quad (1.5.48)$$

Il ne dépend pas du temps et il a de bonnes propriétés spectrales.

On conjecture qu'un théorème analogue devrait être valide pour les breathers. Cependant, l'opérateur linéaire associé (il s'agit de l'opérateur \mathcal{L} où on a remplacé Q par B) est bien moins commode que dans le cas des solitons. Il n'est plus constant en temps à cause de la périodicité du breather, ce qui pose des problèmes, chose qui peut peut-être être résolue en moyennant sur une période. Et il n'a plus les propriétés spectrales qu'on avait dans le cas des solitons. En effet, pour les breathers, on connaît des propriétés spectrales d'opérateurs d'ordre 4 et ce n'est pas pour rien que l'étude de leur stabilité requiert l'utilisation de l'énergie seconde. En effet, on ne connaît pas d'opérateurs obtenus seulement à partir de la masse et de l'énergie qui auraient des bonnes propriétés spectrales avec les breathers.

Une autre piste envisageable est d'utiliser la transformation de Bäcklund pour ramener le problème au voisinage du breather à un problème au voisinage de 0. En effet, elle semble avoir de bonnes propriétés locales, ce qui rend pertinent de s'intéresser à une version du théorème de Liouville au voisinage de 0 considéré dans le chapitre 4. Cependant, l'utilisation de la transformée de Bäcklund contient un certain nombre de difficultés qu'il reste encore à résoudre et à préciser.

Construction d'un train infini de breathers

Le Coz et Tsai [68] ont construit une solution exotique de (NLS) focalisant \dot{H}^1 -sous-critique en établissant qu'il existe une solution qui se présente comme un *train infini* de solitons. Notons qu'un état fondamental Q pour (NLS), déterminant le profil d'un soliton, est déterminé par un paramètre de forme $\omega > 0$ comme l'unique solution strictement positive et à symétrie radiale dans H^1 de $\Delta Q - \omega Q + Q^p = 0$. Etant donné l'état fondamental Q associé à une fréquence ω , on peut définir un soliton de (NLS) qui lui est associé déterminé par les paramètres $x^0 \in \mathbb{R}^d$ (la position initiale), $v \in \mathbb{R}^d$ (la vitesse) et $\gamma \in \mathbb{R}$ (la phase) :

$$R_{Q,\omega,x^0,v,\gamma}(t, x) = e^{i(\omega t + \frac{1}{2}vx - \frac{1}{4}|v|^2 t + \gamma)} Q(x - x^0 - vt). \quad (1.5.49)$$

Enonçons le résultat :

Théorème 1.16 (Le Coz-Tsai [68]). *Soit $d \geq 1$ et $p > 1$ tel que (NLS) est \dot{H}^1 -sous-critique. Soit r_0 tel que $\max\left(1, \frac{d(p-1)}{2}\right) < r_0 < p+1$. Soit $(Q_j, \omega_j)_{j \in \mathbb{N}}$ une suite d'états fondamentaux tels que pour un certain $0 < a < 1$ et $D > 0$ indépendants de j , on ait*

$$\forall x \in \mathbb{R}^d, \forall j \in \mathbb{N}, \quad |Q_j(x)| + \omega_j^{-1/2} |\nabla Q_j(x)| \leq D \omega_j^{1/(p-1)} e^{-a \omega_j^{1/2} |x|}, \quad \text{et} \quad A_1 := \sum_{j \in \mathbb{N}} \omega_j^{\frac{1}{p-1} - \frac{d}{2r_0}} < +\infty. \quad (1.5.50)$$

Alors il existe des constantes $c_1 > 0$ et $v_\# \gg 1$ tels que pour tout profil de train infini de solitons R_∞ donné par

$$R_\infty := \sum_{j \in \mathbb{N}} R_j, \quad R_j(t, x) := R_{Q_j, \omega_j, 0, v_j, \gamma_j}(t, x), \quad j \in \mathbb{N}, \quad (1.5.51)$$

avec des paramètres $v_j \in \mathbb{R}^d$ et $\gamma_j \in \mathbb{R}$ qui vérifient $v_* = \inf_{j, k \in \mathbb{N}, j \neq k} \sqrt{\omega_j} |v_k - v_j| \geq v_\#$, il existe une solution u de (NLS) sur $[0, +\infty)$ vérifiant

$$\forall t \geq 0, \quad \|(u - R_\infty)(t)\|_{L^{p+1}} + \|u - R_\infty\|_{L_t^\infty L_x^2([t, \infty))} \leq e^{-c_1 v_* t}. \quad (1.5.52)$$

Elle est unique dans la classe des solutions vérifiant la dernière estimée.

On conjecture l'existence d'un train infini de breathers pour (mKdV). Cependant, le résultat pour (NLS) se démontre avec un argument de point fixe réalisé grâce à des estimées de Strichartz. On est tenté de mettre la même stratégie en oeuvre pour (mKdV). Cependant, il est plus compliqué de borner la nonlinéarité dans ce cadre en restant dans le même espace. En effet, la nonlinéarité pour (NLS) ne comporte pas de dérivées, alors que celle de (mKdV) si. C'est pourquoi les estimées de Strichartz en vue d'un argument de point fixe ne marchent pas très bien pour (mKdV). Pour dépasser ce problème, on est tenté d'utiliser les estimées de Kenig, Ponce et Vega [59] énoncées dans le Théorème 1.2. Bien qu'elles fournissent un bon point de départ pour éliminer le problème lié à la dérivée dans la nonlinéarité, elles restent insuffisantes pour adapter l'argument de Le Coz et Tsai [68].

Un train infini de solitons a également été utilisé par Martel et Merle [78] pour déplacer logarithmiquement un soliton R par rapport à son déplacement usuel déterminé par sa vitesse dans le cadre de (KdV) : ils construisent une solution $u(t)$ telle que $\|u(t) - Q(x - t - \ln(t))\|_{H^1} \rightarrow 0$ quand $t \rightarrow +\infty$, où Q est l'état fondamental de paramètre de forme $c = 1$. L'idée est que la collision avec chaque soliton du train engendre une translation chez R , ce qui en somme donne lieu à un déplacement logarithmique. Un tel résultat est possible car la collision avec un soliton de paramètre c est explicite et provoque une translation proportionnelle à \sqrt{c} chez R alors que la masse du soliton de paramètre c est proportionnelle à $c^{3/2}$ pour (KdV). Ainsi, on peut avoir un train de solitons qui existe dans un espace de Sobolev et qui provoque un déplacement infini chez R . Malheureusement, d'après l'analyse des ordres de grandeur des décalages fournis par la Proposition 1.14, on ne peut pas espérer un tel résultat pour (mKdV) car la translation provoquée par un soliton de paramètre c est proportionnelle à sa masse.

Dipôles, 2-breather exceptionnel

Les raisonnements effectués dans cette thèse ne peuvent pas être facilement étendus aux dipôles de (mKdV), qui ont été découverts par Wadati et Okhuma [118] par scattering inverse. En effet, les propriétés variationnelles des breathers qu'on utilise ne passent pas à la limite $\alpha \rightarrow 0$. On ne s'attend même pas à ce qu'un dipôle soit stable dans le même sens qu'un breather, car il peut à la fois être approché par des 2-solitons et par des breathers ayant un paramètre α petit.

L'autre raison pour laquelle les raisonnements effectués dans cette thèse ne s'étendent pas aux dipôles est que la localisation exponentielle de type (1.5.18) vérifiée par les solitons et les breathers ne s'étend pas aux dipôles. Ceci dit, ce deuxième problème semble moins sérieux car, même si on ne peut pas localiser exponentiellement un dipôle autour d'un point, on peut le localiser autour d'une région dont la taille croît logarithmiquement en temps.

On peut ainsi se demander si les résultats d'existence et d'unicité pour les multi-breathers démontrés dans cette thèse s'étendent à une collection d'objets constituée de solitons, de breathers et de dipôles. Pour cela, il faudrait comprendre mieux la structure variationnelle vérifiée par les dipôles. On peut également chercher la notion de stabilité adaptée aux dipôles, qui doit forcément être moins stricte que la stabilité orbitale qu'on connaît.

Nguyen [96, 95] a démontré l'existence de dipôles dans un cadre non intégrable pour (NLS) et (gKdV). Cela a pu être fait en déterminant le caractère répulsif ou attractif de l'interaction entre deux solitons selon leur signe.

Théorème 1.17 (Nguyen [95]). *Soit $p \geq 3$ un entier différent de 5. Il existe une solution H^1 de (gKdV) $u(t)$ définie pour tout temps suffisamment grand telle que*

$$\|u(t) - (Q(\cdot - t - \ln(vt)) + \sigma Q(\cdot - t + \ln(vt)))\|_{H^1} \rightarrow 0 \quad \text{lorsque } t \rightarrow +\infty, \quad (1.5.53)$$

où Q désigne l'état fondamental de paramètre $c = 1$,

$$v := v(p) = \sqrt{\frac{8(p-1)}{|5-p|}} (2p+2)^{\frac{1}{p-1}} \|Q\|_{L^2}^{-1} > 0, \quad (1.5.54)$$

$\sigma = -1$ pour $p = 3, 4$ et $\sigma = 1$ pour $p \geq 6$.

On peut donc aussi se demander s'il est possible d'adapter l'approche de Nguyen pour construire une solution de (mKdV) constituée par deux breathers de mêmes paramètres de forme et d'étudier l'interaction entre eux. Notons que dans un premier temps, on peut commencer par chercher l'allure d'un 2-breather constitué de deux breathers ayant des paramètres de forme différents mais la même vitesse, grâce à la formule obtenue par scattering inverse pour les multi-breathers.

1.5.6 Organisation des chapitres

Nous terminons cette introduction générale en décrivant les chapitres qui suivent.

Le chapitre 2 est consacré au développement des résultats de cette thèse [108] liés à l'existence, l'unicité et la régularité des multi-breathers de (mKdV), ainsi que des résultats liés à la vitesse de convergence du multi-breather vers la somme de solitons et de breathers associée. On y démontre les théorèmes 1.9 et 1.10, ainsi que la proposition 1.11.

Le chapitre 3 est consacré au développement des résultats de cette thèse [109] liés à la stabilité orbitale d'une somme de solitons et de breathers de (mKdV). Compte tenu du chapitre 4, on y déduit aussi une amélioration du résultat d'unicité des multi-breathers énoncé dans le chapitre 2. On y démontre le théorème 1.12, ainsi que la proposition 1.13.

Le chapitre 4 regroupe des calculs effectués sur les breathers et les multi-breathers. En particulier, il est consacré à l'étude de la formule des multi-breathers obtenue par scattering inverse. On s'y convainc que la formule implique la proposition 1.14. On y calcule également le décalage subi par deux breathers après interaction, ce qui nous permet de comparer l'ordre de grandeur de ce décalage par rapport à la taille des breathers.

Chapitre 2

Existence et unicité des multi-breathers de (mKdV)

Abstract

We consider the modified Korteweg-de Vries equation (mKdV) and prove that given any sum P of solitons and breathers of (mKdV) (with distinct velocities), there exists a solution p of (mKdV) such that $p(t) - P(t) \rightarrow 0$ when $t \rightarrow +\infty$, which we call multi-breather. In order to do this, we work at the H^2 level (even if usually solitons are considered at the H^1 level). We will show that this convergence takes place in any H^s space and that this convergence is exponentially fast in time.

We also show that the constructed multi-breather is unique in two cases : in the class of solutions which converge to the profile P faster than the inverse of a polynomial of a large enough degree in time (we will call this a super polynomial convergence), or (without hypothesis on the convergence rate), when all the velocities, except possibly one, are positive.¹

2.1 Introduction

2.1.1 Setting of the problem

We consider the modified Korteweg-de Vries equation on \mathbb{R} :

$$\begin{cases} u_t + (u_{xx} + u^3)_x = 0 & (t, x) \in \mathbb{R}^2 \\ u(0) = u_0 & u(t, x) \in \mathbb{R} \end{cases} \quad (\text{mKdV})$$

The (mKdV) equation appears as a model of some physical problems as plasma physics [106, 20], electrodynamics [97], fluid mechanics [50], ferromagnetic vortices [122], and more.

In [58], Kenig, Ponce and Vega established local well-posedness in H^s , for $s \geq \frac{1}{4}$, of the Cauchy problem for (mKdV), by fixed point argument in $L_x^p L_t^q$ type spaces. Moreover, if $s > \frac{1}{4}$, the Cauchy problem is globally well posed [23]. Recently, Harrop-Griffiths, Killip and Visan [46] proved local well-posedness in H^s for $s > -1/2$. However, in this paper, we will only use the global well-posedness in H^2 .

1. This chapter corresponds to the preprint [108] that is accepted for publication in the Revista Matemática Iberoamericana.

(mKdV) is an integrable equation (like the original Korteweg-de Vries equation) and thus it has an infinity of conservation laws, see [92, 2]. We will use three of them (the first two of them are called *mass* and *energy*; the third is sometimes called *second energy*) :

$$M[u](t) := \frac{1}{2} \int_{\mathbb{R}} u^2(t, x) dx, \quad (2.1.1)$$

$$E[u](t) := \frac{1}{2} \int_{\mathbb{R}} u_x^2(t, x) dx - \frac{1}{4} \int_{\mathbb{R}} u^4(t, x) dx, \quad \text{and} \quad (2.1.2)$$

$$F[u](t) := \frac{1}{2} \int_{\mathbb{R}} u_{xx}^2(t, x) dx - \frac{5}{2} \int_{\mathbb{R}} u^2(t, x) u_x^2(t, x) dx + \frac{1}{4} \int_{\mathbb{R}} u^6(t, x) dx. \quad (2.1.3)$$

Observe that if u is a solution of (mKdV) then $-u$ and, for any $x_0 \in \mathbb{R}$, $(t, x) \mapsto u(t, x - x_0)$ are solutions of (mKdV) too.

(mKdV) is a dispersive nonlinear equation that is a special case of a more general class of equations : the general Korteweg-de Vries equation (gKdV), where the nonlinearity u^3 is replaced by $f(u)$ for some real valued function f . The particularity of (mKdV) in comparison to other (gKdV) equation is that it admits special non linear solutions, namely breather solutions.

The most simple nonlinear solutions of (mKdV) are solitons, i.e. a bump of a constant shape that translates with a constant velocity without deformation, that is, solutions of the form $u(t, x) = Q_c(x - ct)$, where c is the velocity and Q_c is the profile function that depends only on one variable. $Q_c \in H^1(\mathbb{R})$ should solve the elliptic equation :

$$Q_c'' = cQ_c - Q_c^3. \quad (2.1.4)$$

We can show that necessarily $c > 0$ and that, if $c > 0$, (2.1.4) has a unique solution in $H^1(\mathbb{R})$, up to translations and reflexion with respect to the x -axis. Actually, one has the explicit formula :

$$Q_c(x) := \left(\frac{2c}{\cosh^2(c^{1/2}x)} \right)^{\frac{1}{2}}. \quad (2.1.5)$$

Observe that we chose Q_c so that it is even and positive.

A *soliton* is a solution of (mKdV), parameterized by a velocity parameter $c > 0$, a sign parameter $\kappa \in \{-1, 1\}$ and a translation parameter $x_0 \in \mathbb{R}$ (it corresponds to the initial position of the soliton) that has the following expression :

$$R_{c,\kappa}(t, x; x_0) := \kappa Q_c(x - x_0 - ct). \quad (2.1.6)$$

When $\kappa = -1$, this object is sometimes called *antisoliton*. Notice that solitons are smooth and decaying. The generalized Korteweg-de Vries equation (gKdV) also admit soliton type solutions, and the focusing nonlinear Schrödinger equation (NLS) as well. Solitons have been extensively studied, in particular their stability. Cazenave, Lions and Weinstein in [121, 14, 19, 120] were interested in orbital stability of (gKdV) and (NLS) solitons in H^1 . A soliton of (mKdV) is indeed orbitally stable, i.e. if a solution is initially close to a soliton in $H^1(\mathbb{R})$, then it stays close to the soliton, up to a space translation defined for any time, in $H^1(\mathbb{R})$. General results about orbital stability of nonlinear dispersive solitons are presented by Grillakis, Shatah and Strauss in [44]. The result about orbital stability of a soliton can be improved in a result of asymptotic stability, as it was done in the works by Martel and Merle [73, 80, 78], see also [41].

A *breather* is a solution of (mKdV), parameterized by $\alpha, \beta > 0$, $x_1, x_2 \in \mathbb{R}$ that has the following expression :

$$B_{\alpha,\beta}(t, x; x_1, x_2) := 2\sqrt{2}\partial_x \left[\arctan \left(\frac{\beta \sin(\alpha y_1)}{\alpha \cosh(\beta y_2)} \right) \right], \quad (2.1.7)$$

where

$$\begin{aligned} y_1 &:= x + \delta t + x_1 \quad \text{and} \quad y_2 := x + \gamma t + x_2, \\ \text{with } \delta &:= \alpha^2 - 3\beta^2 \quad \text{and} \quad \gamma := 3\alpha^2 - \beta^2. \end{aligned}$$

It corresponds to a localized periodic in time function (with frequency α , and exponential localization with decay rate β) that propagates at a constant velocity $-\gamma$ in time. Like solitons, breathers are smooth and decaying in space. Unlike solitons, breather's velocities can be positive, zero or negative. α, β are the shape parameters and x_1, x_2 are the translation parameters of a breather. Note that if we replace the parameter x_1 by $x_1 + \frac{\pi}{\alpha}$, we transform $B_{\alpha, \beta}(\cdot, \cdot; x_1, x_2)$ in $-B_{\alpha, \beta}(\cdot, \cdot; x_1, x_2)$ (therefore, we do not need to talk about “antibreathers”).

Breathers were first introduced by Wadati in [117], and they were already used by Kenig, Ponce and Vega in [60] to prove that the flowmap associated to (mKdV) equation is *not* uniformly continuous in H^s for $s < \frac{1}{4}$: the point is that two breathers with close velocities can be very close at $t = 0$ and can separate as fast as we want in H^s with $s < \frac{1}{4}$, if α is taken large enough.

(mKdV) breathers and their properties, as well as breathers for other equations, are well studied by Alejo and Muñoz and co-authors in [8, 7, 10, 11, 9].

Let us singularize a result of H^2 orbital stability for breathers established in [8], and improved to H^1 orbital stability in [9]. In this last paper, a partial result of asymptotic stability is also given, for breathers traveling to the right only, with positive velocity $-\gamma > 0$; asymptotic stability for breathers in full generality is still an open problem.

When $\alpha \rightarrow 0$, $B_{\alpha, \beta}$ tends to a solution of (mKdV) called *double-pole solution* [118], the methods employed in this article as well as the proof of orbital stability made by Alejo and Muñoz seem not to apply for this limit, which is expected to be unstable according to the numerical computations in [42].

An important result regarding the long time dynamics of (mKdV) is the soliton-breather resolution [21]: it asserts that any generic solution can be approached by a sum of solitons and breathers when $t \rightarrow +\infty$ (up to a dispersive and a self-similar term). Together with their stability properties, the soliton-breather resolution shows why solitons and breathers are essential objects to study. This resolution was established for initial conditions in a weighted Sobolev space in [21] (see also Schuur [107]) by inverse scattering method; see also [107] for the soliton resolution for (KdV). Observe that (mKdV) breathers do not decouple into simple solitons for large time (it is a *fully bounded state* as it is called in [8]); therefore, it must appear in the resolution. The soliton-breather resolution is one of the motivations of the study of multi-breathers, which we define below.

There are works in the literature about a more complicated object obtained from several solitons: a *multi-soliton*. A multi-soliton is a solution $r(t)$ of (mKdV) such that there exists $0 < c_1 < c_2 < \dots < c_N$, $\kappa_1, \dots, \kappa_N \in \{-1, 1\}$ and $x_1, \dots, x_N \in \mathbb{R}$, such that

$$\lim_{t \rightarrow +\infty} \left\| r(t) - \sum_{j=1}^N R_{c_j, \kappa_j}(t, \cdot; x_j) \right\|_{H^1(\mathbb{R})} = 0. \quad (2.1.8)$$

This definition is not specific to (mKdV) and makes sense for many other nonlinear dispersive PDEs as soon as they admit solitons. This object is introduced by Schuur [107] and Lamb [64], see also Miura [91], where explicit formulas are given: these were obtained by inverse scattering method thanks to the integrability of the equation. Multi-solitons were first constructed in a non integrable context by Merle [88] for the mass critical (NLS). Martel [71] constructed multi-solitons for mass-subcritical and critical (gKdV) equations and proved that they are unique in $H^1(\mathbb{R})$, smooth and converge exponentially fast to their profile in any Sobolev space H^s . Similar studies were done for other nonlinear dispersive PDEs. Martel and Merle [79] have proved the existence of multi-solitons for (NLS) in H^1 , Côte, Martel and Merle extended this construction to mass supercritical (gKdV) and (NLS) in [30]. Friederich and Côte in [28] proved smoothness, and uniqueness in a class of algebraic convergence. Côte and Muñoz constructed

in [31] multi-solitons for the nonlinear Klein-Gordon equation. Ming, Rousset and Tzvetkov have constructed multi-solitons for the water-waves systems in [89]. Valet has proved in [115] the existence and uniqueness of multi-solitons in H^1 for the Zakharov-Kuznetsov equation, which generalizes (gKdV) to higher dimension.

2.1.2 Main results

We prove in this article that given any sum of solitons and breathers with distinct velocities, there exists a solution of (mKdV) whose difference with this sum tends to zero when time goes to infinity. This solution will be called a multi-breather. Let us make the definition more precise.

Let $J \in \mathbb{N}$ and $K, L \in \mathbb{N}$ such that $J = K + L$. We will consider a set of L solitons and K breathers :

- the breather parameters are $\alpha_k > 0$, $\beta_k > 0$, $x_{1,k}^0 \in \mathbb{R}$ and $x_{2,k}^0 \in \mathbb{R}$ for $1 \leq k \leq K$.
- the solitons parameters are $c_l > 0$, $\kappa_l \in \{-1, 1\}$ and $x_{0,l}^0 \in \mathbb{R}$ for $1 \leq l \leq L$.

We define for $1 \leq k \leq K$, the k th breather :

$$B_k(t, x) := B_{\alpha_k, \beta_k}(t, x; x_{1,k}^0, x_{2,k}^0); \quad (2.1.9)$$

and for $1 \leq l \leq L$, the l th soliton :

$$R_l(t, x) := R_{c_l, \kappa_l}(t, x; x_{0,l}^0). \quad (2.1.10)$$

We now define the *velocity* of our objects. Recall that for $1 \leq k \leq K$, the velocity of B_k is

$$v_k^b := -\gamma_k = \beta_k^2 - 3\alpha_k^2, \quad (2.1.11)$$

and for $1 \leq l \leq L$, the velocity of R_l is

$$v_l^s := c_l. \quad (2.1.12)$$

The most important assumption we make is that all these velocities are distinct :

$$\forall k \neq k' \quad v_k^b \neq v_{k'}^b, \quad \forall l \neq l' \quad v_l^s \neq v_{l'}^s, \quad \forall k, l \quad v_k^b \neq v_l^s. \quad (2.1.13)$$

These implies for any two of these objects to be far from each other when time is large, and this assumption is essential in our analysis.

It will be useful to order our breathers and solitons by increasing velocities. As these are distinct, we can define an increasing function :

$$\underline{v} : \{1, \dots, J\} \longrightarrow \{v_k^b, 1 \leq k \leq K\} \cup \{v_l^s, 1 \leq l \leq L\}. \quad (2.1.14)$$

The set $\{v_1, \dots, v_J\}$ is thus the (ordered) set of all possible velocities of our objects. We define P_j , for $1 \leq j \leq J$, as the object (either a soliton R_l or a breather B_k) that corresponds to the velocity v_j . Hence, P_1, \dots, P_J are the considered objects ordered by increasing velocity.

We will need both notations : the indexation by k and l , and the indexation by j , and we will keep these notations to avoid ambiguity.

We will denote by x_j the center of mass of P_j , that is

- if $P_j = B_k$ is a breather, we set $x_j(t) := -x_{2,k}^0 + v_j t$;
- if $P_j = R_l$ is a soliton, we set $x_j(t) := x_{0,l}^0 + v_j t$.

We denote :

$$R = \sum_{l=1}^L R_l, \quad B = \sum_{k=1}^K B_k, \quad P = R + B = \sum_{j=1}^J P_j. \quad (2.1.15)$$

We can now define a multi-breather : as solitons are objects which can be studied naturally in $H^1(\mathbb{R})$, it turns out that breathers are best studied in $H^2(\mathbb{R})$; therefore, it is in this latter space that we develop our analysis.

Definition 2.1. A *multi-breather* associated to the sum P given in (2.1.15) of solitons and breathers is a solution $p \in \mathcal{C}([T^*, +\infty), H^2(\mathbb{R}))$, for a constant $T^* > 0$, of (mKdV) such that

$$\lim_{t \rightarrow +\infty} \|p(t) - P(t)\|_{H^2} = 0. \quad (2.1.16)$$

We will prove two results in this article. The first one is the existence and the regularity of a multi-breather, the second one is the uniqueness of a multi-breather. The uniqueness is established in two settings : in the case when all velocities (except possibly one) are positive, and without any assumption on the sign of the considered velocities. However, in the last case, the uniqueness is obtained in a narrower class of functions.

Theorem 2.2. *Given solitons and breathers (2.1.9), (2.1.10) whose velocities (2.1.11) and (2.1.12) satisfy (2.1.13), there exists a multi-breather p associated to P given in (2.1.15). Moreover,*

$$p \in \mathcal{C}^\infty(\mathbb{R} \times \mathbb{R}) \cap \mathcal{C}^\infty(\mathbb{R}, H^s(\mathbb{R}))$$

for any $s \geq 0$ and there exists $\theta > 0$ such that for any $s \geq 0$, there exists $A_s \geq 1$ and $T^* > 0$ such that

$$\forall t \geq T^*, \quad \|p(t) - P(t)\|_{H^s} \leq A_s e^{-\theta t}. \quad (2.1.17)$$

Remark 2.3. We will also show that θ does only depend on the shape parameters of our objects : α_k, β_k, c_l . Moreover, if there exists $D > 0$ such that for all $j \geq 2$, $x_j(0) \geq x_{j-1}(0) + D$, then A_s and T^* do not depend on $x_{1,k}^0, x_{2,k}^0, x_{0,l}^0$ but only on α_k, β_k, c_l and D . Finally, if $D > 0$ is large enough with respect to the problem data, then (2.1.17) is true for $T^* = 0$. See Section 2.3.2 for further details.

Theorem 2.4. *Given the same set of solitons and breathers as in Theorem 2.2 whose velocities satisfy (2.1.13) and $v_2 > 0$ (so that all the velocities, except possibly one, are positive), the multi-breather p associated to P by Theorem 2.2, in the sense of Definition 2.1, is unique.*

Proposition 2.5. *Given the same set of solitons and breathers as in Theorem 2.2 whose velocities satisfy (2.1.13), there exists $N > 0$ large enough such that the multi-breather p associated to P by Theorem 2.2 is the unique solution $u \in \mathcal{C}([T_0, +\infty), H^2(\mathbb{R}))$ of (mKdV) such that*

$$\|u(t) - P(t)\|_{H^2} = O\left(\frac{1}{t^N}\right), \quad \text{as } t \rightarrow +\infty. \quad (2.1.18)$$

In [118], there exists a formula for a multi-breather, obtained by inverse scattering method, that in some sense already gives the existence of a multi-breather. However, the proof of the theorem 2.2 from this formula is rather involved.

In this paper, we give here a different approach to prove the existence of a multi-breather and we clearly show that we have convergence of the constructed multi-breather to the corresponding sum of solitons and breathers in H^s , that this convergence is exponentially fast in time and that the constructed multi-breather is smooth. To do this, we use the variational structure of solitons and breathers. This is why, we give a proof that is potentially generalizable to non integrable equations, and that uses similar type of techniques as in the proof of the uniqueness (the latter cannot be deduced from the formula). In any case, uniqueness of multi-breathers is new.

In this paper, we adapt the arguments given by Martel and Merle [79], by Martel [71] and by Côte and Friederich [28] to the context of breathers. To do so, one needs to understand the variational structure of breathers, in the same fashion as Weinstein did in [121] for (NLS) solitons. Such results were obtained by Alejo and Muñoz in [8] : a breather is a critical point of a Lyapunov functional at the H^2 level, whose Hessian is coercive up to several (but finitely many) orthogonal conditions, see Section 2.2 for details. As we see from [8], the H^2 regularity level is the

most natural setting to study breathers, and the H^1 regularity level is natural for the study of solitons (as we see in [71, 79]). One important issue we face is therefore to understand soliton variational structure at H^2 level, and to adapt the Lyapunov functional in [8] to accommodate for a sum of breathers (and solitons). Notice that arguments based on monotonicity may be adapted only if we suppose that all the considered velocities, except possibly one, are positive. Because [79, 28] are not based on monotonicity (these are results for (NLS) which is not well suited for monotonicity), we can adapt their arguments to obtain existence and uniqueness results for our case without any condition on the sign of velocities. The uniqueness result obtained in this setting is however weaker than the one that is obtained with monotonicity arguments.

2.1.3 Outline of the proof

The proof of Theorem 2.2 (the existence of multi-breathers) is split into two main parts : the construction of an H^2 multi-breather and the proof that this multi-breather is smooth.

An H^2 multi-breather

Let us start with the first part. We consider an increasing sequence (T_n) of \mathbb{R}_+ with $T_n \rightarrow +\infty$, and for $n \in \mathbb{N}$, let p_n the unique global H^2 solution of (mKdV) such that $p_n(T_n) = P(T_n)$ (recall that the Cauchy problem for (mKdV) is globally well-posed in H^2).

We will prove the following *uniform estimate* :

Proposition 2.6. *There exists $T^* > 0$, $A > 0$, $\theta > 0$ such that, for any $n \in \mathbb{N}$ such that $T_n \geq T^*$,*

$$\forall t \in [T^*, T_n], \quad \|p_n(t) - P(t)\|_{H^2} \leq Ae^{-\theta t}. \quad (2.1.19)$$

With this proposition in hand, we can construct an H^2 multi-breather which converges exponentially fast to its profile, which is the first part of Theorem 2.2, as stated below.

Proposition 2.7. *There exists $T^* \in \mathbb{R}$, $A > 0$, $\theta > 0$ and a solution $p \in C([T^*, +\infty), H^2(\mathbb{R}))$ of (mKdV) such that*

$$\forall t \geq T^*, \quad \|p(t) - P(t)\|_{H^2} \leq Ae^{-\theta t}. \quad (2.1.20)$$

Proof of Proposition 2.7 assuming Proposition 2.6. We show that the sequence $(p_n(T^*))$ is L^2 -compact, in the following sense :

Lemma 2.8. *For any $\varepsilon > 0$, there exists $R > 0$ such that*

$$\forall n \in \mathbb{N} \quad \int_{|x| > R} p_n^2(T^*, x) dx < \varepsilon. \quad (2.1.21)$$

An analogous lemma has already been proved on p. 1111 of [71], which is the proof of formula (14) (and can also be found in [79]). The same proof works here. We need to use Proposition 2.6 for T_n large enough and then make a time variation to obtain the result in T^* . We can first find R that works for $P^2(t_0)$ instead of $p_n^2(T^*)$ for a fixed $t_0 > T^*$ large enough. From Proposition 2.6, we see that if we take t_0 large enough, we obtain the desired lemma for $p_n^2(t_0)$ instead of $p_n^2(T^*)$. To finish, with the help of a cut-off function, we control time variations of $\int_{|x| > R} p_n^2(t) dx$, where R is taken larger if needed. This is why, we obtain the result at $t = T^*$.

As a consequence of the Proposition 2.6 above, $(\|p_n(T^*)\|_{H^2})$ is a bounded sequence. Thus, there exists $p^* \in H^2(\mathbb{R})$ such that, up to a subsequence,

$$p_n(T^*) \rightharpoonup p^* \quad \text{in } H^2. \quad (2.1.22)$$

Thus, from Lemma 2.8, there holds the strong convergence :

$$p_n(T^*) \rightarrow p^* \quad \text{in } L^2. \quad (2.1.23)$$

Therefore, we obtain by interpolation :

$$p_n(T^*) \rightarrow p^* \quad \text{in } H^1. \quad (2.1.24)$$

Now, let us consider the global H^1 (even H^2) solution p of (mKdV) such that $p(T^*) = p^*$. As shown in [71], the Cauchy problem for (mKdV) has a continuous dependence in H^1 on compact sets of time. Let $t \geq T^*$. By continuous dependence, we deduce that $p_n(t) \rightarrow p(t)$ in H^1 . $(p_n(t) - P(t))$ is a bounded sequence in H^2 , which admits a unique weak limit and so

$$p_n(t) - P(t) \rightharpoonup p(t) - P(t) \quad \text{in } H^2. \quad (2.1.25)$$

By weak convergence and from Proposition 2.6, we obtain :

$$\|p(t) - P(t)\|_{H^2} \leq \liminf_{n \rightarrow +\infty} \|p_n(t) - P(t)\|_{H^2} \leq Ae^{-\theta t}. \quad (2.1.26)$$

As this is true for any $t \geq T^*$. This ends the proof of the Proposition 2.7. \square

It remains to prove Proposition 2.6, for which we rest on a bootstrap argument. More precisely, we will reduce the proof to the following proposition :

Proposition 2.9. *There exists $T^* > 0$, $A > 0$, $\theta > 0$, such that for any $n \in \mathbb{N}$ such that $T_n \geq T^*$, for any $t^* \in [T^*, T_n]$, if*

$$\forall t \in [t^*, T_n], \quad \|p_n(t) - P(t)\|_{H^2} \leq Ae^{-\theta t}, \quad (2.1.27)$$

then

$$\forall t \in [t^*, T_n], \quad \|p_n(t) - P(t)\|_{H^2} \leq \frac{A}{2}e^{-\theta t}. \quad (2.1.28)$$

The proof of Proposition 2.6 then follows from a simple continuity argument.

Proof of Proposition 2.6 assuming Proposition 2.9. We define t_n^* in the following way :

$$t_n^* := \inf\{t^* \in [T^*, T_n], \quad \forall t \in [t^*, T_n], \quad \|p_n(t) - P(t)\|_{H^2} \leq Ae^{-\theta t}\}. \quad (2.1.29)$$

The map $t \mapsto \|p_n(t) - P(t)\|_{H^2}$ is a continuous function and $\|p_n(T_n) - P(T_n)\|_{H^2} = 0$. This means that there exists $T^* \leq t^* < T_n$ such that

$$\forall t \in [t^*, T_n], \quad \|p_n(t) - P(t)\|_{H^2} \leq Ae^{-\theta t}. \quad (2.1.30)$$

Therefore, we have that

$$T^* \leq t_n^* < T_n. \quad (2.1.31)$$

We would like to prove that $t_n^* = T^*$. Let us argue by contradiction and assume that $t_n^* > T^*$. The Proposition 2.9 allows us to deduce that

$$\forall t \in [t_n^*, T_n], \quad \|p_n(t) - P(t)\|_{H^2} \leq \frac{A}{2}e^{-\theta t}. \quad (2.1.32)$$

This means that

$$\|p_n(t_n^*) - P(t_n^*)\|_{H^2} \leq \frac{A}{2}e^{-\theta t_n^*}, \quad (2.1.33)$$

which means that t_n^* could be chosen smaller, by continuity. This is a contradiction. \square

Hence, we are left to prove Proposition 2.9, which will be done in Section 2.2.

The H^2 multi-breather is smooth

We now turn to the second part of Theorem 2.2, which is strongly adapted from [71]. The heart of this part is to prove uniform estimates in H^s for $p_n - P$, for any $s \geq 0$:

Proposition 2.10. *There exists $T^* > 0$, $\theta > 0$, such that for any $s \geq 0$, there exists $A_s \geq 1$ such that for any $n \in \mathbb{N}$ such that $T_n \geq T^*$,*

$$\forall t \in [T^*, T_n], \quad \|p_n(t) - P(t)\|_{H^s} \leq A_s e^{-\theta t}. \quad (2.1.34)$$

With this improved version of Proposition 2.6, one can prove by the same reasoning as in the proof of the Proposition 2.7, that for any $s \geq 0$, p actually belongs to $L^\infty([T^*, +\infty), H^s(\mathbb{R}))$ and that the convergence of $p(t) - P(t)$ occurs in H^s with an exponential decay rate. More precisely,

Theorem 2.11. *For any $s \geq 2$, we have that $p \in \mathcal{C}([T^*, +\infty), H^s(\mathbb{R}))$, and furthermore,*

$$\forall t \geq T^*, \quad \|p(t) - P(t)\|_{H^s} \leq A_s e^{-\theta t}. \quad (2.1.35)$$

It remains to prove Proposition 2.10, which will be done in Section 2.3.

The uniqueness result

We denote p the multi-breather constructed in the previous sections, the existence of which is established. Let u be a solution of (mKdV) such that

$$\|u - P\|_{H^2} \rightarrow_{t \rightarrow +\infty} 0. \quad (2.1.36)$$

Equivalently, there holds :

$$\|u - p\|_{H^2} \rightarrow_{t \rightarrow +\infty} 0. \quad (2.1.37)$$

We denote

$$z := u - p. \quad (2.1.38)$$

The goal is to prove that $z = 0$. We prove it in two configurations : when all the velocities, except possibly one, are positive (Theorem 2.4), and without any assumption on velocities (Proposition 2.5), but in this last case we need to assume a stronger convergence than given in (2.1.36).

The proof of Theorem 2.4 will be carried out in two steps.

We start with Proposition 2.5, which is adapted from [28]. For this, we do not study $u - P$ anymore, we deal only with $z = u - p$. z is the difference of two solutions of (mKdV), which is much more precise than $u - P$. Thus, we do not modulate parameters of the solitons, as it is needed in other parts of the proof in order to deal with the soliton part of the linear part of the Lyapunov functional, and we avoid some difficulty. In order to prove our inequalities, we need again to use coercivity of the same type of quadratic forms. In order to do this, we replace z by $\tilde{z} = z + \sum_{j=1}^J c_j K_j$, where K_j , $j = 1, \dots, J$ is a well chosen basis of the kernel of the quadratic form, in order to have \tilde{z} orthogonal to any K_j . A important idea is to use slow variations of localized functionals with adapted cut-off functions of the form $\varphi\left(\frac{x-vt}{\delta t}\right)$, which provides an extra $O(1/t)$ decay when derivatives fall on the cut-off, and ultimately explain why algebraic decay comes into play.

In the context of Theorem 2.4, we actually prove that

$$v := u - P \quad (2.1.39)$$

converges exponentially fast to 0 : this is the purpose of Proposition 2.43, which uses some ideas of [71]. Due to Proposition 2.5, we deduce immediately from there that an exponential convergence is trivial, that is $z = 0$.

To prove Proposition 2.43, we use monotonicity properties combined with coercivity of an energy type functional very similar to that used for the existence result. This is why, we also need to modulate, and the choice of the orthogonality condition is essential : it allows to bound linear terms in w that appear in the computations. An issue of the mixed breathers/solitons context is that one cannot build a functional adapted to all the nonlinear objects at once, as it is done in [71]. Instead, we carry out an induction and we argue successively around each object, soliton or breather, separately.

Organisation of the paper

Sections 2.2 and 2.3 are devoted to the proof of the existence of a multi-breather : Proposition 2.9 is proved in Section 2.2, Proposition 2.10 is proved in Section 2.3. Section 2.4 gathers the proofs of the uniqueness results : Section 2.4.1 is devoted to the proof of Proposition 2.5, and Sections 2.4.2 and 2.4.3 are devoted to the proof of Theorem 2.4.

2.1.4 Acknowledgments

The author would like to thank his supervisor Raphaël Côte for suggesting the idea of this work, for fruitful discussions and for his useful advice.

2.2 Construction of a multi-breather in $H^2(\mathbb{R})$

We set

$$\begin{aligned}\beta &:= \min\{\beta_k, 1 \leq k \leq K\} \cup \{\sqrt{c_l}, 1 \leq l \leq L\}, \\ \tau &:= \min\{v_{j+1} - v_j, 1 \leq j \leq J-1\}.\end{aligned}\tag{2.2.1}$$

Our goal in this section is to prove Proposition 2.9.

2.2.1 Elementary results

Let us first collect a few basic facts that will be used throughout the article. One may check an exponential decay result for any of our objects :

Proposition 2.12. *Let $j = 1, \dots, J$, $n, m \in \mathbb{N}$. Then, there exists a constant $C > 0$ such that for any $t, x \in \mathbb{R}$,*

$$|\partial_x^n \partial_t^m P_j(t, x)| \leq C e^{-\beta|x-v_j t|}.\tag{2.2.2}$$

Corollary 2.13. *Let $r > 0$. For t, x such that $v_j t + r < x < v_{j+1} t - r$, we have that*

$$|P(t, x)| \leq C e^{-\beta r}.\tag{2.2.3}$$

The same is true for any space or time derivative of P .

We will also use the following cross-product result :

Proposition 2.14. *Let $i \neq j \in \{1, \dots, J\}$ and $m, n \in \mathbb{N}$. There exists a constant C that depends only on P , such that for any $t \in \mathbb{R}$,*

$$\left| \int \partial_x^m P_i \partial_x^n P_j \right| \leq C e^{-\beta \tau t/2}.\tag{2.2.4}$$

There is also an orthogonality result for breathers that will be useful :

Lemma 2.15. *Let $B := B_{\alpha,\beta}$ be a breather. We denote $B_1 := \partial_{x_1} B$ and $B_2 := \partial_{x_2} B$. Then,*

$$\int BB_1 = \int BB_2 = 0. \quad (2.2.5)$$

Proof. Note that $\text{Span}(B_1, B_2) = \text{Span}(B_x, B_t)$. Therefore, it is enough to prove that

$$\int BB_x = \int BB_t = 0. \quad (2.2.6)$$

Firstly,

$$\int BB_x = \frac{1}{2} \int (B^2)_x = 0. \quad (2.2.7)$$

Secondly,

$$\int BB_t = \frac{1}{2} \int (B^2)_t = \frac{1}{2} \frac{d}{dt} \int B^2 = 0, \quad (2.2.8)$$

by mass conservation and because a breather is a solution of (mKdV). \square

2.2.2 Almost-conservation of localized conservation laws

From now on, we will fix $n \in \mathbb{N}$. This is why, for the simplicity of notations, we can write T for T_n , and p for p_n . The goal will be to find constants $T^*, A > 1, \theta$ that do not depend on n , nor on the translation parameters of the given objects, and that will be chosen later (T^* will depend on A and θ), such that Proposition 2.9 is verified. We will take $t^* \in [T^*, T]$, and we will make the following bootstrap assumption for the remaining of the article :

$$\forall t \in [t^*, T], \quad \|p(t) - P(t)\|_{H^2} \leq Ae^{-\theta t}, \quad (2.2.9)$$

where $p(T) = P(T)$.

Remark 2.16. We have the following property for solutions of (mKdV) : there exists $C_0 > 0$ such that for any solution w of (mKdV), w is global and

$$\forall t \in \mathbb{R}, \quad \|w(t)\|_{H^2} \leq C_0 \|w(T)\|_{H^2}. \quad (2.2.10)$$

Therefore,

$$\forall t \in \mathbb{R}, \quad \|p(t)\|_{H^2} \leq C_0 \|P(T)\|_{H^2} \leq C_0 \sum_{j=1}^J \|P_j(T)\|_{H^2} \leq C_0 C, \quad (2.2.11)$$

where C is a constant that depends only on the problem data (because the H^s -norm of solitons or breathers can be easily bounded).

Let

$$\theta := \frac{\beta\tau}{32}. \quad (2.2.12)$$

Let $\min(1, \frac{\tau}{4}) > \delta > 0$ be a constant to be chosen later.

This part of the proof is adapted from [79].

Let $\psi(x)$ be a C^3 function such that

$$\begin{aligned} 0 &\leq \psi \leq 1 \quad \text{on } \mathbb{R}, \quad \psi' \geq 0 \quad \text{on } \mathbb{R}, \\ \psi(x) &= 0 \quad \text{for } x \leq -1, \quad \psi(x) = 1 \quad \text{for } x \geq 1, \end{aligned} \quad (2.2.13)$$

and satisfying, for a constant $C > 0$, for any $x \in \mathbb{R}$,

$$(\psi'(x))^{4/3} \leq C\psi(x), \quad (\psi'(x))^{4/3} \leq C(1 - \psi(x)), \quad |\psi''(x)|^{3/2} \leq C\psi'(x). \quad (2.2.14)$$

Note that it is enough to take ψ that is equal to $(1+x)^4$ on a neighbourhood of -1 and equal to $1 - (-1+x)^4$ on a neighbourhood of 1 .

These conditions on ψ will be needed for the proof of Proposition 2.30.

For any $j = 2, \dots, J$, let

$$\sigma_j := \frac{1}{2}(v_{j-1} + v_j). \quad (2.2.15)$$

For any $j = 2, \dots, J-1$, let

$$\varphi_j(t, x) := \psi\left(\frac{x - \sigma_j t}{\delta t}\right) - \psi\left(\frac{x - \sigma_{j+1} t}{\delta t}\right), \quad (2.2.16)$$

$$\varphi_1(t, x) := 1 - \psi\left(\frac{x - \sigma_2 t}{\delta t}\right), \quad \varphi_J(t, x) := \psi\left(\frac{x - \sigma_J t}{\delta t}\right), \quad (2.2.17)$$

so that the function φ_j corresponds obviously to the object P_j . We will also use notations φ_l^s and φ_k^b , which represent the same functions, and where φ_l^s corresponds to the soliton R_l and φ_k^b corresponds to the breather B_k .

We will also denote, for $j = 2, \dots, J-1$,

$$\varphi_{1,j}(t, x) := \psi'\left(\frac{x - \sigma_j t}{\delta t}\right) - \psi'\left(\frac{x - \sigma_{j+1} t}{\delta t}\right), \quad (2.2.18)$$

$$\varphi_{1,1}(t, x) := -\psi'\left(\frac{x - \sigma_2 t}{\delta t}\right), \quad \varphi_{1,J}(t, x) := \psi'\left(\frac{x - \sigma_J t}{\delta t}\right). \quad (2.2.19)$$

Of course, notations $\varphi_{1,k}^b$, $\varphi_{1,l}^s$ or $\varphi_{2,j}$ will be used, with similar obvious definitions.

We have that, for $j = 1, \dots, J$,

$$|\varphi_{1,j}| \leq C\varphi_j^{3/4}. \quad (2.2.20)$$

Remark 2.17. If $\delta \leq \frac{\tau}{4}$,

$$\begin{aligned} \int_{-\infty}^{\sigma_j t + \delta t} e^{-2\beta|x - v_j t|} dx &= e^{-2\beta v_j t} \int_{-\infty}^{\sigma_j t + \delta t} e^{2\beta x} dx \\ &= \frac{1}{2\beta} e^{-2\beta v_j t} e^{\beta(v_j + v_{j-1})t} e^{2\beta \delta t} \\ &\leq C e^{-\beta \tau t} e^{2\beta \delta t} \leq C e^{-\beta \tau t/2}, \end{aligned} \quad (2.2.21)$$

and

$$\int_{\sigma_{j+1} t - \delta t}^{+\infty} e^{-2\beta|x - v_j t|} dx \leq C e^{-\beta \tau t/2}, \quad (2.2.22)$$

for the same reason, and if $i \neq j$, e.g. $j > i$,

$$\begin{aligned} \int_{\sigma_j t - \delta t}^{\sigma_{j+1} t + \delta t} e^{-2\beta|x - v_i t|} dx &= e^{2\beta v_i t} \int_{\sigma_j t - \delta t}^{\sigma_{j+1} t + \delta t} e^{-2\beta x} dx \\ &\leq \frac{1}{2\beta} e^{2\beta v_i t} e^{-\beta(v_j + v_{j-1})t} e^{2\beta \delta t} \\ &\leq C e^{-\beta \tau t} e^{2\beta \delta t} \leq C e^{-\beta \tau t/2}. \end{aligned} \quad (2.2.23)$$

And finally, we set for all $j = 1, \dots, J$:

$$\begin{aligned} M_j(t) &:= \int \frac{1}{2} p^2(t, x) \varphi_j(t, x) dx =: M_j[p](t), \\ E_j(t) &:= \int \left(\frac{1}{2} p_x^2(t, x) - \frac{1}{4} p^4(t, x) \right) \varphi_j(t, x) dx =: E_j[p](t). \end{aligned} \quad (2.2.24)$$

Notations $M_l^s, M_k^b, E_l^s, E_k^b$ will also be used.

These are local versions of the mass and the energy of the solution p considered (localized around each breather or soliton). We will prove the following result for the localized mass and energy :

Lemma 2.18. *There exists $C > 0$ and $T_1^* := T_1^*(A)$ such that, if $T^* \geq T_1^*$, then for any $j = 1, \dots, J$, for any $t \in [t^*, T]$,*

$$|M_j(T) - M_j(t)| + |E_j(T) - E_j(t)| \leq \frac{C}{\delta^2 t} A^2 e^{-2\theta t}. \quad (2.2.25)$$

Proof. We will use the results of the computations made on the bottom of page 1115 and on the bottom of page 1116 of [71] to claim the following facts :

$$\begin{aligned} \frac{d}{dt} \frac{1}{2} \int p^2 f &= \int \left(-\frac{3}{2} p_x^2 + \frac{3}{4} p^4 \right) f' - \int p_x p f'', \\ \frac{d}{dt} \int \left[\frac{1}{2} p_x^2 - \frac{1}{4} p^4 \right] f &= \int \left[-\frac{1}{2} (p_{xx} + p^3)^2 - p_{xx}^2 + 3p_x^2 p^2 \right] f' - \int p_{xx} p_x f'', \end{aligned} \quad (2.2.26)$$

where f is a C^2 function that does not depend on time.

$M_j(t)$ is a sum of quantities of the form $\frac{1}{2} \int p^2 \psi\left(\frac{x - \sigma_j t}{\delta t}\right)$. This is why, we compute :

$$\begin{aligned} \frac{d}{dt} \frac{1}{2} \int p^2 \psi\left(\frac{x - \sigma_j t}{\delta t}\right) &= \frac{1}{\delta t} \int \left(-\frac{3}{2} p_x^2 + \frac{3}{4} p^4 \right) \psi'\left(\frac{x - \sigma_j t}{\delta t}\right) \\ &\quad - \frac{1}{(\delta t)^2} \int p_x p \psi''\left(\frac{x - \sigma_j t}{\delta t}\right) - \frac{1}{2} \int p^2 \frac{x}{\delta t^2} \psi'\left(\frac{x - \sigma_j t}{\delta t}\right). \end{aligned} \quad (2.2.27)$$

$\psi'\left(\frac{x - \sigma_j t}{\delta t}\right)$ is zero outside of $\Omega_j(t) := (-\delta t + \sigma_j t, \delta t + \sigma_j t)$. Thus, for $x \in \Omega_j(t)$, $\left|\frac{x}{t}\right| \leq |\sigma_j| + |\delta| \leq |\sigma_j| + 1$, this means that $\left|\frac{x}{t}\right|$ is bounded by a constant (that depends only on the given parameters). We can deduce that

$$\left| \frac{d}{dt} \frac{1}{2} \int p^2 \psi\left(\frac{x - \sigma_j t}{\delta t}\right) \right| \leq \frac{C}{\delta^2 t} \left(\int_{\Omega_j(t)} p_x^2 + \int_{\Omega_j(t)} p^4 + \int_{\Omega_j(t)} p^2 \right). \quad (2.2.28)$$

We bound $\int_{\Omega_j(t)} p^4$ by the following way :

$$\begin{aligned} \int_{\Omega_j(t)} p^4 &\leq \|p\|_{L^\infty}^2 \int_{\Omega_j(t)} p^2 \\ &\leq C \|p\|_{H^1}^2 \int_{\Omega_j(t)} p^2 \quad \text{by Sobolev embedding} \\ &\leq C \int_{\Omega_j(t)} p^2 \quad \text{by Remark 2.16.} \end{aligned} \quad (2.2.29)$$

Thus, we have for any $t \in [t^*, T]$,

$$\left| \frac{d}{dt} \frac{1}{2} \int p^2 \psi \left(\frac{x - \sigma_j t}{\delta t} \right) \right| \leq \frac{C}{\delta^2 t} \left(\int_{\Omega_j(t)} p_x^2 + \int_{\Omega_j(t)} p^2 \right). \quad (2.2.30)$$

$E_j(t)$ is a sum of quantities of the form $\int [\frac{1}{2}p_x^2 - \frac{1}{4}p^4] \psi(\frac{x - \sigma_j t}{\delta t})$. So, we compute :

$$\begin{aligned} & \frac{d}{dt} \int \left[\frac{1}{2}p_x^2 - \frac{1}{4}p^4 \right] \psi \left(\frac{x - \sigma_j t}{\delta t} \right) \\ &= \frac{1}{\delta t} \int \left[-\frac{1}{2}(p_{xx} + p^3)^2 - p_{xx}^2 + 3p_x^2 p \right] \psi' \left(\frac{x - \sigma_j t}{\delta t} \right) \\ & \quad - \frac{1}{(\delta t)^2} \int p_{xx} p_x \psi'' \left(\frac{x - \sigma_j t}{\delta t} \right) - \int \left[\frac{1}{2}p_x^2 - \frac{1}{4}p^4 \right] \frac{x}{\delta t^2} \psi' \left(\frac{x - \sigma_j t}{\delta t} \right). \end{aligned} \quad (2.2.31)$$

We deduce from this, by using similar arguments as for the mass, that for any $t \in [t^*, T]$,

$$\left| \frac{d}{dt} \int \left[\frac{1}{2}p_x^2 - \frac{1}{4}p^4 \right] \psi \left(\frac{x - \sigma_j t}{\delta t} \right) \right| \leq \frac{C}{\delta^2 t} \left(\int_{\Omega_j(t)} p^2 + \int_{\Omega_j(t)} p_x^2 + \int_{\Omega_j(t)} p_{xx}^2 \right). \quad (2.2.32)$$

Now, we write $p(t) = P(t) + (p(t) - P(t))$, and we use the triangular inequality :

$$\int_{\Omega_j(t)} (p^2 + p_x^2 + p_{xx}^2) \leq 2 \int_{\Omega_j(t)} (P^2 + P_x^2 + P_{xx}^2) + 2\|p - P\|_{H^2}^2. \quad (2.2.33)$$

We have assumed that $\|p - P\|_{H^2}^2 \leq A^2 e^{-2\theta t}$, so we need to study P on $\Omega_j(t)$. The following computations work also for the derivatives of P :

$$\begin{aligned} \int_{\Omega_j(t)} P^2 &= \int_{\Omega_j(t)} \left(\sum_{m=1}^J P_m(t, x) \right)^2 dx = \sum_{1 \leq m, l \leq J} \int_{\Omega_j(t)} P_m(t, x) P_l(t, x) dx \\ &\leq C \sum_{1 \leq m, l \leq J} \int_{\Omega_j(t)} e^{-\beta|x-v_m t|} e^{-\beta|x-v_l t|} dx, \end{aligned} \quad (2.2.34)$$

where we use the Proposition 2.12.

We assume that $m \geq j$ (we argue similarly if $m \leq j-1$). Then,

$$\begin{aligned} x \in \Omega_j(t) &\Leftrightarrow -\delta t + \sigma_j t \leq x \leq \delta t + \sigma_j t \\ &\Leftrightarrow -\delta t + (\sigma_j - v_m)t \leq x - v_m t \leq \delta t + (\sigma_j - v_m)t. \end{aligned} \quad (2.2.35)$$

We note that $\sigma_j - v_m \leq -\frac{1}{2}\tau < 0$, we can thus deduce from the condition on δ that $\sigma_j - v_m + \delta \leq -\frac{1}{4}\tau < 0$. We deduce that $x - v_m t$ is negative for $x \in \Omega_j(t)$. Similarly, if $m \leq j-1$, $x - v_m t$ is positive for $x \in \Omega_j(t)$. We will now make calculations for different cases. If $m, l \leq j-1$,

$$\begin{aligned} \int_{\Omega_j(t)} e^{-\beta|x-v_m t|} e^{-\beta|x-v_l t|} dx &\leq \int_{\Omega_j(t)} e^{-\beta(x-v_m t)} e^{-\beta(x-v_l t)} dx \\ &= \frac{1}{2\beta} e^{\beta t(-v_j - v_{j-1} + v_m + v_l)} (e^{2\beta\delta t} - e^{-2\beta\delta t}) \\ &\leq C e^{\beta t(-v_j - v_{j-1} + v_m + v_l + 2\delta)} \leq C e^{-\beta\tau t/2}. \end{aligned} \quad (2.2.36)$$

Similarly, if $m, l \geq j$,

$$\int_{\Omega_j(t)} e^{-\beta|x-v_mt|} e^{-\beta|x-v_lt|} dx \leq C e^{-\beta\tau t/2}. \quad (2.2.37)$$

And, if $m \leq j-1, l \geq j$,

$$\begin{aligned} \int_{\Omega_j(t)} e^{-\beta|x-v_mt|} e^{-\beta|x-v_lt|} dx &\leq \int_{\Omega_j(t)} e^{-\beta(x-v_mt)} e^{\beta(x-v_lt)} dx \\ &\leq 2\delta t e^{\beta t(v_m-v_l)} \leq C e^{-\frac{\beta\tau t}{2}}. \end{aligned} \quad (2.2.38)$$

Thus,

$$\int_{\Omega_j(t)} P^2 \leq C e^{-\frac{\beta\tau t}{2}}, \quad (2.2.39)$$

and the same is valid for the derivatives of P .

Thus, for $t \in [t^*, T]$,

$$\begin{aligned} \sum_{j=1}^J \left| \frac{d}{dt} \frac{1}{2} \int p^2 \psi \left(\frac{x - \sigma_j t}{\delta t} \right) \right| + \left| \frac{d}{dt} \int \left[\frac{1}{2} p_x^2 - \frac{1}{4} p^4 \right] \psi \left(\frac{x - \sigma_j t}{\delta t} \right) \right| \\ \leq \frac{C}{\delta^2 t} A^2 e^{-2\theta t} + \frac{C}{\delta^2 t} e^{-\frac{\beta\tau t}{2}} \leq \frac{C}{\delta^2 t} (A^2 + e^{-2\theta t}) e^{-2\theta t} \leq \frac{C}{\delta^2 t} A^2 e^{-2\theta t}. \end{aligned} \quad (2.2.40)$$

Thus, for $j = 1, \dots, J, t \in [t^*, T]$,

$$\begin{aligned} |M_j(T) - M_j(t)| + |E_j(T) - E_j(t)| &\leq \int_t^T \frac{C}{\delta^2 s} A^2 e^{-2\theta s} ds \leq \frac{C}{\delta^2 t} A^2 \int_t^T e^{-2\theta s} ds \\ &= \frac{C}{\delta^2 t} A^2 \frac{1}{2\theta} (e^{-2\theta t} - e^{-2\theta T}) \leq \frac{C}{\delta^2 t} A^2 e^{-2\theta t}. \end{aligned} \quad (2.2.41)$$

□

2.2.3 Modulation

Lemma 2.19. *There exists $C > 0$, $T_2^* = T_2^*(A)$ such that, if $T^* > T_2^*$, then there exist unique C^1 functions $x_{1,k} : [t^*, T] \rightarrow \mathbb{R}$, $x_{2,k} : [t^*, T] \rightarrow \mathbb{R}$ for $1 \leq k \leq K$ and $x_{0,l} : [t^*, T] \rightarrow \mathbb{R}$, $c_{0,l} : [t^*, T] \rightarrow \mathbb{R}$, such that if we set*

$$\varepsilon(t, x) = p(t, x) - \widetilde{B}(t, x) - \widetilde{R}(t, x) = p(t, x) - \widetilde{P}(t, x), \quad (2.2.42)$$

where, for $1 \leq k \leq K$,

$$\widetilde{B}(t, x) = \sum_{k=1}^K \widetilde{B}_k(t), \quad \widetilde{B}_k(t, x) = B_{\alpha_k, \beta_k}(t, x; x_{1,k}^0 + x_{1,k}(t), x_{2,k}^0 + x_{2,k}(t)), \quad (2.2.43)$$

for $1 \leq l \leq L$,

$$\widetilde{R}(t, x) := \sum_{l=1}^L \widetilde{R}_l(t), \quad \widetilde{R}_l(t, x) := \kappa_l Q_{c_l + c_{0,l}(t)}(x - x_{0,l}^0 + x_{0,l}(t) - c_l t), \quad (2.2.44)$$

$$\widetilde{P}(t) := \widetilde{R}(t) + \widetilde{B}(t), \quad (2.2.45)$$

and

$$\widetilde{P}(t) := \sum_{j=1}^J \widetilde{P}_j(t), \quad (2.2.46)$$

where there is the usual correspondence between \widetilde{P}_j and \widetilde{B}_k or \widetilde{R}_l ,
then, $\varepsilon(t)$ satisfies, for any $k = 1, \dots, K$, for any $l = 1, \dots, L$ and for any $t \in [t^*, T]$,

$$\begin{aligned} \int \widetilde{R}_l(t) \varepsilon(t) \sqrt{\varphi_l^s(t)} &= \int \partial_x \widetilde{R}_l(t) \varepsilon(t) \sqrt{\varphi_l^s(t)} = 0, \\ \int \partial_{x_1} \widetilde{B}_k(t) \varepsilon(t) \sqrt{\varphi_k^b(t)} &= \int \partial_{x_2} \widetilde{B}_k(t) \varepsilon(t) \sqrt{\varphi_k^b(t)} = 0. \end{aligned} \quad (2.2.47)$$

Moreover, for any $t \in [t^*, T]$,

$$\|\varepsilon(t)\|_{H^2} + \sum_{k=1}^K (|x_{1,k}(t)| + |x_{2,k}(t)|) + \sum_{l=1}^L (|x_{0,l}(t)| + |c_{0,l}(t)|) \leq CAe^{-\theta t}, \quad (2.2.48)$$

and

$$\sum_{k=1}^K (|x'_{1,k}(t)| + |x'_{2,k}(t)|) + \sum_{l=1}^L (|x'_{0,l}(t)| + |c'_{0,l}(t)|) \leq C\|\varepsilon(t)\|_{L^2} + Ce^{-\theta t}. \quad (2.2.49)$$

Finally, $p(T) = P(T) = \widetilde{P}(T)$ and $\varepsilon(T) = x_{0,l}(T) = x_{1,k}(T) = x_{2,k}(T) = c_{0,l}(T) = 0$.

Proof : see for example [24] for reference. Let, for $t \in [t^*, T]$,

$$F_t : L^2(\mathbb{R}) \times \mathbb{R}^{2K} \times \mathbb{R}^{2L} \rightarrow \mathbb{R}^{2K+2L}, \quad (2.2.50)$$

such that

$$\begin{aligned} (w, x_{1,k}, x_{2,k}, x_{0,l}, c_{0,l}) &\longmapsto \left(\int \sqrt{\varphi_k^b(t, x)} \partial_{x_1} B_{\alpha_k, \beta_k}(t, x; x_{1,k}^0 + x_{1,k}, x_{2,k}^0 + x_{2,k}) \epsilon, \right. \\ &\int \sqrt{\varphi_k^b(t, x)} \partial_{x_2} B_{\alpha_k, \beta_k}(t, x; x_{1,k}^0 + x_{1,k}, x_{2,k}^0 + x_{2,k}) \epsilon, \\ &\int \sqrt{\varphi_l^s(t, x)} \partial_x \kappa_l Q_{c_l + c_{0,l}}(x - x_{0,l}^0 + x_{0,l} - c_l t) \epsilon, \\ &\left. \int \sqrt{\varphi_l^s(t, x)} \kappa_l Q_{c_l + c_{0,l}}(x - x_{0,l}^0 + x_{0,l} - c_l t) \epsilon \right), \end{aligned} \quad (2.2.51)$$

where

$$\epsilon := w - \sum_{m=1}^K B_{\alpha_m, \beta_m}(t, x; x_{1,m}^0 + x_{1,m}, x_{2,m}^0 + x_{2,m}) - \sum_{n=1}^L \kappa_n Q_{c_n + c_{0,n}}(x - x_{0,n}^0 + x_{0,n} - c_n t). \quad (2.2.52)$$

We observe that F_t is a C^1 function and that $F_t(P(t), 0, 0, 0, 0) = 0$. Now, let us consider the matrix which gives the differential of F_t (with respect to $x_{1,k}, x_{2,k}, x_{0,l}, c_{0,l}$) in $(P(t), 0, 0, 0, 0)$ (we consider diagonal and extra-diagonal

terms for each bloc) :

$$DF_t = \begin{pmatrix} B_{k,k}^1 & B_{k,k}^3 & \times & \times & \times & \times & \times & \times \\ B_{k,k}^3 & B_{k,k}^2 & \times & \times & \times & \times & \times & \times \\ \times & \times & B_{k',k'}^1 & B_{k',k'}^3 & \times & \times & \times & \times \\ \times & \times & B_{k',k'}^3 & B_{k',k'}^2 & \times & \times & \times & \times \\ \times & \times & \times & \times & R_{l,l}^1 & R_{l,l}^4 & \times & \times \\ \times & \times & \times & \times & R_{l,l}^3 & R_{l,l}^2 & \times & \times \\ \times & \times & \times & \times & \times & \times & R_{l',l'}^1 & R_{l',l'}^4 \\ \times & \times & \times & \times & \times & \times & R_{l',l'}^3 & R_{l',l'}^2 \end{pmatrix}, \quad (2.2.53)$$

where

$$\begin{aligned} B_{k,k}^1 &:= - \int (\partial_{x_1} B_{\alpha_k, \beta_k})^2 \sqrt{\varphi_k^b}, & B_{k,k}^2 &:= - \int (\partial_{x_2} B_{\alpha_k, \beta_k})^2 \sqrt{\varphi_k^b}, \\ B_{k,k}^3 &:= - \int \partial_{x_1} B_{\alpha_k, \beta_k} \partial_{x_2} B_{\alpha_k, \beta_k} \sqrt{\varphi_k^b}, \end{aligned} \quad (2.2.54)$$

$$\begin{aligned} R_{l,l}^1 &:= - \int (\partial_x Q_{c_l}(y_{0,l}^0))^2 \sqrt{\varphi_l^s}, & R_{l,l}^2 &:= - \frac{1}{2c_l} \int Q_{c_l}(y_{0,l}^0) \left(Q_{c_l}(y_{0,l}^0) + y_{0,l}^0 \partial_x Q_{c_l}(y_{0,l}^0) \right) \sqrt{\varphi_l^s}, \\ R_{l,l}^3 &:= - \int Q_{c_l}(y_{0,l}^0) \partial_x Q_{c_l}(y_{0,l}^0) \sqrt{\varphi_l^s}, & R_{l,l}^4 &:= - \frac{1}{2c_l} \int \partial_x Q_{c_l}(y_{0,l}^0) \left(Q_{c_l}(y_{0,l}^0) + y_{0,l}^0 \partial_x Q_{c_l}(y_{0,l}^0) \right) \sqrt{\varphi_l^s}, \end{aligned} \quad (2.2.55)$$

denoting $y_{0,l}^0 := x - x_{0,l}^0 - c_l t$, and crosses stand for exponentially decaying terms when $t \rightarrow +\infty$, and where we consider variables in the following order : $x_{1,1}, x_{2,1}, x_{1,2}, x_{2,2}, x_{1,3}, x_{2,3}, \dots, x_{1,K}, x_{2,K}, x_{0,1}, c_{0,1}, \dots, x_{0,L}, c_{0,L}$ and we order the coefficients of the function in the similar way. This is a matrix with dominant diagonal blocs.

Note that $B_{k,k}^1$ is exponentially close to $-\int (\partial_{x_1} B_{\alpha_k, \beta_k})^2$, because if $P_j = B_k$ is a breather,

$$\begin{aligned} \int (\partial_{x_1} B_{\alpha_k, \beta_k})^2 (1 - \sqrt{\varphi_k^b}) &\leq \int_{-\infty}^{\sigma_j t + \delta t} (\partial_{x_1} B_{\alpha_k, \beta_k})^2 + \int_{\sigma_{j+1} t + \delta t}^{+\infty} (\partial_{x_1} B_{\alpha_k, \beta_k})^2 \\ &\leq C \int_{-\infty}^{\sigma_j t + \delta t} e^{-2\beta|x-v_j t|} + \int_{\sigma_{j+1} t + \delta t}^{+\infty} e^{-2\beta|x-v_j t|} \leq C e^{-\frac{\beta}{2}t}, \end{aligned} \quad (2.2.56)$$

and the same is true for the other dominant diagonal terms of the matrix (we can get rid of φ_s).

Therefore, the determinant of the matrix is exponentially close to :

$$\begin{aligned} \det(DF_t) &= \prod_{k=1}^K \left(\int (\partial_{x_1} B_{\alpha_k, \beta_k}(t, x; x_{1,k}^0, x_{2,k}^0))^2 \int (\partial_{x_2} B_{\alpha_k, \beta_k}(t, x; x_{1,k}^0, x_{2,k}^0))^2 \right. \\ &\quad \left. - \left(\int \partial_{x_1} B_{\alpha_k, \beta_k} \partial_{x_2} B_{\alpha_k, \beta_k} \right)^2 \right) \\ &\quad \cdot \prod_{l=1}^L \left(\frac{1}{2c_l} \int Q_{c_l}(y_{0,l}^0) \left(Q_{c_l}(y_{0,l}^0) + y_{0,l}^0 \partial_x Q_{c_l}(y_{0,l}^0) \right) \int (\partial_x Q_{c_l}(y_{0,l}^0))^2 \right), \end{aligned} \quad (2.2.57)$$

because $\int Q_{c_l}(y_{0,l}^0) \partial_x Q_{c_l}(y_{0,l}^0) dx = 0$.

By Cauchy-Schwarz inequality and the fact that

$$\partial_{x_1} B_{\alpha_k, \beta_k}(t, x; x_{1,k}^0, x_{2,k}^0) \quad \text{and} \quad \partial_{x_2} B_{\alpha_k, \beta_k}(t, x; x_{1,k}^0, x_{2,k}^0)$$

are linearly independent as functions of the x variable, for any time t fixed, we see that the first product is positive. Since each member of the product is periodic in time, then the first product is bounded below by a positive constant independent from time and translation parameters.

For the second product, by translation of the variable in the integrations, for any time t fixed, we see that we can replace $y_{0,l}^0$ by x . Then, by integration by parts,

$$\int x Q_{c_l}(x) \partial_x Q_{c_l}(x) dx = -\frac{1}{2} \int Q_{c_l}(x)^2 dx. \quad (2.2.58)$$

By scaling, if q denotes the soliton with $c = 1$, i.e. $q = Q_1$,

$$\int Q_{c_l}^2 = \sqrt{c_l} \int q^2, \quad \int \partial_x Q_{c_l}^2 = c_l^{\frac{3}{2}} \int q_x^2. \quad (2.2.59)$$

Therefore,

$$\begin{aligned} & \frac{1}{2c_l} \int Q_{c_l}(y_{0,l}^0) \left(Q_{c_l}(y_{0,l}^0) + y_{0,l}^0 \partial_x Q_{c_l}(y_{0,l}^0) \right) \int (\partial_x Q_{c_l}(y_{0,l}^0))^2 \\ &= \frac{1}{4} c_l \int q^2 \int (q_x)^2 \\ &\geq \frac{1}{4} \min\{c_n, 1 \leq n \leq L\} \int q^2 \int q_x^2. \end{aligned} \quad (2.2.60)$$

This means that the second product is bounded below by a positive constant independent from time and translation parameters.

This means that if T_2^* is large enough, the considered matrix is invertible.

Now, we may use the implicit function theorem (actually, we use a quantitative version of the implicit function theorem, see [22, Section 2.2] for a precise statement). If w is close enough to $P(t)$, then there exists

$$(x_{1,k}, x_{2,k}, x_{0,l}, c_{0,l}) \quad (2.2.61)$$

such that

$$F_t(w, x_{1,k}, x_{2,k}, x_{0,l}, c_{0,l}) = 0, \quad (2.2.62)$$

where (2.2.61) depends in a regular C^1 way on w . It is possible to show that the “close enough” in the previous sentence does not depend on t ; for this, it is required to use a uniform implicit function theorem. This means that for T_2^* large enough (depending on A), $Ae^{-\theta t}$ is small enough for $t \in [t^*, T]$, thus for $t \in [t^*, T]$, $p(t)$ is close enough to $P(t)$ in order to apply the implicit function theorem. Therefore, we have for $t \in [t^*, T]$, the existence of $x_{1,k}(t)$, $x_{2,k}(t)$, $x_{0,l}(t)$ and $c_{0,l}(t)$. It is possible to show that these functions are C^1 in time. Basically, this comes from the fact that they are C^1 in $p(t)$ and that $p(t)$ has a similar regularity in time (see [24] for more details).

Now, we prove the inequalities (2.2.48) and (2.2.49). We can take the differential of the implicit functions with respect to $p(t)$ for $t \in [t^*, T]$. For this, we differentiate the following equation with respect to $p(t)$:

$$F_t(p(t), x_{1,k}(p(t)), x_{2,k}(p(t)), x_{0,l}(p(t)), c_{0,l}(p(t))) = 0. \quad (2.2.63)$$

We know that the matrix that gives the differential of F_t (with respect to $x_{1,k}$, $x_{2,k}$, $x_{0,l}$, $c_{0,l}$) in

$$(p(t), x_{1,k}(p(t)), x_{2,k}(p(t)), x_{0,l}(p(t)), c_{0,l}(p(t))) \quad (2.2.64)$$

is invertible and its inverse is bounded in time (from the formula giving the inverse of a matrix from the comatrix and the determinant). The differential of F_t with respect to the first variable is also bounded. Thus, by the mean-value theorem :

$$|x_{1,k}| \leq C \|p - P\| \leq CAe^{-\theta t}. \quad (2.2.65)$$

The same is true for $x_{2,k}$, $x_{0,l}$ and $c_{0,l}$.

By applying the mean-value theorem (inequality) for Q_{c_l} with respect to $x_{0,l}$ and $c_{0,l}$ or for B_{α_k, β_k} with respect to $x_{1,k}$ and $x_{2,k}$, we deduce that

$$\|P_j(t) - \widetilde{P}_j(t)\|_{H^2} \leq C(|x_{1,k}(t)| + |x_{2,k}(t)|), \quad (2.2.66)$$

if $P_j = B_k$ is a breather, and

$$\|P_j(t) - \widetilde{P}_j(t)\|_{H^2} \leq C(|x_{0,l}(t)| + |c_{0,l}(t)|), \quad (2.2.67)$$

if $P_j = R_l$ is a soliton.

Finally, by triangular inequality,

$$\begin{aligned} \|\varepsilon(t)\|_{H^2} &\leq \|p(t) - P(t)\|_{H^2} + \|P(t) - \widetilde{P}(t)\|_{H^2} \leq \|p(t) - P(t)\|_{H^2} \\ &\quad + C \left(\sum_{k=1}^K (|x_{1,k}(t)| + |x_{2,k}(t)|) + \sum_{l=1}^L (|x_{0,l}(t)| + |c_{0,l}(t)|) \right) \\ &\leq C\|p(t) - P(t)\|_{H^2} \leq CAe^{-\theta t}. \end{aligned} \quad (2.2.68)$$

This completes the proof of (2.2.48).

For (2.2.49), we will take time derivatives of the equations (2.2.47). From now on, we write \widetilde{B}_{k1} for $\partial_{x_1} \widetilde{B}_k$ and \widetilde{B}_{k2} for $\partial_{x_2} \widetilde{B}_k$. Firstly, we write the PDE verified by ε (knowing that $p, B_1, \dots, B_K, R_1, \dots, R_L$ are solutions of (mKdV)) :

$$\begin{aligned} \partial_t \varepsilon &= -\varepsilon_{xxx} - \left[\varepsilon \left(\varepsilon^2 + 3\varepsilon \sum_{j=1}^J \widetilde{P}_j + 3 \sum_{i,j=1}^J \widetilde{P}_i \widetilde{P}_j \right) \right]_x \\ &\quad - \sum_{k=1}^K x'_{1,k}(t) \widetilde{B}_{k1} - \sum_{k=1}^K x'_{2,k}(t) \widetilde{B}_{k2} - \sum_{l=1}^L x'_{0,l}(t) \widetilde{R}_{lx} \\ &\quad - \sum_{l=1}^L \frac{c'_{0,l}(t)}{2(c_l + c_{0,l}(t))} (\widetilde{R}_l + y_{0,l}(t) \widetilde{R}_{lx}) - \sum_{h \neq i \text{ or } i \neq j} (\widetilde{P}_h \widetilde{P}_i \widetilde{P}_j)_x, \end{aligned} \quad (2.2.69)$$

where $y_{0,l}(t) := x - x_{0,l}^0 + x_{0,l}(t) - c_l t$. Now, we will take the time derivative of the equation $\int \widetilde{B}_{k1} \varepsilon \sqrt{\varphi_k^b} = 0$ (and perform an integration by parts) :

$$\begin{aligned} & - \int (\widetilde{B}_k^3)_{1x} \varepsilon \sqrt{\varphi_k^b} - \int \widetilde{B}_{k1} \sum_{h \neq i \text{ or } g \neq h} (\widetilde{P}_h \widetilde{P}_i \widetilde{P}_g)_x \sqrt{\varphi_k^b} \\ & + x'_{2,k}(t) \int \widetilde{B}_{k12} \varepsilon \sqrt{\varphi_k^b} + \frac{1}{2\delta t} \int \widetilde{B}_{k1} \varepsilon \left(\varepsilon^2 + 3\varepsilon \sum_{i=1}^J \widetilde{P}_i + 3 \sum_{h,i=1}^J \widetilde{P}_h \widetilde{P}_i \right) \frac{\varphi_{1,k}^b}{\sqrt{\varphi_k^b}} \\ & + \int \widetilde{B}_{k1x} \varepsilon \left(\varepsilon^2 + 3\varepsilon \sum_{i=1}^J \widetilde{P}_i + 3 \sum_{h,i=1}^J \widetilde{P}_h \widetilde{P}_i \right) \sqrt{\varphi_k^b} - \frac{1}{2\delta t^2} \int \widetilde{B}_{k1} \varepsilon x \frac{\varphi_{1,k}^b}{\sqrt{\varphi_k^b}} \\ & + \frac{1}{2\delta t} \int \widetilde{B}_{k1} \varepsilon_{xx} \frac{\varphi_{1,k}^b}{\sqrt{\varphi_k^b}} - \frac{1}{2\delta t} \int \widetilde{B}_{k1x} \varepsilon x \frac{\varphi_{1,k}^b}{\sqrt{\varphi_k^b}} + \frac{1}{2\delta t} \int \widetilde{B}_{k1xx} \varepsilon \frac{\varphi_{1,k}^b}{\sqrt{\varphi_k^b}} \end{aligned} \quad (2.2.70)$$

$$\begin{aligned}
& + x'_{1,k}(t) \int \widetilde{B}_{k11} \varepsilon \sqrt{\varphi_k^b} = \sum_{m=1}^K x'_{1,m}(t) \int \widetilde{B}_{k1} \widetilde{B}_{m1} \sqrt{\varphi_k^b} \\
& + \sum_{m=1}^K x'_{2,m}(t) \int \widetilde{B}_{k1} \widetilde{B}_{m2} \sqrt{\varphi_k^b} + \sum_{n=1}^L x'_{0,n}(t) \int \widetilde{B}_{k1} \widetilde{R}_{nx} \sqrt{\varphi_k^b} \\
& + \sum_{n=1}^L \frac{c'_{0,n}(t)}{2(c_n + c_{0,n}(t))} \int \widetilde{B}_{k1} (\widetilde{R}_n + y_{0,n}(t) \widetilde{R}_{nx}) \sqrt{\varphi_k^b}.
\end{aligned}$$

Similarly, taking the time derivative of $\int \widetilde{B}_{k2} \varepsilon \sqrt{\varphi_k^b} = 0$:

$$\begin{aligned}
& - \int (\widetilde{B}_k^3)_{2x} \varepsilon \sqrt{\varphi_k^b} - \int \widetilde{B}_{k2} \sum_{h \neq i \text{ or } g \neq h} (\widetilde{P}_h \widetilde{P}_i \widetilde{P}_g)_x \sqrt{\varphi_k^b} \\
& + x'_{2,k}(t) \int \widetilde{B}_{k22} \varepsilon \sqrt{\varphi_k^b} + \frac{1}{2\delta t} \int \widetilde{B}_{k2} \varepsilon \left(\varepsilon^2 + 3\varepsilon \sum_{i=1}^J \widetilde{P}_i + 3 \sum_{h,i=1}^J \widetilde{P}_h \widetilde{P}_i \right) \frac{\varphi_{1,k}^b}{\sqrt{\varphi_k^b}} \\
& + \int \widetilde{B}_{k2x} \varepsilon \left(\varepsilon^2 + 3\varepsilon \sum_{i=1}^J \widetilde{P}_i + 3 \sum_{h,i=1}^J \widetilde{P}_h \widetilde{P}_i \right) \sqrt{\varphi_k^b} + \frac{1}{2\delta t} \int \widetilde{B}_{k2} \varepsilon_{xx} \frac{\varphi_{1,k}^b}{\sqrt{\varphi_k^b}} \\
& - \frac{1}{2\delta t} \int \widetilde{B}_{k2x} \varepsilon_x \frac{\varphi_{1,k}^b}{\sqrt{\varphi_k^b}} + \frac{1}{2\delta t} \int \widetilde{B}_{k2xx} \varepsilon \frac{\varphi_{1,k}^b}{\sqrt{\varphi_k^b}} - \frac{1}{2\delta t^2} \int \widetilde{B}_{k2} \varepsilon_{xx} \frac{\varphi_{1,k}^b}{\sqrt{\varphi_k^b}} \\
& + x'_{1,k}(t) \int \widetilde{B}_{k12} \varepsilon \sqrt{\varphi_k^b} = \sum_{m=1}^K x'_{1,m}(t) \int \widetilde{B}_{k2} \widetilde{B}_{m1} \sqrt{\varphi_k^b} \\
& + \sum_{m=1}^K x'_{2,m}(t) \int \widetilde{B}_{k2} \widetilde{B}_{m2} \sqrt{\varphi_k^b} + \sum_{n=1}^L x'_{0,n}(t) \int \widetilde{B}_{k2} \widetilde{R}_{nx} \sqrt{\varphi_k^b} \\
& + \sum_{n=1}^L \frac{c'_{0,n}(t)}{2(c_n + c_{0,n}(t))} \int \widetilde{B}_{k2} (\widetilde{R}_n + y_{0,n}(t) \widetilde{R}_{nx}) \sqrt{\varphi_k^b}.
\end{aligned} \tag{2.2.71}$$

Similarly, taking the time derivative of $\int \widetilde{R}_{lx}(t) \varepsilon(t) \sqrt{\varphi_l^s} = 0$:

$$\begin{aligned}
& - \int (\widetilde{R}_l^3)_{xx} \varepsilon \sqrt{\varphi_l^s} + \frac{c'_{0,l}(t)}{2(c_l + c_{0,l}(t))} \int (\widetilde{R}_{lx} + y_{0,l}(t) \widetilde{R}_{lxx}) \varepsilon \sqrt{\varphi_l^s} \\
& + x'_{0,l}(t) \int \widetilde{R}_{lxx} \varepsilon \sqrt{\varphi_l^s} + \frac{1}{2\delta t} \int \widetilde{R}_{lx} \varepsilon \left(\varepsilon^2 + 3\varepsilon \sum_{i=1}^J \widetilde{P}_i + 3 \sum_{h,i=1}^J \widetilde{P}_h \widetilde{P}_i \right) \frac{\varphi_{1,l}^s}{\sqrt{\varphi_l^s}} \\
& + \int \widetilde{R}_{lxx} \varepsilon \left(\varepsilon^2 + 3\varepsilon \sum_{i=1}^J \widetilde{P}_i + 3 \sum_{h,i=1}^J \widetilde{P}_h \widetilde{P}_i \right) \sqrt{\varphi_l^s} - \frac{1}{2\delta t^2} \int \widetilde{R}_{lx} \varepsilon_{xx} \frac{\varphi_{1,l}^s}{\sqrt{\varphi_l^s}} \\
& + \frac{1}{2\delta t} \int \widetilde{R}_{lx} \varepsilon_{xx} \frac{\varphi_{1,l}^s}{\sqrt{\varphi_l^s}} - \frac{1}{2\delta t} \int \widetilde{R}_{lxx} \varepsilon_x \frac{\varphi_{1,l}^s}{\sqrt{\varphi_l^s}} + \frac{1}{2\delta t} \int \widetilde{R}_{lxxx} \varepsilon \frac{\varphi_{1,l}^s}{\sqrt{\varphi_l^s}}
\end{aligned} \tag{2.2.72}$$

$$\begin{aligned}
& - \int \widetilde{R}_{lx} \sum_{h \neq i \text{ or } g \neq h} \left(\widetilde{P}_h \widetilde{P}_i \widetilde{P}_g \right)_x \sqrt{\varphi_l^s} = \sum_{n=1}^L x'_{0,n}(t) \int \widetilde{R}_{lx} \widetilde{R}_{nx} \sqrt{\varphi_l^s} \\
& + \sum_{n=1}^L \frac{c'_{0,n}(t)}{2(c_n + c_{0,n}(t))} \int \widetilde{R}_{lx} \left(\widetilde{R}_n + y_{0,n}(t) \widetilde{R}_{nx} \right) \sqrt{\varphi_l^s} \\
& + \sum_{m=1}^K x'_{1,m}(t) \int \widetilde{R}_{lx} \widetilde{B}_{m1} \sqrt{\varphi_l^s} + \sum_{m=1}^K x'_{2,m}(t) \int \widetilde{R}_{lx} \widetilde{B}_{m2} \sqrt{\varphi_l^s}.
\end{aligned}$$

Finally, taking the time derivative of $\int \widetilde{R}_l \varepsilon \sqrt{\varphi_l^s} = 0$:

$$\begin{aligned}
& - \int \left(\widetilde{R}_l^3 \right)_x \varepsilon \sqrt{\varphi_l^s} + \frac{c'_{0,l}(t)}{2(c_l + c_{0,l}(t))} \int \left(\widetilde{R}_l + y_{0,l}(t) \widetilde{R}_{lx} \right) \varepsilon \sqrt{\varphi_l^s} \\
& + x'_{0,l}(t) \int \widetilde{R}_{lx} \varepsilon \sqrt{\varphi_l^s} + \frac{1}{2\delta t} \int \widetilde{R}_l \varepsilon \left(\varepsilon^2 + 3\varepsilon \sum_{i=1}^J \widetilde{P}_i + 3 \sum_{h,i=1}^J \widetilde{P}_h \widetilde{P}_i \right) \frac{\varphi_{1,l}^s}{\sqrt{\varphi_l^s}} \\
& + \int \widetilde{R}_{lx} \varepsilon \left(\varepsilon^2 + 3\varepsilon \sum_{i=1}^J \widetilde{P}_i + 3 \sum_{h,i=1}^J \widetilde{P}_h \widetilde{P}_i \right) \sqrt{\varphi_l^s} - \frac{1}{2\delta t^2} \int \widetilde{R}_l \varepsilon x \frac{\varphi_{1,l}^s}{\sqrt{\varphi_l^s}} \\
& + \frac{1}{2\delta t} \int \widetilde{R}_l \varepsilon_{xx} \frac{\varphi_{1,l}^s}{\sqrt{\varphi_l^s}} - \frac{1}{2\delta t} \int \widetilde{R}_{lx} \varepsilon_x \frac{\varphi_{1,l}^s}{\sqrt{\varphi_l^s}} + \frac{1}{2\delta t} \int \widetilde{R}_{lx} \varepsilon \frac{\varphi_{1,l}^s}{\sqrt{\varphi_l^s}} \\
& - \int \widetilde{R}_l \sum_{h \neq i \text{ or } g \neq h} \left(\widetilde{P}_h \widetilde{P}_i \widetilde{P}_g \right)_x \sqrt{\varphi_l^s} = \sum_{n=1}^L x'_{0,n}(t) \int \widetilde{R}_l \widetilde{R}_{nx} \sqrt{\varphi_l^s} \\
& + \sum_{n=1}^L \frac{c'_{0,n}(t)}{2(c_n + c_{0,n}(t))} \int \widetilde{R}_l \left(\widetilde{R}_n + y_{0,n}(t) \widetilde{R}_{nx} \right) \sqrt{\varphi_l^s} \\
& + \sum_{m=1}^K x'_{1,m}(t) \int \widetilde{R}_l \widetilde{B}_{m1} \sqrt{\varphi_l^s} + \sum_{m=1}^K x'_{2,m}(t) \int \widetilde{R}_l \widetilde{B}_{m2} \sqrt{\varphi_l^s}.
\end{aligned} \tag{2.2.73}$$

By the Proposition 2.21 below (that follows from the first part of the lemma we prove) and its corollary, several terms of the equalities (2.2.70), (2.2.71), (2.2.72) and (2.2.73) are bounded by $Ce^{-\theta t}$; other terms are $O(\|\varepsilon\|_{L^2})$. We remind that $O(\|\varepsilon\|_{L^2}) \leq CAe^{-\theta t}$. From the basic properties of φ_j (see Section 2.2.2), $\frac{\varphi_{1,j}}{\sqrt{\varphi_j}}$ is bounded. Because of the compact support of φ_j , $\frac{x}{t} \frac{\varphi_{1,j}}{\sqrt{\varphi_j}}$ is bounded independently on x and t . Using these bounds, and after several linear combinations, we obtain the desired inequalities. \square

Remark 2.20. As a consequence of Lemma 2.19, there exists a constant $C > 0$ such that

$$\forall t \in [t^*, T] \quad \sum_{k=1}^K (|x_{1,k}(t)| + |x_{2,k}(t)|) + \sum_{l=1}^L (|x_{0,l}(t)| + |c_{0,l}(t)|) \leq CAe^{-\theta T^*}. \tag{2.2.74}$$

This means, that if we take T_2^* eventually larger (which we will assume in the following of the article), we may extend Proposition 2.12 to \widetilde{P}_j in the following way, by integration of the bounds given by modulation (the constant C is a bit larger in a controlled way, we write $\frac{\beta}{2}$ because the shape of the solitons is a bit modified in a controlled way) :

Proposition 2.21. *Let $j = 1, \dots, J$, $n \in \mathbb{N}$. If $T^* > T_2^*$, then there exists a constant $C > 0$ such that for any $t, x \in \mathbb{R}$,*

$$|\partial_x^n \widetilde{P}_j(t, x)| \leq C e^{-\frac{\beta}{2}|x-v_j t|}. \quad (2.2.75)$$

We will also use that any $\|\partial_x^n \widetilde{P}_j\|_{H^2}$ is bounded by C .

Corollary 2.22. *Let $i \neq j \in \{1, \dots, J\}$ and $m, n \in \mathbb{N}$. If $T^* > T_2^*$, then there exists a constant C that depends only on P , such that for any $t \in \mathbb{R}$,*

$$\left| \int \partial_x^m \widetilde{P}_i \partial_x^n \widetilde{P}_j \right| \leq C e^{-\beta \tau t / 8}. \quad (2.2.76)$$

2.2.4 Study of coercivity

In [8], the Lyapunov functional that was introduced to study the orbital stability of a breather was the following conserved-in-time functional :

$$F[p](t) + 2(\beta^2 - \alpha^2)E[p](t) + (\alpha^2 + \beta^2)^2 M[p](t). \quad (2.2.77)$$

The functional that we will consider here is adapted from the latter. For $t \in [t^*, T]$, we set

$$\mathcal{H}[p](t) := F[p](t) + \sum_{k=1}^K \left(2(\beta_k^2 - \alpha_k^2) E_k^b[p](t) + (\alpha_k^2 + \beta_k^2)^2 M_k^b[p](t) \right) + \sum_{l=1}^L \left(2c_l E_l^s[p](t) + c_l^2 M_l^s[p](t) \right). \quad (2.2.78)$$

For the simplicity of notations, for $j \in \{1, \dots, J\}$, a_j will denote α_k if P_j is the breather B_k or 0 if P_j is a soliton, and b_j will denote β_k if P_j is the breather B_k or $c_l^{1/2}$ if P_j is the soliton R_l . With these notations, we may write :

$$\mathcal{H}[p](t) = F[p](t) + \sum_{j=1}^J \left(2(b_j^2 - a_j^2) E_j[p](t) + (a_j^2 + b_j^2)^2 M_j[p](t) \right). \quad (2.2.79)$$

We would like to study locally this functional around the considered sum of breathers and solitons. The aim of this section will be to prove two following propositions :

Proposition 2.23 (Expansion of H^2 conserved quantity). *There exists $T_4^* > 0$ such that if $T^* \geq T_4^*$, for all $t \in [t^*, T]$, we have that*

$$\begin{aligned} \mathcal{H}[p](t) &= \sum_{j=1}^J \left(F[\widetilde{P}_j](t) + 2(b_j^2 - a_j^2) E[\widetilde{P}_j](t) + (a_j^2 + b_j^2)^2 M[\widetilde{P}_j](t) \right) \\ &\quad + H_2[\varepsilon](t) + O(\|\varepsilon(t)\|_{H^2}^3) + O(e^{-2\theta t} \|\varepsilon(t)\|_{H^2}) + O(e^{-2\theta t}), \end{aligned} \quad (2.2.80)$$

where

$$\begin{aligned} H_2[\varepsilon](t) &:= \frac{1}{2} \int \varepsilon_{xx}^2 - \frac{5}{2} \int \widetilde{P}^2 \varepsilon_x^2 + \frac{5}{2} \int \widetilde{P}_x^2 \varepsilon^2 + 5 \int \widetilde{P} \widetilde{P}_{xx} \varepsilon^2 + \frac{15}{4} \int \widetilde{P}^4 \varepsilon^2 \\ &\quad + \sum_{j=1}^J (b_j^2 - a_j^2) \left(\int \varepsilon_x^2 \varphi_j - 3 \int \widetilde{P}^2 \varepsilon^2 \varphi_j \right) + \sum_{j=1}^J (a_j^2 + b_j^2)^2 \frac{1}{2} \int \varepsilon^2 \varphi_j. \end{aligned} \quad (2.2.81)$$

Proposition 2.24 (Coercivity of H_2). *There exists $\mu > 0$, $T_3^* = T_3^*(A)$ such that, if $T^* \geq T_3^*$, we have for any $t \in [t^*, T]$,*

$$H_2[\varepsilon](t) \geq \mu \|\varepsilon(t)\|_{H^2}^2 - \frac{1}{\mu} \sum_{k=1}^K \left(\int \varepsilon \widetilde{B}_k \sqrt{\varphi_k^b} \right)^2. \quad (2.2.82)$$

The Propositions 2.23 and 2.24 will be used in the next concluding subsection to prove the Proposition 2.9. Firstly, let us prove the Proposition 2.23.

Proof of Proposition 2.23. We would like to compare $\mathcal{H}[\tilde{P} + \varepsilon](t)$ and $\mathcal{H}[\tilde{P}](t)$ (recall that $p = \tilde{P} + \varepsilon$) by studying the difference asymptotically when ε is small. Firstly, let us see how we could simplify the expression of $\mathcal{H}[\tilde{P}](t)$.

Step 1 :

Claim 2.25. If T^* is large enough, for all $t \in [t^*, T]$, we have that

$$\mathcal{H}[\tilde{P}](t) = \sum_{j=1}^J \left(F[\tilde{P}_j](t) + 2(b_j^2 - a_j^2)E[\tilde{P}_j](t) + (a_j^2 + b_j^2)^2 M[\tilde{P}_j](t) \right) + O(e^{-2\theta t}). \quad (2.2.83)$$

Proof. We prove that, for $t \in [t^*, T]$,

$$\left| \mathcal{H}[\tilde{P}] - \sum_{j=1}^J \left(F[\tilde{P}_j] + 2(b_j^2 - a_j^2)E[\tilde{P}_j] + (a_j^2 + b_j^2)^2 M[\tilde{P}_j] \right) \right| \leq C e^{-2\theta t}. \quad (2.2.84)$$

Let us compare $F_j[\tilde{P}]$ and $F[\tilde{P}_j]$:

$$F_j[\tilde{P}] = \int \left(\frac{1}{2} \tilde{P}_{xx}^2 - \frac{5}{2} \tilde{P}^2 \tilde{P}_x^2 + \frac{1}{4} \tilde{P}^6 \right) \varphi_j(t, x) dx, \quad (2.2.85)$$

$$F[\tilde{P}_j] = \int \left(\frac{1}{2} \tilde{P}_{jxx}^2 - \frac{5}{2} \tilde{P}_j^2 \tilde{P}_{jx}^2 + \frac{1}{4} \tilde{P}_j^6 \right) dx. \quad (2.2.86)$$

We compare the corresponding terms of these equalities. Let us start by the first one :

$$\begin{aligned} \left| \int \left(\tilde{P}_{xx}^2 \varphi_j(t, x) - \tilde{P}_{jxx}^2 \right) \right| &\leq \int \tilde{P}_{jxx}^2 |1 - \varphi_j(t, x)| + \sum_{(r,s) \neq (j,j)} \int \left| \tilde{P}_{rxx} \tilde{P}_{sxx} \right| \varphi_j(t, x) \\ &\leq C \int e^{-\frac{\beta}{2}|x-v_j t|} e^{\frac{\beta\tau}{32}t} |1 - \varphi_j(t, x)| dx + C \sum_{i \neq j} \int e^{-\frac{\beta}{2}|x-v_i t|} e^{\frac{\beta\tau}{32}t} \varphi_j(t, x) dx \\ &\leq C e^{\frac{\beta\tau}{32}t} \left[\left(\int_{-\infty}^{\sigma_j t + \delta t} + \int_{\sigma_{j+1} t - \delta t}^{+\infty} \right) e^{-\frac{\beta}{2}|x-v_j t|} dx + \sum_{i \neq j} \int_{\sigma_j t - \delta t}^{\sigma_{j+1} t + \delta t} e^{-\frac{\beta}{2}|x-v_i t|} dx \right] \leq C e^{-\beta\tau t/16}, \end{aligned} \quad (2.2.87)$$

by Proposition 2.21 and Remark 2.17. For the other terms of the difference to be bounded, we reason in a similar way. This completes the proof of the claim. \square

Step 2 :

Therefore, when we manage to compare $\mathcal{H}[p](t)$ and $\mathcal{H}[\tilde{P}](t)$, we are also able to compare $\mathcal{H}[p](t)$ and

$$\sum_{j=1}^J \left(F[\tilde{P}_j](t) + 2(b_j^2 - a_j^2)E[\tilde{P}_j](t) + (a_j^2 + b_j^2)^2 M[\tilde{P}_j](t) \right). \quad (2.2.88)$$

We compute the Taylor expansion of $\mathcal{H}[p] = \mathcal{H}[\tilde{P} + \varepsilon]$:

$$\mathcal{H}[\tilde{P} + \varepsilon] = \frac{1}{2} \int (\tilde{P} + \varepsilon)_{xx}^2 - \frac{5}{2} \int (\tilde{P} + \varepsilon)^2 (\tilde{P} + \varepsilon)_x^2 + \frac{1}{4} \int (\tilde{P} + \varepsilon)^6$$

$$\begin{aligned}
& + \sum_{j=1}^J \left[(b_j^2 - a_j^2) \left(\int (\tilde{P} + \varepsilon)_x^2 \varphi_j - \frac{1}{2} \int (\tilde{P} + \varepsilon)^4 \varphi_j \right) \right] + \sum_{j=1}^J \left[(a_j^2 + b_j^2)^2 \frac{1}{2} \int (\tilde{P} + \varepsilon)^2 \varphi_j \right] \\
& = \frac{1}{2} \int \tilde{P}_{xx}^2 - \frac{5}{2} \int \tilde{P}^2 \tilde{P}_x^2 + \frac{1}{4} \int \tilde{P}^6 + \int \tilde{P}_{(4x)} \varepsilon + 5 \int \tilde{P} \tilde{P}_x^2 \varepsilon + 5 \int \tilde{P}^2 \tilde{P}_{xx} \varepsilon + \frac{3}{2} \int \tilde{P}^5 \varepsilon + \frac{1}{2} \int \varepsilon_{xx}^2 \\
& \quad - \frac{5}{2} \int \tilde{P}^2 \varepsilon_x^2 + \frac{5}{2} \int \tilde{P}_x^2 \varepsilon^2 + 5 \int \tilde{P} \tilde{P}_{xx} \varepsilon^2 + \frac{15}{4} \int \tilde{P}^4 \varepsilon^2 + O(\|\varepsilon(t)\|_{H^2}^3) \\
& \quad + \sum_{j=1}^J (b_j^2 - a_j^2) \left(\int \tilde{P}_x^2 \varphi_j - \frac{1}{2} \int \tilde{P}^4 \varphi_j - 2 \int \tilde{P}_{xx} \varepsilon \varphi_j - 2 \int \tilde{P}_x \varepsilon \varphi_{j,x} - 2 \int \tilde{P}^3 \varepsilon \varphi_j + \int \varepsilon_x^2 \varphi_j - 3 \int \tilde{P}^2 \varepsilon^2 \varphi_j \right) \\
& \quad + \sum_{j=1}^J (a_j^2 + b_j^2)^2 \frac{1}{2} \left(\int \tilde{P}^2 \varphi_j + 2 \int \tilde{P} \varepsilon \varphi_j + \int \varepsilon^2 \varphi_j \right).
\end{aligned} \tag{2.2.89}$$

We can observe that the sum (2.2.89) is composed of 0-order terms in ε , of 1st-order terms in ε , of 2nd-order terms in ε ; 3rd and larger-order terms in ε are contained in $O(\|\varepsilon(t)\|_{H^2}^3)$. The sum of the 0-order terms is actually $\mathcal{H}[\tilde{P}]$. The sum of 2nd-order terms in ε is $H_2[\varepsilon](t)$.

Let us study more closely the 1st-order terms :

$$\begin{aligned}
H_1 & = \int \tilde{P}_{(4x)} \varepsilon + 5 \int \tilde{P} \tilde{P}_x^2 \varepsilon + 5 \int \tilde{P}^2 \tilde{P}_{xx} \varepsilon + \frac{3}{2} \int \tilde{P}^5 \varepsilon \\
& \quad + \sum_{j=1}^J (b_j^2 - a_j^2) \left(2 \int \tilde{P}_x \varepsilon_x \varphi_j - 2 \int \tilde{P}^3 \varepsilon \varphi_j \right) + \sum_{j=1}^J (a_j^2 + b_j^2)^2 \int \tilde{P} \varepsilon \varphi_j.
\end{aligned} \tag{2.2.90}$$

From [8], we know that a breather $A = A_{\alpha, \beta}$ satisfies for any fixed $t \in \mathbb{R}$, the following nonlinear equation :

$$A_{(4x)} - 2(\beta^2 - \alpha^2)(A_{xx} + A^3) + (\alpha^2 + \beta^2)^2 A + 5AA_x^2 + 5A^2 A_{xx} + \frac{3}{2}A^5 = 0. \tag{2.2.91}$$

This equation is also satisfied for $A = \tilde{B}_k$ with $\alpha = \alpha_k$ and $\beta = \beta_k$ for any $k = 1, \dots, K$ (the shape parameters of a breather are not changed by modulation).

For a soliton $Q = R_{c, \kappa}$, we know from $Q_{xx} = cQ - Q^3$ that Q satisfies for any fixed $t \in \mathbb{R}$, the following nonlinear equation (see Section 2.5.1 (Appendix)) :

$$Q_{(4x)} - 2c(Q_{xx} + Q^3) + c^2 Q + 5QQ_x^2 + 5Q^2 Q_{xx} + \frac{3}{2}Q^5 = 0. \tag{2.2.92}$$

This equation is not exactly satisfied for $Q = \tilde{R}_l$ for any $l = 1, \dots, L$ (the shape parameters of a soliton are changed by modulation). The exact equation satisfied by $Q = \tilde{R}_l$ is :

$$\begin{aligned}
& Q_{(4x)} - 2c_l(Q_{xx} + Q^3) + c_l^2 Q + 5QQ_x^2 + 5Q^2 Q_{xx} + \frac{3}{2}Q^5 \\
& = 2c_{0,l}(t)(Q_{xx} + Q^3) - 2c_l c_{0,l}(t)Q - c_{0,l}(t)^2 Q.
\end{aligned} \tag{2.2.93}$$

We will compare H_1 and

$$\begin{aligned}
H'_1 & := \int \tilde{P}_{(4x)} \varepsilon + 5 \sum_{j=1}^J \int \tilde{P}_j \tilde{P}_{j,x}^2 \varepsilon + 5 \sum_{j=1}^J \int \tilde{P}_{j,x}^2 \tilde{P}_{j,xx} \varepsilon + \frac{3}{2} \sum_{j=1}^J \int \tilde{P}_j^5 \varepsilon \\
& \quad - 2 \sum_{j=1}^J (b_j^2 - a_j^2) \left(\int \tilde{P}_{j,xx} \varepsilon + \int \tilde{P}_j^3 \varepsilon \right) + \sum_{j=1}^J (a_j^2 + b_j^2)^2 \int \tilde{P}_j \varepsilon.
\end{aligned} \tag{2.2.94}$$

Firstly, let us compare $\int \widetilde{P} \widetilde{P}_x^2 \varepsilon$ and $\sum_{j=1}^J \int \widetilde{P}_j \widetilde{P}_{j_x}^2 \varepsilon$:

$$\begin{aligned} \int \widetilde{P} \widetilde{P}_x^2 \varepsilon &= \int \left(\sum_{j=1}^J \widetilde{P}_j \right) \left(\sum_{j=1}^J \widetilde{P}_{j_x} \right)^2 \varepsilon \\ &= \sum_{j=1}^J \int \widetilde{P}_j \widetilde{P}_{j_x}^2 \varepsilon + \sum_{h \neq i \text{ or } i \neq j} \int \widetilde{P}_h \widetilde{P}_{i_x} \widetilde{P}_{j_x} \varepsilon. \end{aligned} \quad (2.2.95)$$

To succeed, we need to find a bound for a term of the type $\int \widetilde{P}_h \widetilde{P}_{i_x} \widetilde{P}_{j_x} \varepsilon$ where $h \neq i$ or $i \neq j$. We can perform the following upper bounding (where without loss of generality, we suppose that $i \neq j$) :

$$\begin{aligned} \left| \int \widetilde{P}_h \widetilde{P}_{i_x} \widetilde{P}_{j_x} \varepsilon \right| &\leq C e^{\frac{\beta \tau}{16} t} \int e^{-\frac{\beta}{2} |x-v_i t|} e^{-\frac{\beta}{2} |x-v_j t|} |\varepsilon| \\ &\leq C \|\varepsilon\|_{L^\infty} e^{\frac{\beta \tau}{16} t} \int e^{-\frac{\beta}{2} |x-v_i t|} e^{-\frac{\beta}{2} |x-v_j t|} \\ &\leq C \|\varepsilon\|_{H^2} e^{-\beta \tau t / 8}, \end{aligned} \quad (2.2.96)$$

by Sobolev embeddings and Proposition 2.14.

The bounding is quite similar for $\int \widetilde{P}^2 \widetilde{P}_{xx} \varepsilon$ and $\int \widetilde{P}^5 \varepsilon$. We observe that $-\int \widetilde{P}_{xx} \varepsilon = \int \widetilde{P}_{j_x} \varepsilon_x$. To compare $\int \widetilde{P}_x \varepsilon_x \varphi_j$ and $\int_{\mathbb{R}} \widetilde{P}_{j_x} \varepsilon_x$, and for similar terms, we can use computations that we have already performed at the beginning of this proof. Therefore,

$$\left| \int \widetilde{P}_x \varepsilon_x \varphi_j - \int_{\mathbb{R}} \widetilde{P}_{j_x} \varepsilon_x \right| \leq C \|\varepsilon\|_{H^2} e^{-\frac{\beta \tau t}{16}}. \quad (2.2.97)$$

This enables us to bound the difference between H_1 and H'_1 :

$$|H_1 - H'_1| \leq C \|\varepsilon(t)\|_{H^2} e^{-\frac{\beta \tau t}{16}}. \quad (2.2.98)$$

Now, because our objects are not only breathers, H'_1 is not equal to 0. Actually, we have that

$$\begin{aligned} H'_1 &= 2 \sum_{l=1}^L c_{0,l}(t) \left(\int \widetilde{R}_{lxx} \varepsilon + \int \widetilde{R}_l^3 \varepsilon \right) \\ &\quad - 2 \sum_{l=1}^L c_l c_{0,l}(t) \int \widetilde{R}_l \varepsilon - \sum_{l=1}^L c_{0,l}(t)^2 \int \widetilde{R}_l \varepsilon. \end{aligned} \quad (2.2.99)$$

Now, we introduce :

$$\begin{aligned} H''_1 &= 2 \sum_{l=1}^L c_{0,l}(t) \left(\int \widetilde{R}_{lxx} \varepsilon \sqrt{\varphi_l^s} + \int \widetilde{R}_l^3 \varepsilon \sqrt{\varphi_l^s} \right) \\ &\quad - 2 \sum_{l=1}^L c_l c_{0,l}(t) \int \widetilde{R}_l \varepsilon \sqrt{\varphi_l^s} - \sum_{l=1}^L c_{0,l}(t)^2 \int \widetilde{R}_l \varepsilon \sqrt{\varphi_l^s}. \end{aligned} \quad (2.2.100)$$

By reasoning the same way as for H_1 and H'_1 , we see that

$$|H'_1 - H''_1| \leq C \|\varepsilon(t)\|_{H^2} e^{-2\theta t}. \quad (2.2.101)$$

Because of (2.2.47) and because of the elliptic equation satisfied by a soliton, we have that

$$H_1'' = 0. \quad (2.2.102)$$

Thus,

$$|H_1| = |H_1 - H_1'| + |H_1' - H_1''| + |H_1''| \leq C\|\varepsilon(t)\|_{H^2} e^{-2\theta t}. \quad (2.2.103)$$

The proof of Proposition 2.23 is now completed. \square

Now, we would like to study the quadratic terms in ε of the development of $\mathcal{H}[\tilde{P} + \varepsilon]$. They are contained in $H_2[\varepsilon](t)$.

Let $A = B_{\alpha,\beta}$ be a breather (we note $A_1 := \partial_{x_1} A$ and $A_2 := \partial_{x_2} A$). We define a quadratic form associated to this breather :

$$\begin{aligned} \mathcal{Q}_{\alpha,\beta}^b[\varepsilon] &:= \frac{1}{2} \int \varepsilon_{xx}^2 - \frac{5}{2} \int A^2 \varepsilon_x^2 + \frac{5}{2} \int A_x^2 \varepsilon^2 + 5 \int A A_{xx} \varepsilon^2 + \frac{15}{4} \int A^4 \varepsilon^2 \\ &\quad + (\beta^2 - \alpha^2) \left(\int \varepsilon_x^2 - 3 \int A^2 \varepsilon^2 \right) + (\alpha^2 + \beta^2)^2 \frac{1}{2} \int \varepsilon^2 =: \mathcal{Q}_{\alpha,\beta}[\varepsilon]. \end{aligned} \quad (2.2.104)$$

From [8], we know that the kernel of this quadratic form is of dimension 2 and is spanned by $\partial_{x_1} B_{\alpha,\beta}$ and $\partial_{x_2} B_{\alpha,\beta}$, and that this quadratic form has only one negative eigenvalue that is of multiplicity 1 :

Proposition 2.26 (Proposition 4.11, [79]). *There exists $\mu_{\alpha,\beta}^b > 0$ that depends only on α and β (and does not depend on time), such that if $\varepsilon \in H^2(\mathbb{R})$ is such that*

$$\int A_1 \varepsilon = \int A_2 \varepsilon = 0, \quad (2.2.105)$$

then

$$\mathcal{Q}_{\alpha,\beta}^b[\varepsilon] \geq \mu_{\alpha,\beta}^b \|\varepsilon\|_{H^2}^2 - \frac{1}{\mu_{\alpha,\beta}^b} \left(\int \varepsilon A \right)^2. \quad (2.2.106)$$

Remark 2.27. $\mu_{\alpha,\beta}^b$ is continuous in α, β . Note that translation parameters are implicit in $\mathcal{Q}_{\alpha,\beta}^b$.

Let $Q = R_{c,\kappa}$ be a soliton. We define a quadratic form associated to this soliton :

$$\begin{aligned} \mathcal{Q}_c^s[\varepsilon] &:= \frac{1}{2} \int \varepsilon_{xx}^2 - \frac{5}{2} \int Q^2 \varepsilon_x^2 + \frac{5}{2} \int Q_x^2 \varepsilon^2 + 5 \int Q Q_{xx} \varepsilon^2 + \frac{15}{4} \int Q^4 \varepsilon^2 \\ &\quad + c \left(\int \varepsilon_x^2 - 3 \int Q^2 \varepsilon^2 \right) + c^2 \frac{1}{2} \int \varepsilon^2 =: \mathcal{Q}_{0,\sqrt{c}}[\varepsilon]. \end{aligned} \quad (2.2.107)$$

By the same techniques, such as those presented in [8], adapted to the quadratic form of a soliton, we may establish that the kernel of this quadratic form is of dimension 2, and is spanned by $\partial_x Q$ and $\partial_c Q$, and that this quadratic form does not have any negative eigenvalue (see Section 2.5.2 (Appendix)). After that, from Section 2.5.3 (Appendix), we deduce that the coercivity still works when ε is orthogonal to Q and $\partial_x Q$. More precisely :

Proposition 2.28. *There exists $\mu_c^s > 0$ that depends only on c (and does not depend on time), such that if $\varepsilon \in H^2(\mathbb{R})$ is such that*

$$\int Q \varepsilon = \int Q_x \varepsilon = 0, \quad (2.2.108)$$

then

$$\mathcal{Q}_c^s[\varepsilon] \geq \mu_c^s \|\varepsilon\|_{H^2}^2. \quad (2.2.109)$$

Remark 2.29. μ_c^s is continuous in c . Note that translation and sign parameters are implicit in the notation \mathcal{Q}_c^s .

We would like to find a similar minoration for H_2 (which is a generalization of \mathcal{Q}).

For $j = 1, \dots, J$, let us define for $\epsilon \in H^2$,

$$\begin{aligned} \mathcal{Q}_j[\epsilon] &:= \frac{1}{2} \int \epsilon_{xx}^2 \varphi_j - \frac{5}{2} \int \widetilde{P}_j^2 \epsilon_x^2 \varphi_j + \frac{5}{2} \int \widetilde{P}_{jx}^2 \epsilon^2 \varphi_j \\ &\quad + 5 \int \widetilde{P}_j \widetilde{P}_{jxx} \epsilon^2 \varphi_j + \frac{15}{4} \int \widetilde{P}_j^4 \epsilon^2 \varphi_j \\ &\quad + (b_j^2 - a_j^2) \left(\int \epsilon_x^2 \varphi_j - 3 \int \widetilde{P}_j^2 \epsilon^2 \varphi_j \right) + (a_j^2 + b_j^2)^2 \frac{1}{2} \int \epsilon^2 \varphi_j, \end{aligned} \quad (2.2.110)$$

and

$$\begin{aligned} \mathcal{Q}'_j[\epsilon] &:= \frac{1}{2} \int \epsilon_{xx}^2 \varphi_j - \frac{5}{2} \int \widetilde{P}^2 \epsilon_x^2 \varphi_j + \frac{5}{2} \int \widetilde{P}_x^2 \epsilon^2 \varphi_j \\ &\quad + 5 \int \widetilde{P} \widetilde{P}_{xx} \epsilon^2 \varphi_j + \frac{15}{4} \int \widetilde{P}^4 \epsilon^2 \varphi_j \\ &\quad + (b_j^2 - a_j^2) \left(\int \epsilon_x^2 \varphi_j - 3 \int \widetilde{P}^2 \epsilon^2 \varphi_j \right) + (a_j^2 + b_j^2)^2 \frac{1}{2} \int \epsilon^2 \varphi_j. \end{aligned} \quad (2.2.111)$$

We have that

$$H_2[\varepsilon(t)] = \sum_{j=1}^J \mathcal{Q}'_j[\varepsilon(t)]. \quad (2.2.112)$$

Notations \mathcal{Q}_k^b , $(\mathcal{Q}_k^b)'$, \mathcal{Q}_l^s and $(\mathcal{Q}_l^s)'$ will also be used.

We note that the support of φ_j increases with time, so that \mathcal{Q}_j is near a $\mathcal{Q}_{\alpha_k, \beta_k}^b$ or a $\mathcal{Q}_{c_l}^s$ when time is large (note that $\mathcal{Q}_{\alpha_k, \beta_k}^b$ is the canonical quadratic form associated to the breather \widetilde{B}_k , but the canonical quadratic form associated to the soliton \widetilde{R}_c is $\mathcal{Q}_{c_l + c_{0,l}(t)}^s$). However, firstly, let us study the difference between \mathcal{Q}_j and \mathcal{Q}'_j . Using the computations carried out at the beginning of this part (those done for the linear part) and Sobolev inequalities, we obtain :

$$|\mathcal{Q}_j[\epsilon] - \mathcal{Q}'_j[\epsilon]| \leq C e^{-2\theta t} \|\epsilon\|_{H^2(\mathbb{R})}^2. \quad (2.2.113)$$

Lemma 2.30. *There exists $\mu > 0$ such that for $\rho > 0$, there exists T_3^* such that, if $T^* \geq T_3^*$, for any $\epsilon \in H^2(\mathbb{R})$, for any $t \in [t^*, T]$,*

if

$$\int \widetilde{B}_{k1}(t) \epsilon \sqrt{\varphi_k^b(t)} = \int \widetilde{B}_{k2}(t) \epsilon \sqrt{\varphi_k^b(t)} = 0, \quad (2.2.114)$$

then

$$\mathcal{Q}_k^b[\epsilon] \geq \mu \int (\epsilon^2 + \epsilon_x^2 + \epsilon_{xx}^2) \varphi_k^b(t) - \frac{1}{\mu} \left(\int \epsilon \widetilde{B}_k(t) \sqrt{\varphi_k^b(t)} \right)^2 - \rho \|\epsilon\|_{H^2}^2. \quad (2.2.115)$$

Proof of Lemma 2.30. The idea is to write $\mathcal{Q}_k^b[\epsilon]$ as $\mathcal{Q}_{\alpha_k, \beta_k}[\epsilon \sqrt{\varphi_k^b}]$ plus several error terms. Let j such that $\widetilde{P}_j = \widetilde{B}_k$. We will denote $\varphi_{1,j} := \psi'(\frac{x-\sigma_j t}{\delta t}) - \psi'(\frac{x-\sigma_{j+1} t}{\delta t})$ and $\varphi_{2,j} := \psi''(\frac{x-\sigma_j t}{\delta t}) - \psi''(\frac{x-\sigma_{j+1} t}{\delta t})$, as defined by (2.2.18) and (2.2.19), which will be useful to write the derivatives of φ_j . We recall that they have the same support and bounding

properties as φ_j . We have that

$$\begin{aligned} \int (\epsilon \sqrt{\varphi_j})_{xx}^2 &= \int \epsilon_{xx}^2 \varphi_j + \int \frac{\epsilon_x^2}{(\delta t)^2} \frac{\varphi_{1,j}^2}{\varphi_j} + \frac{1}{4} \int \frac{\epsilon^2}{(\delta t)^4} \frac{\varphi_{2,j}^2}{\varphi_j} + \frac{1}{16} \int \frac{\epsilon^2}{(\delta t)^4} \frac{\varphi_{1,j}^4}{\varphi_j^3} \\ &\quad - \frac{1}{4} \int \frac{\epsilon^2}{(\delta t)^4} \frac{\varphi_{2,j} \varphi_{1,j}^2}{\varphi_j^2} + 2 \int \frac{\epsilon_{xx} \epsilon_x}{\delta t} \varphi_{1,j} + \int \frac{\epsilon_{xx} \epsilon}{(\delta t)^2} \varphi_{2,j} \\ &\quad - \frac{1}{2} \int \frac{\epsilon_{xx} \epsilon}{(\delta t)^2} \frac{\varphi_{1,j}^2}{\varphi_j} + \int \frac{\epsilon_x \epsilon}{(\delta t)^3} \frac{\varphi_{1,j} \varphi_{2,j}}{\varphi_j} - \frac{1}{2} \int \frac{\epsilon_x \epsilon}{(\delta t)^3} \frac{\varphi_{1,j}^3}{\varphi_j^2}. \end{aligned} \quad (2.2.116)$$

We observe that, for T_3^* large enough, and by using the inequalities that define ψ , the error terms can be bounded by $\frac{C}{\delta t} \|\epsilon\|_{H^2}^2 \leq \frac{\rho}{100} \|\epsilon\|_{H^2}^2$. The computation for the other terms is similar and the same bound can be used for the error terms.

Because $\epsilon \sqrt{\varphi_k^b}$ satisfies the orthogonality conditions, we can apply Proposition 2.26, and obtain that

$$\mathcal{Q}_{\alpha_k, \beta_k} [\epsilon \sqrt{\varphi_k^b}] \geq \mu_k^b \left\| \epsilon \sqrt{\varphi_k^b} \right\|_{H^2}^2 - \frac{1}{\mu_k^b} \left(\int \epsilon \sqrt{\varphi_k^b} \widetilde{B}_k \right)^2. \quad (2.2.117)$$

To finish, $\|\epsilon \sqrt{\varphi_k^b}\|_{H^2}^2$ is $\int (\epsilon^2 + \epsilon_x^2 + \epsilon_{xx}^2) \varphi_k^b(t)$ plus several error terms as in (2.2.116). \square

Lemma 2.31. *There exists $\mu > 0$ such that for $\rho > 0$, there exists T_3^* such that if $T^* \geq T_3^*$, then for any $\epsilon \in H^2(\mathbb{R})$, for any $t \in [t^*, T]$, we have that*

if

$$\int \widetilde{R}_l(t) \epsilon \sqrt{\varphi_l^s(t)} = \int \widetilde{R}_{lx}(t) \epsilon \sqrt{\varphi_l^s(t)} = 0, \quad (2.2.118)$$

then

$$\mathcal{Q}_l^s[\epsilon] \geq \mu \int \left(\epsilon^2 + \epsilon_x^2 + \epsilon_{xx}^2 \right) \varphi_l^s(t) - \rho \|\epsilon\|_{H^2}^2. \quad (2.2.119)$$

Proof. As in the previous proof, we write $\mathcal{Q}_l^s[\epsilon]$ as $\mathcal{Q}_{c_l}^s[\epsilon \sqrt{\varphi_l^s}]$ (with $Q = \widetilde{R}_l$) plus several error terms, that are all bounded by $\rho \|\epsilon\|_{H^2}^2$ if T_3^* is chosen large enough. However, $\mathcal{Q}_{c_l}^s[\epsilon \sqrt{\varphi_l^s}]$ is not appropriate in order to have coercivity, the appropriate quadratic form is $\mathcal{Q}_{c_l + c_{0,l}(t)}^s[\epsilon \sqrt{\varphi_l^s}]$. This is why, we need to bound the difference between $\mathcal{Q}_{c_l}^s[\epsilon \sqrt{\varphi_l^s}]$ and $\mathcal{Q}_{c_l + c_{0,l}(t)}^s[\epsilon \sqrt{\varphi_l^s}]$. This difference is

$$c_{0,l}(t) \left(\int \epsilon_x^2 \varphi_j - 3 \int \widetilde{R}_l^2 \epsilon^2 \varphi_j \right) + c_l c_{0,l}(t) \int \epsilon^2 \varphi_j + c_{0,l}(t)^2 \frac{1}{2} \int \epsilon^2 \varphi_j, \quad (2.2.120)$$

which can, because of the bound for $c_{0,l}(t)$, for T_3^* large enough (depending on A), be bounded by $\rho \|\epsilon\|_{H^2}^2$.

Now, $\epsilon \sqrt{\varphi_l^s}$ satisfies the orthogonality conditions we need, and as in the previous proof we may apply coercivity. \square

Proof of Proposition 2.24. We will now use the Lemma 2.30 and its version for solitons (Lemma 2.31) for $\epsilon = \varepsilon(t)$. From this, we deduce that for $\rho > 0$ small enough, we have that

$$\sum_{j=1}^J \mathcal{Q}_j[\varepsilon(t)] \geq \mu \|\varepsilon(t)\|_{H^2}^2 - \frac{1}{\mu} \sum_{k=1}^K \left(\int \varepsilon(t) \widetilde{B}_k \sqrt{\varphi_k^b} \right)^2, \quad (2.2.121)$$

for a suitable constant $\mu > 0$. This means that for T_3^* large enough, by taking, if needed, a smaller constant μ ,

$$H_2[\varepsilon(t)] \geq \mu \|\varepsilon\|_{H^2}^2 - \frac{1}{\mu} \sum_{k=1}^K \left(\int \varepsilon \widetilde{B}_k \sqrt{\varphi_k^b} \right)^2. \quad (2.2.122)$$

The proof of Proposition 2.24 is now completed. \square

2.2.5 Proof of Proposition 2.9 (Bootstrap)

We recall that p_n from Proposition 2.9 is denoted by p and T_n is denoted by T in what follows, in order to simplify the notations. We do the proof that follows under the assumption (2.2.9), so that the Propositions proved above are true for $t \in [t^*, T]$.

The aim of this subsection is to complete the proof of Proposition 2.9 by using the Propositions 2.23 and 2.24.

We note that by Lemma 2.18, the conservation of $F[p](t)$ and the definition of $\mathcal{H}[p]$, we have for any $t \in [t^*, T]$, that

$$|\mathcal{H}[p](T) - \mathcal{H}[p](t)| \leq \frac{CA^2}{\delta^2 t} e^{-2\theta t}. \quad (2.2.123)$$

Thus, for any $t \in [t^*, T]$,

$$\mathcal{H}[p](t) \leq \mathcal{H}[p](T) + \frac{CA^2}{\delta^2 t} e^{-2\theta t}. \quad (2.2.124)$$

From Proposition 2.23,

$$\begin{aligned} & \left| \mathcal{H}[\widetilde{P} + \varepsilon](t) - H_2[\varepsilon](t) \right. \\ & \quad \left. - \sum_{j=1}^J \left(F[\widetilde{P}_j](t) + 2(b_j^2 - a_j^2)E[\widetilde{P}_j](t) + (a_j^2 + b_j^2)^2 M[\widetilde{P}_j](t) \right) \right| \\ & \leq C e^{-2\theta t} + C \|\varepsilon\|_{H^2} e^{-2\theta t} + C \|\varepsilon\|_{H^2}^3 \leq C e^{-2\theta t} + \frac{\mu}{100} \|\varepsilon\|_{H^2}^2. \end{aligned} \quad (2.2.125)$$

In order to obtain the last line, we use the fact that $\|\varepsilon(t)\|_{H^2} \leq CAe^{-\theta t}$, and we take $T^* \geq T_5^*$ for T_5^* large enough (depending on A) so that $\|\varepsilon\|_{H^2} \leq C$ and $C\|\varepsilon(t)\|_{H^2} \leq \frac{\mu}{100}$, and thus $C\|\varepsilon(t)\|_{H^2}^3 \leq \frac{\mu}{100} \|\varepsilon(t)\|_{H^2}^2$.

We remark that if $P_j = B_k$ is a breather, then $F[\widetilde{P}_j]$, $E[\widetilde{P}_j]$ and $M[\widetilde{P}_j]$ are all constants in time. If $P_j = R_l$ is a soliton and we denote q the basic ground state (i.e. the ground state for $c = 1$), we have the following :

$$M[\widetilde{R}_l](t) = (c_l + c_{0,l}(t))^{1/2} M[q], \quad (2.2.126)$$

$$E[\widetilde{R}_l](t) = (c_l + c_{0,l}(t))^{\frac{3}{2}} E[q], \quad (2.2.127)$$

$$F[\widetilde{R}_l](t) = (c_l + c_{0,l}(t))^{5/2} F[q]. \quad (2.2.128)$$

Using that, we can simplify $\mathcal{R}_l(t) := F[\widetilde{R}_l](t) + 2c_l E[\widetilde{R}_l](t) + c_l^2 M[\widetilde{R}_l](t)$ as follows :

$$\begin{aligned} \mathcal{R}_l(t) &= (c_l + c_{0,l}(t))^{5/2} F[q] + 2c_l (c_l + c_{0,l}(t))^{\frac{3}{2}} E[q] \\ &\quad + c_l^2 (c_l + c_{0,l}(t))^{1/2} M[q] \\ &= c_l^{5/2} \left(1 + \frac{c_{0,l}(t)}{c_l} \right)^{5/2} F[q] + 2c_l^{5/2} \left(1 + \frac{c_{0,l}(t)}{c_l} \right)^{\frac{3}{2}} E[q] \\ &\quad + c_l^{5/2} \left(1 + \frac{c_{0,l}(t)}{c_l} \right)^{1/2} M[q]. \end{aligned} \quad (2.2.129)$$

Note that from Lemma 2.19, $|c_{0,l}(t)|^3 \leq CA^3 e^{-\theta t} e^{-2\theta t}$. That is why, if we take T_5^* eventually larger, $|c_{0,l}(t)|^3 \leq Ce^{-2\theta t}$. For this reason, we will do Taylor expansions of order 2 of (2.2.129) :

$$\left(1 + \frac{c_{0,l}(t)}{c_l}\right)^{5/2} = 1 + \frac{5}{2} \frac{c_{0,l}(t)}{c_l} + \frac{15}{8} \frac{c_{0,l}(t)^2}{c_l^2} + O(e^{-2\theta t}), \quad (2.2.130)$$

$$\left(1 + \frac{c_{0,l}(t)}{c_l}\right)^{3/2} = 1 + \frac{3}{2} \frac{c_{0,l}(t)}{c_l} + \frac{3}{8} \frac{c_{0,l}(t)^2}{c_l^2} + O(e^{-2\theta t}), \quad (2.2.131)$$

$$\left(1 + \frac{c_{0,l}(t)}{c_l}\right)^{1/2} = 1 + \frac{1}{2} \frac{c_{0,l}(t)}{c_l} - \frac{1}{8} \frac{c_{0,l}(t)^2}{c_l^2} + O(e^{-2\theta t}). \quad (2.2.132)$$

This allows us to write :

$$\begin{aligned} \mathcal{R}_l(t) &= c_l^{5/2} (F[q] + 2E[q] + M[q]) + c_l^{3/2} c_{0,l}(t) \left(\frac{5}{2} F[q] + 3E[q] + \frac{1}{2} M[q] \right) \\ &\quad + c_l^{1/2} c_{0,l}(t)^2 \left(\frac{15}{8} F[q] + \frac{3}{4} E[q] - \frac{1}{8} M[q] \right) + O(e^{-2\theta t}). \end{aligned} \quad (2.2.133)$$

Now, $c_l^{5/2} (F[q] + 2E[q] + M[q])$ is constant in time. For both other terms, we use that $M[q] = 2$, $E[q] = -\frac{2}{3}$ and $F[q] = \frac{2}{5}$, and we see that $\frac{5}{2} F[q] + 3E[q] + \frac{1}{2} M[q] = 0$ and $\frac{15}{8} F[q] + \frac{3}{4} E[q] - \frac{1}{8} M[q] = 0$. This allows us to write :

$$\mathcal{R}_l(t) = \frac{16}{15} c_l^{5/2} + O(e^{-2\theta t}). \quad (2.2.134)$$

From this, we deduce that

$$\mathcal{R}_l(t) - \mathcal{R}_l(T) = O(e^{-2\theta t}). \quad (2.2.135)$$

By using that $\mathcal{H}[p](T) = \mathcal{H}[P](T) = \mathcal{H}[\tilde{P}](T)$, the equations (2.2.125) and (2.2.124), Claim 2.25, and the fact that for $t \geq T_4^*$, $O(\|\varepsilon(t)\|_{H^2}^3) \leq \frac{\mu}{100} \|\varepsilon\|_{H^2}^2$, we have that

$$\begin{aligned} H_2[\varepsilon](t) &\leq \mathcal{H}[p](t) + Ce^{-2\theta t} + \frac{\mu}{100} \|\varepsilon(t)\|_{H^2}^2 \\ &\quad - \sum_{j=1}^J \left(F[\tilde{P}_j](t) + 2(b_j^2 - a_j^2) E[\tilde{P}_j](t) + (a_j^2 + b_j^2)^2 M[\tilde{P}_j](t) \right) \\ &\leq \mathcal{H}[\tilde{P}](T) + C \left(\frac{A^2}{\delta^2 t} + 1 \right) e^{-2\theta t} + \frac{\mu}{100} \|\varepsilon(t)\|_{H^2}^2 \\ &\quad - \sum_{j=1}^J \left(F[\tilde{P}_j](t) + 2(b_j^2 - a_j^2) E[\tilde{P}_j](t) + (a_j^2 + b_j^2)^2 M[\tilde{P}_j](t) \right) \\ &\leq \mathcal{H}[\tilde{P}](T) + C \left(\frac{A^2}{\delta^2 t} + 1 \right) e^{-2\theta t} + \frac{\mu}{100} \|\varepsilon(t)\|_{H^2}^2 + \sum_{l=1}^L (\mathcal{R}_l(T) - \mathcal{R}_l(t)) \\ &\quad - \sum_{j=1}^J \left(F[\tilde{P}_j](T) + 2(b_j^2 - a_j^2) E[\tilde{P}_j](T) + (a_j^2 + b_j^2)^2 M[\tilde{P}_j](T) \right) \\ &\leq C \left(\frac{A^2}{\delta^2 t} + 1 \right) e^{-2\theta t} + \frac{\mu}{100} \|\varepsilon(t)\|_{H^2}^2. \end{aligned} \quad (2.2.136)$$

From Proposition 2.24, we deduce (by taking a smaller constant μ) that

$$\mu \|\varepsilon\|_{H^2}^2 \leq C \left(\frac{A^2}{\delta^2 t} + 1 \right) e^{-2\theta t} + \frac{1}{\mu} \sum_{k=1}^K \left(\int \varepsilon \widetilde{B}_k \sqrt{\varphi_k^b} \right)^2. \quad (2.2.137)$$

We will now need to establish a result close to Lemma 2.18. We set for any $j = 1, \dots, J$:

$$m_j(t) := \int \frac{1}{2} p^2(t, x) \sqrt{\varphi_j(t, x)} dx := m_j[p](t). \quad (2.2.138)$$

Lemma 2.32. *There exists $C > 0$, $T_6^* = T_6^*(A)$ such that, if $T^* \geq T_6^*$, for any $j = 1, \dots, J$, for any $t \in [t^*, T]$,*

$$|m_j(T) - m_j(t)| \leq \frac{C}{\delta^2 t} A^2 e^{-2\theta t}. \quad (2.2.139)$$

Proof. We compute :

$$\begin{aligned} & \frac{d}{dt} \int \frac{1}{2} p^2(t, x) \sqrt{\varphi_j(t, x)} dx \\ &= \frac{1}{2\delta t} \int \left(-\frac{3}{2} p_x^2 + \frac{3}{4} p^4 \right) \frac{\varphi_{1,j}}{\sqrt{\varphi_j}} - \frac{1}{2(\delta t)^2} \int p_x p \frac{\varphi_{2,j}}{\sqrt{\varphi_j}} \\ & \quad + \frac{1}{4(\delta t)^2} \int p_x p \frac{\varphi_{1,j}^2}{\varphi_j^{\frac{3}{2}}} - \frac{1}{4} \int p^2 \frac{x}{\delta t^2} \frac{\varphi_{1,j}}{\sqrt{\varphi_j}}. \end{aligned} \quad (2.2.140)$$

From the inequalities that define ψ , we find that

$$\left| \frac{d}{dt} \int \frac{1}{2} p^2(t, x) \sqrt{\varphi_j(t, x)} dx \right| \leq \frac{C}{\delta^2 t} \int_{\Omega_j(t) \cup \Omega_{j+1}(t)} (p_x^2 + p^2 + p^4). \quad (2.2.141)$$

From now on, we can follow the proof of Lemma 2.18. \square

Now, we observe the following :

$$\int (\widetilde{P} + \varepsilon)^2 \sqrt{\varphi_k^b} = \int \widetilde{B}_k^2 + 2 \int \widetilde{B}_k \varepsilon \sqrt{\varphi_k^b} + \int \varepsilon^2 \sqrt{\varphi_k^b} + \text{Err}, \quad (2.2.142)$$

where Err stands for the other terms of the sum, which we consider as error terms, and we will show that they are bounded by $Ce^{-\theta t}$.

For $i \neq j$ and any h (if $P_j = B_k$ is a breather),

$$\left| \int \widetilde{P}_i \widetilde{P}_h \sqrt{\varphi_j} \right| \leq C \int_{-\delta t + \sigma_j t}^{\delta t + \sigma_{j+1} t} e^{-\frac{\theta}{2}|x - v_i t|} dx \leq C e^{-\theta t}, \quad (2.2.143)$$

and

$$\begin{aligned} \left| \int \widetilde{P}_i \varepsilon \sqrt{\varphi_j} \right| &\leq \sqrt{\left(\int \widetilde{P}_i^2 \varphi_j \right) \left(\int \varepsilon^2 \right)} \\ &\leq C e^{-\frac{\theta}{2} t} \|\varepsilon\|_{H^2} \leq C A e^{-\theta t} e^{-\frac{\theta}{2} t} \leq C e^{-\theta t}, \end{aligned} \quad (2.2.144)$$

where $T^* \geq T_7^*$ with T_7^* being large enough depending on A . If we use the calculations we have made in the proof of Claim 2.25, we see that

$$\left| \int \widetilde{P}_j^2 - \int \widetilde{P}_j^2 \sqrt{\varphi_j} \right| \leq C e^{-\theta t}. \quad (2.2.145)$$

This proves the bound for the error terms.

Now, we study the variations of (2.2.142). We know that $\int \widetilde{P}_j^2 = \int \widetilde{B}_k^2$ has no variations. We can apply Lemma 2.32 for $\int (\widetilde{P} + \varepsilon)^2 \sqrt{\varphi_j}$. By writing the difference of the equation (2.2.142) between t and T , and using that $\varepsilon(T) = 0$, we deduce, for $T^* \geq \max(T_6^*, T_7^*)$, that

$$\begin{aligned} \left| \int \widetilde{P}_j \varepsilon \sqrt{\varphi_j}(t) \right| &\leq C \left(\frac{A^2}{\delta^2 t} + 1 \right) e^{-\theta t} + \|\varepsilon\|_{H^2}^2 \\ &\leq C \left(\frac{A^2}{\delta^2 t} + 1 \right) e^{-\theta t} + \frac{\mu}{100} \|\varepsilon(t)\|_{H^2}. \end{aligned} \quad (2.2.146)$$

Thus,

$$\begin{aligned} \mu \|\varepsilon\|_{H^2}^2 &\leq C \left(\frac{A^2}{\delta^2 t} + 1 \right) e^{-2\theta t} + \frac{1}{\mu} \sum_{j=1}^J \left(\int \varepsilon \widetilde{P}_j \sqrt{\varphi_j} \right)^2 \\ &\leq C \left(\frac{A^4}{\delta^4 t} + 1 \right) e^{-2\theta t} + \frac{\mu}{100} \|\varepsilon(t)\|_{H^2}^2. \end{aligned} \quad (2.2.147)$$

Therefore,

$$\|\varepsilon(t)\|_{H^2}^2 \leq C \left(\frac{A^4}{\delta^4 t} + 1 \right) e^{-2\theta t}. \quad (2.2.148)$$

By using (2.2.148), the mean-value theorem and Lemma 2.19, we deduce that for $t \in [t^*, T]$,

$$\begin{aligned} \|p(t) - P(t)\|_{H^2} &\leq \|\varepsilon(t)\|_{H^2} + \|\widetilde{P}(t) - P(t)\|_{H^2} \\ &\leq C \left(\sqrt{\frac{A^4}{\delta^4 t} + 1} \right) e^{-\theta t} \\ &\quad + C \left(\sum_{k=1}^K (|x_{1,k}(t)| + |x_{2,k}(t)|) + \sum_{l=1}^L (|x_{0,l}(t)| + |c_{0,l}(t)|) \right) \\ &\leq C \left(\sqrt{\frac{A^4}{\delta^4 t} + 1} \right) e^{-\theta t} + C \sum_{k=1}^K \left(\left| \int_t^T x'_{1,k}(s) ds \right| + \left| \int_t^T x'_{2,k}(s) ds \right| \right) \\ &\quad + C \sum_{l=1}^L \left(\left| \int_t^T x'_{0,l}(s) ds \right| + \left| \int_t^T c'_{0,l}(s) ds \right| \right) \\ &\leq C \left(\frac{A^4}{\delta^4 t} + 1 \right) e^{-\theta t} + C \left(\int_t^T \|\varepsilon(s)\|_{H^2} ds + \int_t^T e^{-\theta s} ds \right) \\ &\leq C \left(\frac{A^4}{\delta^4 t} + 1 \right) e^{-\theta t}. \end{aligned} \quad (2.2.149)$$

We take $A = 4C$ (where C is a constant that can be used anywhere in the proof above) and

$$T^* := \max(T_1^*, T_2^*, T_3^*, T_4^*, T_5^*, T_6^*, T_7^*, T_8^*) \quad (2.2.150)$$

(depending on A), where $T_8^* := T_8^*(A)$ is such that for $t \geq T_8^*$, $\frac{A^4}{\delta^4 t} \leq 1$. And thus, for any $t \in [t^*, T]$,

$$C \left(\frac{A^4}{\delta^4 t} + 1 \right) \leq 2C = \frac{A}{2}, \quad (2.2.151)$$

which is exactly what we wanted to prove.

2.3 p is a smooth multi-breather

Our goal here is to prove Proposition 2.10.

2.3.1 Estimates in higher order Sobolev norms

Firstly, we notice that the proposition is already established for $s = 2$. We note also that if this proposition is proved for an $s \geq 2$ with a corresponding constant A_s , then this proposition is also valid for any $s' \leq s$ with the same constant A_s . This means that A_s can possibly increase with s and that this proposition is already established for $0 \leq s \leq 2$. From now on, we will denote (as before) p_n by p , T_n by T and $p_n - P$ by v , and make sure that the constant A_s that we will obtain in the proof does not depend on n (although it will depend on s). For the constant θ , we will take the usual value : $\theta := \frac{\beta\tau}{32}$. For the constant T^* , we will also take the value that works for Proposition 2.6.

We will prove the proposition by induction on s (it is sufficient to prove the proposition for any integer s). Let $s \geq 3$. We will prove the proposition for s , assuming that the proposition is true for any $0 \leq s' \leq s - 1$.

Let us deduce from the (mKdV) equation the equation satisfied by v :

$$\begin{aligned}
 v_t &= p_t - \sum_{j=1}^J P_{jt} \\
 &= - \left(p_{xx} + p^3 - \sum_{j=1}^J P_{jxx} - \sum_{j=1}^J P_j^3 \right)_x \\
 &= - \left(v_{xx} + (v + P)^3 - \sum_{j=1}^J P_j^3 \right)_x \\
 &= - \left(v_{xx} + v^3 + 3v^2P + 3vP^2 + P^3 - \sum_{j=1}^J P_j^3 \right)_x.
 \end{aligned} \tag{2.3.1}$$

Firstly, we compute $\frac{d}{dt} \int (\partial_x^s v)^2$ by integration by parts :

$$\begin{aligned}
 \frac{d}{dt} \int (\partial_x^s v)^2 &= 2 \int (\partial_x^s v_t) (\partial_x^s v) \\
 &= -2 \int \partial_x^{s+1} \left(v_{xx} + v^3 + 3v^2P + 3vP^2 + P^3 - \sum_{j=1}^J P_j^3 \right) (\partial_x^s v) \\
 &= 2(-1)^{s+1} \int \partial_x^{2s+1} \left(P^3 - \sum_{j=1}^J P_j^3 \right) v - 2 \int \partial_x^{s+1} (v^3) (\partial_x^s v) \\
 &\quad - 6 \int \partial_x^{s+1} (v^2P) (\partial_x^s v) - 6 \int \partial_x^{s+1} (vP^2) (\partial_x^s v),
 \end{aligned} \tag{2.3.2}$$

because $\int (\partial_x^{s+3} v) (\partial_x^s v) = - \int (\partial_x^{s+2} v) (\partial_x^{s+1} v) = 0$.

We will now bound above each of the terms of the obtained sum. By Sobolev embedding, Proposition 2.14 and Proposition 2.6,

$$\left| \int \partial_x^{2s+1} \left(P^3 - \sum_{j=1}^J P_j^3 \right) v \right| \leq \|v\|_{L^\infty} \int \left| \partial_x^{2s+1} \left(P^3 - \sum_{j=1}^J P_j^3 \right) \right|$$

$$\begin{aligned}
&\leq C\|v\|_{H^1}e^{-\beta\tau t/2} \\
&\leq CAe^{-\theta t}e^{-\beta\tau t/2} \\
&\leq CAe^{-2\theta t} \leq CA_{s-1}^2e^{-2\theta t},
\end{aligned} \tag{2.3.3}$$

where $C \geq 0$ is a constant that depends only on s .

We observe that

$$\begin{aligned}
\partial_x^{s+1}(v^3) &= 3(\partial_x^{s+1}v)v^2 + 6(s+1)(\partial_x^s v)v_x v + Z_1(v, v_x, \dots, \partial_x^{s-1}v), \\
\partial_x^{s+1}(v^2 P) &= 2(\partial_x^{s+1}v)vP + 2(s+1)(\partial_x^s v)(vP)_x \\
&\quad + Z_2(v, v_x, \dots, \partial_x^{s-1}v, P, P_x, \dots, \partial_x^{s+1}P),
\end{aligned} \tag{2.3.4}$$

where Z_1 and Z_2 are homogeneous polynomials of degree 3 with constant coefficients.

Now, let us look for a bound for $\int \partial_x^{s+1}(v^3)(\partial_x^s v)$. Firstly, by integration by parts,

$$\begin{aligned}
\int \partial_x^{s+1}(v^3)(\partial_x^s v) &= \frac{3}{2} \int \left((\partial_x^s v)^2 \right)_x v^2 + 3(s+1) \int (\partial_x^s v)^2 (v^2)_x + \int (\partial_x^s v) Z_1 \\
&= \frac{6(s+1)-3}{2} \int (\partial_x^s v)^2 (v^2)_x + \int (\partial_x^s v) Z_1.
\end{aligned} \tag{2.3.5}$$

Then, we bound above each of the terms of the obtained sum :

$$\begin{aligned}
\left| \int (\partial_x^s v)^2 (v^2)_x \right| &\leq C\|v\|_{L^\infty} \|v_x\|_{L^\infty} \int (\partial_x^s v)^2 \\
&\leq C\|v\|_{H^2}^2 \int (\partial_x^s v)^2 \\
&\leq C(\|p\|_{H^2} + \|P\|_{H^2}) A e^{-\theta t} \int (\partial_x^s v)^2 \\
&\leq CC_0 A e^{-\theta t} \int (\partial_x^s v)^2 \leq CA_{s-1} e^{-\theta t} \int (\partial_x^s v)^2.
\end{aligned} \tag{2.3.6}$$

We have actually shown in the computation above that $\|v\|_{H^2}^2$ can be bounded above by $\|v\|_{H^2}$ (with a constant that depends only on problem data), and therefore the degree of $\|v\|_{H^2}$ can be lowered without harm in the upper bound. We will use this fact again for the rest of the proof. In fact, all what it means is that, for several terms, what we have is more than what we need.

By the Cauchy-Schwarz and Gagliardo-Nirenberg-Sobolev inequalities,

$$\begin{aligned}
\left| \int (\partial_x^s v) Z_1 \right| &\leq C \int |\partial_x^s v| \left(\sum_{s'=0}^{s-1} |\partial_x^{s'} v|^3 \right) \\
&\leq C \left(\int |\partial_x^s v|^2 \right)^{1/2} \sum_{s'=0}^{s-1} \left(\int |\partial_x^{s'} v|^6 \right)^{1/2} \\
&\leq C \left(\int |\partial_x^s v|^2 \right)^{1/2} \sum_{s'=0}^{s-1} \left(\int |\partial_x^{s'} v|^2 \right) \left(\int |\partial_x^{s'+1} v|^2 \right)^{1/2} \\
&\leq C \sum_{s'=0}^{s-1} \left(\int |\partial_x^{s'} v|^2 \right) \left(\int |\partial_x^s v|^2 + \int |\partial_x^{s'+1} v|^2 \right)
\end{aligned} \tag{2.3.7}$$

$$\leq CA_{s-1}^2 e^{-2\theta t} + CA_{s-1} e^{-\theta t} \int |\partial_x^s v|^2.$$

Similarly, we bound $\int \partial_x^{s+1}(v^2 P)(\partial_x^s v)$. By integration by parts,

$$\begin{aligned} \int \partial_x^{s+1}(v^2 P)(\partial_x^s v) &= \int \left((\partial_x^s v)^2 \right)_x v P + 2(s+1) \int (\partial_x^s v)^2 (vP)_x + \int (\partial_x^s v) Z_2 \\ &= (2s+1) \int (\partial_x^s v)^2 (vP)_x + \int (\partial_x^s v) Z_2. \end{aligned} \quad (2.3.8)$$

We bound above each of the terms of the obtained sum, starting by

$$\begin{aligned} \left| \int (\partial_x^s v)^2 (vP)_x \right| &\leq C(\|v\|_{L^\infty} + \|v_x\|_{L^\infty}) \int (\partial_x^s v)^2 \\ &\leq CAe^{-\theta t} \int (\partial_x^s v)^2. \end{aligned} \quad (2.3.9)$$

The upper bound of $\left| \int (\partial_x^s v) Z_2 \right|$ is similar to (2.3.7) above :

$$\left| \int (\partial_x^s v) Z_2 \right| \leq CA_{s-1}^2 e^{-2\theta t} + CA_{s-1} e^{-\theta t} \int |\partial_x^s v|^2. \quad (2.3.10)$$

$\int \partial_x^{s+1}(vP^2)(\partial_x^s v)$ remains to be bounded. By integration by parts,

$$\begin{aligned} \int \partial_x^{s+1}(vP^2)(\partial_x^s v) &= - \int \partial_x^{s+2}(vP^2)(\partial_x^{s-1} v) \\ &= - \int (\partial_x^{s+2} v)(\partial_x^{s-1} v) P^2 - (s+2) \int (\partial_x^{s+1} v)(\partial_x^{s-1} v)(P^2)_x \\ &\quad - \frac{(s+2)(s+1)}{2} \int (\partial_x^s v)(\partial_x^{s-1} v)(P^2)_{xx} + \int (\partial_x^{s-1} v) Z_3^0(v, v_x, \dots, \partial_x^{s-1} v) \\ &= \frac{1}{2} \int \left((\partial_x^s v)^2 \right)_x P^2 + (s+1) \int (\partial_x^s v)^2 (P^2)_x \\ &\quad - \frac{s(s+1)}{4} \int \left((\partial_x^{s-1} v)^2 \right)_x (P^2)_{xx} + \int (\partial_x^{s-1} v) Z_3^0(v, v_x, \dots, \partial_x^{s-1} v) \\ &= \frac{2s+1}{2} \int (\partial_x^s v)^2 (P^2)_x + \int (\partial_x^{s-1} v) Z_3(v, v_x, \dots, \partial_x^{s-1} v), \end{aligned} \quad (2.3.11)$$

where Z_3^0 and Z_3 are homogeneous polynomials of degree 1 whose coefficients are polynomials in P and its space derivatives. We have that $|Z_3| \leq C(\sum_{s'=0}^{s-1} |\partial_x^{s'} v|)$. Therefore,

$$\left| \int (\partial_x^{s-1} v) Z_3 \right| \leq CA_{s-1}^2 e^{-2\theta t}. \quad (2.3.12)$$

Thus, by taking the sum of all those inequalities, we obtain :

$$\left| \frac{d}{dt} \int (\partial_x^s v)^2 + 3(2s+1) \int (\partial_x^s v)^2 (P^2)_x \right| \leq CA_{s-1}^2 e^{-2\theta t} + CA_{s-1} e^{-\theta t} \int |\partial_x^s v|^2. \quad (2.3.13)$$

Next, we perform similar computations for $\frac{d}{dt} \int (\partial_x^{s-1} v)^2 P^2$:

$$\begin{aligned} \frac{d}{dt} \int (\partial_x^{s-1} v)^2 P^2 &= 2 \int (\partial_x^{s-1} v_t) (\partial_x^{s-1} v) P^2 + 2 \int (\partial_x^{s-1} v)^2 P_t P \\ &= -2 \int \partial_x^s \left(v_{xx} + v^3 + 3v^2 P + 3v P^2 + P^3 - \sum_{j=1}^J P_j^3 \right) (\partial_x^{s-1} v) P^2 \\ &\quad - 2 \int (\partial_x^{s-1} v)^2 \left(P_{xx} + \sum_{j=1}^J P_j^3 \right)_x P. \end{aligned} \quad (2.3.14)$$

Let us study each of the obtained terms.

Firstly,

$$\begin{aligned} -2 \int (\partial_x^{s+2} v) (\partial_x^{s-1} v) P^2 &= 2 \int (\partial_x^{s+1} v) (\partial_x^s v) P^2 + 2 \int (\partial_x^{s+1} v) (\partial_x^{s-1} v) (P^2)_x \\ &= -3 \int (\partial_x^s v)^2 (P^2)_x - 2 \int (\partial_x^s v) (\partial_x^{s-1} v) (P^2)_{xx} \\ &= -3 \int (\partial_x^s v)^2 (P^2)_x + \int (\partial_x^{s-1} v)^2 (P^2)_{xxx}. \end{aligned} \quad (2.3.15)$$

Indeed,

$$\left| \int (\partial_x^{s-1} v)^2 (P^2)_{xxx} \right| \leq C A^2 e^{-2\theta t}. \quad (2.3.16)$$

Secondly,

$$\left| \int \partial_x^s \left(P^3 - \sum_{j=1}^J P_j^3 \right) (\partial_x^{s-1} v) P^2 \right| \leq C A_{s-1}^2 e^{-2\theta t} \quad (2.3.17)$$

can be obtained similarly to the first part of the proof (starting by an integration by parts to have $\partial_x^{s-2} v$ at the place of $\partial_x^{s-1} v$).

Thirdly,

$$\begin{aligned} \int \partial_x^s (v^3) (\partial_x^{s-1} v) P^2 &= 3 \int (\partial_x^s v) (\partial_x^{s-1} v) v^2 P^2 + \int Z_4(v, v_x, \dots, \partial_x^{s-1} v) P^2 \\ &= -\frac{3}{2} \int (\partial_x^{s-1} v)^2 (v^2 P^2)_x + \int Z_4 P^2, \end{aligned} \quad (2.3.18)$$

where Z_4 is a homogeneous polynomial of degree 4 with constant coefficients. Both terms are easily bounded by $C A_{s-1}^2 e^{-2\theta t}$.

Fourthly, for $\int \partial_x^s (v^2 P) (\partial_x^{s-1} v) P^2$ and $\int \partial_x^s (v P^2) (\partial_x^{s-1} v) P^2$, we reason similarly.

Fifthly,

$$\left| \int (\partial_x^{s-1} v)^2 \left(P_{xx} + \sum_{j=1}^J P_j^3 \right)_x P \right| \leq C A_{s-1}^2 e^{-2\theta t} \quad (2.3.19)$$

is clear.

Therefore,

$$\left| \frac{d}{dt} \int (\partial_x^{s-1} v)^2 P^2 + 3 \int (\partial_x^s v)^2 (P^2)_x \right| \leq C A_{s-1}^2 e^{-2\theta t}. \quad (2.3.20)$$

We set

$$F(t) := \int (\partial_x^s v)^2 - (2s+1) \int (\partial_x^{s-1} v)^2 P^2. \quad (2.3.21)$$

By putting the both parts of the proof together :

$$\left| \frac{d}{dt} F(t) \right| \leq C A_{s-1}^2 e^{-2\theta t} + C A_{s-1} e^{-\theta t} \int |\partial_x^s v|^2. \quad (2.3.22)$$

Because $\left| \int (\partial_x^{s-1} v)^2 P^2 \right| \leq C A^2 e^{-2\theta t}$, we can write the following upper bound :

$$\int (\partial_x^s v)^2 \leq |F(t)| + C A_{s-1}^2 e^{-2\theta t}. \quad (2.3.23)$$

Therefore, we have, for a suitable constant $C > 0$ that depends only on s ,

$$\left| \frac{d}{dt} F(t) \right| \leq C A_{s-1}^2 e^{-2\theta t} + C A_{s-1} e^{-\theta t} |F(t)|. \quad (2.3.24)$$

For $t \in [T^*, T]$, by integration between t and T (we recall that $F(T) = 0$),

$$\begin{aligned} |F(t)| &= |F(T) - F(t)| = \left| \int_t^T \frac{d}{dt} F(\sigma) d\sigma \right| \leq \int_t^T \left| \frac{d}{dt} F(\sigma) \right| d\sigma \\ &\leq C A_{s-1}^2 \int_t^T e^{-2\theta\sigma} d\sigma + C A_{s-1} \int_t^T e^{-\theta\sigma} |F(\sigma)| d\sigma \\ &\leq C A_{s-1}^2 e^{-2\theta t} + C A_{s-1} \int_t^T e^{-\theta\sigma} |F(\sigma)| d\sigma. \end{aligned} \quad (2.3.25)$$

By Gronwall lemma, for all $t \in [T^*, T]$,

$$\begin{aligned} |F(t)| &\leq C A_{s-1}^2 e^{-2\theta t} \\ &\quad + C A_{s-1} \int_t^T e^{-\theta\sigma} C A_{s-1}^2 e^{-2\theta\sigma} \exp \left(\int_t^\sigma C A_{s-1} e^{-\theta u} du \right) d\sigma \\ &\leq C A_{s-1}^2 e^{-2\theta t} \\ &\quad + C A_{s-1}^3 \exp \left(\frac{C A_{s-1}}{\theta} e^{-\theta t} \right) \int_t^T e^{-3\theta\sigma} \exp \left(-\frac{C A_{s-1}}{\theta} e^{-\theta\sigma} \right) d\sigma \\ &\leq C A_{s-1}^2 e^{-2\theta t} + C A_{s-1}^3 \exp \left(\frac{C A_{s-1}}{\theta} \right) \int_t^T e^{-3\theta\sigma} d\sigma \\ &\leq C A_{s-1}^2 e^{-2\theta t} + C A_{s-1}^3 \exp \left(\frac{C A_{s-1}}{\theta} \right) e^{-3\theta t} \\ &\leq C A_{s-1}^3 \exp \left(\frac{C A_{s-1}}{\theta} \right) e^{-2\theta t}. \end{aligned} \quad (2.3.26)$$

Therefore,

$$\int (\partial_x^s v)^2 \leq A_s e^{-2\theta t}, \quad (2.3.27)$$

where $A_s := C A_{s-1}^3 \exp \left(\frac{C A_{s-1}}{\theta} \right)$ and C is a constant large enough that depends only on s . This conclude the proof of Proposition 2.10, and so of Theorem 2.2.

2.3.2 Uniformity of constants

We conclude this section with an explanation regarding Remark 2.3.

In the proof above, the constants that we obtain A, T^*, θ do depend on $P_j(0)$ ($1 \leq j \leq J$). Actually, we may characterize this dependence. In fact, they do not depend on the initial positions of our objects in the case when our objects are initially ordered in the right order and sufficiently far from each other.

Theorem 2.33. *Given parameters (2.1.9), (2.1.10), (2.1.11) and (2.1.12) which satisfy (2.1.13), there exists $D > 0$ large enough that depends only on α_k, β_k, c_l such that if*

$$\forall j \geq 2, \quad x_j(0) \geq x_{j-1}(0) + D, \quad (2.3.28)$$

then the following holds. We set $\theta := \frac{\beta\tau}{32}$, with β and τ given by (2.2.1) and $p(t)$ the multi-breather associated to P by Proposition 2.7. There exists $A_s \geq 1$ for any $s \geq 2$ that depends only on α_k, β_k, c_l and D such that

$$\forall t \geq 0, \quad \|p(t) - P(t)\|_{H^s} \leq A_s e^{-\theta t}. \quad (2.3.29)$$

Firstly, we will prove that for any $D > 0$, if (2.3.28) is satisfied, then the constants A_s and T^* do only depend on α_k, β_k, c_l and D . Finally, we will prove that if $D > 0$ is large enough with respect to the given parameters, then we can take $T^* = 0$.

To establish the validity of this theorem, it is enough to read again the whole article and to make sure that on any step of the proof, there is no dependence on initial positions of our objects when our objects are initially far from each other for the constant C . This will allow to claim the same for the constants A and T^* (but, these constants may depend on D). This works, but we should change a bit the way we write our results.

For Proposition 2.12, we should write :

$$|\partial_x^n \partial_t^m P_j(t, x)| \leq C e^{-\beta|x-v_j t-x_j(0)|}. \quad (2.3.30)$$

Therefore, in Proposition 2.14, we have nothing to change, but the constant C do depend on D . This will also be the case in the following propositions and lemmas of this proof.

We should replace $\sigma_j t$ for the definition of φ_j in (2.2.16) and (2.2.17) by $\sigma_j t + \frac{x_{j-1}(0)+x_j(0)}{2}$ to take into account of initial positions. More precisely, we will have for any $j = 2, \dots, J-1$,

$$\varphi_j(t, x) := \psi\left(\frac{x - \sigma_j t - \frac{x_{j-1}(0)+x_j(0)}{2}}{\delta t}\right) - \psi\left(\frac{x - \sigma_{j+1} t - \frac{x_j(0)+x_{j+1}(0)}{2}}{\delta t}\right), \quad (2.3.31)$$

and similarly for other definitions.

After having done the modulation with C and T^* depending on D , for Proposition 2.21, we should write :

$$|\partial_x^n \widetilde{P}_j(t, x)| \leq C e^{-\frac{\beta}{2}|x-v_j t-x_j(0)|} e^{\frac{\beta\tau}{32}t}. \quad (2.3.32)$$

Therefore, with these adaptations, the same proof works to prove that A_s and T^* do depend only on α_k, β_k, c_l and D .

Now, given α_k, β_k, c_l , we choose $D_0 > 0$ in an arbitrary maner. Therefore, we get $A_s(D_0)$ and $T^*(D_0)$ associated to D_0 . Let $\Lambda := v_J - v_1$ the maximal difference between two velocities. We set $D := D_0 + \Lambda \cdot T^*(D_0)$. Therefore, if we suppose (2.3.28) in $t = 0$ for D , then we have (2.3.28) in $t = -T^*(D_0)$ for D_0 . Therefore, by applying the intermediate result for D_0 , we obtain the desired conclusion with D and A_s that depend on D_0 .

2.4 Uniqueness

p is the multi-breather constructed in the existence part. The goal here is to prove that if a solution u converges to p when $t \rightarrow +\infty$ (in some sense), then $u = p$ (under well chosen assumptions).

We prove here two propositions. For both of them, we assume that the velocities of all our objects are distinct (this was also an assumption for the existence). The first proposition does not make more assumptions on velocities of our objects, but it is a partial uniqueness result as we restrict ourselves to the class of super polynomial convergence to the multi-breather. The second proposition assumes that the velocities of all our objects, except possibly one, are positive (this is a new assumption and it is needed because this proof uses monotonicity arguments).

2.4.1 A solution converging super polynomially to a multi-breather is this multi-breather

The goal of this subsection is to prove Proposition 2.5.

Remark 2.34. Note that in Proposition 2.5, we don't make any assumptions on the sign of v_1 or v_2 . This uniqueness proposition has the same degree of generality as Theorem 2.2.

Proof of Proposition 2.5. Let $p(t)$ be the multi-breather associated to P by Theorem 2.2. Recall that for any s ,

$$\|p(t) - P(t)\|_{H^s} = O(e^{-\theta t}), \quad (2.4.1)$$

for a suitable $\theta > 0$.

Let $N > 2$ to be chosen later. We take $u(t)$ an H^2 solution of (mKdV) such that there exists $C_0 > 0$ such that for t large enough,

$$\|u(t) - P(t)\|_{H^2} \leq \frac{C_0}{t^N}. \quad (2.4.2)$$

From that, we may deduce that for t large enough (namely, $t \geq 2C_0$ along with the previous condition),

$$\|u(t) - P(t)\|_{H^2} \leq \frac{1}{2} \frac{1}{t^{N-1}}. \quad (2.4.3)$$

Our goal is to find a condition on N that do not depend on u , such that the condition (2.4.3) on u for t large enough implies that $u \equiv p$.

Because of (2.4.1), the condition (2.4.3) for t large enough is equivalent to : for t large enough,

$$\|u(t) - p(t)\|_{H^2} \leq \frac{1}{t^{N-1}}. \quad (2.4.4)$$

We denote $z(t) := u(t) - p(t)$. Our goal is to find N large enough that do not depend on z , for which we will be able to prove that $z \equiv 0$, given

$$\|z(t)\|_{H^2} \leq \frac{1}{t^{N-1}}, \quad (2.4.5)$$

for t large enough. Because z is a difference of two solutions of (mKdV), we may derive the following equation for z :

$$z_t + (z_{xx} + (z + p)^3 - p^3)_x = 0. \quad (2.4.6)$$

We divide our proof in several steps.

Step 1. Modulation on z .

For $j = 1, \dots, J$, if $P_j = B_k$ is a breather, we denote

$$K_j := \begin{pmatrix} \partial_{x_1} B_k \\ \partial_{x_2} B_k \end{pmatrix}, \quad (2.4.7)$$

and if $P_j = R_l$ is a soliton, we denote :

$$K_j = \partial_x R_l. \quad (2.4.8)$$

We may derive the following equation for K_j :

$$(K_j)_t + ((K_j)_{xx} + 3P_j^2 K_j)_x = 0. \quad (2.4.9)$$

For $j = 1, \dots, J$, if $P_j = B_k$ is a breather, let $c_j(t) \in \mathbb{R}^2$ defined for t large enough and if $P_j = R_l$ is a soliton, let $c_j(t) \in \mathbb{R}$ defined for t large enough such that for

$$\tilde{z}(t) := z(t) + \sum_{j=1}^J c_j(t) K_j(t) \sqrt{\varphi_j(t)}, \quad (2.4.10)$$

where $c_j K_j$ is either a product of two numbers of \mathbb{R} or a scalar product of two vectors of \mathbb{R}^2 , the following condition is satisfied : for any $j = 1, \dots, J$, for t large enough,

$$\int \tilde{z}(t) K_j(t) \sqrt{\varphi_j(t)} = 0, \quad (2.4.11)$$

where φ_j is defined in Section 2.2.2 (in this proof, it is OK to take $\delta = 1$). It is possible to do so in a unique way, because the Gram matrix associated to $K_j(t) \sqrt{\varphi_j(t)}$, $1 \leq j \leq J$, is invertible; which is the case because $K_j(t) \sqrt{\varphi_j(t)}$, $1 \leq j \leq J$, are linearly independent. This is why $c_j(t)$, $1 \leq j \leq J$, are defined in a unique way. For the same reason, $c_j(t)$ is obtained linearly from $\int K_k(t) z(t) \sqrt{\varphi_k(t)}$, $1 \leq k \leq J$, with coefficients that depend only on K_k , $1 \leq k \leq J$. This is why, from Cauchy-Schwarz, we may deduce the following lemma.

Lemma 2.35. *For any $j = 1, \dots, J$, for t large enough, there exists $C > 0$ that do not depend on z , such that*

$$|c_j(t)| \leq C \|z(t)\|_{L^2}, \quad (2.4.12)$$

$$\|\tilde{z}(t)\|_{H^2} \leq C \|z(t)\|_{H^2}. \quad (2.4.13)$$

The Gram matrix is C^1 in time and invertible. This is why, its inverse is C^1 in time. Because $\int K_j z \sqrt{\varphi_j}$ are C^1 in time, we deduce by multiplication that $c_j(t)$ are C^1 in time.

By differentiating in time the linear relation that defines $c_j(t)$, we see that $c'_j(t)$ is obtained linearly from $\int K_k(t) z(t) \sqrt{\varphi_k(t)}$, $1 \leq k \leq J$, and from $\frac{d}{dt} \int K_k(t) z(t) \sqrt{\varphi_k(t)}$, $1 \leq k \leq J$, with coefficients that depend on K_k , $1 \leq k \leq J$ (and their derivatives). Because it is easy to see that $\frac{d}{dt} \int K_k(t) z(t) \sqrt{\varphi_k(t)}$ may still be bounded by $C \|z(t)\|_{L^2}$, we deduce that for any $j = 1, \dots, J$, for t large enough, there exists $C > 0$ that do not depend on z , such that

$$|c'_j(t)| \leq C \|z(t)\|_{L^2}. \quad (2.4.14)$$

We may derive the following equation for \tilde{z} :

$$\tilde{z}_t + (\tilde{z}_{xx} + 3\tilde{z}p^2)_x = -(3z^2p + z^3)_x + \sum_{k=1}^J c'_k(t) K_k - 3 \sum_{k=1}^J c_k(t) ((P_k^2 - p^2) K_k)_x. \quad (2.4.15)$$

Step 2. A bound for $|c'_j(t)|$.

The goal here is to improve (2.4.14).

Lemma 2.36. *For any $j = 1, \dots, J$, for t large enough, there exists $C > 0$ and $\theta > 0$ that do not depend on z , such that*

$$|c'_j(t)| \leq C\|\tilde{z}(t)\|_{H^2} + Ce^{-\theta t}\|z(t)\|_{H^2} + C\|z(t)\|_{H^2}^2. \quad (2.4.16)$$

Proof. We may differentiate (2.4.11) :

$$\begin{aligned} 0 &= \frac{d}{dt} \int \tilde{z} K_j \sqrt{\varphi_j} \\ &= \int \tilde{z}_t K_j \sqrt{\varphi_j} + \int \tilde{z} (K_j)_t \sqrt{\varphi_j} + \int \tilde{z} K_j (\sqrt{\varphi_j})_t \\ &= - \int (\tilde{z}_{xx} + 3\tilde{z}p^2)_x K_j \sqrt{\varphi_j} - \int (3z^2p + z^3)_x K_j \sqrt{\varphi_j} \\ &\quad + \sum_{k=1}^J \int (c'_k(t) \cdot K_k) K_j \sqrt{\varphi_j} - 3 \sum_{k=1}^J c_k(t) \int \left(c_k(t) \cdot ((P_k^2 - p^2)K_k)_x \right) K_j \sqrt{\varphi_j} \\ &\quad - \int \tilde{z} ((K_j)_{xx} + 3K_j P_j^2)_x \sqrt{\varphi_j} + \int \tilde{z} K_j (\sqrt{\varphi_j})_t. \end{aligned} \quad (2.4.17)$$

We know that $(\sqrt{\varphi_j})_x$ and $(\sqrt{\varphi_j})_t$ are bounded (from inequalities established in Section 2.2.2). This is why, for any t large enough,

$$\left| \int \tilde{z} K_j (\sqrt{\varphi_j})_t \right| \leq C\|\tilde{z}(t)\|_{H^2}. \quad (2.4.18)$$

For the same reason, after eventually doing an integration by parts, for any t large enough,

$$\left| \int (\tilde{z}_{xx} + 3\tilde{z}p^2)_x K_j \sqrt{\varphi_j} \right| + \left| \int \tilde{z} ((K_j)_{xx} + 3K_j P_j^2)_x \sqrt{\varphi_j} \right| \leq C\|\tilde{z}(t)\|_{H^2}. \quad (2.4.19)$$

$\int (3z^2p + z^3)_x K_j \sqrt{\varphi_j}$ is clearly bounded by $C\|z(t)\|_{H^2}^2$. Finally, we see that $(P_k^2 - p^2)K_k$ is exponentially bounded in time (in Sobolev or L^∞ norm), and using Lemma 2.35, we deduce that

$$\int \left(c_k(t) \cdot ((P_k^2 - p^2)K_k)_x \right) K_j \sqrt{\varphi_j} \leq Ce^{-\theta t}\|z(t)\|_{H^2}, \quad (2.4.20)$$

for a suitable $\theta > 0$ that do not depend on z . This is why, we deduce that for any $j = 1, \dots, J$, for t large enough, there exists $C > 0$ and $\theta > 0$ that do not depend on z , such that

$$\left| \sum_{k=1}^J \int (c'_k(t) \cdot K_k) K_j \sqrt{\varphi_j} \right| \leq C\|\tilde{z}(t)\|_{H^2} + Ce^{-\theta t}\|z(t)\|_{H^2} + C\|z(t)\|_{H^2}^2. \quad (2.4.21)$$

We recall that for any $(e_1, e_2) \in (\mathbb{R})^2$ or $(\mathbb{R}^2)^2$, $e_3 \in \mathbb{R}$ or \mathbb{R}^2 , we have the following equality between two elements of \mathbb{R} or \mathbb{R}^2 (where vectors are denoted as a colon)

$$(e_1 \cdot e_2)e_3 = \left(e_1^T (e_2 e_3^T) \right)^T, \quad (2.4.22)$$

where T denotes the transpose.

First of all, because $\int K_k K_j^T \sqrt{\varphi_j}$ converges exponentially to $\int K_k K_j^T$, for $k \neq j$, $\int K_k K_j^T$ is exponentially decreasing, and from (2.4.14), we may write that for any $j = 1, \dots, J$, for t large enough, there exists $C > 0$ and $\theta > 0$ that do not depend on z , such that

$$\left| \left(c'_j(t)^T \int K_j K_j^T \right)^T \right| \leq C\|\tilde{z}(t)\|_{H^2} + Ce^{-\theta t}\|z(t)\|_{H^2} + C\|z(t)\|_{H^2}^2. \quad (2.4.23)$$

Now, in the case when $K_j \in \mathbb{R}^2$, using the fact that its components are linearly independent and Cauchy-Schwarz inequality, we deduce the desired lemma. \square

Step 3. Coercivity.

We define the following functional quadratic in \tilde{z} :

$$\begin{aligned} H(t) = & \frac{1}{2} \int \tilde{z}_{xx}^2 - \frac{5}{2} \int p^2 \tilde{z}_x^2 + \frac{5}{2} \int p_x^2 \tilde{z}^2 + 5 \int p p_{xx} \tilde{z}^2 + \frac{15}{4} \int p^4 \tilde{z}^2 \\ & + \sum_{j=1}^J (b_j^2 - a_j^2) \left(\int \tilde{z}_x^2 \varphi_j - 3 \int p^2 \tilde{z}^2 \varphi_j \right) + \sum_{j=1}^J (a_j^2 + b_j^2)^2 \frac{1}{2} \int \tilde{z}^2 \varphi_j. \end{aligned} \quad (2.4.24)$$

We will prove the following lemma :

Lemma 2.37. *There exists $C > 0$ that do not depend on z , such that for t large enough,*

$$\|\tilde{z}(t)\|_{H^2}^2 \leq C H(t) + C \sum_{j=1}^J \left(\int \tilde{z} P_j \right)^2. \quad (2.4.25)$$

Proof. We denote \mathcal{Q}_j the quadratic form associated to P_j . We remind that

$$\begin{aligned} \mathcal{Q}_j[\varepsilon] := & \frac{1}{2} \int \varepsilon_{xx}^2 - \frac{5}{2} \int P_j^2 \varepsilon_x^2 + \frac{5}{2} \int (P_j)_x^2 \varepsilon^2 + 5 \int P_j (P_j)_{xx} \varepsilon^2 \\ & + \frac{15}{4} \int P_j^4 \varepsilon^2 + (b_j^2 - a_j^2) \left(\int \varepsilon_x^2 - 3 \int P_j^2 \varepsilon^2 \right) + (a_j^2 + b_j^2)^2 \frac{1}{2} \int \varepsilon^2. \end{aligned} \quad (2.4.26)$$

In any case, we have that for any $j = 1, \dots, J$, there exists $\mu_j > 0$, such that if $\varepsilon \in H^2$ satisfies $\int K_j \varepsilon = 0$, then we have

$$\mathcal{Q}_j[\varepsilon] \geq \mu_j \|\varepsilon\|_{H^2}^2 - \frac{1}{\mu_j} \left(\int \varepsilon P_j \right)^2. \quad (2.4.27)$$

Here, we apply this coercivity result with $\varepsilon = \tilde{z} \sqrt{\varphi_j}$ for which the orthogonality conditions (2.4.11) are satisfied. Thus,

$$\|\tilde{z} \sqrt{\varphi_j}\|_{H^2}^2 \leq C \mathcal{Q}_j[\tilde{z} \sqrt{\varphi_j}] + C \left(\int \tilde{z} P_j \sqrt{\varphi_j} \right)^2. \quad (2.4.28)$$

We denote :

$$\begin{aligned} \mathcal{Q}'_j[\varepsilon] := & \frac{1}{2} \int \varepsilon_{xx}^2 \varphi_j - \frac{5}{2} \int p^2 \varepsilon_x^2 \varphi_j + \frac{5}{2} \int p_x^2 \varepsilon^2 \varphi_j + 5 \int p p_{xx} \varepsilon^2 \varphi_j \\ & + \frac{15}{4} \int p^4 \varepsilon^2 \varphi_j + (b_j^2 - a_j^2) \left(\int \varepsilon_x^2 \varphi_j - 3 \int p^2 \varepsilon^2 \varphi_j \right) + (a_j^2 + b_j^2)^2 \frac{1}{2} \int \varepsilon^2 \varphi_j, \end{aligned} \quad (2.4.29)$$

and we observe that

$$H(t) = \sum_{j=1}^J \mathcal{Q}'_j[\tilde{z}(t)]. \quad (2.4.30)$$

In $\mathcal{Q}'_j[\tilde{z}(t)]$, we may replace p by P_j with an error bounded by $C e^{-\theta t} \|\tilde{z}(t)\|_{H^2}^2$, because of (2.4.1) mainly. After that, the expression obtained may be replaced by $\mathcal{Q}_j[\tilde{z}(t) \sqrt{\varphi_j(t)}]$ with an error bounded by $\frac{C}{t} \|\tilde{z}(t)\|_{H^2}^2$ (cf. calculations

done in the proof of Lemma 2.30). For the same reason, $\|\tilde{z}\sqrt{\varphi_j}\|_{H^2}^2$ may be replaced by $\int(\tilde{z}^2 + \tilde{z}_x^2 + \tilde{z}_{xx}^2)\varphi_j$ with an error bounded by $\frac{C}{t}\|\tilde{z}(t)\|_{H^2}^2$. Therefore, because of

$$\|\tilde{z}\|_{H^2}^2 = \sum_{j=1}^J \int (\tilde{z}^2 + \tilde{z}_x^2 + \tilde{z}_{xx}^2)\varphi_j, \quad (2.4.31)$$

the fact that $P_j\sqrt{\varphi_j}$ converges exponentially to P_j , and the fact that $\frac{C}{t}$ may be as small as we want if we take t large enough, we deduce the desired lemma. \square

Step 4. Modification of H for the sake of simplification.

We define :

$$\begin{aligned} \tilde{H}(t) := & \int \left[\frac{1}{2}\tilde{z}_{xx}^2 - \frac{5}{2}((\tilde{z}+p)^2(\tilde{z}+p)_x^2 - p^2p_x^2 - 2\tilde{z}pp_x^2 - 2\tilde{z}_xp^2p_x) \right. \\ & \left. + \frac{1}{4}((\tilde{z}+p)^6 - p^6 - 6\tilde{z}p^5) \right] + \frac{1}{2} \sum_{j=1}^J (a_j^2 + b_j^2)^2 \int \tilde{z}^2 \varphi_j \\ & + 2 \sum_{j=1}^J (b_j^2 - a_j^2) \int \left[\frac{1}{2}\tilde{z}_x^2 - \frac{1}{4}((\tilde{z}+p)^4 - p^4 - 4\tilde{z}p^3) \right] \varphi_j. \end{aligned} \quad (2.4.32)$$

We observe that the difference between H and \tilde{H} is bounded by $O(\|\tilde{z}(t)\|_{H^2}^3)$. We can thus claim :

Lemma 2.38. *There exists $C > 0$ that do not depend on z , such that for t large enough,*

$$\|\tilde{z}(t)\|_{H^2}^2 \leq C\tilde{H}(t) + C \sum_{j=1}^J \left(\int \tilde{z}P_j \right)^2. \quad (2.4.33)$$

Step 5. A bound for $\frac{d\tilde{H}}{dt}$.

Lemma 2.39. *There exists $C > 0$ and $\theta > 0$ that do not depend on z , such that for t large enough,*

$$\left| \frac{d\tilde{H}}{dt} \right| \leq \frac{C}{t} \|\tilde{z}(t)\|_{H^2}^2 + Ce^{-\theta t} \|\tilde{z}(t)\|_{H^2} \|z(t)\|_{H^2} + C \|\tilde{z}(t)\|_{H^2} \|z(t)\|_{H^2}^2. \quad (2.4.34)$$

Proof. We develop the expression of $\tilde{H}(t)$, we differentiate each term obtained and we use (2.4.15), the fact that p is a solution of (mKdV) and the fact that $(\varphi_j)_t = -\frac{x}{t}(\varphi_j)_x$, where $\frac{x}{t}$ is bounded independently from z because of the compact support of φ_j . We obtain several sorts of terms after doing several integrations by parts and several obvious simplifications.

Several terms are clearly bounded by one of the bounds of the lemma, because in these terms, the cumulated degree of z and \tilde{z} is larger than 2. As an example, we show how to deal with $\int z_{xxx}z\tilde{z}_{xx}p$. We use the fact that $z = \tilde{z} - \sum_{j=1}^J c_j K_j$, and we obtain the following :

$$\begin{aligned} \int z_{xxx}z\tilde{z}_{xx}p &= \int \tilde{z}_{xxx}\tilde{z}\tilde{z}_{xx}p - \int \tilde{z}_{xxx} \left(\sum_{j=1}^J c_j K_j \right) \tilde{z}_{xx}p \\ &\quad - \int \left(\sum_{j=1}^J c_j (K_j)_{xxx} \right) \tilde{z}\tilde{z}_{xx}p + \int \left(\sum_{j=1}^J c_j (K_j)_{xxx} \right) \left(\sum_{j=1}^J c_j K_j \right) \tilde{z}_{xx}p. \end{aligned} \quad (2.4.35)$$

It is easy to see that any of these terms is bounded as we want in the lemma (several of them are bounded by $\frac{C}{t}\|\tilde{z}(t)\|_{H^2}^2$, the last one is bounded by $C\|\tilde{z}(t)\|_{H^2}\|z(t)\|_{H^2}^2$), because of Lemma 2.35 and of (2.4.5).

Other terms contain \tilde{z} quadratically and contain $(\varphi_j)_x$. And, $(\varphi_j)_x$ is bounded by $\frac{C}{t}$. This is why, such terms are bounded by $\frac{C}{t}\|\tilde{z}(t)\|_{H^2}^2$.

Several other terms can be, by doing suitable integrations by parts transformed in one of the two following expressions :

$$6 \sum_{j=1}^J \int \tilde{z} \tilde{z}_x p \left[p_{xxxx} - 2(b_j^2 - a_j^2)(p_{xx} + p^3) + (a_j^2 + b_j^2)^2 p + 5pp_x^2 + 5p^2 p_{xx} + \frac{3}{2}p^5 \right] \varphi_j, \quad (2.4.36)$$

$$3 \sum_{j=1}^J \int \tilde{z}^2 p_x \left[p_{xxxx} - 2(b_j^2 - a_j^2)(p_{xx} + p^3) + (a_j^2 + b_j^2)^2 p + 5pp_x^2 + 5p^2 p_{xx} + \frac{3}{2}p^5 \right] \varphi_j. \quad (2.4.37)$$

To deal with these two expressions, we use the elliptic equation satisfied by P_j :

$$(P_j)_{xxxx} - 2(b_j^2 - a_j^2)((P_j)_{xx} + P_j^3) + (a_j^2 + b_j^2)^2 P_j + 5P_j(P_j)_x^2 + 5P_j^2(P_j)_{xx} + \frac{3}{2}P_j^5 = 0, \quad (2.4.38)$$

and the fact that

$$[p_{xxxx} - 2(b_j^2 - a_j^2)(p_{xx} + p^3) + (a_j^2 + b_j^2)^2 p + 5pp_x^2 + 5p^2 p_{xx} + \frac{3}{2}p^5] \varphi_j \quad (2.4.39)$$

converges exponentially to

$$(P_j)_{xxxx} - 2(b_j^2 - a_j^2)((P_j)_{xx} + P_j^3) + (a_j^2 + b_j^2)^2 P_j + 5P_j(P_j)_x^2 + 5P_j^2(P_j)_{xx} + \frac{3}{2}P_j^5, \quad (2.4.40)$$

which is a direct consequence of (2.4.1). This is why, such terms are bounded by $\frac{C}{t}\|\tilde{z}(t)\|_{H^2}^2$.

Other terms contain $(P_j^2 - p^2)K_j$, which is bounded exponentially, with c_j bounded by $\|z\|_{H^2}$. Those terms are obviously bounded by $Ce^{-\theta t}\|\tilde{z}(t)\|_{H^2}\|z(t)\|_{H^2}$.

Other terms contain K_k (or a derivative) and φ_j with $j \neq k$. In this case, this product gives an exponential decreasing, and such a term is bounded by $Ce^{-\theta t}\|\tilde{z}(t)\|_{H^2}\|z(t)\|_{H^2}$, using (2.4.14).

Therefore, we are left with the following terms :

$$\begin{aligned} & \sum_{j=1}^J c'_j(t) \int \left[(K_j)_{xx} \tilde{z}_{xx} - 10K_j \tilde{z}_x p p_x - 5K_j \tilde{z} p_x^2 \right. \\ & \quad \left. - 10(K_j)_x \tilde{z} p p_x - 5(K_j)_x \tilde{z}_x p^2 + \frac{15}{4}K_j \tilde{z} p^4 \right. \\ & \quad \left. + 2(b_j^2 - a_j^2)(K_j)_x \tilde{z}_x - 6(b_j^2 - a_j^2)K_j \tilde{z} p^2 + (a_j^2 + b_j^2)^2 K_j \tilde{z} \right] \varphi_j. \end{aligned} \quad (2.4.41)$$

We may replace p by P_j in the preceeding expression with an error bounded by

$$Ce^{-\theta t}\|\tilde{z}(t)\|_{H^2}\|z(t)\|_{H^2}, \quad (2.4.42)$$

because of (2.4.14) and (2.4.1). This is acceptable, knowing the result we want to prove. By integration by parts, we obtain several terms of the form $c'_j(t) \int (K_j)_{xx} \tilde{z}_x(\varphi_j)_x$, which are bounded by $\frac{C}{t} |c'_j(t)| \|\tilde{z}(t)\|_{H^2}$. Now, from Lemma 2.36, we deduce that they are bounded by

$$\frac{C}{t} \|\tilde{z}(t)\|_{H^2}^2 + C e^{-\theta t} \|\tilde{z}(t)\|_{H^2} \|z(t)\|_{H^2} + C \|\tilde{z}(t)\|_{H^2} \|z(t)\|_{H^2}^2, \quad (2.4.43)$$

which is exactly the bound that we want. And, we are left with the following terms :

$$\begin{aligned} \sum_{j=1}^J c'_j(t) \int & \left[(K_j)_{xxxx} + 10(K_j)_x P_j (P_j)_x + 5K_j (P_j)_x^2 \right. \\ & \left. + 10K_j P_j (P_j)_{xx} + 5(K_j)_{xx} P_j^2 + \frac{15}{2} K_j P_j^4 \right. \\ & \left. - 2(b_j^2 - a_j^2)(K_j)_{xx} - 6(b_j^2 - a_j^2) K_j P_j^2 + (a_j^2 + b_j^2)^2 K_j \right] \tilde{z} \varphi_j. \end{aligned} \quad (2.4.44)$$

The last expression equals zero, because of the elliptic equation satisfied by K_j , which we may derive by differentiating (2.4.38). \square

Step 6. A bound for $\frac{d}{dt} \int \tilde{z} P_j$.

Lemma 2.40. *There exists $C > 0$ and $\theta > 0$ that do not depend on z , such that for t large enough, for any $j = 1, \dots, J$,*

$$\left| \frac{d}{dt} \int \tilde{z} P_j \right| \leq C e^{-\theta t} \|z(t)\|_{H^2} + C \|z(t)\|_{H^2}^2. \quad (2.4.45)$$

Proof. We observe that

$$\int \tilde{z} P_j = \int z P_j + \sum_{k=1}^J c_k(t) \int K_k P_j. \quad (2.4.46)$$

Firstly, for $k = j$,

$$\int K_j P_j = 0, \quad (2.4.47)$$

and for $k \neq j$,

$$\frac{d}{dt} \left[c_k(t) \int K_k P_j \right] = c'_k(t) \int K_k P_j + c_k(t) \int (K_k)_t P_j + c_k(t) \int K_k (P_j)_t, \quad (2.4.48)$$

and it is obvious, from Lemma 2.35 and (2.4.14), that the latter is bounded by $C e^{-\theta t} \|z(t)\|_{H^2}$.

It is left to bound $\frac{d}{dt} \int z P_j$. We use (2.4.6) and we obtain :

$$\frac{d}{dt} \int z P_j = - \int (z_{xx} + (z + p)^3 - p^3)_x P_j - \int z ((P_j)_{xx} + P_j^3)_x. \quad (2.4.49)$$

Several terms are immediately boundable by $C \|z(t)\|_{H^2}^2$, we kill several others by integration by parts and we are left with

$$\int z (p^2 - P_j^2) (P_j)_x, \quad (2.4.50)$$

which is obviously bounded by $C e^{-\theta t} \|z(t)\|_{H^2}$, because of (2.4.1). \square

By differentiation of a square, we obtain that

Lemma 2.41. *There exists $C > 0$ and $\theta > 0$ that do not depend on z , such that for t large enough, for any $j = 1, \dots, J$,*

$$\left| \frac{d}{dt} \left(\int \tilde{z} P_j \right)^2 \right| \leq C e^{-\theta t} \|\tilde{z}(t)\|_{H^2} \|z(t)\|_{H^2} + C \|\tilde{z}(t)\|_{H^2} \|z(t)\|_{H^2}^2. \quad (2.4.51)$$

Step 7. A bound for $\|z(t)\|_{H^2}$ in function of $\tilde{z}(t)$.

Because we have chosen $N > 2$ and because of (2.4.5), we may claim that for t large enough, the integral

$$\int_t^{+\infty} \|z(s)\|_{H^2} ds$$

is finite.

Because of Lemma 2.35 and (2.4.5), we deduce that

$$c_j(t) \rightarrow_{t \rightarrow +\infty} 0. \quad (2.4.52)$$

Knowing this, from Lemma 2.36, we deduce by integration that

$$\begin{aligned} |c_j(t)| &\leq \int_t^{+\infty} |c'_j(s)| ds \\ &\leq C \int_t^{+\infty} \|\tilde{z}(s)\|_{H^2} ds + C \int_t^{+\infty} e^{-\theta s} \|z(s)\|_{H^2} ds + \int_t^{+\infty} \|z(s)\|_{H^2}^2 ds. \end{aligned} \quad (2.4.53)$$

Knowing this and using (2.4.10), we may deduce that

$$\begin{aligned} \|z(t)\|_{H^2} &\leq C \|\tilde{z}(t)\|_{H^2} + C \int_t^{+\infty} \|\tilde{z}(s)\|_{H^2} ds + C \int_t^{+\infty} e^{-\theta s} \|z(s)\|_{H^2} ds \\ &\quad + \int_t^{+\infty} \|z(s)\|_{H^2}^2 ds \\ &\leq C \|\tilde{z}(t)\|_{H^2} + C \int_t^{+\infty} \|\tilde{z}(s)\|_{H^2} ds + C \sup_{s \geq t} \|z(s)\|_{H^2} e^{-\theta t} \\ &\quad + C \sup_{s \geq t} \|z(s)\|_{H^2} \int_t^{+\infty} \|z(s)\|_{H^2} ds, \end{aligned} \quad (2.4.54)$$

which implies, because

$$\int_t^{+\infty} \|\tilde{z}(s)\|_{H^2} ds, \quad \sup_{s \geq t} \|z(s)\|_{H^2} e^{-\theta t}, \quad \sup_{s \geq t} \|z(s)\|_{H^2} \int_t^{+\infty} \|z(s)\|_{H^2} ds \quad (2.4.55)$$

are decreasing in time, that

$$\begin{aligned} \sup_{s \geq t} \|z(s)\|_{H^2} &\leq C \sup_{s \geq t} \|\tilde{z}(s)\|_{H^2} + C \int_t^{+\infty} \|\tilde{z}(s)\|_{H^2} ds + C \sup_{s \geq t} \|z(s)\|_{H^2} e^{-\theta t} \\ &\quad + C \sup_{s \geq t} \|z(s)\|_{H^2} \int_t^{+\infty} \|z(s)\|_{H^2} ds, \end{aligned} \quad (2.4.56)$$

and because $e^{-\theta t}$ and $\int_t^{+\infty} \|z(s)\|_{H^2} ds$ may be as small as we want for t large enough (dependent on z), we may deduce that

Lemma 2.42. *There exists $C > 0$ that do not depend on z , such that for t large enough,*

$$\|z(t)\|_{H^2} \leq \sup_{s \geq t} \|z(s)\|_{H^2} \leq C \sup_{s \geq t} \|\tilde{z}(s)\|_{H^2} + C \int_t^{+\infty} \|\tilde{z}(s)\|_{H^2} ds. \quad (2.4.57)$$

Step 8. Conclusion.

By integration, from Lemmas 2.38, 2.39 and 2.41, for t large enough (depending on z), with constants C and θ that do not depend on z ,

$$\begin{aligned} \|\tilde{z}(t)\|_{H^2}^2 &\leq C \int_t^{+\infty} \frac{1}{s} \|\tilde{z}(s)\|_{H^2}^2 ds + C \int_t^{+\infty} e^{-\theta s} \|\tilde{z}(s)\|_{H^2} \|z(s)\|_{H^2} ds \\ &\quad + C \int_t^{+\infty} \|\tilde{z}(s)\|_{H^2} \|z(s)\|_{H^2}^2 ds \\ &\leq C \sup_{s \geq t} \|\tilde{z}(s)\|_{H^2} \int_t^{+\infty} \left(\frac{1}{s} \|\tilde{z}(s)\|_{H^2} + e^{-\theta s} \|z(s)\|_{H^2} + \|z(s)\|_{H^2}^2 \right) ds. \end{aligned} \quad (2.4.58)$$

Because the right-hand side of the inequality above is decreasing in time, we deduce after taking the supremum of the previous inequality and after simplification, that for t large enough,

$$\begin{aligned} \sup_{s \geq t} \|\tilde{z}(s)\|_{H^2} &\leq C \int_t^{+\infty} \frac{1}{s} \|\tilde{z}(s)\|_{H^2} ds + C \int_t^{+\infty} e^{-\theta s} \|z(s)\|_{H^2} ds \\ &\quad + C \int_t^{+\infty} \|z(s)\|_{H^2}^2 ds \\ &\leq C \int_t^{+\infty} \frac{1}{s} \|\tilde{z}(s)\|_{H^2} ds + C \sup_{s \geq t} \|z(s)\|_{H^2} e^{-\theta t} \\ &\quad + C \sup_{s \geq t} \|z(s)\|_{H^2} \int_t^{+\infty} \|z(s)\|_{H^2} ds. \end{aligned} \quad (2.4.59)$$

And using (2.4.5), the fact that $N - 1 > 1$ and the fact that $e^{-\theta t}$ is decreasing faster than $\frac{1}{t^{N-2}}$, we deduce that for t large enough,

$$\sup_{s \geq t} \|\tilde{z}(s)\|_{H^2} \leq C \int_t^{+\infty} \frac{1}{s} \|\tilde{z}(s)\|_{H^2} ds + C \frac{1}{t^{N-2}} \sup_{s \geq t} \|z(s)\|_{H^2}. \quad (2.4.60)$$

And using Lemma 2.42, we deduce that

$$\begin{aligned} \sup_{s \geq t} \|\tilde{z}(s)\|_{H^2} &\leq C \int_t^{+\infty} \frac{1}{s} \|\tilde{z}(s)\|_{H^2} ds + C \frac{1}{t^{N-2}} \sup_{s \geq t} \|\tilde{z}(s)\|_{H^2} \\ &\quad + C \frac{1}{t^{N-2}} \int_t^{+\infty} \|\tilde{z}(s)\|_{H^2} ds. \end{aligned} \quad (2.4.61)$$

And because $\frac{1}{t^{N-2}}$ can be as small as we want for t large enough, we deduce that for t large enough and for a constant $C > 0$ that do not depend on z or on N ,

$$\|\tilde{z}(t)\|_{H^2} \leq \sup_{s \geq t} \|\tilde{z}(s)\|_{H^2} \leq C \int_t^{+\infty} \frac{1}{s} \|\tilde{z}(s)\|_{H^2} ds + C \frac{1}{t^{N-2}} \int_t^{+\infty} \|\tilde{z}(s)\|_{H^2} ds. \quad (2.4.62)$$

Let us pick $T > 0$ large enough such that for $t \geq T$, the inequality (2.4.62) works (i.e. T is large enough so that every part of the preceeding proof works). From (2.4.10) and Lemma 2.35, we know that for $t \geq T$ (by taking T larger if needed),

$$\|\tilde{z}(t)\|_{H^2} \leq \frac{C}{t^{N-1}}. \quad (2.4.63)$$

This is why, the following quantity is well defined :

$$A := \sup_{t \geq T} \{t^{N-1} \|\tilde{z}(t)\|_{H^2}\}, \quad (2.4.64)$$

which means that for $t \geq T$,

$$\|\tilde{z}(t)\|_{H^2} \leq \frac{A}{t^{N-1}}. \quad (2.4.65)$$

Now, using (2.4.63) and (2.4.65), we deduce from (2.4.62) that for $t \geq T$, with $C > 0$ that do not depend on z , on N or on A ,

$$\|\tilde{z}(t)\|_{H^2} \leq \frac{CA}{N-1} \frac{1}{t^{N-1}} + \frac{CA}{N-2} \frac{1}{t^{2N-4}} \leq \frac{CA}{N-2} \frac{1}{t^{N-1}}, \quad (2.4.66)$$

if we assume that $N > 3$. Now, from (2.4.64), we deduce that there exists $T^* > T$ such that

$$(T^*)^{N-1} \|\tilde{z}(T^*)\|_{H^2} \geq \frac{A}{2}. \quad (2.4.67)$$

This is why, by evaluating (2.4.66) in $t = T^*$, we find that

$$\frac{A}{2(T^*)^{N-1}} \leq \frac{CA}{N-2} \frac{1}{(T^*)^{N-1}}, \quad (2.4.68)$$

which, if we assume that $A > 0$, after simplification yields :

$$N - 2 \leq 2C. \quad (2.4.69)$$

This means that if we assume that $N > 2C + 2$ and $N > 3$, the assumption $A > 0$ leads to a contradiction. Therefore, $A = 0$ under that assumption on N , which implies $\|\tilde{z}(t)\|_{H^2} = 0$, and from Lemma 2.42, this implies that $z \equiv 0$. This means that the condition that we have established for N , namely

$$N > \max(2C + 2, 3), \quad (2.4.70)$$

do not depend on z and allows us to deduce that under (2.4.5), we may establish that $z \equiv 0$. The Proposition 2.5 is now proved. \square

2.4.2 A solution converging to a multi-breather converges exponentially to this multi-breather, if all the velocities (except possibly one) are positive

Proposition 2.43. *Let $u(t)$ be an H^2 solution of (mKdV) on $[T, +\infty)$, for $T \in \mathbb{R}$. We assume that*

$$\|u(t) - p(t)\|_{H^2} \rightarrow_{t \rightarrow +\infty} 0, \quad (2.4.71)$$

where p is the multi-breather constructed in Section 2.2. If

$$v_2 > 0, \quad (2.4.72)$$

then there exists $\varpi > 0$, $T_0 \geq T$ and $C > 0$ such that for any $t \geq T_0$,

$$\|u(t) - p(t)\|_{H^2} \leq C e^{-\varpi t}. \quad (2.4.73)$$

Note that in the formulation of the Proposition above, we may replace p by P without changing its content (it is a consequence from (2.1.17)).

Proof. We set $v(t) := u(t) - P(t)$, such that $\|v(t)\|_{H^2} \rightarrow_{t \rightarrow +\infty} 0$.

We denote :

$$\Psi(x) := \frac{2}{\pi} \arctan(\exp(-\sqrt{\sigma}x/2)), \quad (2.4.74)$$

where $\sigma > 0$ is small enough (with precise conditions that will be mentioned throughout the proof). By direct calculations,

$$\Psi'(x) = \frac{-\sqrt{\sigma}}{2\pi \cosh(\sqrt{\sigma}x/2)}. \quad (2.4.75)$$

Thus,

$$|\Psi'(x)| \leq C \exp(-\sqrt{\sigma}|x|/2). \quad (2.4.76)$$

We have the following properties : $\lim_{+\infty} \Psi = 0$, $\lim_{-\infty} \Psi = 1$, for all $x \in \mathbb{R}$ $\Psi(-x) = 1 - \Psi(x)$, $\Psi'(x) < 0$, $|\Psi''(x)| \leq \frac{\sqrt{\sigma}}{2} |\Psi'(x)|$, $|\Psi'''(x)| \leq \frac{\sqrt{\sigma}}{2} |\Psi''(x)|$, $|\Psi'(x)| \leq \frac{\sqrt{\sigma}}{2} \Psi$ and $|\Psi'(x)| \leq \frac{\sqrt{\sigma}}{2} (1 - \Psi)$.

For $j = 3, \dots, J$, let m_j be such that

$$m_j := \frac{v_{j-1} + v_j}{2}. \quad (2.4.77)$$

If $v_1 \geq 0$, we may also set :

$$m_2 := \frac{v_1 + v_2}{2}, \quad (2.4.78)$$

and if $v_1 < 0$, we set :

$$m_2 := \frac{v_2}{2}, \quad (2.4.79)$$

and so, for any $j = 2, \dots, J$, we have that

$$v_J > m_j > 0, \quad (2.4.80)$$

and we may consider m_j , $j = 2, \dots, J$, as part of problem data from now on. Let us denote $\tau_0 > 0$ the minimal distance between a v_j and a m_j .

From this, we define for $j = 2, \dots, J$,

$$\Phi_j(t, x) := \Psi(x - m_j t). \quad (2.4.81)$$

We may extend this definition to $j = 1$ and $j = J + 1$ in the following way : $\Phi_1 := 0$ and $\Phi_{J+1} := 1$. (2.4.80) allows us to claim that, for any j , Φ_j moves to the right. Thus, the function that allows us to study properties around each object P_j (for $j = 1, \dots, J$) is $\chi_j := \Phi_{j+1} - \Phi_j$.

The goal is to prove that, for t large enough,

$$\|v(t)\|_{H^2} \leq C e^{-\varpi t}, \quad (2.4.82)$$

where $\varpi > 0$ is a constant to be deduced from the constants of the problem. Proposition 2.43 follows from this, because of Theorem 2.2.

Let $\varpi > 0$ to be deduced from the constants of the problem with respect to the needs of the following proof.

We will prove (2.4.82) by induction. Let us denote \mathcal{P}_j the following assertion : $\int (v^2 + v_x^2 + v_{xx}^2) \Phi_j \leq C e^{-2\varpi t}$ for t large enough. We will prove, for $j = 2, \dots, J + 1$, that we have \mathcal{P}_j , knowing that we have \mathcal{P}_{j-1} . Note that the assertion \mathcal{P}_1 is obviously true. This implies the desired inequality. (Note that it is OK if ϖ becomes smaller after a step of this induction, as long as it stays positive.)

Let us write the j -th step of our reasoning by induction (where $j \in \{2, \dots, J+1\}$). Thus, j is fixed in the rest of the proof. We assume that

$$\int (v^2 + v_x^2 + v_{xx}^2) \Phi_{j-1} \leq C e^{-2\varpi t}, \quad (2.4.83)$$

for t large enough.

We divide our proof in several steps.

Step 1. Almost-growth of localized conservation laws.

We define quantities that are similar to quantities defined in Section 2.2.2. We note that we localize around the first $j-1$ objects, not only around the $(j-1)$ -th object. Notations defined in Section 2.2.2 should not be considered in the following proof and should be replaced by notations we define here :

$$M_j(t) := \frac{1}{2} \int u^2(t) \Phi_j(t), \quad (2.4.84)$$

$$E_j(t) := \int \left[\frac{1}{2} u_x^2 - \frac{1}{4} u^4 \right] \Phi_j(t), \quad (2.4.85)$$

$$F_j(t) := \int \left[\frac{1}{2} u_{xx}^2 - \frac{5}{2} u^2 u_x^2 + \frac{1}{4} u^6 \right] \Phi_j(t). \quad (2.4.86)$$

Lemma 2.44. *Let $\omega_2, \omega_6 > 0$, as small as desired. There exists $T_1 \geq T$ and $C > 0$ such that for $t \geq T_1$,*

$$\sum_{i=1}^{j-1} M[P_i] - M_j(t) \geq -C e^{-2\varpi t}, \quad (2.4.87)$$

$$\sum_{i=1}^{j-1} (E[P_i] + \omega_2 M[P_i]) - (E_j(t) + \omega_2 M_j(t)) \geq -C e^{-2\varpi t}, \quad (2.4.88)$$

$$\sum_{i=1}^{j-1} (F[P_i] + \omega_6 M[P_i]) - (F_j(t) + \omega_6 M_j(t)) \geq -C e^{-2\varpi t}. \quad (2.4.89)$$

Proof. We will use the results of the computations made at the bottom of page 1115 and at the bottom of page 1116 of [71], as well as in Section 2.5.5 (Appendix) to claim the three following facts :

$$\frac{d}{dt} \frac{1}{2} \int u^2 f = \int \left(-\frac{3}{2} u_x^2 + \frac{3}{4} u^4 \right) f' + \frac{1}{2} \int u^2 f''', \quad (2.4.90)$$

$$\frac{d}{dt} \int \left[\frac{1}{2} u_x^2 - \frac{1}{4} u^4 \right] f = \int \left[-\frac{1}{2} (u_{xx} + u^3)^2 - u_{xx}^2 + 3u_x^2 u^2 \right] f' + \frac{1}{2} \int u_x^2 f''', \quad (2.4.91)$$

$$\begin{aligned} & \frac{d}{dt} \int \left(\frac{1}{2} u_{xx}^2 - \frac{5}{2} u^2 u_x^2 + \frac{1}{4} u^6 \right) f \\ &= \int \left(-\frac{3}{2} u_{xxx}^2 + 9u_{xx}^2 u^2 + 15u_x^2 u u_{xx} + \frac{9}{16} u^8 + \frac{1}{4} u_x^4 + \frac{3}{2} u_{xx} u^5 \right. \\ & \quad \left. - \frac{45}{4} u^4 u_x^2 \right) f' + 5 \int u^2 u_x u_{xx} f'' + \frac{1}{2} \int u_{xx}^2 f'''. \end{aligned} \quad (2.4.92)$$

where f is a C^3 function that does not depend on time.

For the mass :

If $j \leq J$,

$$2\frac{d}{dt}M_j(t) = - \int \left(3u_x^2 + m_j u^2 - \frac{3}{2}u^4\right) \Phi_{jx}(t) + \int u^2 \Phi_{jxxx}(t). \quad (2.4.93)$$

We recall that

$$|\Phi_{jxx}| \leq \frac{\sqrt{\sigma}}{2} |\Phi_{jx}|, \quad |\Phi_{jxxx}| \leq \frac{\sigma}{4} |\Phi_{jx}|, \quad \Phi_{jx} \leq 0, \quad (2.4.94)$$

where we can choose σ as small as desired. For this proof, we would like to ask for σ :

$$0 < \sigma \leq m_2 \leq m_j. \quad (2.4.95)$$

Thus,

$$2\frac{d}{dt}M_j(t) \geq \int \left(3u_x^2 + \frac{3\sigma}{4}u^2 - \frac{3}{2}u^4\right) |\Phi_{jx}(t)|. \quad (2.4.96)$$

By Corollary 2.13, for $r > 0$, if t, x satisfy $v_{j-1}t + r < x < v_j t - r$, then

$$\begin{aligned} |u(t, x)| &\leq |P(t, x)| + \|v(t)\|_{L^\infty} \\ &\leq Ce^{-\beta r} + C\|v(t)\|_{H^2}, \end{aligned} \quad (2.4.97)$$

the same could be said for u_x .

We can thus deduce that for r large enough and for T_1 large enough, for $x \in (v_{j-1}t + r, v_j t - r)$, we can obtain that $|u|$ is bounded by any fixed constant, that can be taken as small as desired. Here, we will use the latter to bound $\frac{3}{2}u^2$ by $\frac{\sigma}{4}$.

For $t \geq T_1$ and $x \leq v_{j-1}t + r$ or $x \geq v_j t - r$, we have $|x - m_j t| \leq \tau_0 t - r$, and therefore for such t, x :

$$\begin{aligned} |\Phi_{jx}(t, x)| &\leq C \exp(-\sqrt{\sigma}|x - m_j t|/2) \\ &\leq C \exp(-\sqrt{\sigma}\tau_0 t/2) \exp(\sqrt{\sigma}r/2). \end{aligned} \quad (2.4.98)$$

Because $\int u^4$ is bounded by a constant for any time and $\exp(\sqrt{\sigma}r/2)$ is a fixed constant (r is already chosen), we have, for $t \geq T_1$,

$$\frac{d}{dt}M_j(t) \geq \int \left(\frac{3}{2}u_x^2 + \frac{\sigma}{4}u^2\right) |\Phi_{jx}(t)| - Ce^{-2\varpi t} \geq -Ce^{-2\varpi t}, \quad (2.4.99)$$

where ϖ is chosen as a suitable function of σ and τ_0 .

By integration, we deduce that for any $t_1 \geq t$, with a constant $C > 0$ that does not depend on t_1 , we have :

$$M_j(t_1) - M_j(t) \geq -Ce^{-2\varpi t}. \quad (2.4.100)$$

We note that this conclusion is immediate when $j = J + 1$, because we have exactly the conserved quantity.

We have that

$$\begin{aligned} &\left| \sum_{i=1}^{j-1} M[P_i] - M_j(t_1) \right| \\ &\leq \left| \sum_{i=1}^{j-1} \frac{1}{2} \int P_i^2 - \frac{1}{2} \int P^2 \Phi_j(t_1) \right| + \frac{1}{2} \left| \int P^2 \Phi_j(t_1) - \int u^2 \Phi_j(t_1) \right| \\ &\leq Ce^{-\kappa(\beta, \sigma, \tau_0)t_1} + \frac{1}{2} \int |P^2 - u^2| \Phi_j(t_1) \\ &\leq Ce^{-\kappa(\beta, \sigma, \tau_0)t_1} + C \int |P^2 - u^2| \rightarrow_{t_1 \rightarrow +\infty} 0. \end{aligned} \quad (2.4.101)$$

This means that when we take the limit of (2.4.100) when $t_1 \rightarrow +\infty$, we obtain, for $t \geq T_1$,

$$\sum_{i=1}^{j-1} M[P_i] - M_j(t) \geq -Ce^{-2\varpi t}, \quad (2.4.102)$$

which is exactly what we wished to prove.

For the energy :

If $j \leq J$,

$$\begin{aligned} 2\frac{d}{dt}E_j(t) &= \int \left[-(u_{xx} + u^3)^2 - 2u_{xx}^2 + 6u_x^2u^2 \right] \Phi_{jx}(t) \\ &\quad - m_j \int \left(u_x^2 - \frac{1}{2}u^4 \right) \Phi_{jx}(t) + \frac{1}{2} \int u_x^2 \Phi_{jxxx}(t) \\ &\geq \int \left[(u_{xx} + u^3)^2 + 2u_{xx}^2 - 6u_x^2u^2 + \frac{3\sigma}{4}u_x^2 - \frac{m_j}{2}u^4 \right] |\Phi_{jx}(t)|. \end{aligned} \quad (2.4.103)$$

We can do the same reasoning as for the mass to bound above $\frac{m_j}{2}u^2$ by ω_1 , a constant that we can choose as small as desired, and to bound above $6u^2$ by $\frac{\sigma}{4}$. We obtain that if T_1 is large enough (dependently on the chosen constant ω_1),

$$2\frac{d}{dt}E_j(t) \geq \int \left[(u_{xx} + u^3)^2 + 2u_{xx}^2 + \frac{\sigma}{2}u_x^2 - \omega_1u^2 \right] |\Phi_{jx}(t)| - Ce^{-2\varpi t}. \quad (2.4.104)$$

By using what we have performed for the mass, we have that if we take ω_1 small enough with respect to $\frac{\omega_2\sigma}{2}$,

$$\frac{d}{dt}(E_j + \omega_2M_j)(t) \geq -Ce^{-2\varpi t}. \quad (2.4.105)$$

Then, by integration and similarly as for the mass, we obtain the desired conclusion that is true for any j .

For F :

If $j \leq J$,

$$\begin{aligned} 2\frac{d}{dt}F_j(t) &= \int \left(-3u_{xxx}^2 + 18u_{xx}^2u^2 + 30u_x^2uu_{xx} + \frac{9}{8}u^8 + \frac{1}{2}u_x^4 + 3u_{xx}u^5 \right. \\ &\quad \left. - \frac{45}{2}u^4u_x^2 \right) \Phi_{jx}(t) - m_j \int \left(u_{xx}^2 - 5u^2u_x^2 + \frac{1}{2}u^6 \right) \Phi_{jx}(t) \\ &\quad + 10 \int u^2u_{xx}u_x \Phi_{jxx}(t) + \int u_{xx}^2 \Phi_{jxxx}(t) \\ &\geq \int \left(3u_{xxx}^2 + \frac{45}{2}u^4u_x^2 - 18u_{xx}^2u^2 - 15u_x^2u^2 - 15u_x^2u_{xx}^2 - \frac{9}{8}u^8 \right. \\ &\quad \left. - \frac{1}{2}u_x^4 - \frac{3}{2}u_{xx}^2u^4 - \frac{3}{2}u^6 \right) |\Phi_{jx}(t)| \\ &\quad + \int \left(\sigma u_{xx}^2 + \frac{\sigma}{2}u^6 - 5m_ju^2u_x^2 \right) |\Phi_{jx}(t)| - 5 \int u^2u_x^2 |\Phi_{jxx}(t)| \\ &\quad - 5 \int u^2u_{xx}^2 |\Phi_{jxx}(t)| - \int u_{xx}^2 |\Phi_{jxxx}(t)|. \end{aligned} \quad (2.4.106)$$

By the same reasoning as for the energy and the mass, if we set $\omega_3, \omega_4, \omega_5 > 0$ constants that we can take as small as desired, and if T_1 is large enough dependently on these constants, for $t \geq T_1$, we have that

$$\begin{aligned} 2\frac{d}{dt}F_j(t) &\geq \int \left(3u_{xxx}^2 + \frac{45}{2}u^4u_x^2 + \frac{3\sigma}{4}u_{xx}^2 + \frac{\sigma}{2}u^6 - \omega_3u_{xx}^2 - \omega_4u_x^2 \right. \\ &\quad \left. - \omega_5u^2 \right) |\Phi_{jx}(t)| - Ce^{-2\varpi t}. \end{aligned} \quad (2.4.107)$$

By using what we have carried out for the mass, we have that if we take $\omega_3, \omega_4, \omega_5$ small enough (with respect to ω_6),

$$\frac{d}{dt}(F_j + \omega_6 M_j)(t) \geq -C e^{-2\varpi t}. \quad (2.4.108)$$

Then, by integration and similarly as before, we obtain that the desired conclusion true for any j . \square

Remark 2.45. If $j = J + 1$, we have that

$$\sum_{i=1}^J M[P_i] - M_{J+1}(t) = 0, \quad (2.4.109)$$

$$\sum_{i=1}^J E[P_i] - E_{J+1}(t) = 0, \quad (2.4.110)$$

$$\sum_{i=1}^J F[P_i] - F_{J+1}(t) = 0. \quad (2.4.111)$$

Step 2. Almost-growth of the Lyapunov functional.

Let $0 < \nu < 1$ be close enough to 1. We set, for $j = 2, \dots, J + 1$, a functional that is close to the Lyapunov functional that we will consider in the following steps of the proof :

$$\mathcal{F}_j(t) := F_j(t) + 2(b_{j-1}^2 - a_{j-1}^2)E_j(t) + \nu(a_{j-1}^2 + b_{j-1}^2)^2 M_j(t). \quad (2.4.112)$$

The following lemma states the almost-growth of \mathcal{F}_j :

Lemma 2.46. *There exists $0 < \nu < 1$ close enough to 1 such that there exists $T_1 \geq T$ and $C > 0$ such that for any $t \geq T_1$,*

$$\mathcal{F}_j(t) - F[P_{j-1}] - 2(b_{j-1}^2 - a_{j-1}^2)E[P_{j-1}] - \nu(a_{j-1}^2 + b_{j-1}^2)^2 M[P_{j-1}] \leq C e^{-2\varpi t}. \quad (2.4.113)$$

Proof. If $b_{j-1}^2 - a_{j-1}^2 \geq 0$, then the Lemma 2.46 is an immediate consequence of Lemma 2.44.

For the rest of the proof, we consider the case $b_{j-1}^2 - a_{j-1}^2 < 0$ (that can be relevant for $j = 2$). As in the proof of Lemma 2.44, we may choose $r > 0$ large enough in order to be able to bound the terms for which the degree of u is larger than 2 with a coefficient that is as small as we want that will be denoted $\omega > 0$.

We take σ small enough with respect to ω . We may deduce from Lemma 2.44 and its proof that

$$\begin{aligned} \frac{d}{dt}F_j(t) &\geq -C e^{-2\varpi t} + \frac{3}{2} \int u_{xxx}^2 |\Phi_{jx}| + \frac{m_j}{2} \int u_{xx}^2 |\Phi_{jx}| \\ &\quad - \omega \int u_{xx}^2 |\Phi_{jx}| - \omega \int u_x^2 |\Phi_{jx}| - \omega \int u^2 |\Phi_{jx}|, \end{aligned} \quad (2.4.114)$$

$$\begin{aligned} -\frac{d}{dt}E_j(t) &\geq -C e^{-2\varpi t} - \frac{3}{2} \int u_{xx}^2 |\Phi_{jx}| - \frac{m_j}{2} \int u_x^2 |\Phi_{jx}| \\ &\quad - \omega \int u_x^2 |\Phi_{jx}| - \omega \int u^2 |\Phi_{jx}|, \end{aligned} \quad (2.4.115)$$

$$\begin{aligned} \frac{d}{dt}M_j(t) &\geq -C e^{-2\varpi t} + \frac{3}{2} \int u_x^2 |\Phi_{jx}| + \frac{m_j}{2} \int u^2 |\Phi_{jx}| \\ &\quad - \omega \int u^2 |\Phi_{jx}|. \end{aligned} \quad (2.4.116)$$

So,

$$\begin{aligned}
\frac{d}{dt} \mathcal{F}_j(t) &= -Ce^{-2\varpi t} + \frac{3}{2} \int u_{xxx}^2 |\Phi_{jx}| \\
&\quad + \left(3(b_{j-1}^2 - a_{j-1}^2) + \frac{m_j}{2} - \omega \right) \int u_{xx}^2 |\Phi_{jx}| \\
&\quad + \left(\frac{3}{2} \nu (a_{j-1}^2 + b_{j-1}^2)^2 + m_j (b_{j-1}^2 - a_{j-1}^2) - \omega \right) \int u_x^2 |\Phi_{jx}| \\
&\quad + \left(\frac{m_j}{2} \nu (a_{j-1}^2 + b_{j-1}^2)^2 - \omega \right) \int u^2 |\Phi_{jx}|.
\end{aligned} \tag{2.4.117}$$

We may choose m_j small enough (with respect to ν) so that

$$\frac{3}{2} \nu (a_{j-1}^2 + b_{j-1}^2)^2 + m_j (b_{j-1}^2 - a_{j-1}^2) > \frac{3}{2} \nu' (a_{j-1}^2 + b_{j-1}^2)^2, \tag{2.4.118}$$

where $0 < \nu' < \nu$ that is as close to 1 as we need.

We choose ω small enough with respect to the previous choice so that

$$\frac{3}{2} \nu (a_{j-1}^2 + b_{j-1}^2)^2 + m_j (b_{j-1}^2 - a_{j-1}^2) - \omega \geq \frac{3}{2} \nu' (a_{j-1}^2 + b_{j-1}^2)^2. \tag{2.4.119}$$

We may choose ω even smaller (with respect to m_j and ν) so that

$$\frac{m_j}{2} \nu (a_{j-1}^2 + b_{j-1}^2)^2 - \omega \geq 0, \tag{2.4.120}$$

and

$$3(b_{j-1}^2 - a_{j-1}^2) + \frac{m_j}{2} - \omega \geq 3(b_{j-1}^2 - a_{j-1}^2). \tag{2.4.121}$$

In the case when with the chosen values of m_j , ν and ω , $3(b_{j-1}^2 - a_{j-1}^2) + \frac{m_j}{2} - \omega$ is positive, the desired conclusion is straightforward by integration. From now on, we place ourselves in the case when

$$3(b_{j-1}^2 - a_{j-1}^2) + \frac{m_j}{2} - \omega < 0. \tag{2.4.122}$$

Now, we want to bound above $\int u_{xx}^2 |\Phi_{jx}|$. By integration by parts,

$$\begin{aligned}
\int u_{xx}^2 |\Phi_{jx}| &= - \int u_x u_{xxx} |\Phi_{jx}| - \int u_x u_{xx} |\Phi_{jxx}| \\
&\leq \sqrt{\int u_x^2 |\Phi_{jx}| \int u_{xxx}^2 |\Phi_{jx}|} + \frac{\sqrt{\sigma}}{2} \sqrt{\int u_x^2 |\Phi_{jx}| \int u_{xx}^2 |\Phi_{jx}|},
\end{aligned} \tag{2.4.123}$$

because $|\Phi_{jxx}| \leq \frac{\sqrt{\sigma}}{2} |\Phi_{jx}|$.

We denote :

$$X := \sqrt{\int u_{xx}^2 |\Phi_{jx}|}, \tag{2.4.124}$$

and

$$A := \sqrt{\int u_x^2 |\Phi_{jx}| \int u_{xxx}^2 |\Phi_{jx}|}. \tag{2.4.125}$$

So, we have that

$$X^2 \leq A + \varepsilon X, \quad (2.4.126)$$

where

$$\varepsilon := \frac{\sqrt{\sigma}}{2} \sqrt{\int u_x^2 |\Phi_{jx}|} \leq \frac{\sqrt{\sigma}}{2} \|u\|_{H^1}, \quad (2.4.127)$$

which can be as small as we want if we take σ small enough (for a given solution u).

We deduce that

$$X \leq \frac{\varepsilon + \sqrt{\varepsilon^2 + 4A}}{2} \leq \varepsilon + \sqrt{A}. \quad (2.4.128)$$

Thus,

$$\begin{aligned} \int u_{xx}^2 |\Phi_{jx}| &\leq \left(\frac{\sigma}{4} \sqrt{\int u_x^2 |\Phi_{jx}|} + \sqrt{\sigma} \left(\int u_x^2 |\Phi_{jx}| \int u_{xxx}^2 |\Phi_{jx}| \right)^{\frac{1}{4}} \right. \\ &\quad \left. + \sqrt{\int u_{xxx}^2 |\Phi_{jx}|} \right) \sqrt{\int u_x^2 |\Phi_{jx}|}. \end{aligned} \quad (2.4.129)$$

So,

$$\begin{aligned} \left(3(b_{j-1}^2 - a_{j-1}^2) + \frac{m_j}{2} - \omega \right) \int u_{xx}^2 |\Phi_{jx}| &\geq 3(b_{j-1}^2 - a_{j-1}^2) \sqrt{\int u_x^2 |\Phi_{jx}| \int u_{xxx}^2 |\Phi_{jx}|} \\ &\quad + 3(b_{j-1}^2 - a_{j-1}^2) \sqrt{\sigma} \left(\int u_x^2 |\Phi_{jx}| \right)^{\frac{3}{4}} \left(\int u_{xxx}^2 |\Phi_{jx}| \right)^{\frac{1}{4}} \\ &\quad + 3(b_{j-1}^2 - a_{j-1}^2) \frac{\sigma}{4} \int u_x^2 |\Phi_{jx}|. \end{aligned} \quad (2.4.130)$$

On the other hand, we have for a choice of $\nu_1, \nu_2, \nu_3 > 0$ such that $\nu_1 + \nu_2 + \nu_3 = \nu'$ (where ν_1 should be near to ν' and ν_2 and ν_3 should be near to 0) and a choice of $\epsilon_1, \epsilon_2 > 0$ such that $\epsilon_1 + \epsilon_2 = 1$ (where ϵ_1 should be near to 1 and ϵ_2 should be near to 0) that will be made with respect to the needs of the proof, that

$$\begin{aligned} &\frac{3}{2} \int u_{xxx}^2 |\Phi_{jx}| + \left(\frac{3}{2} \nu' (a_{j-1}^2 + b_{j-1}^2)^2 \right) \int u_x^2 |\Phi_{jx}| \\ &\geq \frac{3}{2} \epsilon_1 \int u_{xxx}^2 |\Phi_{jx}| + \left(\frac{3}{2} \nu_1 (a_{j-1}^2 + b_{j-1}^2)^2 \right) \int u_x^2 |\Phi_{jx}| \\ &\quad + \frac{3}{2} \epsilon_2 \int u_{xxx}^2 |\Phi_{jx}| + \left(\frac{3}{2} \nu_2 (a_{j-1}^2 + b_{j-1}^2)^2 \right) \int u_x^2 |\Phi_{jx}| \\ &\quad + \left(\frac{3}{2} \nu_3 (a_{j-1}^2 + b_{j-1}^2)^2 \right) \int u_x^2 |\Phi_{jx}| \\ &\geq 2 \sqrt{\frac{3}{2} \epsilon_1 \left(\frac{3}{2} \nu_1 (a_{j-1}^2 + b_{j-1}^2)^2 \right)} \sqrt{\int u_x^2 |\Phi_{jx}| \int u_{xxx}^2 |\Phi_{jx}|} \\ &\quad + 4 \left(\frac{3}{2} \epsilon_2 \int u_{xxx}^2 |\Phi_{jx}| \right)^{\frac{1}{4}} \left(\left(\frac{1}{2} \nu_2 (a_{j-1}^2 + b_{j-1}^2)^2 \right) \int u_x^2 |\Phi_{jx}| \right)^{\frac{3}{4}} \\ &\quad + \left(\frac{3}{2} \nu_3 (a_{j-1}^2 + b_{j-1}^2)^2 \right) \int u_x^2 |\Phi_{jx}| \end{aligned} \quad (2.4.131)$$

$$\begin{aligned}
&\geq 3\sqrt{\epsilon_1\nu_1}(a_{j-1}^2 + b_{j-1}^2)\sqrt{\int u_x^2|\Phi_{jx}|\int u_{xxx}^2|\Phi_{jx}|} \\
&+ 2 \cdot 3^{\frac{1}{4}}\epsilon_2^{\frac{1}{4}}\nu_2^{\frac{3}{4}}(a_{j-1}^2 + b_{j-1}^2)^{\frac{3}{2}}\left(\int u_x^2|\Phi_{jx}|\right)^{\frac{3}{4}}\left(\int u_{xxx}^2|\Phi_{jx}|\right)^{\frac{1}{4}} \\
&+ \frac{3}{2}\nu_3(a_{j-1}^2 + b_{j-1}^2)^2\int u_x^2|\Phi_{jx}|.
\end{aligned}$$

This is why, we deduce that

$$\begin{aligned}
\frac{d}{dt}\mathcal{F}_j(t) &\geq -Ce^{-2\varpi t} \\
&+ \left(3(b_{j-1}^2 - a_{j-1}^2) + 3\sqrt{\epsilon_1\nu_1}(a_{j-1}^2 + b_{j-1}^2)\right)\sqrt{\int u_x^2|\Phi_{jx}|\int u_{xxx}^2|\Phi_{jx}|} \\
&+ \left(3(b_{j-1}^2 - a_{j-1}^2)\sqrt{\sigma} + 2 \cdot 3^{\frac{1}{4}}\epsilon_2^{\frac{1}{4}}\nu_2^{\frac{3}{4}}(a_{j-1}^2 + b_{j-1}^2)^{\frac{3}{2}}\right)\left(\int u_x^2|\Phi_{jx}|\right)^{\frac{3}{4}}\left(\int u_{xxx}^2|\Phi_{jx}|\right)^{\frac{1}{4}} \\
&+ \left(3(b_{j-1}^2 - a_{j-1}^2)\frac{\sigma}{4} + \frac{3}{2}\nu_3(a_{j-1}^2 + b_{j-1}^2)^2\right)\int u_x^2|\Phi_{jx}|.
\end{aligned} \tag{2.4.132}$$

To finish, we need to state a condition (on $\nu_1, \nu_2, \nu_3, \epsilon_1, \epsilon_2$) that will ensure that

$$E_1 := 3(b_{j-1}^2 - a_{j-1}^2) + 3\sqrt{\epsilon_1\nu_1}(a_{j-1}^2 + b_{j-1}^2) \geq 0, \tag{2.4.133}$$

$$E_2 := 3(b_{j-1}^2 - a_{j-1}^2)\sqrt{\sigma} + 2 \cdot 3^{\frac{1}{4}}\epsilon_2^{\frac{1}{4}}\nu_2^{\frac{3}{4}}(a_{j-1}^2 + b_{j-1}^2)^{\frac{3}{2}} \geq 0, \tag{2.4.134}$$

and

$$E_3 := 3(b_{j-1}^2 - a_{j-1}^2)\frac{\sigma}{4} + \frac{3}{2}\nu_3(a_{j-1}^2 + b_{j-1}^2)^2 \geq 0. \tag{2.4.135}$$

We can choose ϵ_1 and ν_1 so that $\sqrt{\epsilon_1\nu_1} \geq \frac{1}{2}$. So, we have that

$$\begin{aligned}
\frac{E_1}{3} &= (\sqrt{\epsilon_1\nu_1} + 1)b_{j-1}^2 + (\sqrt{\epsilon_1\nu_1} - 1)a_{j-1}^2 \\
&\geq \frac{3}{2}b_{j-1}^2 + (\sqrt{\epsilon_1\nu_1} - 1)a_{j-1}^2.
\end{aligned} \tag{2.4.136}$$

By choosing ϵ_1 and ν_1 wisely (note that we can choose ν' and ν with respect to the condition on ν_1 that we need), we may ensure that $|\sqrt{\epsilon_1\nu_1} - 1|$ is as small as we want with respect to the ratio $\frac{b_{j-1}}{a_{j-1}}$, which will ensure that $E_1 \geq 0$.

In order to have $E_2 > 0$ and $E_3 > 0$, the only requirement is that $\epsilon_2, \nu_2, \nu_3 > 0$. This being set, it is enough to take σ small enough in order to obtain the desired conclusion.

Thus,

$$\frac{d}{dt}\mathcal{F}_j(t) \geq -Ce^{-2\varpi t}. \tag{2.4.137}$$

We obtain the desired conclusion by integration. \square

Step 3. Modulation.

Notations that were defined in Section 2.2.3 should not be taken into consideration in the following proof and should be replaced by notations we define here.

Lemma 2.47. *There exists $C > 0$, $T_2 \geq T$, such that there exist unique C^1 functions $y_1, y_2 : [T_2, +\infty) \rightarrow \mathbb{R}$ such that if we set :*

$$w(t, x) := u - \tilde{P}, \quad (2.4.138)$$

where

$$\tilde{P}(t, x) := \sum_{i=1}^J \tilde{P}_i(t, x), \quad (2.4.139)$$

for $i \neq j-1$,

$$\tilde{P}_i(t, x) := P_i(t, x), \quad (2.4.140)$$

and either,

$$\widetilde{P_{j-1}}(t, x) := \kappa_l Q_{c_l+y_1(t)}(x - x_{0,l}^0 + y_2(t) - c_l t), \quad \text{if } P_{j-1} = R_l \text{ is a soliton}, \quad (2.4.141)$$

or,

$$\widetilde{P_{j-1}}(t, x) := B_{\alpha_k, \beta_k}(t, x; x_{1,k} + y_1(t), x_{2,k} + y_2(t)), \quad \text{if } P_{j-1} = B_k \text{ is a breather}, \quad (2.4.142)$$

then, $w(t)$ satisfies, for any $t \in [T_2, +\infty)$, either,

$$\int \widetilde{P_{j-1_1}}(t) w(t) = \int \widetilde{P_{j-1_2}}(t) w(t) = 0, \quad \text{if } P_{j-1} \text{ is a breather}, \quad (2.4.143)$$

or,

$$\int \widetilde{P_{j-1}}(t) w(t) = \int \widetilde{P_{j-1_x}}(t) w(t) = 0, \quad \text{if } P_{j-1} \text{ is a soliton}, \quad (2.4.144)$$

where in the case when P_{j-1} is a breather we denote :

$$\widetilde{P_{j-1_1}}(t, x) := \partial_{x_1} \widetilde{P_{j-1}}, \quad \widetilde{P_{j-1_2}}(t, x) := \partial_{x_2} \widetilde{P_{j-1}}. \quad (2.4.145)$$

Moreover, for any $t \in [T_2, +\infty)$,

$$\|w(t)\|_{H^2} + |y_1(t)| + |y_2(t)| \leq C \|v(t)\|_{H^2}, \quad (2.4.146)$$

and, if ϖ is small enough,

$$|y_1'(t)| + |y_2'(t)| \leq C \left(\int w(t)^2 \Phi_j \right)^{1/2} + C e^{-\varpi t}. \quad (2.4.147)$$

Proof. The proof that has to be performed is similar to the proof of Lemma 2.19, which is a consequence of a quantitative version of the implicit function theorem. See [22, Section 2.2] for a precise statement. The proof of (2.4.147) is also similar : as in the proof of Lemma 2.19, we take the time derivative of $\int \widetilde{P_{j-1_1}}(t) w(t) = \int \widetilde{P_{j-1_2}}(t) w(t) = 0$. To be complete, let us perform this proof.

For $t \in [T_2, +\infty)$, let

$$F_t : L^2(\mathbb{R}) \times \mathbb{R}^2 \rightarrow \mathbb{R}^2 \quad (2.4.148)$$

be such that if $P_{j-1} = B_k$ is a breather,

$$(U, y_1, y_2) \mapsto \left(\int \partial_{x_1} B_{\alpha_k, \beta_k}(t, x; x_{1,k}^0 + y_1, x_{2,k}^0 + y_2) \epsilon dx, \int \partial_{x_2} B_{\alpha_k, \beta_k}(t, x; x_{1,k}^0 + y_1, x_{2,k}^0 + y_2) \epsilon dx \right), \quad (2.4.149)$$

where

$$\epsilon := U - P + P_{j-1} - B_{\alpha_k, \beta_k}(t, x; x_{1,k}^0 + y_1, x_{2,k}^0 + y_2), \quad (2.4.150)$$

and if $P_{j-1} = R_l$ is a soliton,

$$(U, y_1, y_2) \mapsto \left(\int \kappa_l Q_{c_l+y_1}(x - x_{0,l}^0 + y_2 - c_l t) \epsilon \, dx, \right. \\ \left. \int \partial_x \kappa_l Q_{c_l+y_1}(x - x_{0,l}^0 + y_2 - c_l t) \epsilon \, dx \right), \quad (2.4.151)$$

where

$$\epsilon := U - P + P_{j-1} - \kappa_l Q_{c_l+y_1}(x - x_{0,l}^0 + y_2 - c_l t). \quad (2.4.152)$$

We observe that F_t is a C^1 function and that $F_t(P(t), 0, 0) = 0$. Now, let us consider the matrix which gives the differential of F_t (with respect to y_1, y_2) in $(P(t), 0, 0)$.

In the case when $P_{j-1} = B_k$ is a breather, this matrix is :

$$DF_t = \begin{pmatrix} -\int (\partial_{x_1} B_k)^2 \, dx & -\int \partial_{x_1} B_k \partial_{x_2} B_k \, dx \\ -\int \partial_{x_1} B_k \partial_{x_2} B_k \, dx & -\int (\partial_{x_2} B_k)^2 \, dx \end{pmatrix}, \quad (2.4.153)$$

whose determinant is :

$$\det(DF_t) = \int (\partial_{x_1} B_k)^2 \, dx \int (\partial_{x_2} B_k)^2 \, dx - \left(\int \partial_{x_1} B_k \partial_{x_2} B_k \, dx \right)^2. \quad (2.4.154)$$

By Cauchy-Schwarz inequality and the fact that $\partial_{x_1} B_k$ and $\partial_{x_2} B_k$ are linearly independent as functions of the x variable, for any time t fixed, we see that $\det(DF_t)$ is positive. Since each member of its expression is periodic in time, then $\det(DF_t)$ is bounded below by a positive constant independent on time and translation parameters of B_k .

In the case when $P_{j-1} = R_l$ is a soliton, let us recall, denoting $y_{0,l} := x - x_{0,l}^0 + y_2 - c_l t$, that

$$\partial_{y_1} Q_{c_l+y_1}(y_{0,l}) = \frac{1}{2c_l} (Q_{c_l+y_1}(y_{0,l}) + y_{0,l} \partial_x Q_{c_l+y_1}(y_{0,l})). \quad (2.4.155)$$

Thus, denoting $Q_{c_l}(x - x_{0,l}^0 - c_l t)$ by Q_{c_l} and $x - x_{0,l}^0 - c_l t$ by $y_{0,l}^0$,

$$DF_t = \begin{pmatrix} -\frac{1}{2c_l} \int Q_{c_l} (Q_{c_l} + y_{0,l}^0 \partial_x Q_{c_l}) \, dx & -\int Q_{c_l} \partial_x Q_{c_l} \, dx \\ -\frac{1}{2c_l} \int \partial_x Q_{c_l} (Q_{c_l} + y_{0,l}^0 \partial_x Q_{c_l}) \, dx & -\int (\partial_x Q_{c_l})^2 \, dx \end{pmatrix}, \quad (2.4.156)$$

whose determinant is :

$$\det(DF_t) = \frac{1}{2c_l} \int Q_{c_l} (Q_{c_l} + y_{0,l}^0 \partial_x Q_{c_l}) \, dx \int (\partial_x Q_{c_l})^2 \, dx, \quad (2.4.157)$$

because $\int Q_{c_l} \partial_x Q_{c_l} \, dx = 0$. And, from the computations made to obtain (2.2.60), we have that

$$\det(DF_t) = \frac{1}{4} c_l \int q^2 \int q_x^2, \quad (2.4.158)$$

where q denotes the soliton with $c = 1$, i.e. $q = Q_1$.

This means that $\det(DF_t)$ is bounded below by a positive constant independent on time and translation parameters of R_l . Thus, in any case, DF_t is invertible.

Now, we may use the implicit function theorem. If U is close enough to $P(t)$, then there exists (y_1, y_2) such that $F_t(U, y_1, y_2) = 0$, where (y_1, y_2) depends in a regular C^1 way on U . It is possible to show that the “close enough” in the previous sentence does not depend on t ; for this, it is required to use a uniform implicit function theorem. This means that for T_2 large enough, $\|v(t)\|_{H^2}$ is small enough for $t \in [T_2, +\infty)$, thus for $t \geq T_2$, $u(t)$ is close enough to $P(t)$ in order to apply the implicit function theorem. Therefore, we have for $t \in [T_2, +\infty)$, the existence of $y_1(t)$ and $y_2(t)$. It is possible to show that these functions are C^1 in time. Basically, this comes from the fact that they are C^1 in $u(t)$ and that $u(t)$ has a similar regularity in time (see [24] for more details).

Now, we prove the inequalities (2.4.146) and (2.4.147). We can take the differential of the implicit functions with respect to $u(t)$ for $t \in [T_2, +\infty)$. For this, we differentiate the following equation with respect to $u(t)$:

$$F_t(u(t), y_1(u(t)), y_2(u(t))) = 0. \quad (2.4.159)$$

We know that the matrix that gives the differential of F_t (with respect to y_1, y_2) in

$$(u(t), y_1(u(t)), y_2(u(t)))$$

is invertible and that its inverse is bounded in time. The differential of F_t with respect to the first variable is also bounded (from its expression, F_t being linear in U). Thus, by the mean-value theorem (given $(y_1, y_2)(P(t)) = (0, 0)$) :

$$|y_1(u(t))| + |y_2(u(t))| \leq C\|u(t) - P(t)\| \leq C\|v(t)\|_{H^2}. \quad (2.4.160)$$

By applying the mean-value theorem (inequality) for Q_{c_l} or B_{α_k, β_k} with respect to y_1 and y_2 , we deduce that

$$\|P_{j-1}(t) - \widetilde{P_{j-1}}(t)\|_{H^2} \leq C(|y_1(t)| + |y_2(t)|). \quad (2.4.161)$$

Finally, by triangular inequality,

$$\begin{aligned} \|w(t)\|_{H^2} &\leq \|u(t) - P(t)\|_{H^2} + \|P(t) - \widetilde{P}(t)\|_{H^2} \\ &\leq \|u(t) - P(t)\|_{H^2} + C(|y_1(t)| + |y_2(t)|) \\ &\leq C\|v(t)\|_{H^2}. \end{aligned} \quad (2.4.162)$$

This completes the proof of (2.4.146).

For (2.4.147), we will take time derivatives of the equations (2.4.143) and (2.4.144). Firstly, we may write the PDE verified by w :

$$\begin{aligned} \partial_t w &= -w_{xxx} - \left[w \left(w^2 + 3w \sum_{i=1}^J \widetilde{P}_i + 3 \sum_{i,m=1}^J \widetilde{P}_i \widetilde{P}_m \right) \right]_x \\ &\quad - \sum_{h \neq i \text{ or } i \neq m} (\widetilde{P}_h \widetilde{P}_i \widetilde{P}_m)_x - E, \end{aligned} \quad (2.4.163)$$

where, if $P_{j-1} = B_k$ is a breather,

$$E := y_1'(t) \widetilde{B_{k1}} + y_2'(t) \widetilde{B_{k2}}, \quad (2.4.164)$$

and if $P_{j-1} = R_l$ is a soliton, denoting $y_{0,l}(t) := x - x_{0,l}^0 + y_2(t) - c_l t$,

$$E := \frac{y_1'(t)}{2(c_l + y_1(t))} (\widetilde{R_l} + y_{0,l}(t) \widetilde{R_{lx}}) + y_2'(t) \widetilde{R_{lx}}. \quad (2.4.165)$$

If $P_{j-1} = B_k$, we start by taking the time derivative of $\int \widetilde{B}_{k1} w = 0$ and perform some integrations by parts to obtain :

$$\begin{aligned}
& - \int (\widetilde{B}_k^3)_{1x} w + y'_1(t) \int \widetilde{B}_{k11} w + y'_2(t) \int \widetilde{B}_{k12} w \\
& + \int \widetilde{B}_{k1x} w \left(w^2 + 3w \sum_{i=1}^J \widetilde{P}_i + 3 \sum_{h,i=1}^J \widetilde{P}_h \widetilde{P}_i \right) - \int \widetilde{B}_{k1} \sum_{h \neq i \text{ or } g \neq h} (\widetilde{P}_h \widetilde{P}_i \widetilde{P}_g)_x \\
& = y'_1(t) \int \widetilde{B}_{k1}^2 + y'_2(t) \int \widetilde{B}_{k1} \widetilde{B}_{k2},
\end{aligned} \tag{2.4.166}$$

then, we take the time derivative of $\int \widetilde{B}_{k2} w = 0$:

$$\begin{aligned}
& - \int (\widetilde{B}_k^3)_{2x} w + y'_1(t) \int \widetilde{B}_{k12} w + y'_2(t) \int \widetilde{B}_{k22} w \\
& + \int \widetilde{B}_{k2x} w \left(w^2 + 3w \sum_{i=1}^J \widetilde{P}_i + 3 \sum_{h,i=1}^J \widetilde{P}_h \widetilde{P}_i \right) - \int \widetilde{B}_{k2} \sum_{h \neq i \text{ or } g \neq h} (\widetilde{P}_h \widetilde{P}_i \widetilde{P}_g)_x \\
& = y'_1(t) \int \widetilde{B}_{k1} \widetilde{B}_{k2} + y'_2(t) \int \widetilde{B}_{k2}^2.
\end{aligned} \tag{2.4.167}$$

If $P_{j-1} = R_l$, we start by taking the time derivative of $\int \widetilde{R}_l w = 0$ and perform some integrations by parts to obtain :

$$\begin{aligned}
& - \int (\widetilde{R}_l^3)_x w + \frac{y'_1(t)}{2c_l} \int (\widetilde{R}_l + y_{0,l}(t) \widetilde{R}_{lx}) w + y'_2(t) \int \widetilde{R}_{lx} w \\
& + \int \widetilde{R}_{lx} w \left(w^2 + 3w \sum_{i=1}^J \widetilde{P}_i + 3 \sum_{h,i=1}^J \widetilde{P}_h \widetilde{P}_i \right) - \int \widetilde{R}_l \sum_{h \neq i \text{ or } g \neq h} (\widetilde{P}_h \widetilde{P}_i \widetilde{P}_g)_x \\
& = \frac{y'_1(t)}{2(c_l + y_1(t))} \int \widetilde{R}_l (\widetilde{R}_l + y_{0,l}(t) \widetilde{R}_{lx}) + y'_2(t) \int \widetilde{R}_l \widetilde{R}_{lx},
\end{aligned} \tag{2.4.168}$$

then, we take the time derivative of $\int \widetilde{R}_{lx} w = 0$:

$$\begin{aligned}
& - \int (\widetilde{R}_l^3)_{xx} w + \frac{y'_1(t)}{2c_l} \int (\widetilde{R}_{lx} + y_{0,l}(t) \widetilde{R}_{lxx}) w + y'_2(t) \int \widetilde{R}_{lxx} w \\
& + \int \widetilde{R}_{lxx} w \left(w^2 + 3w \sum_{i=1}^J \widetilde{P}_i + 3 \sum_{h,i=1}^J \widetilde{P}_h \widetilde{P}_i \right) - \int \widetilde{R}_{lx} \sum_{h \neq i \text{ or } g \neq h} (\widetilde{P}_h \widetilde{P}_i \widetilde{P}_g)_x \\
& = \frac{y'_1(t)}{2(c_l + y_1(t))} \int \widetilde{R}_{lx} (\widetilde{R}_l + y_{0,l}(t) \widetilde{R}_{lx}) + y'_2(t) \int (\widetilde{R}_{lx})^2.
\end{aligned} \tag{2.4.169}$$

As a consequence of (2.4.146), we see that $|y_1(t)| + |y_2(t)|$ tends to 0 when $t \rightarrow +\infty$. This is why, we may use Proposition 2.21 and Corollary 2.22 here, if T_2 is large enough. So, several terms of the four equalities above are obviously bounded by $(w(t)^2 \Phi_j)^{1/2}$ or $e^{-\varpi t}$ for $\varpi > 0$, a constant chosen small enough. Using these bounds, and after several linear combinations, we obtain (2.4.147). \square

Step 4. Quadratic approximations of localized conservation laws.

Lemma 2.48. *Let $\omega > 0$ as small as we want. There exists $C > 0, T_3 \geq T$ such that the following holds for $t \geq T_3$:*

$$\left| M_j(t) - \sum_{i=1}^{j-1} M[\tilde{P}_i] - \sum_{i=1}^{j-1} \int \tilde{P}_i w - \frac{1}{2} \int w^2 \Phi_j \right| \leq C e^{-2\varpi t}, \quad (2.4.170)$$

$$\begin{aligned} & \left| E_j(t) - \sum_{i=1}^{j-1} E[\tilde{P}_i] - \sum_{i=1}^{j-1} \int [\tilde{P}_{ix} w_x - \tilde{P}_i^3 w] \right. \\ & \quad \left. - \int \left[\frac{1}{2} w_x^2 - \frac{3}{2} \tilde{P}^2 w^2 \right] \Phi_j \right| \leq C e^{-2\varpi t} + \omega \int w^2 \Phi_j, \end{aligned} \quad (2.4.171)$$

$$\begin{aligned} & \left| F_j(t) - \sum_{i=1}^{j-1} F[\tilde{P}_i] - \sum_{i=1}^{j-1} \int [\tilde{P}_{ixx} w_{xx} - 5\tilde{P}_i \tilde{P}_{ix}^2 w - 5\tilde{P}_i^2 \tilde{P}_{ix} w_x + \frac{3}{2} \tilde{P}_i^5 w] \right. \\ & \quad \left. - \int \left[\frac{1}{2} w_{xx}^2 - \frac{5}{2} w^2 \tilde{P}_x^2 - 10\tilde{P} w \tilde{P}_x w_x - \frac{5}{2} \tilde{P}^2 w_x^2 + \frac{15}{4} \tilde{P}^4 w^2 \right] \Phi_j(t) \right| \\ & \leq C e^{-2\varpi t} + \omega \int (w^2 + w_x^2) \Phi_j. \end{aligned} \quad (2.4.172)$$

Proof. For the mass :

We compute :

$$\begin{aligned} M_j(t) &= \frac{1}{2} \int (\tilde{P} + w)^2 \Phi_j \\ &= \frac{1}{2} \int \tilde{P}^2 \Phi_j + \int \tilde{P} w \Phi_j + \frac{1}{2} \int w^2 \Phi_j. \end{aligned} \quad (2.4.173)$$

As in Step 1, we can show that $\frac{1}{2} \int \tilde{P}^2 \Phi_j$ converges exponentially (we choose ϖ with respect to this exponential convergence) to $\sum_{i=1}^{j-1} M[\tilde{P}_i]$. Similarly, the difference between $\int \tilde{P} w \Phi_j$ and $\sum_{i=1}^{j-1} \int \tilde{P}_i w$ converges exponentially to 0 (the velocity of a soliton is not modified a lot by modulation, this is why it works in any cases).

For E and F , we perform similar basic computations with the only difference that there will also be terms of degree 3 or more in w . We know that $\|w(t)\|_{H^2} \rightarrow_{t \rightarrow +\infty} 0$, this is the reason why for t large enough, such terms are boundable by $\omega \int w^2 \Phi_j$ or $\omega \int w_x^2 \Phi_j$. \square

Step 5. Approximation of the Lyapunov functional.

By analogy with the existence part, we introduce the following Lyapunov functional :

$$\mathcal{H}_j(t) := F_j(t) + 2(b_{j-1}^2 - a_{j-1}^2) E_j(t) + (a_{j-1}^2 + b_{j-1}^2)^2 M_j(t). \quad (2.4.174)$$

We will use the previous steps to approximate $\mathcal{H}_j(t)$.

Lemma 2.49. *There exists $T_4 \geq T$ such that the following holds for $t \geq T_4$:*

$$\begin{aligned} \mathcal{H}_j(t) &= \sum_{i=1}^{j-1} F[\tilde{P}_i] + 2(b_{j-1}^2 - a_{j-1}^2) \sum_{i=1}^{j-1} E[\tilde{P}_i] + (a_{j-1}^2 + b_{j-1}^2)^2 \sum_{i=1}^{j-1} M[\tilde{P}_i] \\ & \quad + H_j(t) + O(e^{-2\varpi t}) + o\left(\int (w^2 + w_x^2) \Phi_j\right), \end{aligned} \quad (2.4.175)$$

where

$$\begin{aligned}
H_j(t) := & \int \left[\frac{1}{2} w_{xx}^2 - \frac{5}{2} w_x^2 \widetilde{P_{j-1}}^2 + \frac{5}{2} w^2 \widetilde{P_{j-1}}^2 + 5 w^2 \widetilde{P_{j-1}} \widetilde{P_{j-1}xx} \right. \\
& \left. + \frac{15}{4} w^2 \widetilde{P_{j-1}}^4 \right] \Phi_j(t) + (b_{j-1}^2 - a_{j-1}^2) \int \left[w_x^2 - 3 w^2 \widetilde{P_{j-1}}^2 \right] \Phi_j(t) \\
& + \frac{1}{2} (a_{j-1}^2 + b_{j-1}^2)^2 \int w^2 \Phi_j(t).
\end{aligned} \tag{2.4.176}$$

Proof. This lemma is obtained from the summation of the facts established in the previous lemma. We get rid of the linear terms in the following way, by integrations by parts :

$$\begin{aligned}
& \sum_{i=1}^{j-1} \int \left(\widetilde{P_{i}xx} w_{xx} - 5 \widetilde{P_i} \widetilde{P_{i}x}^2 w - 5 \widetilde{P_i}^2 \widetilde{P_{i}x} w_x + \frac{3}{2} \widetilde{P_i}^5 w \right) \\
& + 2(b_{j-1}^2 - a_{j-1}^2) \sum_{i=1}^{j-1} \int \left(\widetilde{P_{i}x} w_x - \widetilde{P_i}^3 w \right) + (a_{j-1}^2 + b_{j-1}^2)^2 \sum_{i=1}^{j-1} \widetilde{P_i} w \\
& = \sum_{i=1}^{j-1} \int \left(\widetilde{P_{i}xxxx} + 5 \widetilde{P_i} \widetilde{P_{i}x}^2 + 5 \int \widetilde{P_i}^2 \widetilde{P_{i}xx} + \frac{3}{2} \widetilde{P_i}^5 \right) w \\
& + 2(b_{j-1}^2 - a_{j-1}^2) \sum_{i=1}^{j-1} \int \left(- \widetilde{P_{i}xx} - \widetilde{P_i}^3 \right) w + (a_{j-1}^2 + b_{j-1}^2)^2 \sum_{i=1}^{j-1} \int \widetilde{P_i} w.
\end{aligned} \tag{2.4.177}$$

If we consider that this sum goes from $i = 1$ to $j - 2$, we see that for $1 \leq i \leq j - 2$, this sum is exponentially bounded by induction assumption (we use that for $i \leq j - 2$, a polynomial in $\widetilde{P_i}$ and its derivatives is bounded by $C\Phi_{j-1}$ and that $w = v + (P_{j-1} - \widetilde{P_{j-1}})$). It is left to consider the sum of the terms with $i = j - 1$.

For $i = j - 1$, we have nearly the elliptic equation satisfied by $\widetilde{P_{j-1}}$. It is actually exactly this equation in the case when $\widetilde{P_{j-1}}$ is a breather. When $\widetilde{P_{j-1}}$ is a soliton, its shape parameter is modified by modulation. This is why, in this case, the sum of the terms with $i = j - 1$ is equal to

$$2y_1(t) \int \left(- \widetilde{P_{j-1}xx} - \widetilde{P_{j-1}}^3 \right) w + 2b_{j-1}^2 y_1(t) \int \widetilde{P_{j-1}} w + y_1(t)^2 \int \widetilde{P_{j-1}} w, \tag{2.4.178}$$

which vanishes because of the orthogonality condition from the modulation (Lemma 2.47) and the elliptic equation satisfied by a soliton (2.1.4).

H_j is obtained as the sum of the quadratic parts of the previous lemma on which we have performed some integrations by parts, and some simplifications based on the fact that for $i \geq j$, $\widetilde{P_i}\Phi_j(t)$ is exponentially decreasing, and the fact that for $i \leq j - 2$, $\int \widetilde{P_i} w^2$ is exponentially decreasing by the induction assumption (2.4.83). Therefore, H_j corresponds to the sum of the quadratic parts of previous lemma to which we have to add $5 \int w^2 \widetilde{P_x} \Phi_{jx}$, which is bounded exponentially. \square

Step 6. Bound from above for $H_j(t)$.

From Lemma 2.44, we know that for any $t \geq T_1$,

$$M_j(t) - \sum_{i=1}^{j-1} M[P_i] \leq C e^{-2\varpi t}. \tag{2.4.179}$$

By summing this fact with the fact from the Lemma 2.46, we obtain the following inequality for any t large enough :

$$\begin{aligned} \mathcal{H}_j(t) - \sum_{i=1}^{j-1} F[P_i] - 2(b_{j-1}^2 - a_{j-1}^2) \sum_{i=1}^{j-1} E[P_i] \\ - (a_{j-1}^2 + b_{j-1}^2)^2 \sum_{i=1}^{j-1} M[P_i] \leq C e^{-2\varpi t}. \end{aligned} \quad (2.4.180)$$

From Lemma 2.49, for $t \geq T_3$,

$$\begin{aligned} H_j(t) \leq F[P_{j-1}] - F[\widetilde{P_{j-1}}] + 2(b_{j-1}^2 - a_{j-1}^2) (E[P_{j-1}] - E[\widetilde{P_{j-1}}]) \\ + (a_{j-1}^2 + b_{j-1}^2)^2 (M[P_{j-1}] - M[\widetilde{P_{j-1}}]) + C e^{-2\varpi t} + \omega \int (w^2 + w_x^2) \Phi_j. \end{aligned} \quad (2.4.181)$$

In the case if P_{j-1} is a breather, we obtain immediately that

$$H_j(t) \leq C e^{-2\varpi t} + \omega \int (w^2 + w_x^2) \Phi_j. \quad (2.4.182)$$

The case when P_{j-1} is a soliton needs more inspection. As in the existence part, we have the following relations :

$$M[\widetilde{P_{j-1}}](t) = (b_{j-1}^2 + y_1(t))^{1/2} M[q], \quad (2.4.183)$$

$$E[\widetilde{P_{j-1}}](t) = (b_{j-1}^2 + y_1(t))^{\frac{3}{2}} E[q], \quad (2.4.184)$$

$$F[\widetilde{P_{j-1}}](t) = (b_{j-1}^2 + y_1(t))^{5/2} F[q]. \quad (2.4.185)$$

We set $\mathcal{R}_{j-1}(t) := F[\widetilde{P_{j-1}}](t) + 2b_{j-1}^2 E[\widetilde{P_{j-1}}](t) + b_{j-1}^4 M[\widetilde{P_{j-1}}](t)$, and we simplify it as follows :

$$\begin{aligned} \mathcal{R}_{j-1}(t) = b_{j-1}^5 \left(1 + \frac{y_1(t)}{b_{j-1}^2}\right)^{5/2} F[q] + 2b_{j-1}^5 \left(1 + \frac{y_1(t)}{b_{j-1}^2}\right)^{\frac{3}{2}} E[q] \\ + b_{j-1}^5 \left(1 + \frac{y_1(t)}{b_{j-1}^2}\right)^{1/2} M[q]. \end{aligned} \quad (2.4.186)$$

After making a Taylor expansion as in Section 2.2.5,

$$\mathcal{R}_{j-1}(t) - F[P_{j-1}] - 2b_{j-1}^2 E[P_{j-1}] - b_{j-1}^4 M[P_{j-1}] = O(y_1(t)^3). \quad (2.4.187)$$

Therefore, if T_4 is large enough, $\|v(t)\|_{H^2}$ can be as small as we want, and for $t \geq T_4$, if P_{j-1} a soliton, we may write :

$$H_j(t) \leq C e^{-2\varpi t} + \omega \int (w^2 + w_x^2) \Phi_j + \omega y_1(t)^2. \quad (2.4.188)$$

Step 7. Coercivity.

H_j can be seen as the quadratic form associated to $\widetilde{P_{j-1}}$ and evaluated in $w\sqrt{\Phi_j}$, modulo several terms that can be bounded by $C\sqrt{\sigma} \int (w^2 + w_x^2 + w_{xx}^2) \Phi_j$ (because these terms depend on derivatives of Φ_j). Let us prove that we can apply Section 2.5.4 (Appendix) for $w\sqrt{\Phi_j}$.

More precisely, we need to prove that for $\nu > 0$ small enough (from Section 2.5.4),

$$\left| \int w \sqrt{\Phi_j} \widetilde{P_{j-1_1}} \right| + \left| \int w \sqrt{\Phi_j} \widetilde{P_{j-1_2}} \right| \leq \nu \|w \sqrt{\Phi_j}\|_{H^2}, \quad (2.4.189)$$

if P_{j-1} is a breather or that

$$\left| \int w \sqrt{\Phi_j} \widetilde{P_{j-1}} \right| + \left| \int w \sqrt{\Phi_j} \widetilde{P_{j-1_x}} \right| \leq \nu \|w \sqrt{\Phi_j}\|_{H^2}, \quad (2.4.190)$$

if P_{j-1} is a soliton. In any case, the proof is the same and let us write K at the place of $\widetilde{P_{j-1_1}}$, $\widetilde{P_{j-1_2}}$, $\widetilde{P_{j-1}}$ or $\widetilde{P_{j-1_x}}$. This means that we want to bound $\int w \sqrt{\Phi_j} K$.

From (2.4.143), (2.4.144), we see that it is enough to bound $\int w(1 - \sqrt{\Phi_j})K$ by $\nu \|w \sqrt{\Phi_j}\|_{H^2}$. The reasoning that follows works for $j \leq J$, for $j = J+1$ the result is immediate because $\Phi_{J+1} = 1$. Φ_j is a translate of Ψ , and, using the fact that when $v \rightarrow 0$, $\sqrt{1+v} = 1 + O(v)$,

$$1 - \sqrt{\Psi} = 1 - \sqrt{1 + \Psi - 1} = 1 - \sqrt{1 - \Psi(-x)} = O(\Psi(-x)), \quad (2.4.191)$$

which means that $1 - \sqrt{\Phi_j} \leq C \min(1, \exp(\sqrt{\sigma}(x - m_j t)/2))$. We may deduce now that

$$\begin{aligned} \left| \int w(1 - \sqrt{\Phi_j})K \right| &= \left| \int w \sqrt{\Phi_j} \frac{1 - \sqrt{\Phi_j}}{\sqrt{\Phi_j}} K \right| \\ &\leq \left\| \frac{1 - \sqrt{\Phi_j}}{\sqrt{\Phi_j}} K \right\|_{L^2} \|w \sqrt{\Phi_j}\|_{L^2} \\ &\leq C e^{\sqrt{\sigma}(m_j - v_{j-1})t} \|w \sqrt{\Phi_j}\|_{L^2}, \end{aligned} \quad (2.4.192)$$

if $\frac{\sqrt{\sigma}}{4} < \frac{\beta}{2}$. And so, if t is large enough, we get the bound that we want.

Thus, there exists $\mu > 0$ such that for $t \geq T_5$ (where T_5 is large enough and depends on σ),

$$\begin{aligned} \mu \|w \sqrt{\Phi_j}\|_{H^2}^2 &\leq H_j(t) + C\sqrt{\sigma} \int (w^2 + w_x^2 + w_{xx}^2) \Phi_j + \frac{1}{\mu} \left(\int \widetilde{P_{j-1}} w \sqrt{\Phi_j} \right)^2 \\ &\leq C e^{-2\varpi t} + \omega \int (w^2 + w_x^2) \Phi_j + C\sqrt{\sigma} \int (w^2 + w_x^2 + w_{xx}^2) \Phi_j \\ &\quad + \omega y_1(t)^2 + \frac{1}{\mu} \left(\int \widetilde{P_{j-1}} w \sqrt{\Phi_j} \right)^2, \end{aligned} \quad (2.4.193)$$

where the term $\frac{1}{\mu} \left(\int \widetilde{P_{j-1}} w \sqrt{\Phi_j} \right)^2$ is present only if $\widetilde{P_{j-1}}$ is a breather and the term $\omega y_1(t)^2$ is present only if $\widetilde{P_{j-1}}$ is a soliton.

For σ small enough and ω small enough, we deduce that

$$\int (w^2 + w_x^2 + w_{xx}^2) \Phi_j \leq C e^{-2\varpi t} + \omega y_1(t)^2 + C \left(\int \widetilde{P_{j-1}} w \sqrt{\Phi_j} \right)^2. \quad (2.4.194)$$

We set $T_0 := \max(T_1, T_2, T_3, T_4, T_5)$.

Step 8. Bound for $\left| \int \widetilde{P_{j-1}} w \sqrt{\Phi_j} \right|$ (to do in the case if $\widetilde{P_{j-1}}$ is a breather).

We would like to prove that $\int \widetilde{P_{j-1}w} \sqrt{\Phi_j}$ is exponentially decreasing. To do so, we would like to get rid of $\sqrt{\Phi_j}$. It is clear that $\int \widetilde{P_{j-1}w} (1 - \sqrt{\Phi_j})$ is exponentially decreasing. Thus, it is enough to prove that $\int \widetilde{P_{j-1}w}$ is exponentially decreasing.

If $i \leq j-2$, we know that $\int \widetilde{P_i w}$ is exponentially decreasing by the induction assumption (2.4.83). Thus, it is enough to prove that $\sum_{i=1}^{j-1} \int \widetilde{P_i w}$ is exponentially decreasing.

From the mass approximation of Lemma 2.48 and Lemma 2.44, we observe that, for $t \geq T_0$,

$$\begin{aligned} \sum_{i=1}^{j-1} \int \widetilde{P_i w} &= O(e^{-2\varpi t}) + M_j(t) - \sum_{i=1}^{j-1} M[P_i] - \frac{1}{2} \int w^2 \Phi_j \\ &\leq C e^{-2\varpi t} - \frac{1}{2} \int w^2 \Phi_{j+o(y_1(t)^2)} \leq C e^{-2\varpi t}. \end{aligned} \quad (2.4.195)$$

Now, we use the fact that the sum of the linear parts of our localized conservation laws is exponentially decreasing, which we have established in the proof of Lemma 2.49. Therefore, the linear terms of $F_j + 2(b_{j-1}^2 - a_{j-1}^2)E_j$ are equal to $O(e^{-2\varpi t}) - (a_{j-1}^2 + b_{j-1}^2)^2 \sum_{i=1}^{j-1} \int \widetilde{P_i w}$.

From Lemma 2.44, we observe that, for $t \geq T_0$,

$$\begin{aligned} - (1 - \nu)(a_{j-1}^2 + b_{j-1}^2)^2 \sum_{i=1}^{j-1} \int \widetilde{P_i w} &= O(e^{-2\varpi t}) + O\left(\int (w^2 + w_x^2) \Phi_j\right) \\ &+ \mathcal{F}_j(t) - \sum_{i=1}^{j-1} (F[P_i] + 2(b_{j-1}^2 - a_{j-1}^2)E[P_i] + \nu(a_{j-1}^2 + b_{j-1}^2)^2 M[P_i]). \end{aligned} \quad (2.4.196)$$

And, from Lemma 2.46, we obtain that

$$-(1 - \nu)(a_{j-1}^2 + b_{j-1}^2)^2 \sum_{i=1}^{j-1} \int \widetilde{P_i w} \leq C e^{-2\varpi t} + C \int (w^2 + w_x^2) \Phi_j. \quad (2.4.197)$$

Therefore, by taking the constant $C > 0$ larger if necessarily (in function of ν), we obtain that

$$-\sum_{i=1}^{j-1} \int \widetilde{P_i w} \leq C e^{-2\varpi t} + C \int (w^2 + w_x^2) \Phi_j. \quad (2.4.198)$$

Thus, we deduce the following bound :

$$\left| \int \widetilde{P_{j-1}w} \sqrt{\Phi_j} \right| \leq C e^{-2\varpi t} + C \int (w^2 + w_x^2) \Phi_j. \quad (2.4.199)$$

Because $\|w(t)\|_{H^2} \rightarrow_{t \rightarrow +\infty} 0$, we deduce that

$$\left(\int \widetilde{P_{j-1}w} \sqrt{\Phi_j} \right)^2 = o(e^{-2\varpi t}) + o\left(\int (w^2 + w_x^2) \Phi_j\right). \quad (2.4.200)$$

Step 9. Conclusion.

From (2.4.194) and (2.4.200), we deduce for $t \geq T_0$, that

$$\int (w^2 + w_x^2 + w_{xx}^2) \Phi_j = O(e^{-2\varpi t}) + o(y_1(t)^2) + o\left(\int (w^2 + w_x^2) \Phi_j\right). \quad (2.4.201)$$

This means that if we take T_0 large enough, we have :

$$\int (w^2 + w_x^2 + w_{xx}^2) \Phi_j = o(y_1(t)^2) + O(e^{-2\varpi t}), \quad (2.4.202)$$

where the term $o(y_1(t)^2)$ is present only if P_{j-1} is a soliton.

Before finishing the proof, we need to find a better bound for $y_1(t)$ than just a convergence to 0 given by the modulation (in the case when P_{j-1} is a soliton). For this, we study $M_j(t)$:

$$\begin{aligned} M_j(t) &= \frac{1}{2} \int u^2(t) \Phi_j(t) = \frac{1}{2} \int (\tilde{P}(t) + w(t))^2 \Phi_j(t) \\ &= \frac{1}{2} \int \tilde{P}(t)^2 \Phi_j(t) + \int \tilde{P}(t) w(t) \Phi_j(t) + \frac{1}{2} \int w(t)^2 \Phi_j(t) \\ &= \frac{1}{2} \sum_{i=1}^{j-1} \int \tilde{P}_i(t)^2 + \sum_{i=1}^{j-1} \int \tilde{P}_i(t) w(t) + O(e^{-2\varpi t}) + \frac{1}{2} \int w(t)^2 \Phi_j(t) \\ &= \frac{1}{2} \int \widetilde{P_{j-1}}(t)^2 + \int \widetilde{P_{j-1}}(t) w(t) + O(e^{-2\varpi t}) \\ &\quad + \frac{1}{2} \int w(t)^2 \Phi_j(t) + \frac{1}{2} \sum_{i=1}^{j-2} \int P_i(t)^2, \end{aligned} \quad (2.4.203)$$

by the induction assumption (2.4.83), then

$$M_j(t) = \frac{1}{2} \int \widetilde{P_{j-1}}(t)^2 + O(e^{-2\varpi t}) + \frac{1}{2} \int w(t)^2 \Phi_j(t) + \frac{1}{2} \sum_{i=1}^{j-2} \int P_i(t)^2, \quad (2.4.204)$$

by the orthogonality condition from the modulation (Lemma 2.47). Therefore,

$$\begin{aligned} M_j(t) &= (b_{j-1}^2 + y_1(t))^{1/2} M[q] + O(e^{-2\varpi t}) + \frac{1}{2} \int w(t)^2 \Phi_j(t) \\ &\quad + \frac{1}{2} \sum_{i=1}^{j-2} \int P_i(t)^2. \end{aligned} \quad (2.4.205)$$

Now, if we take $t_1 \geq t$, we obtain from (2.4.202) that

$$\begin{aligned} M_j(t_1) - M_j(t) &= \left[(b_{j-1}^2 + y_1(t_1))^{1/2} - (b_{j-1}^2 + y_1(t))^{1/2} \right] M[q] \\ &\quad + O(e^{-2\varpi t}) + o(y_1(t)^2) + o(y_1(t_1)^2). \end{aligned} \quad (2.4.206)$$

By doing a Taylor expansion of order 1, as in the existence part, we obtain :

$$(b_{j-1}^2 + y_1(t_1))^{1/2} = b_{j-1} \left(1 + \frac{1}{2} \frac{y_1(t_1)}{b_{j-1}^2} + O(y_1(t_1)^2) \right). \quad (2.4.207)$$

Therefore,

$$\begin{aligned} &(b_{j-1}^2 + y_1(t_1))^{1/2} - (b_{j-1}^2 + y_1(t))^{1/2} \\ &= \frac{1}{2b_{j-1}} (y_1(t_1) - y_1(t)) + O(y_1(t_1)^2) + O(y_1(t)^2). \end{aligned} \quad (2.4.208)$$

Now, we recall that when $t_1 \rightarrow +\infty$, we have $y_1(t_1) \rightarrow 0$. Therefore, by taking the limit of the previous formula when $t_1 \rightarrow +\infty$, we obtain :

$$b_{j-1} - (b_{j-1}^2 + y_1(t))^{1/2} = -\frac{y_1(t)}{2b_{j-1}} + O(y_1(t)^2). \quad (2.4.209)$$

Therefore, from (2.4.206), with $t_1 \rightarrow +\infty$,

$$\sum_{i=1}^{j-1} M[P_i] - M_j(t) = -\frac{y_1(t)}{2b_{j-1}} M[q] + O(e^{-2\varpi t}) + O(y_1(t)^2). \quad (2.4.210)$$

The second step is to study $E_j(t)$ (we do the same reasoning as for M_j) :

$$\begin{aligned} E_j(t) &= \int \left[\frac{1}{2} u_x^2 - \frac{1}{4} u^4 \right] \Phi_j(t) \\ &= \int \left[\frac{1}{2} \tilde{P}_x^2 - \frac{1}{4} \tilde{P}^4 \right] \Phi_j(t) + \int \left[\tilde{P}_x w_x - \tilde{P}^3 w \right] \Phi_j(t) + O\left(\int w^2 \Phi_j(t) \right), \end{aligned} \quad (2.4.211)$$

and after simplifications by Φ_j due to exponential convergences, induction assumption (2.4.83) and orthogonality conditions (Lemma 2.47),

$$\begin{aligned} E_j(t) &= E[\widetilde{P_{j-1}}(t)] + \sum_{i=1}^{j-2} E[P_i] + O(e^{-2\varpi t}) + O\left(\int w^2 \Phi_j(t) \right) \\ &= (b_{j-1}^2 + y_1(t))^{\frac{3}{2}} E[q] + \sum_{i=1}^{j-2} E[P_i] + O(e^{-2\varpi t}) + O\left(\int w^2 \Phi_j(t) \right) \\ &= (b_{j-1}^2 + y_1(t))^{\frac{3}{2}} E[q] + \sum_{i=1}^{j-2} E[P_i] + O(e^{-2\varpi t}) + o(y_1(t)^2), \end{aligned} \quad (2.4.212)$$

by (2.4.202). And then, by taking the difference for $t_1 \geq t$,

$$\begin{aligned} E_j(t_1) - E_j(t) &= \left[(b_{j-1}^2 + y_1(t_1))^{\frac{3}{2}} - (b_{j-1}^2 + y_1(t))^{\frac{3}{2}} \right] E[q] \\ &\quad + O(e^{-2\varpi t}) + o(y_1(t_1)^2) + o(y_1(t)^2). \end{aligned} \quad (2.4.213)$$

By taking a Taylor expansion of order 1, we obtain :

$$(b_{j-1}^2 + y_1(t_1))^{\frac{3}{2}} = b_{j-1}^3 \left(1 + \frac{3}{2} \frac{y_1(t_1)}{b_{j-1}^2} + O(y_1(t_1)^2) \right). \quad (2.4.214)$$

Therefore, after taking $t_1 \rightarrow +\infty$, we obtain :

$$\sum_{i=1}^{j-1} E[P_i] - E_j(t) = -\frac{3}{2} b_{j-1} y_1(t) E[q] + O(e^{-2\varpi t}) + O(y_1(t)^2). \quad (2.4.215)$$

This is why, from (2.4.210), (2.4.215) and Lemma 2.44, we obtain :

$$-\frac{y_1(t)}{2b_{j-1}} M[q] + O(e^{-2\varpi t}) + O(y_1(t)^2) \geq -C e^{-2\varpi t}, \quad (2.4.216)$$

and

$$-\frac{3}{2}b_{j-1}y_1(t)E[q] + O(e^{-2\varpi t}) + O(y_1(t)^2) \geq -Ce^{-2\varpi t}. \quad (2.4.217)$$

Because $M[q] = 2$ and $E[q] = -\frac{2}{3}$, we rewrite both previous inequalities (2.4.216) and (2.4.217) in the following way (and we pass $O(e^{-2\varpi t})$ on the other side of each inequality) :

$$-\frac{y_1(t)}{b_{j-1}} + O(y_1(t)^2) \geq -Ce^{-2\varpi t}, \quad (2.4.218)$$

and

$$b_{j-1}y_1(t) + O(y_1(t)^2) \geq -Ce^{-2\varpi t}. \quad (2.4.219)$$

Because $y_1(t) \rightarrow +\infty$, by taking T_0 larger if needed, $O(y_1(t)^2)$ can be bounded above by any positive constant multiplied by $|y_1(t)|$, so by taking this constant small enough (by taking T_0 large enough) and combining both previous inequalities (2.4.218) and (2.4.219), we obtain :

$$|y_1(t)| \leq Ce^{-2\varpi t}. \quad (2.4.220)$$

Therefore, we have obtained a better bound for $y_1(t)$ in the case when P_{j-1} is a soliton. Therefore, we may conclude that in any case, for $t \geq T_0$, for T_0 large enough,

$$\int (w^2 + w_x^2 + w_{xx}^2) \Phi_j(t) = O(e^{-2\varpi t}). \quad (2.4.221)$$

Then, we deduce from (2.4.147) that

$$|y_1'(t)| + |y_2'(t)| = O(e^{-\varpi t}). \quad (2.4.222)$$

Because $|y_1(t)| + |y_2(t)| \rightarrow_{t \rightarrow +\infty} 0$, we obtain by integration :

$$|y_1(t)| + |y_2(t)| = O(e^{-\varpi t}). \quad (2.4.223)$$

And, so, by the mean-value theorem,

$$\|\widetilde{P_{j-1}} - P_{j-1}\|_{H^2} \leq C(|y_1(t)| + |y_2(t)|) \leq Ce^{-\varpi t}. \quad (2.4.224)$$

From $v = w + \widetilde{P_{j-1}} - P_{j-1}$, we deduce :

$$\begin{aligned} \int (v^2 + v_x^2 + v_{xx}^2) \Phi_j &\leq C \int (w^2 + w_x^2 + w_{xx}^2) \Phi_j \\ &\quad + C \int \left[(\widetilde{P_{j-1}} - P_{j-1})^2 + (\widetilde{P_{j-1}} - P_{j-1})_x^2 + (\widetilde{P_{j-1}} - P_{j-1})_{xx}^2 \right] \Phi_j \\ &\leq Ce^{-2\varpi t}, \end{aligned} \quad (2.4.225)$$

and this finishes the induction. □

2.4.3 Proof of Theorem 2.4

Proof of Theorem 2.4. We suppose that $v_2 > 0$. Let p be the associated multi-breather given by Theorem 2.2. Let u be a solution of (mKdV) such that

$$\|u(t) - p(t)\|_{H^2} \rightarrow_{t \rightarrow +\infty} 0. \quad (2.4.226)$$

From Proposition 2.43, we deduce that there exists a constant $C > 0$ and a constant $\varpi > 0$ such that for t large enough

$$\|u(t) - p(t)\|_{H^2} \leq C e^{-\varpi t}. \quad (2.4.227)$$

This implies that u satisfies the assumptions of Proposition 2.5. Thus, $u = p$ and Theorem 2.4 is proved. \square

2.5 Appendix

The first two subsections of the Appendix show that a soliton has similar properties as a “limit breather” of parameter $\alpha = 0$. Firstly, the corresponding elliptic equation is satisfied by a soliton. Secondly, the corresponding quadratic form is coercive for a soliton, and we see that its kernel is spanned by $\partial_x Q$ and $\partial_c Q$. In the third subsection, we prove that it is possible for ϵ to be orthogonal to Q and $\partial_x Q$ (instead of $\partial_x Q$ and $\partial_c Q$) in order to satisfy a coercivity for the quadratic form. We will use this fact for the proof of the existence, as well as for the first part of the proof of the uniqueness. In the fourth subsection, we prove that we can have coercivity for quadratic forms when the orthogonality condition is not exactly satisfied. We will use this result for the proof of the uniqueness. The last subsection is about computations for the third conservation law. It will be useful for the monotonicity property for localized F that we need in the proof of the uniqueness.

2.5.1 Elliptic equation satisfied by a soliton

Lemma 2.50. *A soliton $Q = R_{c,\kappa}$ satisfies for any time $t \in \mathbb{R}$, the following nonlinear elliptic equation :*

$$Q_{(4x)} - 2c(Q_{xx} + Q^3) + c^2 Q + 5Q Q_x^2 + 5Q^2 Q_{xx} + \frac{3}{2} Q^5 = 0. \quad (2.5.1)$$

Proof. In order to derive this equation, we will use the equation that defines a soliton (and that is satisfied by Q at any time) :

$$Q_{xx} = cQ - Q^3. \quad (2.5.2)$$

We will also need the following equation :

$$Q_x^2 = cQ^2 - \frac{1}{2} Q^4, \quad (2.5.3)$$

that can be derived by taking the space derivative of $Q_x^2 - cQ^2 + \frac{1}{2} Q^4$, and by showing that this derivative is zero. From this, we deduce that $Q_x^2 - cQ^2 + \frac{1}{2} Q^4$ is constant, and by taking its limit when $x \rightarrow \pm\infty$, we see that this constant is zero. More precisely, the derivative of $Q_x^2 - cQ^2 + \frac{1}{2} Q^4$ is :

$$2Q_x Q_{xx} - 2cQ Q_x + 2Q^3 Q_x = 2Q_x (Q_{xx} - cQ + Q^3) = 0. \quad (2.5.4)$$

From now on, the derivation of (2.5.1) is straightforward. It is sufficient to take space derivatives of $Q_{xx} = cQ - Q^3$ and to inject them into the right hand side of the equation (2.5.1), which we want to prove to be equal to zero. By doing this, we make the maximal order of a derivative of Q present in the right hand side equation lower. At the end, we have only, zero and first order derivatives. To have only a polynomial in Q , we have to use $Q_x^2 = cQ^2 - \frac{1}{2} Q^4$, and the calculations show that this polynomial is zero. \square

2.5.2 Study of coercivity of the quadratic form associated to a soliton

In this article, we adapt the argument for the breathers in [8] to the soliton case. We consider :

$$\begin{aligned} \mathcal{Q}_c^s[\epsilon] &:= \frac{1}{2} \int \epsilon_{xx}^2 - \frac{5}{2} \int Q^2 \epsilon_x^2 + \frac{5}{2} \int Q_x^2 \epsilon^2 + 5 \int Q Q_{xx} \epsilon^2 + \frac{15}{4} \int Q^4 \epsilon^2 \\ &+ c \left(\int \epsilon_x^2 - 3 \int Q^2 \epsilon^2 \right) + c^2 \frac{1}{2} \int \epsilon^2 =: \mathcal{Q}_{0, \sqrt{c}}[\epsilon]. \end{aligned} \quad (2.5.5)$$

Firstly, we prove, by simple calculations, as in the previous section, that Q_x and $Q + xQ_x$ are in the kernel of this quadratic form. It is easy to see, by asymptotic study that these two functions are linearly independent.

The self-adjoint linear operator associated to this quadratic form is :

$$\begin{aligned} \mathcal{L}_c^s[\epsilon] &:= \epsilon_{(4x)} - 2c\epsilon_{xx} + c^2\epsilon + 5Q^2\epsilon_{xx} + 10QQ_x\epsilon_x \\ &+ \left(5Q_x^2 + 10QQ_{xx} + \frac{15}{2}Q^4 - 6cQ^2 \right) \epsilon, \end{aligned} \quad (2.5.6)$$

so that $\mathcal{Q}_c^s[\epsilon] = \int \epsilon \mathcal{L}_c^s[\epsilon]$. \mathcal{L}_c^s is a compact perturbation of the constant coefficients operator :

$$\mathcal{M}[\epsilon] := \epsilon_{(4x)} - 2c\epsilon_{xx} + c^2\epsilon. \quad (2.5.7)$$

A direct analysis involving ODE shows that the null space of \mathcal{M} is spawned by four linearly independent functions :

$$e^{\pm\sqrt{c}x}, \quad xe^{\pm\sqrt{c}x}. \quad (2.5.8)$$

Among these four functions, there are only two L^2 -integrable ones in the semi-infinite line $[0, +\infty)$. Therefore, the null space of $\mathcal{L}_c^s|_{H^4(\mathbb{R})}$ is spanned by at most two L^2 -functions. Therefore,

$$\ker(\mathcal{L}_c^s) = \text{Span}(\partial_x Q, Q + x\partial_x Q). \quad (2.5.9)$$

Lemma 2.51. *The operator \mathcal{L}_c^s does not have any negative eigenvalue.*

Proof. \mathcal{L}_c^s has

$$\sum_{x \in \mathbb{R}} \dim \ker W[Q_x, Q + xQ_x](t, x) \quad (2.5.10)$$

negative eigenvalues, counting multiplicity, where W is the Wronskian matrix :

$$W[Q_x, Q + xQ_x](t, x) := \begin{bmatrix} Q_x & Q + xQ_x \\ Q_{xx} & (Q + xQ_x)_x \end{bmatrix}. \quad (2.5.11)$$

For this result, see [43], where the finite interval case was considered. As shown in several articles [52, 70], the extension to the real line is direct.

Thus, it is sufficient to see that $\det W[Q_x, Q + xQ_x](t, x)$ is never zero. For this, let us simply calculate this determinant :

$$\begin{aligned} Q_x(2Q_x + xQ_{xx}) - (Q + xQ_x)Q_{xx} &= 2Q_x^2 - QQ_{xx} \\ &= 2cQ^2 - Q^4 - Q(cQ - Q^3) \\ &= cQ^2 > 0. \end{aligned} \quad (2.5.12)$$

□

2.5.3 Coercivity of the quadratic form associated to a soliton

For $Q = R_{c,\kappa}$, let

$$\begin{aligned} \mathcal{Q}_c^s[\epsilon] := & \frac{1}{2} \int \epsilon_{xx}^2 - \frac{5}{2} \int Q^2 \epsilon_x^2 + \frac{5}{2} \int Q_x^2 \epsilon^2 + 5 \int Q Q_{xx} \epsilon^2 + \frac{15}{4} \int Q^4 \epsilon^2 \\ & + c \left(\int \epsilon_x^2 - 3 \int Q^2 \epsilon^2 \right) + c^2 \frac{1}{2} \int \epsilon^2. \end{aligned} \quad (2.5.13)$$

Lemma 2.52. *There exists $\mu_c > 0$ such that for any $\epsilon \in H^2$ such that $\int \epsilon Q = \int \epsilon Q_x = 0$, we have that*

$$\mathcal{Q}_c^s[\epsilon] \geq \mu_c \|\epsilon\|_{H^2}^2. \quad (2.5.14)$$

Proof. From Section 2.5.2, we know that if $\int \epsilon \partial_x Q = \int \epsilon \partial_c Q = 0$, then, for a constant $\nu_c > 0$, we have that

$$\mathcal{Q}_c^s[\epsilon] \geq \nu_c \|\epsilon\|_{H^2}^2. \quad (2.5.15)$$

Let $\epsilon \in H^2$ be such that $\int \epsilon Q = \int \epsilon \partial_x Q = 0$. There exists $a \in \mathbb{R}$ and $\epsilon_\perp \in \text{Span}(\partial_x Q, \partial_c Q)^\perp$ such that

$$\epsilon = a \partial_c Q + \epsilon_\perp. \quad (2.5.16)$$

From $\int \epsilon Q = 0$, we have that

$$a \int \partial_c Q \cdot Q + \int \epsilon_\perp Q = 0, \quad (2.5.17)$$

thus,

$$\frac{a}{2} \int Q^2 + \int \epsilon_\perp Q = 0, \quad (2.5.18)$$

which allows us to derive :

$$a = -2 \frac{\int \epsilon_\perp Q}{\int Q^2}. \quad (2.5.19)$$

Because $\partial_c Q$ is in the kernel of \mathcal{Q}_c^s , we have that

$$\mathcal{Q}_c^s[\epsilon] = \mathcal{Q}_c^s[\epsilon_\perp] \geq \nu_c \|\epsilon_\perp\|_{H^2}^2. \quad (2.5.20)$$

Now, from

$$\epsilon = -2 \frac{\int \epsilon_\perp Q}{\int Q^2} \partial_c Q + \epsilon_\perp, \quad (2.5.21)$$

we have by triangular and Cauchy-Schwarz inequalities that

$$\begin{aligned} \|\epsilon\|_{H^2} & \leq \|\epsilon_\perp\|_{H^2} + 2 \frac{|\int \epsilon_\perp Q|}{\|Q\|_{L^2}^2} \|\partial_c Q\|_{H^2} \\ & \leq \|\epsilon_\perp\|_{H^2} + 2 \frac{\|\partial_c Q\|_{H^2}}{\|Q\|_{L^2}} \|\epsilon_\perp\|_{L^2} \\ & \leq \left(1 + 2 \frac{\|\partial_c Q\|_{H^2}}{\|Q\|_{L^2}}\right) \|\epsilon_\perp\|_{H^2}. \end{aligned} \quad (2.5.22)$$

Therefore, we may derive a constant μ_c (independent on ϵ) such that

$$\mathcal{Q}_c^s[\epsilon] \geq \mu_c \|\epsilon\|_{H^2}^2. \quad (2.5.23)$$

□

2.5.4 Coercivity with almost orthogonality conditions (to be used for the uniqueness)

For $B := B_{\alpha,\beta}$ or any of its translations, we define the canonical quadratic form associated to B :

$$\begin{aligned} \mathcal{Q}_{\alpha,\beta}^b[\epsilon] := & \frac{1}{2} \int \epsilon_{xx}^2 - \frac{5}{2} \int B^2 \epsilon_x^2 + \frac{5}{2} \int B_x^2 \epsilon^2 + 5 \int B B_{xx} \epsilon^2 + \frac{15}{4} \int B^4 \epsilon^2 \\ & + (\beta^2 - \alpha^2) \left(\int \epsilon_x^2 - 3 \int B^2 \epsilon^2 \right) + (\alpha^2 + \beta^2) \frac{1}{2} \int \epsilon^2, \end{aligned} \quad (2.5.24)$$

and we know that $\partial_{x_1} B$ and $\partial_{x_2} B$ span the kernel of $\mathcal{Q}_{\alpha,\beta}^b$. More precisely, there exists $\mu_{\alpha,\beta}^b > 0$ such that if ϵ is orthogonal to $\partial_{x_1} B$ and $\partial_{x_2} B$, we have that

$$\mathcal{Q}_{\alpha,\beta}^b[\epsilon] \geq \mu_{\alpha,\beta}^b \|\epsilon\|_{H^2}^2 - \frac{1}{\mu_{\alpha,\beta}^b} \left(\int \epsilon B \right)^2. \quad (2.5.25)$$

We would like to prove the following lemma (adapted from the Appendix A of [77]) :

Lemma 2.53. *There exists $\nu := \nu_{\alpha,\beta}^b > 0$ such that, for $\epsilon \in H^2(\mathbb{R})$, if*

$$\left| \int (\partial_{x_1} B_{\alpha,\beta}) \epsilon \right| + \left| \int (\partial_{x_2} B_{\alpha,\beta}) \epsilon \right| < \nu \|\epsilon\|_{H^2}, \quad (2.5.26)$$

then

$$\mathcal{Q}_{\alpha,\beta}^b[\epsilon] \geq \frac{\mu_{\alpha,\beta}^b}{4} \|\epsilon\|_{H^2}^2 - \frac{4}{\mu_{\alpha,\beta}^b} \left(\int \epsilon B_{\alpha,\beta} \right)^2, \quad (2.5.27)$$

where $B_{\alpha,\beta}$ denotes the breather of parameters α and β or any of its translations (in space or in time).

Proof. Take $\nu > 0$ (we will find a condition on ν later in the proof) and take ϵ satisfying the assumption of the lemma. Then (denoting $B = B_{\alpha,\beta}$),

$$\epsilon = \epsilon_1 + aB_1 + bB_2 = \epsilon_1 + \epsilon_2, \quad (2.5.28)$$

where $\int \epsilon_1 B_1 = \int \epsilon_1 B_2 = \int \epsilon_1 \epsilon_2 = 0$.

By performing a L^2 -scalar product of (2.5.28) with B_1 and B_2 , we obtain, by assumption, that

$$\left| a \int B_1^2 + b \int B_1 B_2 \right| \leq \nu \|\epsilon\|_{H^2}, \quad (2.5.29)$$

$$\left| a \int B_1 B_2 + b \int B_2^2 \right| \leq \nu \|\epsilon\|_{H^2}. \quad (2.5.30)$$

Therefore, by making linear combinations of these two inequalities, using triangular and Cauchy-Schwarz inequalities, we obtain that

$$|a| + |b| \leq C\nu \|\epsilon\|_{H^2}. \quad (2.5.31)$$

We can take space derivatives of (2.5.28). And thus, we obtain, for ν small enough, that

$$\frac{1}{2} \|\epsilon\|_{H^2}^2 \leq \|\epsilon_1\|_{H^2}^2 \leq 2 \|\epsilon\|_{H^2}^2. \quad (2.5.32)$$

Because of $\int B B_1 = \int B B_2 = 0$,

$$\int \epsilon B = \int \epsilon_1 B. \quad (2.5.33)$$

By bilinearity,

$$\begin{aligned}\mathcal{Q}_{\alpha,\beta}^b[\epsilon] &= \mathcal{Q}_{\alpha,\beta}^b[\epsilon_1] + \mathcal{Q}_{\alpha,\beta}^b[\epsilon_2] + \int \epsilon_{1,xx}\epsilon_{2,xx} - 5 \int B^2\epsilon_{1,x}\epsilon_{2,x} \\ &\quad + 5 \int B_x^2\epsilon_1\epsilon_2 + 10 \int BB_{xx}\epsilon_1\epsilon_2 + \frac{15}{2} \int B^4\epsilon_1\epsilon_2 \\ &\quad + (\beta^2 - \alpha^2) \left(2 \int \epsilon_{1,x}\epsilon_{2,x} - 6 \int B^2\epsilon_1\epsilon_2 \right) + (\alpha^2 + \beta^2)^2 \int \epsilon_1\epsilon_2\end{aligned}\tag{2.5.34}$$

We know from the coercivity of $\mathcal{Q}_{\alpha,\beta}^b$ that

$$\begin{aligned}\mathcal{Q}_{\alpha,\beta}^b[\epsilon_1] &\geq \mu_{\alpha,\beta}^b \|\epsilon_1\|_{H^2}^2 - \frac{1}{\mu_{\alpha,\beta}^b} \left(\int \epsilon_1 B \right)^2 \\ &\geq \frac{\mu_{\alpha,\beta}^b}{2} \|\epsilon\|_{H^2}^2 - \frac{2}{\mu_{\alpha,\beta}^b} \left(\int \epsilon B \right)^2.\end{aligned}\tag{2.5.35}$$

Also, if we denote by $\mathcal{L}_{\alpha,\beta}^b$ the self-adjoint operator associated to the quadratic form $\mathcal{Q}_{\alpha,\beta}^b$,

$$\mathcal{Q}_{\alpha,\beta}^b[\epsilon_2] = a^2 \mathcal{Q}_{\alpha,\beta}^b[B_1] + b^2 \mathcal{Q}_{\alpha,\beta}^b[B_2] + 2ab \int \mathcal{L}_{\alpha,\beta}^b[B_1]B_2 \leq C\nu^2 \|\epsilon\|_{H^2}^2.\tag{2.5.36}$$

Actually, in this case, $\mathcal{Q}_{\alpha,\beta}^b[\epsilon_2] = 0$, because ϵ_2 is in the kernel of $\mathcal{Q}_{\alpha,\beta}^b$ (but, when we adapt this proof for solitons, we can only write the bound).

Now, we recall that $\int \epsilon_1\epsilon_2 = 0$, and study the other terms by using Cauchy-Schwarz :

$$\begin{aligned}&\left| \int \epsilon_{1,xx}\epsilon_{2,xx} - 5 \int B^2\epsilon_{1,x}\epsilon_{2,x} + 5 \int B_x^2\epsilon_1\epsilon_2 + 10 \int BB_{xx}\epsilon_1\epsilon_2 \right. \\ &\quad \left. + \frac{15}{2} \int B^4\epsilon_1\epsilon_2 + (\beta^2 - \alpha^2) \left(2 \int \epsilon_{1,x}\epsilon_{2,x} - 6 \int B^2\epsilon_1\epsilon_2 \right) \right| \\ &\leq C(|a| + |b|) \|\epsilon_1\|_{H^2} \leq C\nu \|\epsilon\|_{H^2(\mathbb{R})}^2.\end{aligned}\tag{2.5.37}$$

We observe that if we take ν small enough, the claim of the lemma is proved. \square

We prove in the same way that we have a similar lemma for solitons :

Lemma 2.54. *There exists $\nu := \nu_c^s > 0$, such that, for $\epsilon \in H^2(\mathbb{R})$, if*

$$\left| \int (\partial_c R_{c,\kappa})\epsilon \right| + \left| \int (\partial_x R_{c,\kappa})\epsilon \right| \leq \nu \|\epsilon\|_{H^2},\tag{2.5.38}$$

then

$$\mathcal{Q}_c^s[\epsilon] \geq \frac{\mu_c^s}{4} \|\epsilon\|_{H^2}^2,\tag{2.5.39}$$

where $R_{c,\kappa}$ denotes the soliton of parameter c and sign κ or any of its translations.

And even,

Lemma 2.55. *There exists $\nu := \nu_c^s > 0$, such that, for $\epsilon \in H^2(\mathbb{R})$, if*

$$\left| \int R_{c,\kappa} \epsilon \right| + \left| \int (\partial_x R_{c,\kappa}) \epsilon \right| \leq \nu \|\epsilon\|_{H^2}, \quad (2.5.40)$$

then

$$\mathcal{Q}_c^s[\epsilon] \geq \frac{\mu_c^s}{4} \|\epsilon\|_{H^2}^2, \quad (2.5.41)$$

where $R_{c,\kappa}$ denotes the soliton of parameter c and sign κ or any of its translations.

2.5.5 Computations for the third localized integral (to be used for the uniqueness)

Lemma 2.56. *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a C^3 function that do not depend on time and u a solution of (mKdV). Then,*

$$\begin{aligned} & \frac{d}{dt} \int \left(\frac{1}{2} u_{xx}^2 - \frac{5}{2} u^2 u_x^2 + \frac{1}{4} u^6 \right) f \\ &= \int \left(-\frac{3}{2} u_{xxx}^2 + 9 u_{xx}^2 u^2 + 15 u_x^2 u u_{xx} + \frac{9}{16} u^8 + \frac{1}{4} u_x^4 + \frac{3}{2} u_{xx} u^5 \right. \\ & \quad \left. - \frac{45}{4} u^4 u_x^2 \right) f' + 5 \int u^2 u_x u_{xx} f'' + \frac{1}{2} \int u_{xx}^2 f'''. \end{aligned} \quad (2.5.42)$$

Proof. We perform by doing integrations by parts when needed and basic calculations :

$$\begin{aligned} & \frac{d}{dt} \int \left(\frac{1}{2} u_{xx}^2 - \frac{5}{2} u^2 u_x^2 + \frac{1}{4} u^6 \right) f \\ &= \int u_{txx} u_{xx} f - 5 \int u_t u u_x^2 f - 5 \int u^2 u_{tx} u_x f + \frac{3}{2} \int u_t u^5 f \\ &= - \int (u_{xx} + u^3)_{xxx} u_{xx} f + 5 \int (u_{xx} + u^3)_x u u_x^2 f \\ &+ 5 \int u^2 (u_{xx} + u^3)_{xx} u_x f - \frac{3}{2} \int (u_{xx} + u^3)_x u^5 f \\ &= \int (u_{xx} + u^3)_{xx} u_{xxx} f + \int (u_{xx} + u^3)_{xx} u_{xx} f' + 5 \int (u_{xx} + u^3)_x u u_x^2 f \\ &+ 5 \int u^2 (u_{xx} + u^3)_{xx} u_x f - \frac{3}{2} \int (u_{xx} + u^3)_x u^5 f \\ &= -\frac{1}{2} \int u_{xxx}^2 f' + \int (u^3)_{xx} u_{xxx} f + \int (u_{xx} + u^3)_{xx} u_{xx} f' + 5 \int u_{xxx} u u_x^2 f + 5 \int (u^3)_x u u_x^2 f \\ &+ 5 \int u^2 u_{xxxx} u_x f + 5 \int u^2 (u^3)_{xx} u_x f - \frac{3}{2} \int u_{xxx} u^5 f - \frac{3}{2} \int (u^3)_x u^5 f \\ &= -\frac{1}{2} \int u_{xxx}^2 f' + \int (u_{xx} + u^3)_{xx} u_{xx} f' + \int (3u_{xx} u^2 + 6u_x^2 u) u_{xxx} f + 5 \int u_{xxx} u u_x^2 f \\ &+ 15 \int u_x^3 u^3 f + 5 \int u^2 u_{xxxx} u_x f + 5 \int u^2 (3u_{xx} u^2 + 6u_x^2 u) u_x f - \frac{3}{2} \int u_{xxx} u^5 f - \frac{9}{2} \int u_x u^7 f \\ &= -\frac{1}{2} \int u_{xxx}^2 f' + \int (u_{xx} + u^3)_{xx} u_{xx} f' + 3 \int u^2 u_{xx} u_{xxx} f + 5 \int u^2 u_{xxxx} u_x f \\ &+ 11 \int u u_x^2 u_{xxx} f + 45 \int u^3 u_x^3 f + 15 \int u^4 u_x u_{xx} f - \frac{3}{2} \int u_{xxx} u^5 f + \frac{9}{16} \int u^8 f' \end{aligned}$$

$$\begin{aligned}
&= -\frac{1}{2} \int u_{xxx}^2 f' + \int (u_{xx} + u^3)_{xx} u_{xx} f' + \frac{9}{16} \int u^8 f' - 2 \int u^2 u_{xx} u_{xxx} f \\
&\quad + \int u u_x^2 u_{xxx} f - 5 \int u^2 u_x u_{xxx} f' + 45 \int u^3 u_x^3 f + 15 \int u^4 u_x u_{xx} f - \frac{3}{2} \int u^5 u_{xxx} f \\
&= -\frac{1}{2} \int u_{xxx}^2 f' + \int (u_{xx} + u^3)_{xx} u_{xx} f' + \frac{9}{16} \int u^8 f' - 5 \int u^2 u_x u_{xxx} f' - \int u^2 (u_{xx}^2)_x f \\
&\quad + \int u u_x^2 u_{xxx} f + 45 \int u^3 u_x^3 f + 15 \int u^4 u_x u_{xx} f - \frac{3}{2} \int u^5 u_{xxx} f \tag{2.5.43} \\
&= -\frac{1}{2} \int u_{xxx}^2 f' + \int (u_{xx} + u^3)_{xx} u_{xx} f' + \frac{9}{16} \int u^8 f' - 5 \int u^2 u_x u_{xxx} f' \\
&\quad + \int u^2 u_{xx}^2 f' + 2 \int u u_x u_{xx}^2 f - \int u_x^3 u_{xx} f - 2 \int u u_x u_{xx}^2 f \\
&\quad - \int u u_x^2 u_{xxx} f' + 45 \int u^3 u_x^3 f + 15 \int u^4 u_x u_{xx} f - \frac{3}{2} \int u^5 u_{xxx} f \\
&= -\frac{1}{2} \int u_{xxx}^2 f' + \int (u_{xx} + u^3)_{xx} u_{xx} f' + \frac{9}{16} \int u^8 f' - 5 \int u^2 u_x u_{xxx} f' + \int u^2 u_{xx}^2 f' \\
&\quad - \int u u_x^2 u_{xxx} f' - \frac{1}{4} \int (u_x^4)_x f + 45 \int u^3 u_x^3 f + \frac{45}{4} \int u^4 (u_x^2)_x f + \frac{3}{2} \int u^5 u_{xxx} f' \\
&= -\frac{1}{2} \int u_{xxx}^2 f' + \int (u_{xx} + u^3)_{xx} u_{xx} f' + \frac{9}{16} \int u^8 f' - 5 \int u^2 u_x u_{xxx} f' + \int u^2 u_{xx}^2 f' \\
&\quad - \int u u_x^2 u_{xxx} f' + \frac{1}{4} \int u_x^4 f' + \frac{3}{2} \int u^5 u_{xxx} f' + 45 \int u^3 u_x^3 f - 45 \int u^3 u_x^3 f - \frac{45}{4} \int u^4 u_x^2 f' \\
&= -\frac{3}{2} \int u_{xxx}^2 f' - \int u_{xxx} u_{xx} f'' + 4 \int u_{xx}^2 u^2 f' + 5 \int u_x^2 u u_{xx} f' - 5 \int u^2 u_x u_{xxx} f' \\
&\quad + \frac{9}{16} \int u^8 f' + \frac{1}{4} \int u_x^4 f' + \frac{3}{2} \int u^5 u_{xxx} f' - \frac{45}{4} \int u^4 u_x^2 f' \\
&= -\frac{3}{2} \int u_{xxx}^2 f' + 9 \int u_{xx}^2 u^2 f' + 15 \int u_x^2 u u_{xx} f' + \frac{9}{16} \int u^8 f' + \frac{1}{4} \int u_x^4 f' \\
&\quad + \frac{3}{2} \int u^5 u_{xxx} f' - \frac{45}{4} \int u^4 u_x^2 f' - \int u_{xxx} u_{xx} f'' + 5 \int u^2 u_x u_{xxx} f'' \\
&= \int \left(-\frac{3}{2} u_{xxx}^2 + 9 u_{xx}^2 u^2 + 15 u_x^2 u u_{xx} + \frac{9}{16} u^8 + \frac{1}{4} u_x^4 + \frac{3}{2} u_{xx} u^5 - \frac{45}{4} u^4 u_x^2 \right) f' \\
&\quad + 5 \int u^2 u_x u_{xx} f'' + \frac{1}{2} \int u_{xx}^2 f'''.
\end{aligned}$$

which is exactly the desired expression. \square

Chapitre 3

Stabilité orbitale d'une somme de solitons et de breathers de (mKdV)

Abstract

In this article, we prove that a sum of solitons and breathers of the modified Korteweg-de Vries equation (mKdV) is orbitally stable. The orbital stability is shown in H^2 . More precisely, we will show that if a solution of (mKdV) is close enough to a sum of solitons and breathers with distinct velocities at $t = 0$ in the H^2 sense, then it stays close to this sum of solitons and breathers for any time $t \geq 0$ in the H^2 sense, up to space translations for solitons or space and phase translations for breathers, provided the condition that the considered solitons and breathers are sufficiently decoupled from each other and that the velocities of the considered breathers are all positive, except possibly one. The constants that appear in this stability result do not depend on translation parameters.

From this, we deduce the orbital stability of any multi-breather of (mKdV), provided the condition that the velocities of the considered breathers are all positive, except possibly one (the condition about the decoupling of the considered solitons and breathers between each other is not required in this setting). The constants that appear in this stability result depend on translation parameters of the considered solitons and breathers.¹

3.1 Introduction

3.1.1 Setting of the problem

We consider the modified Korteweg-de Vries equation :

$$\begin{cases} u_t + (u_{xx} + u^3)_x = 0, & (t, x) \in \mathbb{R} \times \mathbb{R}, \\ u(0, x) = u_0(x), & x \in \mathbb{R}, \end{cases} \quad (\text{mKdV})$$

for $u_0 \in H^2(\mathbb{R})$.

(mKdV) appears as a good approximation of some physical problems as ferromagnetic vortices [122], fluid mechanics [50], electrodynamics [97], plasma physics [106, 20], etc.

The Cauchy problem for (mKdV) is locally well-posed in H^s for $s > -\frac{1}{2}$ [46]. For $s > \frac{1}{4}$, the Cauchy problem is globally well-posed [23]. In this paper, we will use only basic results about the Cauchy problem : the fact that it is globally well-posed in H^1 or H^2 .

1. This chapter corresponds to the preprint [109], that is accepted for publication in Nonlinearity.

Note that the set of solutions of (mKdV) is stable under space or time translations or under reflexions with respect to the x -axis.

We have following conservation laws for a solution $u(t)$ of (mKdV) :

$$M[u](t) := \frac{1}{2} \int u^2(t, x) dx, \quad (3.1.1)$$

$$E[u](t) := \frac{1}{2} \int u_x^2(t, x) dx - \frac{1}{4} \int u^4(t, x) dx, \quad (3.1.2)$$

$$F[u](t) := \frac{1}{2} \int u_{xx}^2(t, x) dx - \frac{5}{2} \int u^2(t, x) u_x^2(t, x) dx + \frac{1}{4} \int u^6(t, x) dx. \quad (3.1.3)$$

Note that (mKdV) has actually infinitely many conservation laws, because it is integrable, like the original Korteweg-de Vries equation (KdV), which has quadratic nonlinearity [92, 2]. It is also a special case of the generalized Korteweg-de Vries equation (gKdV) [71].

(gKdV) belongs to the family of nonlinear (focusing) dispersive equations. Other examples of equations belonging to this family are the nonlinear Schrödinger equation (NLS) [79, 121, 120] and the nonlinear Klein-Gordon equation (KG) [29, 31]. They share a common property : they all admit special solutions called solitons, a bump that translates with a constant velocity without deformation. However, (mKdV) enjoys a specific feature : it admits another class of special solutions called breathers, which we will describe below. We will consider here both specific solutions of (mKdV) together : solitons and breathers.

For $c > 0$, $\kappa \in \{-1, 1\}$ and $x_0 \in \mathbb{R}$, a *soliton* parametrized by c , κ and x_0 is a solution of (mKdV) that corresponds to a bump of constant shape that translates with a constant velocity c without deformation, that is initially centered in x_0 and that has for sign κ . In other words, it is a solution of the form :

$$R_{c,\kappa}(t, x; x_0) := \kappa Q_c(x - ct - x_0), \quad (3.1.4)$$

where Q_c is the profile function that depends only on one variable and that is positive and pair. $Q_c \in H^1(\mathbb{R})$ should solve the following elliptic equation :

$$Q_c'' - cQ_c + Q_c^3 = 0. \quad (3.1.5)$$

It is possible to show that (3.1.5) has a unique solution in H^1 , up to translations and reflexion with respect to the x -axis. One can show that Q_c has the following expression :

$$Q_c(x) := \left(\frac{2c}{\cosh^2(\sqrt{c}x)} \right)^{\frac{1}{2}}. \quad (3.1.6)$$

We denote $Q := Q_1$ the basic ground state.

When $\kappa = -1$, the soliton is sometimes called an *antisoliton*.

The soliton $R_{c,\kappa}(x_0)$ travels with velocity c to the right, the position of its center at a time t is $x_0 + ct$. It is exponentially localized, depending on c and $x_0 + ct$ (the position of the bound depends on $x_0 + ct$, the amplitude and the exponential decay rate depend on c) :

$$|R_{c,\kappa}(t, x; x_0)| \leq \sqrt{2c} \exp(-\sqrt{c}|x - x_0 - ct|). \quad (3.1.7)$$

Analogous bounds are valid for any derivative (with the same decay rate but different amplitude). This motivates the terminology “shape parameter” for c .

Solitons, in particular their stability, have been extensively studied : regarding orbital stability (in H^1), we refer to Cazenave, Lions [19] and Weinstein [121, 120] for (NLS) and Weinstein [121], Bona-Souganidis-Strauss [16] and

Martel-Merle [74] for (mKdV), see also Grillakis-Shatah-Strauss [44] for a result in an abstract setting. Asymptotic stability (in H^1) of (mKdV) solitons was shown by Martel-Merle [73, 80, 78] and refined by Germain-Pusateri-Rousset [41].

For $\alpha, \beta > 0$ and $x_1, x_2 \in \mathbb{R}$, a *breather* parametrized by α, β, x_1, x_2 is a solution of (mKdV) that has the following expression :

$$B_{\alpha,\beta}(t, x; x_1, x_2) := 2\sqrt{2}\partial_x \left[\arctan \left(\frac{\beta \sin(\alpha y_1)}{\alpha \cosh(\beta y_2)} \right) \right], \quad (3.1.8)$$

where $y_1 := x + \delta t + x_1$, $y_2 := x + \gamma t + x_2$, $\delta := \alpha^2 - 3\beta^2$ and $\gamma := 3\alpha^2 - \beta^2$.

The breather $B_{\alpha,\beta}(x_1, x_2)$ travels with velocity $-\gamma$; the position of its center at a time t is $-x_2 - \gamma t$. It is exponentially localized, depending on α, β and $-x_2 - \gamma t$ (the position of the bound depends on $-x_2 - \gamma t$, the coefficient depends on α and β , and the exponential decay rate depends on β) :

$$|B_{\alpha,\beta}(t, x; x_1, x_2)| \leq C(\alpha, \beta) \exp(-\beta |x + x_2 + \gamma t|), \quad (3.1.9)$$

Analogous bounds are valid for any derivative (with the same decay rate but different amplitude). This motivates the terminology “shape” and “frequency” parameters for β and α , respectively.

One doesn’t talk of “antibreathers”, because if we replace x_1 by $x_1 + \frac{\pi}{\alpha}$, then a breather is transformed in its opposite.

Similarly to (3.1.5), it is known that a breather $B = B_{\alpha,\beta}$ satisfies the following elliptic equation on \mathbb{R} :

$$B_{xxxx} + 5BB_x^2 + 5B^2B_{xx} + \frac{3}{2}B^5 - 2(\beta^2 - \alpha^2)(B_{xx} + B^3) + (\alpha^2 + \beta^2)^2 B = 0. \quad (3.1.10)$$

Note that in [108] a similar form of elliptic equation (with a fourth order derivative) is obtained for a soliton $R_{c,\kappa}$ from (3.1.5). This allows us to consider solitons and breathers at the same level of regularity, which is the key of the proof made in this article.

This object was first introduced by Wadati [117], and it was used by Kenig, Ponce and Vega in [60] for the ill-posedness for (mKdV) for rough data. Their properties, in particular their stability, are well studied by Alejo, Muñoz and co-authors [8, 7, 10, 11, 9]. We know that a breather is orbitally stable in H^2 [8]. Afterwards, H^1 orbital stability was proved via Bäcklund transformation [9], and also H^1 asymptotic stability, for breathers moving to the right. Asymptotic stability of breathers in full generality is still an open problem.

When $\alpha \rightarrow 0$, $B_{\alpha,\beta}$ tends to a *double-pole solution* of (mKdV) : it is a couple soliton-antisoliton that move with a constant velocity and that have a repulsive logarithmic interaction [118]. However, this limit, which is somehow at a boundary between solitons and breathers, is expected to be unstable [42]. We do not consider this object in this paper.

Solitons and breathers are important objects to study because of their stability properties and also because of the soliton-breather resolution. The latter is an important result about the long time dynamics of (mKdV), which asserts that any generic solution can be approached by a sum of solitons and breathers when $t \rightarrow +\infty$. It is established for initial conditions in a weighted Sobolev space in [21] (see also Schuur [107]) by inverse scattering method; see also [107] for the soliton resolution for (KdV).

Given a set of basic objects (solitons and breathers), we consider a solution that tends to this sum when $t \rightarrow +\infty$, called *multi-breather*. In [108], we have shown existence, regularity and uniqueness of multi-breathers of (mKdV). There is also a formula for multi-breathers of (mKdV) obtained by Wadati [118], which was derived as a consequence of the integrability of (mKdV); but is it not well suited for our purpose.

Martel, Merle and Tsai [87] proved that a sum of (decoupled and ordered) solitons is orbitally stable in H^1 for (mKdV), and actually asymptotically stable (in the region $x \geq \delta t$ for $\delta > 0$ small). Le Coz [69] has established stability of (mKdV) N -solitons in H^N by modifying the approach used by Maddocks and Sachs [70] for (KdV).

Inspired by [87], similar asymptotic stability results were obtained for sums of (decoupled) solitons for various nonlinear dispersive equations : we refer to El Dika [34] for the Benjamin-Bona-Mahony equation (BBM), Kenig and Martel [57] for the Benjamin-Ono equation (BO), El Dika-Molinet [35] for the Camassa-Holm multipeakon, and Côte-Muñoz-Pilod-Simpson [32] for the Zakharov-Kuznetsov (ZK) equation.

Because breathers have an H^2 structure, we prove orbital stability in H^2 in this paper for a sum of solitons and breathers. One of the difficulties is to obtain H^2 stability results for solitons too, i.e. to study solitons at a H^2 level.

3.1.2 Main results

We prove in this article that given any sum of solitons and breathers with distinct velocities and such that all these velocities except possibly one are positive, a solution u of (mKdV) that is initially close to this sum in H^2 stays close to this sum for any time of a considered time interval up to space translations for solitons or space and phase translations for breathers. This is orbital stability. Let us make the definition of orbital stability more precise.

Let $L \in \mathbb{N}$. We consider a set of L solitons : given $c_l^0 > 0$, $\kappa_l \in \{-1, 1\}$ and $x_{0,l}^0 \in \mathbb{R}$ for $1 \leq l \leq L$, we set, for $1 \leq l \leq L$,

$$R_l(t, x) := R_{c_l^0, \kappa_l}(t, x; x_{0,l}^0). \quad (3.1.11)$$

Let $K \in \mathbb{N}$. We consider a set of K breathers : given $\alpha_k > 0$, $\beta_k > 0$ and $x_{1,k}^0, x_{2,k}^0 \in \mathbb{R}$ for $1 \leq k \leq K$, we set, for $1 \leq k \leq K$,

$$B_k(t, x) := B_{\alpha_k, \beta_k}(t, x; x_{1,k}^0, x_{2,k}^0). \quad (3.1.12)$$

We now define important parameters for each of the objects of the problem. For $1 \leq l \leq L$, the velocity of the l th soliton is

$$v_l^s := c_l^0, \quad (3.1.13)$$

and for $1 \leq k \leq K$, the velocity of the k th breather is

$$v_k^b := \beta_k^2 - 3\alpha_k^2. \quad (3.1.14)$$

For $1 \leq l \leq L$, the center of the l th soliton is

$$x_l^s(t) := x_{0,l}^0 + v_l^s t, \quad (3.1.15)$$

and for $1 \leq k \leq K$, the center of the k th breather is

$$x_k^b(t) := -x_{2,k}^0 + v_k^b t. \quad (3.1.16)$$

We set $J := K + L$ the total number of objects in the problem. We assume that $J \geq 2$, because for $J = 1$ the proof is already done in [8] when the only object is a breather, and in [121] when the only object is a soliton.

We assume that the velocities of our objects are all distinct, this will imply that our objects are far from each other when time is large, which is essential for our analysis. More precisely,

$$\forall k \neq k', \quad v_k^b \neq v_{k'}^b, \quad \forall l \neq l', \quad v_l^s \neq v_{l'}^s, \quad \forall k, l, \quad v_k^b \neq v_l^s. \quad (3.1.17)$$

This allows us to define an increasing function

$$\underline{v} : \{1, \dots, J\} \rightarrow \{v_k^b, 1 \leq k \leq K\} \cup \{v_l^s, 1 \leq l \leq L\}. \quad (3.1.18)$$

The set $\{v_1, \dots, v_J\}$ is thus the set of all the possible velocities of our objects. We have

$$v_1 < v_2 < \dots < v_J. \quad (3.1.19)$$

We define, for $1 \leq j \leq J$, P_j as the object (R_l or B_k) that corresponds to the velocity v_j , i.e. if $v_j = v_l^s$, we set $P_j := R_l$, and if $v_j = v_k^b$, we set $P_j := B_k$. So, P_1, \dots, P_J are the considered objects that are ordered by increasing velocity.

We denote $x_j(t)$ the position (the center of mass) of $P_j(t)$. More precisely, if $P_j = R_l$, we set $x_j(t) := x_l^s(t)$; and if $P_j = B_k$, we set $x_j(t) := x_k^b(t)$.

We set

$$R := \sum_{l=1}^L R_l, \quad B := \sum_{k=1}^K B_k, \quad P := R + B = \sum_{j=1}^J P_j. \quad (3.1.20)$$

We need both notations : indexation by k and l , and indexation by j , and we keep these notations to avoid ambiguity.

The main result that we will prove in this article is the following : a sum of decoupled solitons and breathers, with $v_2 > 0$ (that is, all travel to the right, but at most one can be static or travel to the left), is orbitally stable. The precise statement is as follows.

Theorem 3.1. *Given a set of solitons and breathers (3.1.11), (3.1.12) whose velocities (3.1.13) and (3.1.14) satisfy (3.1.17), and whose positions are set by (3.1.15) and (3.1.16), we define the corresponding sum P in (3.1.20), and we define P_j, v_j, x_j a reindexation of the given set of solitons and breathers such that (3.1.19). We assume that*

$$v_2 > 0. \quad (3.1.21)$$

Then there exists $A_0, \theta_0, D_0, a_0 > 0$, constants (depending on c_l^0, α_k, β_k , but not on $x_{0,l}^0, x_{1,k}^0$ or $x_{2,k}^0$), such that the following is true. Let $D \geq D_0$ and $0 \leq a \leq a_0$ such that

$$\|u(0) - P(0)\|_{H^2} \leq a, \quad \text{and} \quad x_j(0) > x_{j-1}(0) + 2D, \quad \text{for all } j = 2, \dots, J, \quad (3.1.22)$$

for a solution $u \in \mathcal{C}(\mathbb{R}, H^2(\mathbb{R}))$ of (mKdV).

Then, there exist $x_{0,l}(t), x_{1,k}(t), x_{2,k}(t)$ defined for any $t \geq 0$ such that

$$\forall t \geq 0, \quad \left\| u(t) - \sum_{l=1}^L R_{c_l^0, \kappa_l}(t, \cdot; x_{0,l}(t)) - \sum_{k=1}^K B_{\alpha_k, \beta_k}(t, \cdot; x_{1,k}(t), x_{2,k}(t)) \right\|_{H^2} \leq A_0 (a + e^{-\theta_0 D}), \quad (3.1.23)$$

with

$$\forall t \geq 0, \quad \sum_{l=1}^L |x'_{0,l}(t)| + \sum_{k=1}^K (|x'_{1,k}(t)| + |x'_{2,k}(t)|) \leq C A_0 (a + e^{-\theta_0 D}), \quad (3.1.24)$$

for some constant $C > 0$.

Remark 3.2. Let us stress on the fact that the constant A_0 do not depend on the translation parameters of the considered objects.

We also deduce a consequence from Theorem 3.1 for multi-breathers.

Recall that multi-breathers were defined and constructed in [108], and their uniqueness was proved there when $v_2 > 0$. They can be also obtained from the formula from [118] obtained by inverse scattering method by Wadati. Let p be a multi-breather associated to P (in the setting $v_2 > 0$, we know that p is also unique). To emphasize the dependence of p with respect to the parameters, we will write

$$p(t, x) =: p(t, x; \alpha_k, \beta_k, x_{1,k}^0, x_{2,k}^0, \kappa_l, c_l^0, x_{0,l}^0). \quad (3.1.25)$$

The same way, to emphasize the dependence of P with respect to the parameters, we will write

$$P(t, x) =: P(t, x; \alpha_k, \beta_k, x_{1,k}^0, x_{2,k}^0, \kappa_l, c_l^0, x_{0,l}^0). \quad (3.1.26)$$

Theorem 3.1 can be recast as orbital stability of a *multi-breather* :

Theorem 3.3. *Let $\alpha_k, \beta_k, x_{1,k}^0, x_{2,k}^0, \kappa_l, c_l^0, x_{0,l}^0$ and p the multi-breather associated to these parameters with notations as in (3.1.25) given by [108, Theorem 1.2]. We assume that (3.1.17) holds and $v_2 > 0$. There exists $\eta_0 > 0$ small enough, $C_0 > 0$ large enough such that the following is true for $0 < \eta < \eta_0$.*

Let $u(t)$ be a solution of (mKdV), such that

$$\|u(0) - p(0; \alpha_k, \beta_k, x_{1,k}^0, x_{2,k}^0, c_l^0, x_{0,l}^0)\|_{H^2} \leq \eta. \quad (3.1.27)$$

Then there exist C^1 functions $x_{0,l}(t), x_{1,k}(t), x_{2,k}(t)$ defined for any $t \geq 0$ such that

$$\forall t \geq 0, \quad \|u(t) - p(t; \alpha_k, \beta_k, x_{1,k}(t), x_{2,k}(t), c_l^0, x_{0,l}(t))\|_{H^2} \leq C_0 \eta, \quad (3.1.28)$$

with (3.1.24) that is satisfied.

Remark 3.4. Let us stress on the fact the C_0 do depend on the translation parameters of the considered solitons and breathers, i.e. on $x_{0,l}^0, x_{1,k}^0$ and $x_{2,k}^0$. More precisely, it depends on the time we need to wait until the collisions between the considered solitons and breathers are over. This is a fundamental difference between Theorem 3.3 and Theorem 3.1. In fact, in Theorem 3.1, we avoid collisions and this is the reason why we get a more uniform result. It is reasonable to expect that Theorem 3.3 can be improved so that C_0 do not depend on translation parameters.

In this paper, we adapt the arguments given by Martel, Merle and Tsai [87] to the context of breathers. To do so, it is needed to understand the variational structure of breathers, in the same manner as Weinstein did in [121] for (NLS) and (mKdV) solitons. Such results have been obtained by Alejo and Muñoz in [8]. When a soliton is a critical point of a Lyapunov functional at the H^1 level, whose Hessian is coercive up to two orthogonal conditions, a breather is a critical point of a Lyapunov functional at the H^2 level, whose Hessian is coercive up to three orthogonal conditions. One important issue that we address is to understand the variational structure of a soliton at the H^2 level. We do this by modifying the Lyapunov functional from [8], and we will also adapt it for a sum of solitons and breathers. We need to make assumptions on the velocities of our breathers (recall that the velocity of a soliton is always positive), because several arguments are based on monotonicity properties, which hold only on the right.

3.1.3 Organisation of the proof

The proof of Theorem 3.1 is based on two results : a modulation lemma and a bootstrap proposition. We give a detailed outline of both results in Section 3.2, and give in Section 3.3 the proof of the heart of the argument (Proposition 3.10), where we complete the bootstrap via a (finite) induction argument on the an improved bound localized on the last j objects.

In Section 3.4, we prove Theorem 3.3 as a quick consequence of Theorem 3.1.

3.1.4 Acknowledgments

The author would like to thank his supervisor Raphaël Côte for suggesting the idea of the work, for fruitful discussions and his useful advice.

The author would also like to thank Guillaume Ferriere for his useful remarks and suggestions.

3.2 Reduction of the proof to an induction

For the proof, we assume the assumption of the Theorem 3.1 true (i.e. we assume (3.1.22) true) for a solution $u(t)$ and a choice of translation parameters, and the goal is to find the suitable constants A, θ, D_0, a_0 (that do not depend on u nor on translation parameters) so that the Theorem 3.1 holds.

3.2.1 Some useful notations

We set some useful constants for this paper. We define the worst exponential decay rate :

$$\beta := \min\{\beta_k, 1 \leq k \leq K\} \cup \{\sqrt{c_l}, 1 \leq l \leq L\}, \quad (3.2.1)$$

and the worst distance between two consecutive velocities :

$$\tau := \min\{v_{j+1} - v_j, 1 \leq j \leq J-1\}. \quad (3.2.2)$$

For any $j = 2, \dots, J$, we have that

$$\forall t \in I, \quad x_j(t) - x_{j-1}(t) \geq x_j(0) - x_{j-1}(0) + \tau t. \quad (3.2.3)$$

We introduce general parameters, for $j = 1, \dots, J$, (a_j, b_j) . If $P_j = B_k$ is a breather, we set $(a_j, b_j) := (\alpha_k, \beta_k)$. If $P_j = R_l$ is a soliton, we set $(a_j, b_j) := (0, \sqrt{c_l^0})$.

3.2.2 Modulation lemma

We will first state a standard modulation lemma, which can be proved similarly as the modulation lemma in [108]. We need it because we will construct translations of our objects so that they are near to u and some orthogonality conditions are satisfied. These orthogonality conditions will allow us to use coercivity of some quadratic forms in the following of the proof.

Lemma 3.5. *Let $A' > 0$, $\theta > 0$, $t' > 0$ and $y_{1,k}(t)$, $y_{2,k}(t)$, $y_{3,l}(t)$, $y_{0,l}(t)$ defined for $t \in [0, t']$ (with $y_{3,l}(t) > 0$) such that $\forall t \in [0, t']$, $|y_{3,l}(t) - c_l^0| \leq \min\{\frac{\tau}{8}\} \cup \{\frac{c_p^0}{4}, 1 \leq p \leq L\}$. If D_0 is large enough and a_0 is small enough (dependently on A'), there exists a constant $C_2 > 1$ such that the following holds. Let $u(t)$ be a solution of (mKdV) such that for any $t \in [0, t']$,*

$$\left\| u(t) - \sum_{l=1}^L \kappa_l Q_{y_{3,l}(t)}(\cdot + y_{0,l}(t) - c_l^0 t) - \sum_{k=1}^K B_{\alpha_k, \beta_k}(t, \cdot; y_{1,k}(t), y_{2,k}(t)) \right\|_{H^2} \leq A' (a + e^{-\theta D}), \quad (3.2.4)$$

$$\left\| u(0) - \sum_{l=1}^L \kappa_l Q_{y_{3,l}(0)}(\cdot + y_{0,l}(0)) - \sum_{k=1}^K B_{\alpha_k, \beta_k}(0, \cdot; y_{1,k}(0), y_{2,k}(0)) \right\|_{H^2} \leq a, \quad (3.2.5)$$

and if we set $y_j(t) := -y_{0,l}(t) + v_l^s t$ if $P_j = R_l$, and $y_j(t) := -y_{2,k}(t) + v_k^b t$ if $P_j = B_k$, we have, for any $2 \leq j \leq J$,

$$\forall t \in [0, t'], \quad y_j(t) - y_{j-1}(t) \geq D, \quad (3.2.6)$$

then, there exists \mathcal{C}^1 functions $z_{1,k}(t)$, $z_{2,k}(t)$, $z_{3,l}(t)$, $z_{0,l}(t)$ defined for $t \in [0, t']$ (with $z_{3,l}(t) > 0$), such that if we set

$$\varepsilon(t) := u(t) - \widetilde{P}(t), \quad (3.2.7)$$

where

$$\widetilde{R}_l(t, x) := \kappa_l Q_{z_{3,l}(t)}(\cdot + z_{0,l}(t) - c_l^0 t) \quad \text{for } 1 \leq l \leq L, \quad (3.2.8)$$

$$\widetilde{B}_k(t, x) := B_{\alpha_k, \beta_k}(t, x; z_{1,k}(t), z_{2,k}(t)) \quad \text{for } 1 \leq k \leq K, \quad (3.2.9)$$

$$\widetilde{P}_j := \widetilde{R}_l \quad \text{if } P_j = R_l, \quad \widetilde{P}_j := \widetilde{B}_k \quad \text{if } P_j = B_k, \quad (3.2.10)$$

$$\widetilde{R} := \sum_{l=1}^L \widetilde{R}_l, \quad \widetilde{B} := \sum_{k=1}^K \widetilde{B}_k, \quad \widetilde{P} := \widetilde{R} + \widetilde{B} = \sum_{j=1}^J \widetilde{P}_j, \quad (3.2.11)$$

then, for $t \in [0, t']$, for $l = 1, \dots, L$, for $k = 1, \dots, K$,

$$\int \widetilde{R}_l(t) \varepsilon(t) = \int \partial_x \widetilde{R}_l(t) \varepsilon(t) = \int \partial_{x_1} \widetilde{B}_k(t) \varepsilon(t) = \int \partial_{x_2} \widetilde{B}_k(t) \varepsilon(t) = 0. \quad (3.2.12)$$

Moreover, for $t \in [0, t']$, we have

$$\|\varepsilon(t)\|_{H^2} + |z_{1,k}(t) - y_{1,k}(t)| + |z_{2,k}(t) - y_{2,k}(t)| + |z_{3,l}(t) - y_{3,l}(t)| + |z_{0,l}(t) - y_{0,l}(t)| \leq C_2 A' (a + e^{-\theta D}), \quad (3.2.13)$$

and

$$\|\varepsilon(0)\|_{H^2} + |z_{1,k}(0) - y_{1,k}(0)| + |z_{2,k}(0) - y_{2,k}(0)| + |z_{3,l}(0) - y_{3,l}(0)| + |z_{0,l}(0) - y_{0,l}(0)| \leq C_2 a, \quad (3.2.14)$$

and for any $t \in [0, t']$, $(z_{1,k}(t), z_{2,k}(t), z_{3,l}(t), z_{0,l}(t)) \in \mathbb{R}^{2K+2L}$ is unique such that (3.2.12) is satisfied and $(z_{1,k}(t), z_{2,k}(t), z_{3,l}(t), z_{0,l}(t))$ is in a suitable neighbourhood of $(y_{1,k}(t), y_{2,k}(t), y_{3,l}(t), y_{0,l}(t))$ that depends only on $A' (a + e^{-\theta D})$.

We set, for $t \in [0, t']$, $z_j(t) := z_l^s(t) := -z_{0,l}(t) + v_l^s t$ if $P_j = R_l$, and $z_j(t) := z_k^b(t) := -z_{2,k}(t) + v_k^b t$ if $P_j = B_k$. For D_0 large enough and a_0 small enough, if we assume that

$$\forall t \in [0, t'], \quad z_j(t) - z_{j-1}(t) \geq D, \quad (3.2.15)$$

(note that (3.2.15) is a consequence of $\forall t \in [0, t']$, $y_j(t) - y_{j-1}(t) \geq 2D$ if a_0 is small enough and D_0 is large enough), then for any $t \in [0, t']$, we have that
for $k = 1, \dots, K$,

$$|z'_{1,k}(t)| + |z'_{2,k}(t)| \leq C_2 \left(\int e^{-\frac{\beta}{2}|x-z_k^b(t)|} \varepsilon^2 \right)^{1/2} + C_2 e^{-\frac{\beta D}{8}}, \quad (3.2.16)$$

for $l = 1, \dots, L$,

$$|z'_{3,l}(t)| + |z'_{0,l}(t)| \leq C_2 \left(\int e^{-\frac{\beta}{2}|x-z_l^s(t)|} \varepsilon^2 \right)^{1/2} + C_2 e^{-\frac{\beta D}{8}}. \quad (3.2.17)$$

Remark 3.6. We will also use generalized notations for $y_{1,k}(t)$, $y_{2,k}(t)$ and other object-specific notations, in the lemma above.

For $j = 1, \dots, J$, either if $P_j = R_l$ is a soliton, we denote

$$y_{1,j}^*(t) := y_{3,l}(t), \quad z_{1,j}^*(t) := z_{3,l}(t), \quad y_{2,j}^*(t) := y_{0,l}(t), \quad z_{2,j}^*(t) := z_{0,l}(t), \quad (3.2.18)$$

or if $P_j = B_k$ is a breather, we denote

$$y_{1,j}^*(t) := y_{1,k}(t), \quad z_{1,j}^*(t) := z_{1,k}(t), \quad y_{2,j}^*(t) := y_{2,k}(t), \quad z_{2,j}^*(t) := z_{2,k}(t). \quad (3.2.19)$$

Proof. See [108, Lemma 2.8], for the proof of a similar result. We also refer to [87, 8, 22]. \square

3.2.3 Bootstrap

Given Lemma 3.5, we reduce the proof of Theorem 3.1 to the following bootstrap proposition (we will use the notations given in Section 3.1.2; in particular the position $x_j(t)$ of $P_j(t)$ is defined there) :

Proposition 3.7. *There exists $A \geq C_2$ and $\theta > 0$ such that, if D_0 is large enough and a_0 is small enough such that*

$$C_2 a \leq A (a + e^{-\theta D}) \leq \min\{\frac{\tau}{8}\} \cup \{\frac{c_p^0}{4}, 1 \leq p \leq L\}, \quad (3.2.20)$$

for $t^* > 0$, we assume that there exist C^1 functions $x_{1,k}(t), x_{2,k}(t), c_l(t), x_{0,l}(t) \in \mathbb{R}$ (with $c_l(t) > 0$) defined for $t \in [0, t^*]$ such that, if we denote

$$\forall t \in [0, t^*], \quad \varepsilon(t) := u(t) - \sum_{l=1}^L \widetilde{R}_l(t) - \sum_{k=1}^K \widetilde{B}_k(t), \quad (3.2.21)$$

where u is a solution of (mKdV),

$$\forall t \in [0, t^*], \quad \widetilde{R}_l(t) := \kappa_l Q_{c_l(t)}(\cdot + x_{0,l}(t) - c_l^0 t) \quad \text{for } 1 \leq l \leq L, \quad (3.2.22)$$

and

$$\forall t \in [0, t^*], \quad \widetilde{B}_k(t) := B_{\alpha_k, \beta_k}(t, \cdot; x_{1,k}(t), x_{2,k}(t)), \quad \text{for } 1 \leq k \leq K, \quad (3.2.23)$$

$\widetilde{P}_j := \widetilde{R}_l$ if $P_j = R_l$ and $\widetilde{P}_j := \widetilde{B}_k$ if $P_j = B_k$, and $\widetilde{x}_l^s(t) := -x_{0,l}(t) + v_l^s t$, $\widetilde{x}_k^b(t) := -x_{2,k}(t) + v_k^b t$, $\widetilde{x}_j(t) := \widetilde{x}_l^s(t)$ if $P_j = R_l$, and $\widetilde{x}_j(t) := \widetilde{x}_k^b(t)$ if $P_j = B_k$,
and if we assume that

$$\forall 1 \leq j \leq J-1, \quad x_{j+1}(0) - x_j(0) \geq 2D, \quad (3.2.24)$$

$$\forall t \in [0, t^*], \quad \|\varepsilon(t)\|_{H^2} \leq A (a + e^{-\theta D}), \quad \|\varepsilon(0)\|_{H^2} \leq C_2 a, \quad (3.2.25)$$

$$\forall t \in [0, t^*], \quad |c_l(t) - c_l^0| \leq A (a + e^{-\theta D}), \quad |c_l(0) - c_l^0| \leq C_2 a, \quad (3.2.26)$$

$$\sum_{l=1}^L |x_{0,l}(0) + x_{0,l}^0| + \sum_{k=1}^K (|x_{1,k}(0) - x_{1,k}^0| + |x_{2,k}(0) - x_{2,k}^0|) \leq C a, \quad (3.2.27)$$

$$\forall t \in [0, t^*], \quad \sum_{l=1}^L (|c'_l(t)| + |x'_{0,l}(t)|) + \sum_{k=1}^K (|x'_{1,k}(t)| + |x'_{2,k}(t)|) \leq C A (a + e^{-\theta D}), \quad (3.2.28)$$

where $C > 0$ is a large enough constant, and

$$\forall t \in [0, t^*], \quad \int \widetilde{B}_{k1}(t) \varepsilon(t) = \int \widetilde{B}_{k2}(t) \varepsilon(t) = \int \widetilde{R}_l(t) \varepsilon(t) = \int \widetilde{R}_{lx}(t) \varepsilon(t) = 0, \quad (3.2.29)$$

then

$$\forall t \in [0, t^*], \quad \|\varepsilon(t)\|_{H^2} \leq \frac{A}{2} (a + e^{-\theta D}), \quad (3.2.30)$$

$$\forall t \in [0, t^*], \quad |c_l(t) - c_l^0| \leq \frac{A}{2} (a + e^{-\theta D}). \quad (3.2.31)$$

Remark 3.8. We will use generalized notations for $x_{1,k}(t)$, $x_{2,k}(t)$, $c_l(t)$ and $x_{0,l}(t)$.

For $j = 1, \dots, J$, either if $P_j = R_l$ is a soliton, we denote

$$x_{1,j}^*(t) := c_l(t), \quad x_{2,j}^*(t) := x_{0,l}(t), \quad (3.2.32)$$

or if $P_j = B_k$ is a breather, we denote

$$x_{1,j}^*(t) := x_{1,k}(t), \quad x_{2,j}^*(t) := x_{2,k}(t). \quad (3.2.33)$$

We denote

$$\widetilde{P}(t) := \sum_{j=1}^J \widetilde{P}_j(t) = \sum_{k=1}^K \widetilde{B}_k(t) + \sum_{l=1}^L \widetilde{R}_l(t). \quad (3.2.34)$$

Remark 3.9. We note that, for any $j = 1, \dots, J-1$, (3.2.24) and (3.2.27), assuming that a_0 is chosen small enough and D_0 is chosen large enough, imply that

$$\widetilde{x_{j+1}}(0) - \widetilde{x_j}(0) \geq D. \quad (3.2.35)$$

From definitions of $\widetilde{x_j}(t)$, of τ and (3.2.28), assuming that a_0 is chosen small enough and D_0 is chosen large enough, we deduce that for any $t \in [0, t^*]$,

$$\forall 1 \leq j \leq J-1, \quad \widetilde{x_{j+1}}'(t) - \widetilde{x_j}'(t) \geq \frac{\tau}{2}, \quad (3.2.36)$$

and

$$\forall 2 \leq j \leq J, \quad \frac{v_2}{2} \leq \widetilde{x_j}'(t) \leq 2v_J. \quad (3.2.37)$$

From (3.2.35) and (3.2.36), we may deduce that for any $t \in [0, t^*]$,

$$\forall 1 \leq j \leq J, \quad \widetilde{x_{j+1}}(t) - \widetilde{x_j}(t) \geq D + \frac{\tau}{2}t. \quad (3.2.38)$$

The proof of this proposition will be the goal of the following. The proof of Theorem 3.1 then follows from a continuity argument.

Proof of Theorem 3.1 assuming Proposition 3.7. We take A, θ, D_0, a_0 that work for Proposition 3.7, we will show that they will also work for Theorem 3.1. We take $D \geq D_0$ and $0 \leq a \leq a_0$ and we assume that (3.1.22) is true for a solution u of (mKdV). This implies (3.2.24).

We assume that a_0 is small enough and D_0 is large enough such that $A(a + e^{-\theta D}) \leq \min\{\frac{\tau}{8}\} \cup \{\frac{c_p^0}{4}, 1 \leq p \leq L\}$. Because $\frac{A}{C_2} > 1$, by continuity, there exists $t_1 > 0$ such that

$$\forall t \in [0, t_1], \quad \|u(t) - P(t)\|_{H^2} \leq \frac{A}{C_2} (a + e^{-\theta D}). \quad (3.2.39)$$

Of course, we have that

$$\forall t \in [0, t_1], \quad x_j(t) - x_{j-1}(t) \geq 2D + \tau t. \quad (3.2.40)$$

We apply Lemma 3.5 on $[0, t_1]$ with $A' = \frac{A}{C_2}$. We take D_0 larger and a_0 smaller if needed. So, there exist C^1 functions $x_{1,k}(t), x_{2,k}(t), c_l(t), x_{0,l}(t) \in \mathbb{R}$ with $c_l(t) > 0$ defined for $t \in [0, t_1]$, such that if we set

$$\varepsilon(t) := u(t) - \widetilde{P}(t), \quad (3.2.41)$$

with the same notations as usual, we have that

$$\forall t \in [0, t_1], \quad \|\varepsilon(t)\|_{H^2} \leq A(a + e^{-\theta D}), \quad \|\varepsilon(0)\|_{H^2} \leq C_2 a, \quad (3.2.42)$$

$$\forall t \in [0, t_1], \quad |c_l(t) - c_l^0| \leq A(a + e^{-\theta D}), \quad |c_l(0) - c_l^0| \leq C_2 a, \quad (3.2.43)$$

$$\sum_{l=1}^L |x_{0,l}(0) + x_{0,l}^0| + \sum_{k=1}^K (|x_{1,k}(0) - x_{1,k}^0| + |x_{2,k}(0) - x_{2,k}^0|) \leq C a, \quad (3.2.44)$$

$$\forall t \in [0, t_1], \quad \sum_{l=1}^L (|c'_l(t)| + |x'_{0,l}(t)|) + \sum_{k=1}^K (|x'_{1,k}(t)| + |x'_{2,k}(t)|) \leq C A(a + e^{-\theta D}), \quad (3.2.45)$$

if $\theta > 0$ is chosen small enough, and

$$\forall t \in [0, t_1], \quad \int \widetilde{B}_{k_1}(t)\varepsilon(t) = \int \widetilde{B}_{k_2}(t)\varepsilon(t) = \int \widetilde{R}_l(t)\varepsilon(t) = \int \widetilde{R}_{lx}(t)\varepsilon(t) = 0. \quad (3.2.46)$$

Remember that (3.2.38) for any $t \in [0, t_1]$ is a consequence of what is written above.

Let $I^* \subset \mathbb{R}_+$ be the supremum of intervals $I' \supset [0, t_1]$ for inclusion in the set of intervals of \mathbb{R}_+ such that the C^1 functions $x_{1,k}(t)$, $x_{2,k}(t)$, $c_l(t)$, $x_{0,l}(t)$ may be extended on I' and such that (3.2.24), (3.2.25), (3.2.26), (3.2.27), (3.2.28) and (3.2.29) are still satisfied for any $t \in I'$.

In order to make the definition above licit, we need to point out the following fact : by uniqueness in Lemma 3.5, we find that if we have two extensions on I_2 and I_3 , then we find that those extensions coincide on $I_2 \cap I_3$ (we remind that $[0, t_1] \subset I_2 \cap I_3$). This is why, if we have suitable extensions on I_2 and I_3 , then we have a suitable extension on $I_2 \cup I_3$. And, the supremum I^* is simply the union of all the possible extensions I' and the implicit functions may be extended on I^* .

When (3.2.24), (3.2.25), (3.2.26), (3.2.27), (3.2.28) and (3.2.29) are true for $x_{1,k}(t)$, $x_{2,k}(t)$, $c_l(t)$, $x_{0,l}(t)$ with $I = [0, t_2] \subset I^*$ (where $t_2 \geq t_1$), we may extend these implicit functions a bit further (in a random way, that we denote $y_{1,k}(t)$, $y_{2,k}(t)$, $y_{3,l}(t)$ and $y_{0,l}(t)$, and we denote y_j their positions) and have (3.2.24), (3.2.25), (3.2.26), (3.2.27), (3.2.28) and (3.2.29) that are still satisfied, but on the extended interval. The extended interval will be an interval of the form $[0, t'_2]$, where $t'_2 > t_2$. We do it in the following way. First, we apply Proposition 3.7 on I , and that makes (3.2.25) and (3.2.26) a bit improved on I and become (3.2.30) and (3.2.31) on I . And so, (3.2.25) and (3.2.26) are satisfied on an extension of I . After application of Lemma 3.5 with $A' = A$ (where the notation A' is from Lemma 3.5), that we may apply thanks to (3.2.38) (more precisely, from Remark 3.9, (3.2.38) is satisfied on I , from what for any $1 \leq j \leq J-1$, $\widetilde{y_{j+1}}(t) - \widetilde{y_j}(t) \geq D$ is true on an extension of I) we see that (3.2.29) can be also extended (after modification of the implicit functions on the extension of I : we now denote them $x_{1,k}(t)$, $x_{2,k}(t)$, $x_{3,l}(t)$ and $x_{0,l}(t)$, as they coincide with implicit functions given at the beginning of this paragraph on I from uniqueness in Lemma 3.5). Note that after this modification, it is needed to reconsider all the previous sentences of this paragraph, in order to take a smaller extension if needed and have (3.2.25) and (3.2.26) satisfied. Because of Remark 3.9, (3.2.38) is satisfied on I , this is why for any $1 \leq j \leq J-1$, $\widetilde{x_{j+1}}(t) - \widetilde{x_j}(t) \geq D$ is satisfied on an extension of I (eventually smaller, but of the form $[0, t'_2]$ as specified at the beginning of the paragraph). This is why, we may apply consequences (3.2.16) and (3.2.17) of Lemma 3.5 and obtain that (3.2.28) is also satisfied on this extension of I . Thus, we may find an extension $[0, t'_2]$ (with $t'_2 > t_2$) of I and extensions of $x_{1,k}(t)$, $x_{2,k}(t)$, $c_l(t)$, $x_{0,l}(t)$ on this extension such that (3.2.24) (3.2.25), (3.2.26), (3.2.27) (3.2.28) and (3.2.29) are satisfied.

We deduce that the interval I^* is necessarily an open subset of \mathbb{R}_+ . Let us prove that I^* is also a closed subset of \mathbb{R}_+ . This will allow us to conclude that $I^* = \mathbb{R}_+$.

If I^* is not closed, then it is $[0, t_2)$, where $t_2 > 0$. Let (T_n) be a sequence of points of I^* that converges to t_2 . Then, $(x_{1,k}(T_n))$, $(x_{2,k}(T_n))$, $(c_l(T_n))$ and $(x_{0,l}(T_n))$ are Cauchy sequences because of (3.2.28). Thus, they converge

and $x_{1,k}(t)$, $x_{2,k}(t)$, $c_l(t)$ and $x_{0,l}(t)$ may be extended continuously in $t = t_2$. By continuity, it is clear that (3.2.25), (3.2.26), (3.2.29) and (3.2.38) are still satisfied in $t = t_2$. This is why, we may apply Lemma 3.5 on $[0, t_2]$. From uniqueness in Lemma 3.5, we find that $x_{1,k}(t)$, $x_{2,k}(t)$, $c_l(t)$ and $x_{0,l}(t)$ are C^1 on $[0, t_2]$. Thus, (3.2.28) is also satisfied in $t = t_2$. That contradicts the maximality of I^* . Thus, I^* is closed in \mathbb{R}_+ .

So, we deduce that we have C^1 functions $x_{1,k}(t)$, $x_{2,k}(t)$, $c_l(t)$, $x_{0,l}(t)$ defined for any $t \geq 0$ such that,

$$\forall t \geq 0, \quad \|\varepsilon(t)\|_{H^2} \leq A(a + e^{-\theta D}), \quad (3.2.47)$$

$$\forall t \geq 0, \quad |c_l(t) - c_l^0| \leq A(a + e^{-\theta D}), \quad (3.2.48)$$

$$\forall t \geq 0, \quad \sum_{l=1}^L (|c'_l(t)| + |x'_{0,l}(t)|) + \sum_{k=1}^K (|x'_{1,k}(t)| + |x'_{2,k}(t)|) \leq CA(a + e^{-\theta D}), \quad (3.2.49)$$

where $C > 0$ is a large enough constant, and

$$\forall t \geq 0, \quad \int \widetilde{B}_{k1}(t)\varepsilon(t) = \int \widetilde{B}_{k2}(t)\varepsilon(t) = \int \widetilde{R}_l(t)\varepsilon(t) = \int \widetilde{R}_{lx}(t)\varepsilon(t) = 0. \quad (3.2.50)$$

And we set

$$w(t) := u(t) - \sum_{l=1}^L \kappa_l Q_{c_l^0}(\cdot + x_{0,l}(t) - c_l^0 t) - \sum_{k=1}^K B_{\alpha_k, \beta_k}(t, \cdot; x_{1,k}(t), x_{2,k}(t)). \quad (3.2.51)$$

We need to bound w to finish the proof. For $t \geq 0$, we use triangular inequality and we bound the norm between two ground states centered at a same point,

$$\begin{aligned} \|w(t)\|_{H^2} &\leq \|\varepsilon(t)\|_{H^2} + \sum_{l=1}^L \|Q_{c_l(t)} - Q_{c_l^0}\|_{H^2} \\ &\leq \|\varepsilon(t)\|_{H^2} + C \sum_{l=1}^L |c_l(t) - c_l^0| \\ &\leq CA(a + e^{-\theta D}), \end{aligned} \quad (3.2.52)$$

and this is exactly what we wanted to prove. Moreover (3.1.24) is a straightforward consequence of (3.2.49). \square

Hence, we are left to prove Proposition 3.7.

3.2.4 Proof by induction

We will prove Proposition 3.7 by induction. More precisely, we want to find $A > C_2, \theta, D_0, a_0$ so that the Proposition 3.7 holds for any $t^* > 0$.

We assume that there exists functions $x_{1,k}(t)$, $x_{2,k}(t)$, $c_l(t)$, $x_{0,l}(t) \in \mathbb{R}$ defined for $t \in [0, t^*]$ such that, with notations of Proposition 3.7,

$$\forall t \in [0, t^*], \quad \|\varepsilon(t)\|_{H^2} \leq A(a + e^{-\theta D}), \quad \|\varepsilon(0)\|_{H^2} \leq C_2 a, \quad (3.2.53)$$

$$\forall t \in [0, t^*], \quad |c_l(t) - c_l^0| \leq A(a + e^{-\theta D}) \leq \min \left\{ \frac{\tau}{8} \right\} \cup \left\{ \frac{c_p^0}{4}, 0 \leq p \leq L \right\}, \quad |c_l(0) - c_l^0| \leq C_2 a, \quad (3.2.54)$$

$$\forall t \in [0, t^*], \quad \sum_{l=1}^L (|c'_l(t)| + |x'_{0,l}(t)|) + \sum_{k=1}^K (|x'_{1,k}(t)| + |x'_{2,k}(t)|) \leq CA(a + e^{-\theta D}), \quad (3.2.55)$$

$$\forall t \in [0, t^*], \quad \int \widetilde{B}_{k1}(t)\varepsilon(t) = \int \widetilde{B}_{k2}(t)\varepsilon(t) = \int \widetilde{R}_l(t)\varepsilon(t) = \int \widetilde{R}_{lx}(t)\varepsilon(t) = 0, \quad (3.2.56)$$

as well as (3.2.27) and (3.2.24).

As in Remark 3.9, we deduce that (3.2.36), (3.2.37) and (3.2.38) are satisfied.

And the goal is to improve inequalities (3.2.53) and (3.2.54).

We define the average between positions of two consecutive objects. For $j = 3, \dots, J$, we set

$$\forall t \in [0, t^*], \quad m_j(t) := \frac{\widetilde{x_{j-1}}(t) + \widetilde{x_j}(t)}{2}, \quad (3.2.57)$$

and we set

$$\forall t \in [0, t^*], \quad m_2(t) := \frac{\widetilde{x_1}(0) + \widetilde{x_2}(0)}{2} + \int_0^t \max\left(\frac{\widetilde{x_1}'(s) + \widetilde{x_2}'(s)}{2}, \frac{\widetilde{x_2}'(s)}{2}\right) ds. \quad (3.2.58)$$

With these definitions, we make sure that for any $j = 2, \dots, J$, $m_j'(t) > 0$, even if $v_1 < 0$.

By (3.2.36), for $j \geq 3$,

$$\forall t \in [0, t^*], \quad \widetilde{x_j}'(t) - m_j'(t) \geq \frac{\tau}{4}, \quad m_j'(t) - \widetilde{x_{j-1}}'(t) \geq \frac{\tau}{4}, \quad (3.2.59)$$

and for $j = 2$, we have

$$\forall t \in [0, t^*], \quad m_2'(t) = \max\left(\frac{\widetilde{x_1}'(t) + \widetilde{x_2}'(t)}{2}, \frac{\widetilde{x_2}'(t)}{2}\right), \quad (3.2.60)$$

and so,

$$\forall t \in [0, t^*], \quad m_2'(t) - \widetilde{x_1}'(t) \geq \frac{\tau}{4}, \quad \widetilde{x_2}'(t) - m_2'(t) \geq \min\left(\frac{v_2}{4}, \frac{\tau}{4}\right). \quad (3.2.61)$$

This is why, we set

$$\zeta := \min\left(\frac{v_2}{4}, \frac{\tau}{4}\right), \quad (3.2.62)$$

a constant that depends only on problem data, and so for any $j \geq 2$,

$$\forall t \in [0, t^*], \quad \widetilde{x_j}'(t) - m_j'(t) \geq \zeta, \quad m_j'(t) - \widetilde{x_{j-1}}'(t) \geq \zeta. \quad (3.2.63)$$

The latter implies that $\forall j \geq 2, \quad \forall t \in [0, t^*], \quad \widetilde{x_{j-1}}(t) < m_j(t) < \widetilde{x_j}(t)$, and we may deduce by integration and by (3.2.38) and (3.2.63) that

$$\begin{aligned} \widetilde{x_j}(t) - m_j(t) &= \widetilde{x_j}(0) - m_j(0) + \int_0^t (\widetilde{x_j}'(s) - m_j'(s)) ds \\ &= \frac{\widetilde{x_j}(0) - \widetilde{x_{j-1}}(0)}{2} + \int_0^t (\widetilde{x_j}'(s) - m_j'(s)) ds \\ &\geq \frac{D}{2} + \zeta t, \end{aligned} \quad (3.2.64)$$

and similarly,

$$m_j(t) - \widetilde{x_{j-1}}(t) \geq \frac{D}{2} + \zeta t. \quad (3.2.65)$$

We have that (the m_j are chosen for that) for any $j \geq 2$,

$$\forall t \in [0, t^*], \quad 2v_J \geq \widetilde{x_J}'(t) \geq m_J'(t) \geq \frac{v_2}{4} \geq \zeta. \quad (3.2.66)$$

We will reason by induction in order to improve (3.2.53) and (3.2.54). For this, we introduce a cut-off function. Let $\sigma > 0$ be a constant small enough for which the conditions will be fixed in the following of the proof.

We denote :

$$\Psi(x) := \frac{2}{\pi} \arctan(\exp(\sqrt{\sigma}x/2)). \quad (3.2.67)$$

By direct calculations,

$$\Psi'(x) = \frac{\sqrt{\sigma}}{2\pi \cosh(\sqrt{\sigma}x/2)}, \quad (3.2.68)$$

and so,

$$|\Psi'(x)| \leq C \exp(-\sqrt{\sigma}|x|/2). \quad (3.2.69)$$

We have the following properties : $\lim_{+\infty} \Psi = 1$, $\lim_{-\infty} \Psi = 0$, for all $x \in \mathbb{R}$ $\Psi(-x) + \Psi(x) = 1$, $\Psi'(x) > 0$, $|\Psi''(x)| \leq \frac{\sqrt{\sigma}}{2} |\Psi'(x)|$, $|\Psi'''(x)| \leq \frac{\sqrt{\sigma}}{2} |\Psi''(x)|$, $|\Psi'(x)| \leq \frac{\sqrt{\sigma}}{2} \Psi$ and $|\Psi'(x)| \leq \frac{\sqrt{\sigma}}{2} (1 - \Psi)$.

We define cut-off functions filtering P_j and all the objects faster than P_j : for $j = 2, \dots, J$,

$$\Phi_j(t, x) := \Psi(x - m_j(t)). \quad (3.2.70)$$

We have :

$$(\Phi_j)_t = -m'_j (\Phi_j)_x. \quad (3.2.71)$$

We may extend this definition to $j = 1$ and $j = J + 1$ in the following way : $\Phi_1 := 1$ and $\Phi_{J+1} := 0$.

In order to prove Proposition 3.7 by induction, we will find an increasing sequence $(Z_j)_{j=1, \dots, J+1}$ such that $Z_1 := 2$ and $Z_{J+1} := +\infty$ and such that we will be able to prove the following proposition for any $j = 1, \dots, J$. The goal is to obtain inequalities of Proposition 3.7 with better constants. So, in order to achieve this, we do the following induction : we suppose that localized inequalities around P_{j+1}, \dots, P_J are obtained with strongly improved constants (constants divided by Z_{j+1}), and we deduce from them localized inequalities around P_j, \dots, P_J with improved constants, but a little bit less improved than earlier (constants divided by Z_j). We will also assume the bootstrap assumption. This induction is sufficient, because it starts from an assumption on an empty set of objects and ends with a conclusion with inequalities localized around all the objects, i.e. global (not localized at all).

Proposition 3.10. *Assuming that*

$$\forall t \in [0, t^*], \quad \int (\varepsilon^2 + \varepsilon_x^2 + \varepsilon_{xx}^2) \Phi_{j+1} \leq \left(\frac{A}{Z_{j+1}} \right)^2 (a^2 + e^{-2\theta D}), \quad (3.2.72)$$

and for any $j' \geq j + 1$ such that $P_{j'} = R_l$ is a soliton,

$$\forall t \in [0, t^*], \quad |c_l(t) - c_l(0)| \leq \left(\frac{A}{Z_{j+1}} \right)^2 (a^2 + e^{-2\theta D}), \quad (3.2.73)$$

and for any $j' \geq j + 1$ such that $P_{j'}$ is a breather,

$$\forall t \in [0, t^*], \quad \left| \int \widetilde{P}_{j'} \varepsilon(t) - \int \widetilde{P}_{j'} \varepsilon(0) \right| \leq \left(\frac{A}{Z_{j+1}} \right)^2 (a^2 + e^{-2\theta D}), \quad (3.2.74)$$

and

$$\forall t \in [0, t^*], \quad \left| \int [\widetilde{P}_{j'}]_{xx} + \widetilde{P}_{j'}^3 \varepsilon(t) - \int [\widetilde{P}_{j'}]_{xx} + \widetilde{P}_{j'}^3 \varepsilon(0) \right| \leq \left(\frac{A}{Z_{j+1}} \right)^2 (a^2 + e^{-2\theta D}), \quad (3.2.75)$$

we have that

$$\forall t \in [0, t^*], \quad \int (\varepsilon^2 + \varepsilon_x^2 + \varepsilon_{xx}^2) \Phi_j \leq \left(\frac{A}{Z_j} \right)^2 (a^2 + e^{-2\theta D}), \quad (3.2.76)$$

and if $P_j = R_l$ is a soliton,

$$\forall t \in [0, t^*], \quad |c_l(t) - c_l(0)| \leq \left(\frac{A}{Z_j} \right)^2 (a^2 + e^{-2\theta D}), \quad (3.2.77)$$

and if P_j is a breather,

$$\forall t \in [0, t^*], \quad \left| \int \widetilde{P}_j \varepsilon(t) - \int \widetilde{P}_j \varepsilon(0) \right| \leq \left(\frac{A}{Z_j} \right)^2 (a^2 + e^{-2\theta D}), \quad (3.2.78)$$

and

$$\forall t \in [0, t^*], \quad \left| \int [\widetilde{P}_{jxx} + \widetilde{P}_j^3] \varepsilon(t) - \int [\widetilde{P}_{jxx} + \widetilde{P}_j^3] \varepsilon(0) \right| \leq \left(\frac{A}{Z_j} \right)^2 (a^2 + e^{-2\theta D}). \quad (3.2.79)$$

Remark 3.11. For $j = 1, \dots, J$, we may denote by \mathcal{P}_j the following assertion : (3.2.76), (3.2.77), (3.2.78) and (3.2.79). The Proposition 3.10 may be reformulated in the following way :

There exists an increasing sequence $(Z_j)_{j=1, \dots, J+1}$ with $Z_1 = 2$ and $Z_{J+1} = +\infty$, A large enough and $\theta > 0$ such that for D_0 large enough and for a_0 small enough, we have the following : for any $j = 1, \dots, J$,

$$\mathcal{P}_{j+1} \implies \mathcal{P}_j. \quad (3.2.80)$$

Remark 3.12. Note that the inequalities (3.2.78) and (3.2.79) imply the following inequality for any $t \in [0, t^*]$ in the case when P_j is a breather :

$$\begin{aligned} & \left| \int \left[\widetilde{P}_{jxxxx} + 5\widetilde{P}_j \widetilde{P}_{jx}^2 + 5\widetilde{P}_j^2 \widetilde{P}_{jxx} + \frac{3}{2} \widetilde{P}_j^5 \right] \varepsilon(t) \right. \\ & \left. - \int \left[\widetilde{P}_{jxxxx} + 5\widetilde{P}_j \widetilde{P}_{jx}^2 + 5\widetilde{P}_j^2 \widetilde{P}_{jxx} + \frac{3}{2} \widetilde{P}_j^5 \right] \varepsilon(0) \right| \leq C \left(\frac{A}{Z_j} \right)^2 (a^2 + e^{-2\theta D}), \end{aligned} \quad (3.2.81)$$

because of the elliptic equation verified by \widetilde{P}_j , which, in the case when $\widetilde{P}_j = \widetilde{B}_k$ is a breather is the following :

$$\widetilde{B}_{kxxxx} + 5\widetilde{B}_k \widetilde{B}_{kx}^2 + 5\widetilde{B}_k^2 \widetilde{B}_{kxx} + \frac{3}{2} \widetilde{B}_k^5 - 2(\beta_k^2 - \alpha_k^2) (\widetilde{B}_{kxx} + \widetilde{B}_k^3) + (\alpha_k^2 + \beta_k^2)^2 \widetilde{B}_k = 0. \quad (3.2.82)$$

But the inequality (3.2.81) is also true in the case when $\widetilde{P}_j = \widetilde{R}_l$ is a soliton, because we have even better in this case. There are two elliptic equations [108] :

$$\widetilde{R}_{lxxxx} + 5\widetilde{R}_l \widetilde{R}_{lx}^2 + 5\widetilde{R}_l^2 \widetilde{R}_{lxx} + \frac{3}{2} \widetilde{R}_l^5 - 2c_l(t) (\widetilde{R}_{lxx} + \widetilde{R}_l^3) + c_l(t)^2 \widetilde{R}_l = 0, \quad (3.2.83)$$

and

$$(\widetilde{R}_{lxx} + \widetilde{R}_l^3) - c_l(t) \widetilde{R}_l = 0, \quad (3.2.84)$$

which implies that (3.2.56) implies

$$\int (\widetilde{R}_{lxx} + \widetilde{R}_l^3) \varepsilon = 0, \quad (3.2.85)$$

and

$$\int \left(\widetilde{R}_{lxxxx} + 5\widetilde{R}_l \widetilde{R}_{lx}^2 + 5\widetilde{R}_l^2 \widetilde{R}_{lxx} + \frac{3}{2} \widetilde{R}_l^5 \right) \varepsilon = 0, \quad (3.2.86)$$

which implies of course the inequality (3.2.81).

The proof of Proposition 3.10 will be the goal of the Section 3. The proof of Proposition 3.7 follows from it.

Proof of Proposition 3.7 assuming Proposition 3.10. We perform the induction in the decreasing order : $j = J, J-1, \dots, 2, 1$. \mathcal{P}_{J+1} is empty, and Proposition 3.10 gives the (decreasing) induction step. Hence, $\mathcal{P}_1, \dots, \mathcal{P}_J$ are true. Due to \mathcal{P}_1 :

$$\forall t \in [0, t^*], \quad \|\varepsilon(t)\|_{H^2} \leq \frac{A}{2} (a + e^{-\theta D}). \quad (3.2.87)$$

For $l = 1, \dots, L$, we have from \mathcal{P}_1 that for any $t \in [0, t^*]$,

$$\begin{aligned} |c_l(t) - c_l^0| &\leq |c_l(t) - c_l(0)| + |c_l(0) - c_l^0| \\ &\leq \left(\frac{A}{2}\right)^2 (a^2 + e^{-2\theta D}) + C_2 a \\ &\leq \left[\left(\frac{A}{2}\right)^2 (a + e^{-\theta D}) + C_2\right] (a + e^{-\theta D}). \end{aligned} \quad (3.2.88)$$

If we take A large enough with respect to C_2 , and a_0 smaller and D_0 larger if needed with respect to A and θ , then

$$\left(\frac{A}{2}\right)^2 (a + e^{-\theta D}) + C_2 \leq \frac{A}{2}, \quad (3.2.89)$$

and that concludes the proof of Proposition 3.7. □

Hence, we are left to prove Proposition 3.10. We will write the proof for a fixed $j \in \{1, \dots, J\}$. We assume \mathcal{P}_{j+1} with a set of constants $Z_1, \dots, Z_{J+1}, A, \theta, D_0, a_0$. We will establish some conditions for these constants during the proof.

3.3 Orbital stability of a sum of solitons and breathers in $H^2(\mathbb{R})$

In this Section, we prove Proposition 3.10. We assume \mathcal{P}_{j+1} and we prove \mathcal{P}_j .

3.3.1 Almost decay of conservation laws at the right

We localize around the most right objects, starting from and including the j -th. We set :

$$M_j(t) := \frac{1}{2} \int u^2(t) \Phi_j(t) =: M_j[u](t), \quad (3.3.1)$$

$$E_j(t) := \int \left[\frac{1}{2} u_x^2 - \frac{1}{4} u^4 \right] \Phi_j(t) =: E_j[u](t), \quad (3.3.2)$$

$$F_j(t) := \int \left[\frac{1}{2} u_{xx}^2 - \frac{5}{2} u^2 u_x^2 + \frac{1}{4} u^6 \right] \Phi_j(t) =: F_j[u](t). \quad (3.3.3)$$

Lemma 3.13. *Let $0 < \omega_1, \omega_2 < 1$. If $0 < \sigma < \zeta$, $0 < \theta < \frac{\sqrt{\sigma}}{16}$, D_0 is large enough and a_0 is small enough (depending on A, θ, ω_1 and ω_2), then for any $t \in [0, t^*]$,*

$$M_j(t) - M_j(0) \leq C e^{-2\theta D}, \quad (3.3.4)$$

$$(E_j(t) + \omega_1 M_j(t)) - (E_j(0) + \omega_1 M_j(0)) \leq C e^{-2\theta D}, \quad (3.3.5)$$

$$(F_j(t) + \omega_2 M_j(t)) - (F_j(0) + \omega_2 M_j(0)) \leq C e^{-2\theta D}. \quad (3.3.6)$$

Remark 3.14. If $j = 1$, we have $= 0$ at the place of $\leq Ce^{-2\theta D}$, we will need it in the following of the proof.

Proof. If $j = 1$, we have exact conservation laws, so this Lemma is obvious. We assume that $j \geq 2$ for the following of this proof. From Appendix and minoration of m'_j ,

$$\begin{aligned} \frac{d}{dt}M_j(t) &= \int \left[-\frac{3}{2}u_x^2 + \frac{3}{4}u^4 \right] \Phi_{jx} + \frac{1}{2} \int u^2 \Phi_{jxxx} + \frac{1}{2} \int u^2 \Phi_{jt} \\ &= \int \left[-\frac{3}{2}u_x^2 + \frac{3}{4}u^4 \right] \Phi_{jx} + \frac{1}{2} \int u^2 \Phi_{jxxx} - \frac{1}{2}m'_j \int u^2 \Phi_{jx} \\ &\leq \int \left[-\frac{3}{2}u_x^2 + \frac{3}{4}u^4 - \frac{1}{2}\sigma u^2 \right] \Phi_{jx} + \frac{1}{2} \int u^2 \Phi_{jxxx}. \end{aligned} \quad (3.3.7)$$

Now, we use that $|\Phi_{jxxx}| \leq \frac{\sigma}{4}\Phi_{jx}$, and we obtain that

$$2\frac{d}{dt}M_j(t) \leq - \int \left[-3u_x^2 + \frac{3\sigma}{4}u^2 - \frac{3}{2}u^4 \right] \Phi_{jx}. \quad (3.3.8)$$

Now, from Appendix, we know that for $r > 0$, if t, x satisfy $\widetilde{x_{j-1}}(t) + r < x < \widetilde{x_j}(t) - r$, then $|\widetilde{P}(t, x)| \leq Ce^{-\beta r}$. And so, for t, x such that $\widetilde{x_{j-1}}(t) + r < x < \widetilde{x_j}(t) - r$, by Sobolev embedding,

$$\begin{aligned} |u(t, x)| &\leq |\widetilde{P}(t, x)| + C\|\varepsilon(t)\|_{H^2} \\ &\leq Ce^{-\beta r} + CA(a + e^{-\theta D}). \end{aligned} \quad (3.3.9)$$

From that, we can deduce that for r large enough, a_0 small enough and D_0 large enough, for $x \in [\widetilde{x_{j-1}}(t) + r, \widetilde{x_j}(t) - r]$, we can obtain that $|u(t, x)|$ is bounded by any chosen constant. Here, we will use that to bound $\frac{3}{2}u^2$ by $\frac{\sigma}{4}$.

For t, x such that $x < \widetilde{x_{j-1}}(t) + r$ or $x > \widetilde{x_j}(t) - r$:

$$\begin{aligned} |\Phi_{jx}(t, x)| &\leq C \exp\left(-\frac{\sqrt{\sigma}}{2}|x - m_j(t)|\right) \\ &\leq C \exp\left(-\frac{\sqrt{\sigma}}{2}\left(\frac{D}{2} + \zeta t - r\right)\right) \\ &\leq C \exp\left(-\frac{\sqrt{\sigma}\zeta}{2}t\right) \exp\left(-\frac{\sqrt{\sigma}}{4}D\right) \exp\left(\frac{\sqrt{\sigma}}{2}r\right), \end{aligned} \quad (3.3.10)$$

and so, if we choose D_0 large enough (more precisely, $D_0 \geq 4r$), we obtain for $x \notin [\widetilde{x_{j-1}}(t) + r, \widetilde{x_j}(t) - r]$:

$$|\Phi_{jx}(t, x)| \leq C \exp\left(-\frac{\sqrt{\sigma}\zeta}{2}t\right) \exp\left(-\frac{\sqrt{\sigma}}{8}D\right). \quad (3.3.11)$$

Because $\int u^4$ is bounded by a constant for any time (that depends only on problem data), we deduce that :

$$\begin{aligned} \frac{d}{dt}M_j(t) &\leq - \int \left(\frac{3}{2}u_x^2 + \frac{\sigma}{4}u^2 \right) \Phi_{jx}(t) + C \exp\left(-\frac{\sqrt{\sigma}\zeta}{2}t\right) \exp\left(-\frac{\sqrt{\sigma}}{8}D\right) \\ &\leq C \exp\left(-\frac{\sqrt{\sigma}\zeta}{2}t\right) \exp\left(-\frac{\sqrt{\sigma}}{8}D\right). \end{aligned} \quad (3.3.12)$$

We deduce what we want to prove by integration.

For the second inequality, the argument is similar. We start by using Appendix :

$$\begin{aligned} \frac{d}{dt} E_j(t) &= \int \left[-\frac{1}{2} (u_{xx} + u^3)^2 - u_{xx}^2 + 3u^2 u_x^2 \right] \Phi_{jx} + \frac{1}{2} \int u_x^2 \Phi_{jxxx} - m'_j \int \left[\frac{1}{2} u_x^2 - \frac{1}{4} u^4 \right] \Phi_{jx} \\ &\leq - \int \left[\frac{1}{2} (u_{xx} + u^3)^2 + u_{xx}^2 - 3u^2 u_x^2 + \frac{\sigma}{2} u_x^2 - \frac{v_J}{2} u^4 \right] \Phi_{jx} + \frac{1}{2} \int u_x^2 \Phi_{jxxx}, \end{aligned} \quad (3.3.13)$$

and again, because $|\Phi_{jxxx}| \leq \frac{\sigma}{4} \Phi_{jx}$, we obtain that

$$2 \frac{d}{dt} E_j(t) \leq - \int \left[(u_{xx} + u^3)^2 + 2u_{xx}^2 - 6u^2 u_x^2 + \frac{3\sigma}{4} u_x^2 - v_J u^4 \right] \Phi_{jx}, \quad (3.3.14)$$

and by doing a similar argument as for the mass, but to bound $6u^2$ by $\frac{\sigma}{4}$ and to bound $v_J u^2$ by $\omega_3 > 0$, a constant as small as we need, we obtain

$$\begin{aligned} \frac{d}{dt} E_j(t) &\leq - \int \left[\frac{1}{2} (u_{xx} + u^3)^2 + u_{xx}^2 + \frac{\sigma}{4} u_x^2 - \frac{\omega_3}{2} u^2 \right] \Phi_{jx} + C \exp \left(-\frac{\sqrt{\sigma}\zeta}{2} t \right) \exp \left(-\frac{\sqrt{\sigma}}{8} D \right) \\ &\leq \frac{\omega_3}{2} \int u^2 \Phi_{jx} + C \exp \left(-\frac{\sqrt{\sigma}\zeta}{2} t \right) \exp \left(-\frac{\sqrt{\sigma}}{8} D \right), \end{aligned} \quad (3.3.15)$$

and so, if we choose ω_3 such that $\frac{\omega_3}{2} \leq \omega_1 \frac{\sigma}{4}$,

$$\begin{aligned} \frac{d}{dt} (E_j(t) + \omega_1 M_j(t)) &\leq \frac{\omega_3}{2} \int u^2 \Phi_{jx} + C \exp \left(-\frac{\sqrt{\sigma}\zeta}{2} t \right) \exp \left(-\frac{\sqrt{\sigma}}{8} D \right) - \frac{\omega_1 \sigma}{4} \int u^2 \Phi_{jx} \\ &\leq C \exp \left(-\frac{\sqrt{\sigma}\zeta}{2} t \right) \exp \left(-\frac{\sqrt{\sigma}}{8} D \right), \end{aligned} \quad (3.3.16)$$

and we deduce what we want to prove by integration.

For the third inequality, the argument is similar. We start by using Appendix :

$$\begin{aligned} \frac{d}{dt} F_j(t) &= \int \left(-\frac{3}{2} u_{xxx}^2 + 9u_{xx}^2 u^2 + 15u_x^2 u u_{xx} + \frac{9}{16} u^8 + \frac{1}{4} u_x^4 + \frac{3}{2} u_{xx} u^5 - \frac{45}{4} u^4 u_x^2 \right) \Phi_{jx} \\ &\quad + 5 \int u^2 u_x u_{xx} \Phi_{jxx} + \frac{1}{2} \int u_{xx}^2 \Phi_{jxxx} - m'_j \int \left[\frac{1}{2} u_{xx}^2 - \frac{5}{2} u^2 u_x^2 + \frac{1}{4} u^6 \right] \Phi_{jx} \\ &\leq - \int \left(\frac{3}{2} u_{xxx}^2 - 9u_{xx}^2 u^2 - 15u_x^2 u u_{xx} - \frac{9}{16} u^8 - \frac{1}{4} u_x^4 - \frac{3}{2} u_{xx} u^5 + \frac{45}{4} u^4 u_x^2 + \frac{\sigma}{2} u_{xx}^2 \right. \\ &\quad \left. - 5v_J u^2 u_x^2 + \frac{\sigma}{4} u^6 \right) \Phi_{jx} + 5 \int u^2 u_x u_{xx} \Phi_{jxx} + \frac{1}{2} \int u_{xx}^2 \Phi_{jxxx}, \end{aligned} \quad (3.3.17)$$

and again, because $|\Phi_{jxx}| \leq \frac{\sqrt{\sigma}}{2} \Phi_{jx}$ and $|\Phi_{jxxx}| \leq \frac{\sigma}{4} \Phi_{jx}$, we obtain that

$$\begin{aligned} 2 \frac{d}{dt} F_j(t) &\leq - \int \left(3u_{xxx}^2 - \left(18 + \frac{5}{2} \sqrt{\sigma} \right) u_{xx}^2 u^2 - 30u_x^2 u u_{xx} - \frac{9}{8} u^8 - \frac{1}{2} u_x^4 - 3u_{xx} u^5 \right. \\ &\quad \left. + \frac{45}{2} u^4 u_x^2 + \frac{3\sigma}{4} u_{xx}^2 - \left(10v_J + \frac{5}{2} \sqrt{\sigma} \right) u^2 u_x^2 + \frac{\sigma}{2} u^6 \right) \Phi_{jx}, \end{aligned} \quad (3.3.18)$$

and by doing a similar argument as for the mass, but to bound $(18 + \frac{5}{2}\sqrt{\sigma})u^2$ by $\frac{\sigma}{4}$, to bound $30uu_{xx}$ by ω_4 , to bound $\frac{9}{8}u^6$ by ω_5 , to bound $\frac{1}{2}u_x^2$ by ω_4 , to bound $3u_{xx}u^3$ by ω_5 and to bound $(10v_J + \frac{5}{2}\sqrt{\sigma})u^2$ by ω_4 , where $\omega_3, \omega_4 > 0$ are constants that we can choose as small as we want. And we obtain

$$\begin{aligned} \frac{d}{dt}F_j(t) &\leq - \int \left[\frac{3}{2}u_{xxx}^2 + \frac{45}{4}u^4u_x^2 + \frac{\sigma}{4}u^6 + \frac{\sigma}{4}u_{xx}^2 - C\omega_4u_x^2 - C\omega_5u^2 \right] \Phi_{jx} \\ &\quad + C \exp\left(-\frac{\sqrt{\sigma}\zeta}{2}t\right) \exp\left(-\frac{\sqrt{\sigma}}{8}D\right) \\ &\leq C\omega_4 \int u_x^2 \Phi_{jx} + C\omega_5 \int u^2 \Phi_{jx} + C \exp\left(-\frac{\sqrt{\sigma}\zeta}{2}t\right) \exp\left(-\frac{\sqrt{\sigma}}{8}D\right), \end{aligned} \quad (3.3.19)$$

and so, if we choose ω_4, ω_5 such that $C\omega_4 \leq \omega_1 \frac{3}{2}$ and $C\omega_5 \leq \omega_1 \frac{\sigma}{4}$,

$$\frac{d}{dt}(F_j(t) + \omega_2 M_j(t)) \leq C \exp\left(-\frac{\sqrt{\sigma}\zeta}{2}t\right) \exp\left(-\frac{\sqrt{\sigma}}{8}D\right), \quad (3.3.20)$$

and we deduce what we want to prove by integration. \square

3.3.2 Quadratic approximation for conservation laws at the right

We can write the following Taylor expansions with $u = \tilde{P} + \varepsilon$ for any $t \in [0, t^*]$:

$$M_j(t) - M_j[\tilde{P}](t) - \int \tilde{P}\varepsilon\Phi_j - \frac{1}{2} \int \varepsilon^2\Phi_j = 0, \quad (3.3.21)$$

$$\left| E_j(t) - E_j[\tilde{P}](t) - \int [\tilde{P}_x\varepsilon_x - \tilde{P}^3\varepsilon] \Phi_j - \int \left[\frac{1}{2}\varepsilon_x^2 - \frac{3}{2}\tilde{P}^2\varepsilon^2 \right] \Phi_j \right| \leq CA(a + e^{-\theta D}) \int \varepsilon^2\Phi_j, \quad (3.3.22)$$

$$\begin{aligned} \left| F_j(t) - F_j[\tilde{P}](t) - \int \left[\tilde{P}_{xx}\varepsilon_{xx} - 5\tilde{P}\tilde{P}_x^2\varepsilon - 5\tilde{P}^2\tilde{P}_x\varepsilon_x + \frac{3}{2}\tilde{P}^5\varepsilon \right] \Phi_j \right. \\ \left. - \int \left[\frac{1}{2}\varepsilon_{xx}^2 - \frac{5}{2}\tilde{P}_x^2\varepsilon^2 - 10\tilde{P}\tilde{P}_x\varepsilon\varepsilon_x - \frac{5}{2}\tilde{P}^2\varepsilon_x^2 + \frac{15}{4}\tilde{P}^4\varepsilon^2 \right] \Phi_j \right| \leq CA(a + e^{-\theta D}) \int (\varepsilon^2 + \varepsilon_x^2) \Phi_j. \end{aligned} \quad (3.3.23)$$

Now, we want to simplify each term of the Taylor expansion.

Constant terms of the Taylor expansion

We obtain the following lemma dealing with variations of the constant parts of each Taylor expansion of conservation laws at the right. We reduce each variation to the variation of each conservation law of P_j . Note that if P_j is a breather, the variation of any conservation law of \tilde{P}_j is 0. But, if $P_j = R_l$, we may express in the following way $M[\tilde{R}_l]$, $E[\tilde{R}_l]$ and $F[\tilde{R}_l]$ with respect to Q , the ground state of parameter $c = 1$ (the basic ground state) :

$$M[\tilde{R}_l](t) = c_l(t)^{1/2}M[Q], \quad (3.3.24)$$

$$E[\tilde{R}_l](t) = c_l(t)^{3/2}E[Q], \quad (3.3.25)$$

$$F[\tilde{R}_l](t) = c_l(t)^{5/2}F[Q]. \quad (3.3.26)$$

Lemma 3.15. *If $\sigma < 4\beta^2$, $\theta < \min\left(\frac{\beta}{4}, \frac{\sqrt{\sigma}}{8}\right)$, then for any $t \in [0, t^*]$,*

$$\left| M_j[\tilde{P}](t) - M_j[\tilde{P}](0) - \left(M[\tilde{P}_j](t) - M[\tilde{P}_j](0) \right) \right| \leq Ce^{-2\theta D} + C \left(\frac{A}{Z_{j+1}} \right)^2 (a^2 + e^{-2\theta D}), \quad (3.3.27)$$

$$\left| E_j[\tilde{P}](t) - E_j[\tilde{P}](0) - \left(E[\tilde{P}_j](t) - E[\tilde{P}_j](0) \right) \right| \leq Ce^{-2\theta D} + C \left(\frac{A}{Z_{j+1}} \right)^2 (a^2 + e^{-2\theta D}), \quad (3.3.28)$$

$$\left| F_j[\tilde{P}](t) - F_j[\tilde{P}](0) - \left(F[\tilde{P}_j](t) - F[\tilde{P}_j](0) \right) \right| \leq Ce^{-2\theta D} + C \left(\frac{A}{Z_{j+1}} \right)^2 (a^2 + e^{-2\theta D}). \quad (3.3.29)$$

Remark 3.16. In the case when P_j is a breather, we have that

$$M[\tilde{P}_j](t) - M[\tilde{P}_j](0) = E[\tilde{P}_j](t) - E[\tilde{P}_j](0) = F[\tilde{P}_j](t) - F[\tilde{P}_j](0) = 0. \quad (3.3.30)$$

It is not true in the case when P_j is a soliton.

Proof. When we develop using $\tilde{P} = \sum_{i=1}^J \tilde{P}_i$, we obtain terms with $\tilde{P}_i \tilde{P}_j$ with $i \neq j$ and the other terms that have all the same index. For the first type of terms, it is enough to bound $\int \tilde{P}_i \tilde{P}_j$ for $i \neq j$:

$$\left| \int \tilde{P}_i \tilde{P}_j \right| \leq Ce^{-\frac{\beta D}{2}}. \quad (3.3.31)$$

Now, we look on the terms with the same index, for example $\int \tilde{P}_i^2 \Phi_j$. We will distinguish several cases. If $i < j$,

$$\begin{aligned} \int \tilde{P}_i^2 \Phi_j &\leq C \int e^{-2\beta|x-\tilde{x}_i(t)|} e^{\frac{\sqrt{\sigma}}{2}(x-m_j(t))} dx \\ &= C \int_{-\infty}^{\tilde{x}_i(t)} e^{(2\beta+\sqrt{\sigma}/2)x-2\beta\tilde{x}_i(t)-\frac{\sqrt{\sigma}}{2}m_j(t)} dx + C \int_{\tilde{x}_i(t)}^{+\infty} e^{(-2\beta+\sqrt{\sigma}/2)x+2\beta\tilde{x}_i(t)-\frac{\sqrt{\sigma}}{2}m_j(t)} dx \\ &\leq \frac{C}{\sqrt{\sigma}} e^{\frac{\sqrt{\sigma}}{2}(\tilde{x}_i(t)-m_j(t))} \\ &\leq Ce^{-\frac{\sqrt{\sigma}D}{4}}. \end{aligned} \quad (3.3.32)$$

For the same reason and properties of Ψ , for $i \geq j$,

$$\int \tilde{P}_i^2 (1 - \Phi_j) \leq Ce^{-\frac{\sqrt{\sigma}D}{4}}. \quad (3.3.33)$$

For $i > j$, we may use \mathcal{P}_i ; and for $i = j$, we cannot. So, if for $i \geq j+1$, $\tilde{P}_i = \tilde{R}_l$ is a soliton, we have by the mean-value theorem, using (3.2.73),

$$\begin{aligned} \left| M[\tilde{P}_i](t) - M[\tilde{P}_i](0) \right| &= \left| c_l(t)^{1/2} - c_l(0)^{1/2} \right| |M[Q]| \\ &\leq C |c_l(t) - c_l(0)| \\ &\leq C \left(\frac{A}{Z_{j+1}} \right)^2 (a^2 + e^{-2\theta D}), \end{aligned} \quad (3.3.34)$$

and, by the same way, we may obtain,

$$\left| E[\tilde{P}_i](t) - E[\tilde{P}_i](0) \right| \leq C \left(\frac{A}{Z_{j+1}} \right)^2 (a^2 + e^{-2\theta D}), \quad (3.3.35)$$

and

$$\left| F[\tilde{P}_i](t) - F[\tilde{P}_i](0) \right| \leq C \left(\frac{A}{Z_{j+1}} \right)^2 (a^2 + e^{-2\theta D}). \quad (3.3.36)$$

□

Linear terms of the Taylor expansion

We denote :

$$m[X] := \int X \varepsilon, \quad (3.3.37)$$

$$m_j[X] := \int X \varepsilon \Phi_j, \quad (3.3.38)$$

$$e[X] := \int [X_x \varepsilon_x - X^3 \varepsilon] = - \int [X_{xx} + X^3] \varepsilon, \quad (3.3.39)$$

$$e_j[X] := \int [X_x \varepsilon_x - X^3 \varepsilon] \Phi_j, \quad (3.3.40)$$

$$f[X] := \int \left[X_{xx} \varepsilon_{xx} - 5X X_x^2 \varepsilon - 5X^2 X_x \varepsilon_x + \frac{3}{2} X^5 \varepsilon \right] = \int \left[X_{xxx} + 5X X_x^2 + 5X^2 X_{xx} + \frac{3}{2} X^5 \right] \varepsilon, \quad (3.3.41)$$

$$f_j[X] := \int \left[X_{xx} \varepsilon_{xx} - 5X X_x^2 \varepsilon - 5X^2 X_x \varepsilon_x + \frac{3}{2} X^5 \varepsilon \right] \Phi_j. \quad (3.3.42)$$

Lemma 3.17. *If $\sigma < \beta^2$, $\theta < \min\left(\frac{\beta}{4}, \frac{\sqrt{\sigma}}{8}\right)$, if D_0 is large enough and a_0 is small enough, then for any $t \in [0, t^*]$,*

$$\left| m_j[\tilde{P}](t) - m_j[\tilde{P}](0) - \left(m[\tilde{P}_j](t) - m[\tilde{P}_j](0) \right) \right| \leq C e^{-2\theta D} + C \left(\frac{A}{Z_{j+1}} \right)^2 (a^2 + e^{-2\theta D}), \quad (3.3.43)$$

$$\left| e_j[\tilde{P}](t) - e_j[\tilde{P}](0) - \left(e[\tilde{P}_j](t) - e[\tilde{P}_j](0) \right) \right| \leq C e^{-2\theta D} + C \left(\frac{A}{Z_{j+1}} \right)^2 (a^2 + e^{-2\theta D}), \quad (3.3.44)$$

$$\left| f_j[\tilde{P}](t) - f_j[\tilde{P}](0) - \left(f[\tilde{P}_j](t) - f[\tilde{P}_j](0) \right) \right| \leq C e^{-2\theta D} + C \left(\frac{A}{Z_{j+1}} \right)^2 (a^2 + e^{-2\theta D}). \quad (3.3.45)$$

Proof. We develop using $\tilde{P} = \sum_{i=1}^J \tilde{P}_i$. We obtain terms with $\tilde{P}_i \tilde{P}_j$ with $i \neq j$ and the other terms that have all the same index. Knowing that ε bounded for D_0 large enough and a_0 small enough (with respect to A), we obtain the same bounds in the same way as for the constant part.

Now, if, for $i \geq j+1$, \tilde{P}_i is a soliton, then we have simply : $m[\tilde{P}_i] = e[\tilde{P}_i] = f[\tilde{P}_i] = 0$.

If, for $i \geq j+1$, \tilde{P}_i is a breather, we have a bound for the variation of these quantities by \mathcal{P}_i . □

Quadratic part of the Taylor expansion

We set :

$$\mathcal{M}_j[X] := \frac{1}{2} \int \varepsilon^2 \Phi_j, \quad (3.3.46)$$

$$\mathcal{E}_j[X] := \int \left[\frac{1}{2} \varepsilon_x^2 - \frac{3}{2} X^2 \varepsilon^2 \right] \Phi_j, \quad (3.3.47)$$

$$\mathcal{F}_j[X] := \int \left[\frac{1}{2} \varepsilon_{xx}^2 - \frac{5}{2} X_x^2 \varepsilon^2 - 10 X X_x \varepsilon \varepsilon_x - \frac{5}{2} X^2 \varepsilon_x^2 + \frac{15}{4} X^4 \varepsilon^2 \right] \Phi_j, \quad (3.3.48)$$

and $\mathcal{M}_j(t) := \mathcal{M}_j[\tilde{P}](t)$, ...

Lemma 3.18. *If $\sigma < \beta^2$, $\theta < \frac{\sqrt{\sigma}}{8}$, if D_0 is large enough and a_0 is small enough, then for any $t \in [0, t^*]$,*

$$\left| \mathcal{M}_j[\tilde{P}](t) - \mathcal{M}_j[\tilde{P}_j](t) \right| \leq C e^{-2\theta D} + C \left(\frac{A}{Z_{j+1}} \right)^2 (a^2 + e^{-2\theta D}), \quad (3.3.49)$$

$$\left| \mathcal{E}_j[\tilde{P}](t) - \mathcal{E}_j[\tilde{P}_j](t) \right| \leq C e^{-2\theta D} + C \left(\frac{A}{Z_{j+1}} \right)^2 (a^2 + e^{-2\theta D}), \quad (3.3.50)$$

$$\left| \mathcal{F}_j[\tilde{P}](t) - \mathcal{F}_j[\tilde{P}_j](t) \right| \leq C e^{-2\theta D} + C \left(\frac{A}{Z_{j+1}} \right)^2 (a^2 + e^{-2\theta D}). \quad (3.3.51)$$

Proof. We develop using $\tilde{P} = \sum_{i=1}^J \tilde{P}_i$. For terms with \tilde{P}_i , with $i > j$, we use the induction assumption for ε . For terms with \tilde{P}_i , with $i < j$, we do as in the previous sections. \square

Note that (3.3.49) is useless, because $\mathcal{M}_j[\tilde{P}](t) - \mathcal{M}_j[\tilde{P}_j](t) = 0$ since $\mathcal{M}_j[X]$ do not depend on X , but we write it in order to argue in the same way for the three conserved quantities.

3.3.3 Lyapunov functional and simplifications

We introduce the following Lyapunov functional :

$$\mathcal{H}_j(t) := F_j(t) + 2(b_j^2 - a_j^2) E_j(t) + (a_j^2 + b_j^2)^2 M_j(t). \quad (3.3.52)$$

We set :

$$\mathcal{K}(t) := F[\tilde{P}_j](t) + 2(b_j^2 - a_j^2) E[\tilde{P}_j](t) + (a_j^2 + b_j^2)^2 M[\tilde{P}_j](t), \quad (3.3.53)$$

$$\mathcal{L}(t) := f[\tilde{P}_j](t) + 2(b_j^2 - a_j^2) e[\tilde{P}_j](t) + (a_j^2 + b_j^2)^2 m[\tilde{P}_j](t), \quad (3.3.54)$$

$$\mathcal{Q}(t) := \mathcal{F}_j[\tilde{P}_j](t) + 2(b_j^2 - a_j^2) \mathcal{E}_j[\tilde{P}_j](t) + (a_j^2 + b_j^2)^2 \mathcal{M}_j[\tilde{P}_j](t). \quad (3.3.55)$$

We have the following :

Lemma 3.19. *If $\sigma < \beta^2$, $\theta < \min\left(\frac{\beta}{4}, \frac{\sqrt{\sigma}}{8}\right)$, if D_0 is large enough and a_0 is small enough, then for any $t \in [0, t^*]$,*

$$|\mathcal{H}_j(t) - \mathcal{H}_j(0) - (\mathcal{Q}(t) - \mathcal{Q}(0))| \leq C e^{-2\theta D} + C \left(\frac{A}{Z_{j+1}} \right)^2 (a^2 + e^{-2\theta D}). \quad (3.3.56)$$

Proof. From (3.3.21), (3.3.22), (3.3.23), Lemmas 3.15, 3.17 and 3.18 we deduce that :

if $\sigma < \beta^2$, $\theta > \min\left(\frac{\beta}{4}, \frac{\sqrt{\sigma}}{8}\right)$, if D_0 is large enough and a_0 is small enough, then for any $t \in [0, t^*]$,

$$\begin{aligned} \mathcal{H}_j(t) - \mathcal{H}_j(0) &= \mathcal{K}(t) - \mathcal{K}(0) + \mathcal{L}(t) - \mathcal{L}(0) + \mathcal{Q}(t) - \mathcal{Q}(0) + O\left(A(a + e^{-\theta D}) \int (\varepsilon^2 + \varepsilon_x^2) \Phi_j\right) \\ &\quad + O(e^{-2\theta D}) + O\left(\left(\frac{A}{Z_{j+1}}\right)^2 (a^2 + e^{-2\theta D})\right). \end{aligned} \quad (3.3.57)$$

Now, from the bootstrap assumption (3.2.25), we see that if we take D_0 large enough and a_0 small enough, then we can bound $A(a + e^{-\theta D}) \int (\varepsilon^2 + \varepsilon_x^2) \Phi_j$ by $\left(\frac{A}{Z_{j+1}}\right)^2 (a^2 + e^{-2\theta D})$. So, for any $t \in [0, t^*]$,

$$\mathcal{H}_j(t) - \mathcal{H}_j(0) = \mathcal{K}(t) - \mathcal{K}(0) + \mathcal{L}(t) - \mathcal{L}(0) + \mathcal{Q}(t) - \mathcal{Q}(0) + O(e^{-2\theta D}) + O\left(\left(\frac{A}{Z_{j+1}}\right)^2 (a^2 + e^{-2\theta D})\right). \quad (3.3.58)$$

Now, we will simplify $\mathcal{K}(t) - \mathcal{K}(0)$ and $\mathcal{L}(t) - \mathcal{L}(0)$.

Simplification of $\mathcal{K}(t) - \mathcal{K}(0)$:

If \widetilde{P}_j is a breather, then $\mathcal{K}(t) - \mathcal{K}(0) = 0$. If $\widetilde{P}_j = \widetilde{R}_l$ is a soliton, then we have

$$\begin{aligned} \mathcal{K}(t) &= F[\widetilde{R}_l](t) + 2c_l^0 E[\widetilde{R}_l](t) + (c_l^0)^2 M[\widetilde{R}_l](t) \\ &= c_l(t)^{5/2} F[Q] + 2c_l^0 c_l(t)^{3/2} E[Q] + (c_l^0)^2 c_l(t)^{1/2} M[Q]. \end{aligned} \quad (3.3.59)$$

Now, we observe that $c_l(t) = c_l^0 + (c_l(t) - c_l^0)$, this is why we want to do a Taylor expansion for each power function. We recall that by (3.2.54), for D_0 large enough and a_0 small enough (with respect to Z_{j+1}), $|c_l(t) - c_l^0|^3$ is bounded by $C\left(\frac{A}{Z_{j+1}}\right)^2 (a^2 + e^{-2\theta D})$. This is why we may approximate $\mathcal{K}(t)$ by a Taylor expansion of order 2 :

$$\begin{aligned} (c_l^0)^{5/2} &\left(1 + \frac{5}{2} \frac{c_l(t) - c_l^0}{c_l^0} + \frac{15}{8} \left(\frac{c_l(t) - c_l^0}{c_l^0}\right)^2\right) F[Q] + 2(c_l^0)^{5/2} \left(1 + \frac{3}{2} \frac{c_l(t) - c_l^0}{c_l^0} + \frac{3}{8} \left(\frac{c_l(t) - c_l^0}{c_l^0}\right)^2\right) E[Q] \\ &\quad + (c_l^0)^{5/2} \left(1 + \frac{1}{2} \frac{c_l(t) - c_l^0}{c_l^0} - \frac{1}{8} \left(\frac{c_l(t) - c_l^0}{c_l^0}\right)^2\right) M[Q] \\ &= (c_l^0)^{5/2} (F[Q] + 2E[Q] + M[Q]) + (c_l(t) - c_l^0) (c_l^0)^{3/2} \left(\frac{5}{2} F[Q] + 3E[Q] + \frac{1}{2} M[Q]\right) \\ &\quad + (c_l(t) - c_l^0)^2 (c_l^0)^{1/2} \left(\frac{15}{8} F[Q] + \frac{3}{4} E[Q] - \frac{1}{8} M[Q]\right). \end{aligned} \quad (3.3.60)$$

Now, we use the fact that $M[Q] = 2$, $E[Q] = -\frac{2}{3}$ and $F[Q] = \frac{2}{5}$, and we obtain that the Taylor expression of order 2 is in fact :

$$\frac{16}{15} (c_l^0)^{5/2} + 0 (c_l(t) - c_l^0) (c_l^0)^{3/2} + 0 (c_l(t) - c_l^0)^2 (c_l^0)^{1/2} = \frac{16}{15} (c_l^0)^{5/2}. \quad (3.3.61)$$

Thus,

$$\left| \mathcal{K}(t) - \frac{16}{15} (c_l^0)^{5/2} \right| \leq C \left(\frac{A}{Z_{j+1}}\right)^2 (a^2 + e^{-2\theta D}), \quad (3.3.62)$$

and so,

$$|\mathcal{K}(t) - \mathcal{K}(0)| \leq C \left(\frac{A}{Z_{j+1}} \right)^2 (a^2 + e^{-2\theta D}). \quad (3.3.63)$$

Simplification of $\mathcal{L}(t) - \mathcal{L}(0)$:

If \widetilde{P}_j is a breather, we have, by the elliptic equation verified by a breather, that $\mathcal{L}(t) = 0$.

If \widetilde{P}_j is a soliton, we have, by (3.2.56), (3.2.85) and (3.2.86), that $\mathcal{L}(t) = 0$ (we have simply $m[\widetilde{P}_j] = e[\widetilde{P}_j] = f[\widetilde{P}_j] = 0$). \square

3.3.4 Coercivity

Lemma 3.20. *If σ is small enough (with respect to constants that depend only on problem data),*

$$\theta < \min \left(\frac{\beta}{4}, \frac{\sqrt{\sigma}}{8} \right), \quad (3.3.64)$$

if D_0 is large enough and a_0 is small enough, then for any $t \in [0, t^]$,*

$$\int (\varepsilon^2 + \varepsilon_x^2 + \varepsilon_{xx}^2) \Phi_j \leq C\mathcal{Q}(t) + C \left(\int \varepsilon \widetilde{P}_j \right)^2 + Ce^{-2\theta D}. \quad (3.3.65)$$

Proof. We notice that $\int (\varepsilon^2 + \varepsilon_x^2 + \varepsilon_{xx}^2) \Phi_j$ is $\|\varepsilon \sqrt{\Phi_j}\|_{H^2}^2$ modulo some terms that can be bounded by

$$C\sqrt{\sigma} \int (\varepsilon^2 + \varepsilon_x^2 + \varepsilon_{xx}^2) \Phi_j. \quad (3.3.66)$$

This is why we will bound $\|\varepsilon \sqrt{\Phi_j}\|_{H^2}^2$. We can bound it by the canonical quadratic form associated to \widetilde{P}_j and evaluated in $\varepsilon \sqrt{\Phi_j}$, if $\varepsilon \sqrt{\Phi_j}$ satisfies quite well the orthogonality conditions, which is the case (see Sections 5.4 and (372) in [108]).

So, we obtain that, if \widetilde{P}_j is a breather, the canonical quadratic form is $\mathcal{Q}(t)$ modulo (3.3.66), and we have

$$\int (\varepsilon^2 + \varepsilon_x^2 + \varepsilon_{xx}^2) \Phi_j \leq C\mathcal{Q}(t) + C \left(\int \varepsilon \sqrt{\Phi_j} \widetilde{P}_j \right)^2 + C\sqrt{\sigma} \int (\varepsilon^2 + \varepsilon_x^2 + \varepsilon_{xx}^2) \Phi_j, \quad (3.3.67)$$

which means that if σ is small enough,

$$\int (\varepsilon^2 + \varepsilon_x^2 + \varepsilon_{xx}^2) \Phi_j \leq C\mathcal{Q}(t) + C \left(\int \varepsilon \sqrt{\Phi_j} \widetilde{P}_j \right)^2, \quad (3.3.68)$$

and we check that $\int \varepsilon \sqrt{\Phi_j} \widetilde{P}_j$ is $\int \varepsilon \widetilde{P}_j$ modulo $Ce^{-\theta D}$. So,

$$\int (\varepsilon^2 + \varepsilon_x^2 + \varepsilon_{xx}^2) \Phi_j \leq C\mathcal{Q}(t) + C \left(\int \varepsilon \widetilde{P}_j \right)^2 + Ce^{-2\theta D}. \quad (3.3.69)$$

If $\widetilde{P}_j = \widetilde{R}_l$ is a soliton, the canonical quadratic form is, modulo $C\sqrt{\sigma} \int (\varepsilon^2 + \varepsilon_x^2 + \varepsilon_{xx}^2) \Phi_j$,

$$\mathcal{Q}_0(t) := \mathcal{F}_j[\widetilde{R}_l](t) + 2c_l(t)\mathcal{E}_j[\widetilde{R}_l](t) + c_l(t)^2 \mathcal{M}_j[\widetilde{R}_l](t). \quad (3.3.70)$$

This is why, for the same reasons as above, we have

$$\int (\varepsilon^2 + \varepsilon_x^2 + \varepsilon_{xx}^2) \Phi_j \leq C \mathcal{Q}_0(t). \quad (3.3.71)$$

This is why, we need to be able to bound $|\mathcal{Q}_0(t) - \mathcal{Q}(t)|$. We have

$$\mathcal{Q}_0(t) - \mathcal{Q}(t) = 2(c_l(t) - c_l^0) \mathcal{E}_j[\widetilde{R}_l](t) + 2c_l^0(c_l(t) - c_l^0) \mathcal{M}_j[\widetilde{R}_l](t) + (c_l(t) - c_l^0)^2 \mathcal{M}_j[\widetilde{R}_l](t), \quad (3.3.72)$$

and so, because \mathcal{M}_j and \mathcal{E}_j are quadratic in ε , we have that

$$|\mathcal{Q}_0(t) - \mathcal{Q}(t)| \leq CA(a + e^{-\theta D}) \int (\varepsilon^2 + \varepsilon_x^2) \Phi_j, \quad (3.3.73)$$

and so, if we take a_0 small enough and D_0 large enough, we obtain

$$\int (\varepsilon^2 + \varepsilon_x^2 + \varepsilon_{xx}^2) \Phi_j \leq C \mathcal{Q}(t). \quad (3.3.74)$$

□

3.3.5 Proof of \mathcal{P}_j

Now, we are left to prove \mathcal{P}_j . More precisely, if P_j is a soliton, we will prove (3.2.76) and (3.2.77); if P_j is a breather, we will prove (3.2.76), (3.2.78) and (3.2.79). We will distinguish several cases. We assume that $\sigma \leq \min(\zeta, \beta^2)$, $\theta \leq \min(\frac{\beta}{4}, \frac{\sqrt{\sigma}}{16})$, and that D_0 is large enough and a_0 is small enough (depending on A, θ), so that all the previous lemmas are verified.

Case when P_j is a soliton

Proof of (3.2.76) By Lemma 3.13 and the fact that $b_j^2 - a_j^2 > 0$ for $j \geq 2$ (which is a direct consequence of $v_2 > 0$), we have that, if $j \geq 2$, for any $t \in [0, t^*]$,

$$\mathcal{H}_j(t) - \mathcal{H}_j(0) \leq C e^{-2\theta D}. \quad (3.3.75)$$

(3.3.75) is also true for $j = 1$, because of Remark 3.14.

We have, by (3.3.56), (3.3.75), the definition of $\mathcal{Q}(0)$ and (3.2.53) for $\varepsilon(0)$, for any $t \in [0, t^*]$,

$$\mathcal{Q}(t) = [\mathcal{Q}(t) - \mathcal{Q}(0) - (\mathcal{H}_j(t) - \mathcal{H}_j(0))] + [\mathcal{H}_j(t) - \mathcal{H}_j(0)] + \mathcal{Q}(0) \quad (3.3.76)$$

$$\leq C e^{-2\theta D} + C \left(\frac{A}{Z_{j+1}} \right)^2 (a^2 + e^{-2\theta D}) + C \|\varepsilon(0)\|_{H^2}^2 \quad (3.3.77)$$

$$\leq C (a^2 + e^{-2\theta D}) + C \left(\frac{A}{Z_{j+1}} \right)^2 (a^2 + e^{-2\theta D}). \quad (3.3.78)$$

And so, by (3.3.65) and (3.3.78), we have for any $t \in [0, t^*]$,

$$\int (\varepsilon^2 + \varepsilon_x^2 + \varepsilon_{xx}^2) \Phi_j \leq C \mathcal{Q}(t) + C \left(\int \varepsilon \widetilde{P}_j \right)^2 + C e^{-2\theta D} \quad (3.3.79)$$

$$\leq C (a^2 + e^{-2\theta D}) + C \left(\frac{A}{Z_{j+1}} \right)^2 (a^2 + e^{-2\theta D}) + C \left(\int \varepsilon \widetilde{P}_j \right)^2. \quad (3.3.80)$$

Because of (3.2.56) and \widetilde{P}_j is a soliton, we have that

$$\int \varepsilon \widetilde{P}_j = 0. \quad (3.3.81)$$

So, the proof of (3.2.76) is completed for a suitable constant Z_j that will be precised later.

Proof of (3.2.77) From (3.3.21), (3.3.24), (3.3.27), (3.3.43), (3.2.56) and (3.3.80), we have for any $t \in [0, t^*]$:

$$\begin{aligned} M_j(0) - M_j(t) &= M_j[\widetilde{P}](0) - M_j[\widetilde{P}](t) + \int \widetilde{P} \varepsilon \Phi_j(0) - \int \widetilde{P} \varepsilon \Phi_j(t) + \frac{1}{2} \int \varepsilon^2 \Phi_j(0) - \frac{1}{2} \int \varepsilon^2 \Phi_j(t) \\ &= \left(c_l(0)^{1/2} - c_l(t)^{1/2} \right) M[Q] + O\left((a^2 + e^{-2\theta D})\right) + O\left(\left(\frac{A}{Z_{j+1}}\right)^2 (a^2 + e^{-2\theta D})\right). \end{aligned} \quad (3.3.82)$$

Similarly, from (3.3.22), (3.3.25), (3.3.28), (3.3.44), (3.2.85) and (3.3.80), by taking a_0 smaller and D_0 larger with respect to A if needed, we have for any $t \in [0, t^*]$:

$$E_j(0) - E_j(t) = \left(c_l(0)^{3/2} - c_l(t)^{3/2} \right) E[Q] + O\left((a^2 + e^{-2\theta D})\right) + O\left(\left(\frac{A}{Z_{j+1}}\right)^2 (a^2 + e^{-2\theta D})\right). \quad (3.3.83)$$

For $\eta = \frac{1}{2}, \frac{3}{2}$, because we know that $c_l(t)$ is not too far from $c_l(0)$ (and the both are not too far from c_l^0 , by (3.2.54)), we can write for any $t \in [0, t^*]$:

$$c_l(t)^\eta = (c_l(0))^\eta \left(1 + \eta \frac{c_l(t) - c_l(0)}{c_l(0)} + O\left((c_l(t) - c_l(0))^2\right) \right), \quad (3.3.84)$$

and so for any $t \in [0, t^*]$,

$$c_l(t)^\eta - c_l(0)^\eta = \eta c_l(0)^{\eta-1} (c_l(t) - c_l(0)) + O\left((c_l(t) - c_l(0))^2\right), \quad (3.3.85)$$

and if a_0 is small enough and D_0 is large enough, we have by (3.2.26), for any $t \in [0, t^*]$:

$$\left| O\left((c_l(t) - c_l(0))^2\right) \right| \leq \frac{1}{2} \eta c_l(0)^{\eta-1} |c_l(t) - c_l(0)|. \quad (3.3.86)$$

Thus, for any $t \in [0, t^*]$,

$$2\eta c_l(0)^{\eta-1} |c_l(t) - c_l(0)| \geq |c_l(t)^\eta - c_l(0)^\eta| \geq \frac{\eta c_l(0)^{\eta-1}}{2} |c_l(t) - c_l(0)|, \quad (3.3.87)$$

where $c_l(0)^{\eta-1}$ is between $\min\{\frac{c_p^0}{2}, 1 \leq p \leq L\}^{\eta-1}$ and $\max\{2c_p^0, 1 \leq p \leq L\}^{\eta-1}$ by (3.2.54), and so is bounded above and below by a constant that depends only on the shape parameters of the solitons. In order to bound $|c_l(t) - c_l(0)|$ for a given $t \in [0, t^*]$, we will distinguish two cases.

Case when $c_l(t) - c_l(0) \geq 0$ From (3.3.82) and (3.3.87) for $\eta = \frac{1}{2}$, we can say that

$$|c_l(t) - c_l(0)| \leq C \left(c_l(t)^{1/2} - c_l(0)^{1/2} \right) \quad (3.3.88)$$

$$\leq C \frac{M_j(t) - M_j(0)}{M[Q]} + C (a^2 + e^{-2\theta D}) + C \left(\frac{A}{Z_{j+1}} \right)^2 (a^2 + e^{-2\theta D}).$$

Now, $M[Q] > 0$ and from Lemma 3.13,

$$|c_l(t) - c_l(0)| \leq C (a^2 + e^{-2\theta D}) + C \left(\frac{A}{Z_{j+1}} \right)^2 (a^2 + e^{-2\theta D}). \quad (3.3.89)$$

Thus, (3.2.77) is established for a suitable constant Z_j that will be precised later.

Case when $c_l(t) - c_l(0) \leq 0$ From (3.3.83) and (3.3.87) for $\eta = \frac{3}{2}$, we can say that

$$\begin{aligned} |c_l(t) - c_l(0)| &\leq C (c_l(0)^{3/2} - c_l(t)^{3/2}) \\ &\leq C \frac{E_j(0) - E_j(t)}{E[Q]} + C (a^2 + e^{-2\theta D}) + C \left(\frac{A}{Z_{j+1}} \right)^2 (a^2 + e^{-2\theta D}). \end{aligned} \quad (3.3.90)$$

Now, $E[Q] < 0$ and from Lemma 3.13, (3.3.82) and (3.3.87) for $\eta = \frac{1}{2}$,

$$\begin{aligned} |c_l(t) - c_l(0)| &\leq C\omega_1 (M_j(0) - M_j(t)) + C (a^2 + e^{-2\theta D}) + C \left(\frac{A}{Z_{j+1}} \right)^2 (a^2 + e^{-2\theta D}) \\ &\leq C\omega_1 (c_l(0)^{1/2} - c_l(t)^{1/2}) + C (a^2 + e^{-2\theta D}) + C \left(\frac{A}{Z_{j+1}} \right)^2 (a^2 + e^{-2\theta D}) \\ &\leq C\omega_1 |c_l(t) - c_l(0)| + C (a^2 + e^{-2\theta D}) + C \left(\frac{A}{Z_{j+1}} \right)^2 (a^2 + e^{-2\theta D}), \end{aligned} \quad (3.3.91)$$

and so, by taking ω_1 small enough, we may deduce the desired inequality :

$$|c_l(t) - c_l(0)| \leq C (a^2 + e^{-2\theta D}) + C \left(\frac{A}{Z_{j+1}} \right)^2 (a^2 + e^{-2\theta D}). \quad (3.3.92)$$

Thus, (3.2.77) is established for a suitable constant Z_j that will be precised later.

Case when P_j is a breather

Preliminaries By the same argument as in the case when P_j is a soliton, we establish (3.3.80). However, we are not able to prove (3.2.76) immediately, because $\int \widetilde{P}_j \varepsilon$ is not necessarily equal to 0 in the case when \widetilde{P}_j is a breather.

From Lemma 3.17, (3.3.21), Lemma 3.15 and Remark 3.16, we have that for any $t \in [0, t^*]$,

$$\begin{aligned} \int \widetilde{P}_j \varepsilon(t) - \int \widetilde{P}_j \varepsilon(0) &= M_j(t) - M_j(0) - \frac{1}{2} \int \varepsilon^2 \Phi_j(t) + \frac{1}{2} \int \varepsilon^2 \Phi_j(0) \\ &\quad + O(e^{-2\theta D}) + O \left(\left(\frac{A}{Z_{j+1}} \right)^2 (a^2 + e^{-2\theta D}) \right). \end{aligned} \quad (3.3.93)$$

then we use (3.3.4) and (3.2.53) for $\varepsilon(0)$, we have that for any $t \in [0, t^*]$,

$$\int \widetilde{P}_j \varepsilon(t) - \int \widetilde{P}_j \varepsilon(0) \leq -\frac{1}{2} \int \varepsilon^2 \Phi_j(t) + \frac{1}{2} \int \varepsilon^2 \Phi_j(0) + O(e^{-2\theta D}) + O \left(\left(\frac{A}{Z_{j+1}} \right)^2 (a^2 + e^{-2\theta D}) \right)$$

$$\leq C(a^2 + e^{-2\theta D}) + C\left(\frac{A}{Z_{j+1}}\right)^2 (a^2 + e^{-2\theta D}). \quad (3.3.94)$$

Now, from (3.2.82), Lemma 3.17, (3.3.21), Lemma 3.15, Remark 3.16 and (3.2.25), we have for any $t \in [0, t^*]$,

$$\begin{aligned} \left((a_j^2 + b_j^2)^2\right) \left(\int \widetilde{P}_j \varepsilon(0) - \int \widetilde{P}_j \varepsilon(t)\right) &= 2(b_j^2 - a_j^2) \left(e[\widetilde{P}_j](t) - e[\widetilde{P}_j](0)\right) + \left(f[\widetilde{P}_j](t) - f[\widetilde{P}_j](0)\right) \\ &= 2(b_j^2 - a_j^2) (E_j(t) - E_j(0) - \mathcal{E}_j(t) + \mathcal{E}_j(0)) \\ &\quad + (F_j(t) - F_j(0) - \mathcal{F}_j(t) + \mathcal{F}_j(0)) \\ &\quad + O(e^{-2\theta D}) + O\left(\left(\frac{A}{Z_{j+1}}\right)^2 (a^2 + e^{-2\theta D})\right). \end{aligned} \quad (3.3.95)$$

Recall that if $j \geq 2$, $b_j^2 - a_j^2 > 0$. This is why, from Lemma 3.13 and (3.2.53) for $\varepsilon(0)$, we have that, if $j \geq 2$, for any $t \in [0, t^*]$:

$$\begin{aligned} \int \widetilde{P}_j \varepsilon(0) - \int \widetilde{P}_j \varepsilon(t) &\leq \frac{-2(b_j^2 - a_j^2) \mathcal{E}_j(t) - \mathcal{F}_j(t)}{(a_j^2 + b_j^2)^2} \\ &\quad + C(a^2 + e^{-2\theta D}) + C\left(\frac{A}{Z_{j+1}}\right)^2 (a^2 + e^{-2\theta D}) \\ &\quad + \frac{2(b_j^2 - a_j^2) \omega_1 + \omega_2}{(a_j^2 + b_j^2)^2} (M_j(0) - M_j(t)). \end{aligned} \quad (3.3.96)$$

(3.3.96) is also true if $j = 1$ because of Remark 3.14.

And, from (3.3.21), Lemma 3.15, Remark 3.16, Lemma 3.17 and (3.2.53) for $\varepsilon(0)$, we have for any $t \in [0, t^*]$,

$$\begin{aligned} M_j(0) - M_j(t) &= M_j[\widetilde{P}](0) - M_j[\widetilde{P}](t) + \int \widetilde{P} \varepsilon \Phi_j(0) - \int \widetilde{P} \varepsilon \Phi_j(t) + \frac{1}{2} \int \varepsilon^2 \Phi_j(0) - \frac{1}{2} \int \varepsilon^2 \Phi_j(t) \\ &= O(a^2 + e^{-2\theta D}) + O\left(\left(\frac{A}{Z_{j+1}}\right)^2 (a^2 + e^{-2\theta D})\right) + \int \widetilde{P}_j \varepsilon(0) - \int \widetilde{P}_j \varepsilon(t) - \mathcal{M}_j(t). \end{aligned} \quad (3.3.97)$$

And so, if we choose ω_1 and ω_2 small enough with respect to the problem constants, we obtain for any $t \in [0, t^*]$:

$$\begin{aligned} \int \widetilde{P}_j \varepsilon(0) - \int \widetilde{P}_j \varepsilon(t) &\leq \frac{-(2(b_j^2 - a_j^2) \omega_1 + \omega_2) \mathcal{M}_j(t) - 2(b_j^2 - a_j^2) \mathcal{E}_j(t) - \mathcal{F}_j(t)}{(a_j^2 + b_j^2)^2} \\ &\quad + C(a^2 + e^{-2\theta D}) + C\left(\frac{A}{Z_{j+1}}\right)^2 (a^2 + e^{-2\theta D}). \end{aligned} \quad (3.3.98)$$

Because $|\mathcal{M}_j(t)| + |\mathcal{E}_j(t)| + |\mathcal{F}_j(t)| \leq \int (\varepsilon^2 + \varepsilon_x^2 + \varepsilon_{xx}^2) \Phi_j$, we deduce that for any $t \in [0, t^*]$,

$$\int \widetilde{P}_j \varepsilon(0) - \int \widetilde{P}_j \varepsilon(t) \leq C \int (\varepsilon^2 + \varepsilon_x^2 + \varepsilon_{xx}^2) \Phi_j + C(a^2 + e^{-2\theta D}) + C\left(\frac{A}{Z_{j+1}}\right)^2 (a^2 + e^{-2\theta D}). \quad (3.3.99)$$

And so, by putting (3.3.94) and (3.3.99) together, we have that for any $t \in [0, t^*]$:

$$\left| \int \widetilde{P}_j \varepsilon(0) - \int \widetilde{P}_j \varepsilon(t) \right| \leq C \int (\varepsilon^2 + \varepsilon_x^2 + \varepsilon_{xx}^2) \Phi_j + C(a^2 + e^{-2\theta D}) + C\left(\frac{A}{Z_{j+1}}\right)^2 (a^2 + e^{-2\theta D}). \quad (3.3.100)$$

Proof of (3.2.76) From (3.2.53), (3.3.100), we deduce for any $t \in [0, t^*]$:

$$\begin{aligned} \left| \int \widetilde{P}_j \varepsilon(t) \right| &\leq \left| \int \widetilde{P}_j \varepsilon(0) - \int \widetilde{P}_j \varepsilon(t) \right| + \left| \int \widetilde{P}_j \varepsilon(0) \right| \\ &\leq C \int (\varepsilon^2 + \varepsilon_x^2 + \varepsilon_{xx}^2) \Phi_j + C (a + e^{-2\theta D}) + C \left(\frac{A}{Z_{j+1}} \right)^2 (a^2 + e^{-2\theta D}) \\ &\leq CA^2 (a^2 + e^{-2\theta D}) + C (a + e^{-2\theta D}). \end{aligned} \quad (3.3.101)$$

And so, if a_0 is small enough and D_0 is large enough, for any $t \in [0, t^*]$,

$$\left(\int \widetilde{P}_j \varepsilon(t) \right)^2 \leq C \left(\frac{A}{Z_{j+1}} \right)^2 (a^2 + e^{-2\theta D}) + C (a^2 + e^{-2\theta D}). \quad (3.3.102)$$

This is why, from (3.3.80), for any $t \in [0, t^*]$,

$$\int (\varepsilon^2 + \varepsilon_x^2 + \varepsilon_{xx}^2) \Phi_j \leq C (a^2 + e^{-2\theta D}) + C \left(\frac{A}{Z_{j+1}} \right)^2 (a^2 + e^{-2\theta D}), \quad (3.3.103)$$

and (3.2.76) is established for a suitable constant Z_j that will be precised later.

Proof of (3.2.78) From (3.3.103) and (3.3.100), for any $t \in [0, t^*]$:

$$\left| \int \widetilde{P}_j \varepsilon(0) - \int \widetilde{P}_j \varepsilon(t) \right| \leq C \left(\frac{A}{Z_{j+1}} \right)^2 (a^2 + e^{-2\theta D}) + C (a^2 + e^{-2\theta D}). \quad (3.3.104)$$

Thus, (3.2.78) is established for a suitable constant Z_j that will be precised later.

Proof of (3.2.79) From Lemma 3.17, (3.3.22), Lemma 3.15, Remark 3.16, (3.3.100) and (3.3.103), for any $t \in [0, t^*]$,

$$\begin{aligned} e[\widetilde{P}_j](t) - e[\widetilde{P}_j](0) &= E_j(t) - E_j(0) - \mathcal{E}_j(t) + \mathcal{E}_j(0) \\ &\quad + O(e^{-2\theta D}) + O\left(\left(\frac{A}{Z_{j+1}}\right)^2 (a^2 + e^{-2\theta D})\right) \\ &= E_j(t) - E_j(0) + O(a^2 + e^{-2\theta D}) + O\left(\left(\frac{A}{Z_{j+1}}\right)^2 (a^2 + e^{-2\theta D})\right). \end{aligned} \quad (3.3.105)$$

Now, from Lemma 3.13, for any $t \in [0, t^*]$,

$$e[\widetilde{P}_j](t) - e[\widetilde{P}_j](0) \leq \omega_1 (M_j(0) - M_j(t)) + O(a^2 + e^{-2\theta D}) + O\left(\left(\frac{A}{Z_{j+1}}\right)^2 (a^2 + e^{-2\theta D})\right). \quad (3.3.106)$$

And from (3.3.97) and (3.3.104), for any $t \in [0, t^*]$,

$$e[\widetilde{P}_j](t) - e[\widetilde{P}_j](0) \leq C \left(\frac{A}{Z_{j+1}} \right)^2 (a^2 + e^{-2\theta D}) + C (a^2 + e^{-2\theta D}). \quad (3.3.107)$$

To bound above $e[\widetilde{P}_j](0) - e[\widetilde{P}_j](t)$, we do as in (3.3.95), (3.3.96), (3.3.97), (3.3.98) and (3.3.99), but with the energy instead of the mass, and after using (3.3.103), we obtain for any $t \in [0, t^*]$,

$$\left| e[\widetilde{P}_j](t) - e[\widetilde{P}_j](0) \right| \leq C \left(\frac{A}{Z_{j+1}} \right)^2 (a^2 + e^{-2\theta D}) + C (a^2 + e^{-2\theta D}). \quad (3.3.108)$$

Thus, (3.2.79) is established for a suitable constant Z_j that will be precised later.

3.3.6 Choice of suitable A and Z_j

The induction holds if

$$C \left(1 + \left(\frac{A}{Z_{j+1}} \right)^2 \right) \leq \left(\frac{A}{Z_j} \right)^2. \quad (3.3.109)$$

We can set

$$A := \sqrt{2} (2C)^{\frac{J}{2}}, \quad (3.3.110)$$

and for $1 \leq j \leq J$,

$$Z_j := 2 (2C)^{\frac{j-1}{2}}. \quad (3.3.111)$$

And, if $C > 1$, the induction holds.

The proof of Proposition 3.10 is now complete.

3.4 A consequence of Theorem 3.1 : orbital stability of a multi-breather

We assume the Theorem 3.1 proved, let us prove Theorem 3.3.

Proof of Theorem 3.3. Let $A_0 > 0$, $\theta_0 > 0$, $D_0 > 0$ and $a_0 > 0$ from Theorem 3.1 (these constants do only depend on the shape/frequency parameters of our objects and not on their initial positions). Let $\eta_0 > \eta > 0$ with $\eta_0 < \frac{a_0}{2C_1}$ and C_1 defined in the following. Let $a < a_0$ and $D > D_0$ such that $A_0 (a + e^{-\theta_0 D}) < 4A_0 C_1 \eta$ and $a = 2C_1 \eta$. We may take D even larger so that $T^* = 0$ where T^* is defined in [108, Theorem 1.2]. Let θ_1 associated to shape/frequency parameters $\alpha_k, \beta_k, \kappa_l, c_l^0$ by [108, Theorem 1.2]. Let A_2 associated to D by [108, Theorem 1.2].

Let $\tau > 0$ be the minimal difference between two velocities.

Let p be the multi-breather associated to $\alpha_k, \beta_k, x_{1,k}^0, x_{2,k}^0, c_l^0, x_{0,l}^0$ by [108, Theorem 1.2] with notations as in (3.1.25) and P the corresponding sum with notations as in (3.1.26). We may choose $T \geq 0$ large enough such that

$$\forall t \geq T, \quad \|p(t) - P(t)\|_{H^2} \leq a/2, \quad (3.4.1)$$

which is possible from [108], and such that

$$\forall j \geq 2, \quad x_j(T) - x_{j-1}(T) > 2D, \quad (3.4.2)$$

which is possible because the distance between two objects is increasing with a speed that is at least τ .

By [58] we know that we have continuous dependence of the solution of (mKdV) with respect to the initial data. And so, there exists $C_1 > 0$ (that depends on T) such that if $\|u(0) - p(0)\|_{H^2} \leq \eta$ and η_0 is small enough, then

$$\forall t \in [0, T], \quad \|u(t) - p(t)\|_{H^2} \leq C_1 \eta. \quad (3.4.3)$$

Therefore, by triangular inequality,

$$\|u(T) - P(T)\|_{H^2} \leq a = 2C_1 \eta. \quad (3.4.4)$$

This means that the assumptions of Theorem 3.1 are all satisfied in T instead of 0. And so, this means that there exists $x_{0,l}(t), x_{1,k}(t), x_{2,k}(t)$ defined for any $t \geq T$ such that

$$\forall t \geq T, \quad \|u(t) - P(t; \alpha_k, \beta_k, x_{1,k}(t), x_{2,k}(t), \kappa_l, c_l^0, x_{0,l}(t))\|_{H^2} \leq A_0(a + e^{-\theta_0 D}) < 4A_0C_1\eta. \quad (3.4.5)$$

Now, we see that the assumptions of [108, Theorem 1.2, Remark 1.3] are all satisfied in any $t \geq T$ instead of 0 for the sum $P(\alpha_k, \beta_k, x_{1,k}(t), x_{2,k}(t), \kappa_l, c_l^0, x_{0,l}(t))$. Indeed, if we denote $\tilde{x}_j(t)$ the position of

$$P_j(t; \alpha_k, \beta_k, x_{1,k}(t), x_{2,k}(t), \kappa_l, c_l^0, x_{0,l}(t)), \quad (3.4.6)$$

we know from Remark 3.2.38 that for any $t \geq T$ and $j \geq 2$,

$$\tilde{x}_j(t) - \widetilde{x_{j-1}}(t) \geq D. \quad (3.4.7)$$

By taking T larger if needed, we may insure that $e^{-\theta_1 T} < C_1\eta$.

Therefore, for any $t \geq T$,

$$\begin{aligned} \|p(t; \alpha_k, \beta_k, x_{1,k}(t), x_{2,k}(t), \kappa_l, c_l^0, x_{0,l}(t)) - P(t; \alpha_k, \beta_k, x_{1,k}(t), x_{2,k}(t), \kappa_l, c_l^0, x_{0,l}(t))\|_{H^2} \\ \leq A_2e^{-\theta_1 t} \leq A_2e^{-\theta_1 T} \leq A_2C_1\eta. \end{aligned}$$

And so, by triangular inequality,

$$\forall t \geq T, \quad \|u(t) - p(t; \alpha_k, \beta_k, x_{1,k}(t), x_{2,k}(t), \kappa_l, c_l^0, x_{0,l}(t))\|_{H^2} \leq (4A_0 + A_2)C_1\eta. \quad (3.4.8)$$

The latter proves the Theorem for $t \geq T$ with $C_0 = (4A_0 + A_2)C_1$. For $0 \leq t \leq T$, it is enough to use (3.4.3). \square

Appendix : Equations for localized conservation laws

Lemma 3.21. *Let f be a C^3 function that do not depend on time and u a solution of (mKdV). Then,*

$$\frac{d}{dt} \frac{1}{2} \int u^2 f = \int \left[-\frac{3}{2} u_x^2 + \frac{3}{4} u^4 \right] f_x + \frac{1}{2} \int u^2 f_{xxx}, \quad (3.4.9)$$

$$\frac{d}{dt} \int \left[\frac{1}{2} u_x^2 - \frac{1}{4} u^4 \right] f = \int \left[-\frac{1}{2} (u_{xx} + u^3)^2 - u_{xx}^2 + 3u^2 u_x^2 \right] f_x + \frac{1}{2} \int u_x^2 f_{xxx}, \quad (3.4.10)$$

$$\begin{aligned} \frac{d}{dt} \int \left[\frac{1}{2} u_{xx}^2 - \frac{5}{2} u^2 u_x^2 + \frac{1}{4} u^6 \right] f \\ = \int \left(-\frac{3}{2} u_{xxx}^2 + 9u_{xx}^2 u^2 + 15u_x^2 u u_{xx} + \frac{9}{16} u^8 + \frac{1}{4} u_x^4 + \frac{3}{2} u_{xx} u^5 - \frac{45}{4} u^4 u_x^2 \right) f' \\ + 5 \int u^2 u_x u_{xx} f'' + \frac{1}{2} \int u_{xx}^2 f'''. \end{aligned} \quad (3.4.11)$$

Proof. see the bottom of the page 1115 and the bottom of the page 1116 of [71] and Section 5.5 in [108]. \square

Lemma 3.22. *Let $r > 0$. If t, x satisfy $\widetilde{x_{j-1}}(t) + r < x < \tilde{x}_j(t) - r$, then*

$$|\tilde{P}(t, x)| \leq Ce^{-\beta r}. \quad (3.4.12)$$

Proof. immediate consequence of the exponential majoration of each object. \square

Chapitre 4

Calculs sur les breathers et les multi-breathers

In this chapter, we compute the mass, the energy and the second energy of any soliton and any breather. Then, we study the consequences of the integrability of (mKdV). More precisely, we show the consequences of the formula obtained for multi-breathers by inverse scattering method. We prove that this formula implies the existence and the regularity of multi-breathers in H^s , and that it is a multi-breather at the same time for $t \rightarrow +\infty$ and for $t \rightarrow -\infty$ with the same shape parameters. And, we compute the consequences of collisions between solitons or breathers on their translation parameters. Finally, we consider the Liouville theorem at a neighborhood of 0, which could be a first step to prove the Liouville theorem at a neighborhood of a breather.

4.1 Mass, energy and second energy of a soliton and of a breather

We denote Q_c a soliton of (mKdV) of velocity $c > 0$ and $B_{\alpha,\beta}$ a breather of (mKdV) of parameters $\alpha, \beta > 0$. The translation parameters do not affect the computations of the constants of motion of a soliton or of a breather, because the conserved integrals do not depend on time or space translations. We will denote here $M := I_2$ the mass, $E := I_3$ the energy and $F := I_4$ the second energy (where I_2, I_3, I_4 are notations from the Introduction).

We already know the mass and the energy of a soliton and of a breather of (mKdV) [8] :

Proposition 4.1 (Alejo, Muñoz). *Let $c > 0$ and $\alpha, \beta > 0$, then*

$$M[Q_c] = 2c^{1/2}, \quad E[Q_c] = -\frac{2}{3}c^{3/2}, \quad (4.1.1)$$

$$M[B_{\alpha,\beta}] = 4\beta, \quad E[B_{\alpha,\beta}] = -\frac{4}{3}\beta(\beta^2 - 3\alpha^2). \quad (4.1.2)$$

We compute now the second energy of a soliton :

Proposition 4.2. *Let $c > 0$, then*

$$F[Q_c] = \frac{2}{5}c^{5/2}. \quad (4.1.3)$$

Proof. Let $c > 0$. We denote Q for Q_c in this proof. Using the definition of the second energy and integrations by parts :

$$\partial_c F[Q] = \int \partial_c Q_{xx} \cdot Q_{xx} - 5 \int \partial_c Q \cdot Q Q_x^2 - 5 \int Q^2 \partial_c Q_x \cdot Q_x + \frac{3}{2} \int \partial_c Q \cdot Q^5 \quad (4.1.4)$$

$$= \int \partial_c Q \cdot \left(Q_{xxxx} + 5Q Q_x^2 + 5Q^2 Q_{xx} + \frac{3}{2} Q^5 \right). \quad (4.1.5)$$

But, Q satisfies the elliptic equation (1.5.34).

So, we integrate it after having multiplied it by $\partial_c Q$, and we obtain :

$$\partial_c F[Q] + 2c \partial_c E[Q] + c^2 \partial_c M[Q] = 0. \quad (4.1.6)$$

Thus,

$$\partial_c F[Q] = c^{3/2}. \quad (4.1.7)$$

By integration,

$$F[Q] = \frac{2}{5} c^{5/2} + cst. \quad (4.1.8)$$

We remark that

$$\|Q_c\|_{H^2} \rightarrow_{c \rightarrow 0} 0, \quad (4.1.9)$$

thus

$$F[Q_c] \rightarrow_{c \rightarrow 0} 0, \quad (4.1.10)$$

this is why the constant is null. \square

Finally, we compute the second energy of a breather :

Proposition 4.3. *Let $\alpha, \beta > 0$, then*

$$F[B_{\alpha, \beta}] = \frac{4}{5} \beta (\beta^4 - 10\beta^2 \alpha^2 + 5\alpha^4). \quad (4.1.11)$$

Proof. Let $\alpha, \beta > 0$. We proceed in a similar manner to the previous proof. We multiply (1.4.29) by $\partial_\alpha B_{\alpha, \beta}$ and we integrate it to obtain :

$$\partial_\alpha F[B_{\alpha, \beta}] + 2(\beta^2 - \alpha^2) \partial_\alpha E[B_{\alpha, \beta}] + (\alpha^2 + \beta^2) \partial_\alpha M[B_{\alpha, \beta}] = 0. \quad (4.1.12)$$

Thus,

$$\partial_\alpha F[B_{\alpha, \beta}] = -16(\beta^2 - \alpha^2) \alpha \beta = -16\alpha \beta^3 + 16\alpha^3 \beta. \quad (4.1.13)$$

So, by integration,

$$F[B_{\alpha, \beta}] = -8\alpha^2 \beta^3 + 4\alpha^4 \beta + f(\beta), \quad (4.1.14)$$

where $f(\beta)$ denotes an expression that depends only on β .

But, when $\alpha \rightarrow 0$, $B_{\alpha, \beta}$ converges in H^2 to a dipole $B_{0, \beta}$ that consists of a soliton and an antisoliton, the both of them having for parameter β^2 , that move away logarithmically from each other. So,

$$F[B_{0, \beta}] = 2F[Q_{\beta^2}] = \frac{4}{5} \beta^5. \quad (4.1.15)$$

Thus, $f(\beta) = \frac{4}{5} \beta^5$, which concludes the proof. \square

4.2 Consequences of the formula for multi-breathers

The goal here is to prove that the formula giving a multi-breather found by Wadati [117] corresponds to the natural definition of a multi-breather in H^2 . We will also prove that this solution is also a multi-breather when $t \rightarrow -\infty$.

Let α_k, β_k , $1 \leq k \leq K$ be the shape parameters of the breathers, and $x_{1,k}, x_{2,k}$ their translation parameters. We note B_k the corresponding breather. More precisely,

$$B_k(t, x) = 2\sqrt{2} \frac{\partial}{\partial x} \arctan \left[\frac{\beta_k \sin(\alpha_k (x + \delta_k t + x_{1,k}))}{\alpha_k \cosh(\beta_k (x + \gamma_k t + x_{2,k}))} \right], \quad (4.2.1)$$

with $\delta_k := \alpha_k^2 - 3\beta_k^2$ and $\gamma_k := 3\alpha_k^2 - \beta_k^2$. We can also write it in an equivalent way as follows :

$$B_k(t, x) = 2\sqrt{2} \frac{\partial}{\partial x} \arg \left[\frac{\cosh(\beta_k (x + \gamma_k t + x_{2,k}))}{\beta_k} + i \frac{\sin(\alpha_k (x + \delta_k t + x_{1,k}))}{\alpha_k} \right]. \quad (4.2.2)$$

Let c_l , $1 \leq l \leq L$ be the shape parameters of the solitons, $\kappa_l \in \{-1, 1\}$ their signs, and $x_{0,l}$ their translation parameters. We note R_l the corresponding soliton. More precisely,

$$R_l(t, x) = \kappa_l 2\sqrt{2} \frac{\partial}{\partial x} \arctan [\exp(\sqrt{c_l} (x - c_l t - x_{0,l}))]. \quad (4.2.3)$$

The velocity of B_k is

$$v_k^b := \beta_k^2 - 3\alpha_k^2 = -\gamma_k. \quad (4.2.4)$$

The velocity of R_l is

$$v_l^s := c_l. \quad (4.2.5)$$

We suppose that the velocities v_k^b , $1 \leq k \leq K$, v_l^s , $1 \leq l \leq L$, are all distinct.

We can arrange the objects B_k, R_l in order of increasing velocities by defining P_j , $1 \leq j \leq J = K + L$. And we note

$$v_1 < v_2 < \dots < v_J \quad (4.2.6)$$

their velocities. Moreover, we suppose that B_k are arranged in order of increasing velocities, and so are the solitons R_l .

The goal is to prove that we can give by a formula a multi-breather corresponding to

$$P := \sum_{j=1}^J P_j \quad (4.2.7)$$

when $t \rightarrow +\infty$.

Let us formulate exactly the proposition that we will prove :

Proposition 4.4. *Let us suppose (4.2.6) for a set P_1, \dots, P_J of solitons and breathers de (mKdV), and let us use the notations given above. There exists $\theta > 0$ (that depends only on shape parameters of P_1, \dots, P_J) and a multi-breather*

p associated to P_1, \dots, P_J such that for any $s \geq 0$, there exist $A_s > 0$ and $T > 0$ (that depend on translation and shape parameters of P_1, \dots, P_J) such that

$$\forall t \geq T, \quad \|p(t) - P(t)\|_{H^s} \leq A_s \exp(-\theta t), \quad (4.2.8)$$

and

$$\forall t \leq -T, \quad \left\| p(t) - \sum_{j=1}^J P_j^{-\infty}(t) \right\|_{H^2} \leq A_s \exp(-\theta |t|), \quad (4.2.9)$$

where, for $j = 1, \dots, J$, if $P_j = B_k$ is a breather, denoting $l := \max(\{0\} \cup \{n, v_n^s < v_j\})$,

$$P_j^{-\infty}(t, x) := B_{\alpha_k, \beta_k}(t, x; x_{1,k}^{-\infty}, x_{2,k}^{-\infty}), \quad (4.2.10)$$

where

$$\begin{aligned} x_{1,k}^{-\infty} := & x_{1,k} - \frac{2}{\alpha_k} \left[\sum_{n=1}^{k-1} \arg \left(\frac{(\alpha_k - \alpha_n + i(\beta_k - \beta_n))(\alpha_k + \alpha_n + i(\beta_k - \beta_n))}{(\alpha_k + \alpha_n + i(\beta_k + \beta_n))(\alpha_k - \alpha_n + i(\beta_k + \beta_n))} \right) + \sum_{n=1}^l \arg \left(\frac{\alpha_k + i(\beta_k - \sqrt{c_l})}{\alpha_k + i(\beta_k + \sqrt{c_l})} \right) \right] \\ & + \frac{2}{\alpha_k} \left[\sum_{n=k+1}^K \arg \left(\frac{(\alpha_k - \alpha_n + i(\beta_k - \beta_n))(\alpha_k + \alpha_n + i(\beta_k - \beta_n))}{(\alpha_k + \alpha_n + i(\beta_k + \beta_n))(\alpha_k - \alpha_n + i(\beta_k + \beta_n))} \right) + \sum_{n=l+1}^L \arg \left(\frac{\alpha_k + i(\beta_k - \sqrt{c_l})}{\alpha_k + i(\beta_k + \sqrt{c_l})} \right) \right], \end{aligned} \quad (4.2.11)$$

and

$$\begin{aligned} x_{2,k}^{-\infty} := & x_{2,k} \\ & - \frac{1}{\beta_k} \left[\sum_{n=1}^{k-1} \ln \left(\frac{((\alpha_k - \alpha_n)^2 + (\beta_k - \beta_n)^2)((\alpha_k + \alpha_n)^2 + (\beta_k - \beta_n)^2)}{((\alpha_k + \alpha_n)^2 + (\beta_k + \beta_n)^2)((\alpha_k - \alpha_n)^2 + (\beta_k + \beta_n)^2)} \right) + \sum_{n=1}^l \ln \left(\frac{\alpha_k^2 + (\beta_k - \sqrt{c_l})^2}{\alpha_k^2 + (\beta_k + \sqrt{c_l})^2} \right) \right] \\ & + \frac{1}{\beta_k} \left[\sum_{n=k+1}^K \ln \left(\frac{((\alpha_k - \alpha_n)^2 + (\beta_k - \beta_n)^2)((\alpha_k + \alpha_n)^2 + (\beta_k - \beta_n)^2)}{((\alpha_k + \alpha_n)^2 + (\beta_k + \beta_n)^2)((\alpha_k - \alpha_n)^2 + (\beta_k + \beta_n)^2)} \right) + \sum_{n=l+1}^L \ln \left(\frac{\alpha_k^2 + (\beta_k - \sqrt{c_l})^2}{\alpha_k^2 + (\beta_k + \sqrt{c_l})^2} \right) \right], \end{aligned} \quad (4.2.12)$$

and if $P_j = R_l$ is a soliton, denoting $k := \max(\{0\} \cup \{n, v_n^b < v_j\})$,

$$P_j^{-\infty} := \kappa_l R_{c_l}(t, x; x_{0,l}^{-\infty}), \quad (4.2.13)$$

where

$$\begin{aligned} x_{0,l}^{-\infty} := & x_{0,l} - \frac{2}{\sqrt{c_l}} \left[\sum_{n=k+1}^K \ln \left(\frac{\alpha_n^2 + (\beta_n - \sqrt{c_l})^2}{\alpha_n^2 + (\beta_n + \sqrt{c_l})^2} \right) + \sum_{n=l+1}^L \ln \left(\frac{\sqrt{c_n} - \sqrt{c_l}}{\sqrt{c_n} + \sqrt{c_l}} \right) \right] \\ & + \frac{2}{\sqrt{c_l}} \left[\sum_{n=1}^k \ln \left(\frac{\alpha_n^2 + (\beta_n - \sqrt{c_l})^2}{\alpha_n^2 + (\beta_n + \sqrt{c_l})^2} \right) + \sum_{n=1}^{l-1} \ln \left(\frac{\sqrt{c_l} - \sqrt{c_n}}{\sqrt{c_n} + \sqrt{c_l}} \right) \right]. \end{aligned} \quad (4.2.14)$$

The following is dedicated to the proof of this Proposition.

Let $N = 2K + L$. We define $\zeta_1, \dots, \zeta_N \in \mathbb{C}$. For $1 \leq k \leq K$, we set

$$\zeta_{2k-1} := \frac{\alpha_k}{2} + i\frac{\beta_k}{2}, \quad (4.2.15)$$

$$\zeta_{2k} := -\frac{\alpha_k}{2} + i\frac{\beta_k}{2}. \quad (4.2.16)$$

For $1 \leq l \leq L$, we set

$$\zeta_{2K+l} := i\frac{\sqrt{c_l}}{2}. \quad (4.2.17)$$

We also define parameters that will determine translation parameters of our set of objects : $\mathcal{A}_1, \dots, \mathcal{A}_N \in \mathbb{R}$ and $\mathcal{B}_1, \dots, \mathcal{B}_N \in \mathbb{R}$ such that

$$\forall n \in \llbracket 1, N \rrbracket, \quad (\mathcal{A}_n, \mathcal{B}_n) \neq (0, 0). \quad (4.2.18)$$

For $1 \leq k \leq K$, we also ask that

$$\mathcal{A}_{2k} = -\mathcal{A}_{2k-1}, \quad (4.2.19)$$

and

$$\mathcal{B}_{2k} = \mathcal{B}_{2k-1}. \quad (4.2.20)$$

For $1 \leq l \leq L$, we also ask that

$$\mathcal{A}_{2K+l} = 0. \quad (4.2.21)$$

We define $V(x, t)$ a matrix $N \times N$ whose coefficients are given by the following formula :

$$V_{n,m}(t, x) := \frac{i(\mathcal{A}_n + i\mathcal{B}_n) \exp(8i\zeta_n^3 t)}{\zeta_n + \zeta_m} \exp(i(\zeta_n + \zeta_m)x). \quad (4.2.22)$$

We note that, for $1 \leq k \leq K$,

$$\begin{aligned} 8\zeta_{2k-1}^3 &= \alpha_k(\alpha_k^2 - 3\beta_k^2) + i\beta_k(3\beta_k^2 - \alpha_k^2) \\ &= \alpha_k\delta_k + i\beta_k\gamma_k, \end{aligned} \quad (4.2.23)$$

$$8\zeta_{2k}^3 = -\alpha_k\delta_k + i\beta_k\gamma_k, \quad (4.2.24)$$

and for $1 \leq l \leq L$,

$$8\zeta_{2K+l}^3 = -i\sqrt{c_l}c_l. \quad (4.2.25)$$

The solution that we consider is given by the following formula [117] :

$$p(t, x) := 2\sqrt{2} \frac{\partial}{\partial x} \arctan \left[\frac{\Im \det(I + V)}{\Re \det(I + V)} \right]. \quad (4.2.26)$$

We can present it in an equivalent way as follows :

$$p(t, x) := 2\sqrt{2} \frac{\partial}{\partial x} \arg [\det (I + V)]. \quad (4.2.27)$$

We have in general, that

$$\det(I + V) = \sum_{E \subset [1, N]} \sum_{\sigma \in \mathfrak{S}(E)} \varepsilon(\sigma) \prod_{k \in E} V_{k, \sigma(k)}, \quad (4.2.28)$$

where ε stands for the signature of a permutation.

4.2.1 Formula for a 2-soliton

As an example, let us give the formula for a 2-soliton of (mKdV). We will see that even in this case, the explicit expression is quite long. That is why, even if the formula given in (4.2.26) is quite abstract, we cannot give a more explicit formula for p in general.

In the case of a 2-soliton, we have $N = 2$. So, we have two positive shape parameters $c_1 \neq c_2$ and two nonzero translation parameters $\mathcal{B}_1, \mathcal{B}_2 \in \mathbb{R}$. We will suppose that $\mathcal{B}_1, \mathcal{B}_2 > 0$ in the computations that we make in the following (in the other cases, the computations are similar, but we need to distinguish them). As we will see in the following subsections, this case corresponds to the case when both of the solitons are antisolitons. To obtain the case when both of the solitons are positive, it is enough to take the opposite of the formula given in (4.2.26).

The matrix V is the following :

$$V(t, x) = \begin{pmatrix} \frac{i\mathcal{B}_1 \exp(\sqrt{c_1}c_1 t)}{\sqrt{c_1}} \exp(-\sqrt{c_1}x) & \frac{2i\mathcal{B}_1 \exp(\sqrt{c_1}c_1 t)}{\sqrt{c_1} + \sqrt{c_2}} \exp\left(-\frac{\sqrt{c_1} + \sqrt{c_2}}{2}x\right) \\ \frac{2i\mathcal{B}_2 \exp(\sqrt{c_2}c_2 t)}{\sqrt{c_1} + \sqrt{c_2}} \exp\left(-\frac{\sqrt{c_1} + \sqrt{c_2}}{2}x\right) & \frac{i\mathcal{B}_2 \exp(\sqrt{c_2}c_2 t)}{\sqrt{c_2}} \exp(-\sqrt{c_2}x) \end{pmatrix}. \quad (4.2.29)$$

So,

$$\begin{aligned} \det(I + V) &= \mathcal{B}_1 \mathcal{B}_2 \left(-\frac{1}{\sqrt{c_1}\sqrt{c_2}} + \frac{4}{(\sqrt{c_1} + \sqrt{c_2})^2} \right) \exp((\sqrt{c_1}c_1 + \sqrt{c_2}c_2)t) \exp(-(\sqrt{c_1} + \sqrt{c_2})x) \\ &\quad + \frac{i\mathcal{B}_1 \exp(\sqrt{c_1}c_1 t)}{\sqrt{c_1}} \exp(-\sqrt{c_1}x) + \frac{i\mathcal{B}_2 \exp(\sqrt{c_2}c_2 t)}{\sqrt{c_2}} \exp(-\sqrt{c_2}x) + 1. \end{aligned} \quad (4.2.30)$$

Then,

$$\begin{aligned} &\frac{\Im \det(I + V)}{\Re \det(I + V)} \\ &= \frac{\frac{\mathcal{B}_1 \exp(\sqrt{c_1}c_1 t)}{\sqrt{c_1}} \exp(-\sqrt{c_1}x) + \frac{\mathcal{B}_2 \exp(\sqrt{c_2}c_2 t)}{\sqrt{c_2}} \exp(-\sqrt{c_2}x)}{1 + \mathcal{B}_1 \mathcal{B}_2 \left(-\frac{1}{\sqrt{c_1}\sqrt{c_2}} + \frac{4}{(\sqrt{c_1} + \sqrt{c_2})^2} \right) \exp((\sqrt{c_1}c_1 + \sqrt{c_2}c_2)t) \exp(-(\sqrt{c_1} + \sqrt{c_2})x)} \\ &= \frac{\frac{\mathcal{B}_1^{1/2} \mathcal{B}_2^{1/4}}{\mathcal{B}_2^{1/2} c_1^{1/4}} \exp\left(\frac{\sqrt{c_1}c_1 - \sqrt{c_2}c_2}{2}t\right) \exp\left(-\frac{\sqrt{c_1} - \sqrt{c_2}}{2}x\right) + \frac{\mathcal{B}_2^{1/2} c_1^{1/4}}{\mathcal{B}_1^{1/2} c_2^{1/4}} \exp\left(\frac{\sqrt{c_2}c_2 - \sqrt{c_1}c_1}{2}t\right) \exp\left(-\frac{\sqrt{c_2} - \sqrt{c_1}}{2}x\right)}{\frac{c_1^{1/4} c_2^{1/4}}{\mathcal{B}_1^{1/2} \mathcal{B}_2^{1/2}} \exp\left(-\frac{\sqrt{c_1}c_1 + \sqrt{c_2}c_2}{2}t\right) \exp\left(\frac{\sqrt{c_1} + \sqrt{c_2}}{2}x\right) - \frac{\mathcal{B}_1^{1/2} \mathcal{B}_2^{1/2} (\sqrt{c_1} - \sqrt{c_2})^2}{(\sqrt{c_1} + \sqrt{c_2})^2 c_1^{1/4} c_2^{1/4}} \exp\left(\frac{\sqrt{c_1}c_1 + \sqrt{c_2}c_2}{2}t\right) \exp\left(-\frac{\sqrt{c_1} + \sqrt{c_2}}{2}x\right)} \\ &= \frac{(\sqrt{c_1} + \sqrt{c_2})}{|\sqrt{c_1} - \sqrt{c_2}|} \frac{\cosh\left(\frac{\sqrt{c_1}c_1 - \sqrt{c_2}c_2}{2}t - \frac{\sqrt{c_1} - \sqrt{c_2}}{2}x + \ln\left(\frac{\mathcal{B}_1^{1/2} c_2^{1/4}}{\mathcal{B}_2^{1/2} c_1^{1/4}}\right)\right)}{\sinh\left(-\frac{\sqrt{c_1}c_1 + \sqrt{c_2}c_2}{2}t + \frac{\sqrt{c_1} + \sqrt{c_2}}{2}x + \ln\left(\frac{c_1^{1/4} c_2^{1/4}}{\mathcal{B}_1^{1/2} \mathcal{B}_2^{1/2}} \frac{\sqrt{c_1} + \sqrt{c_2}}{|\sqrt{c_1} - \sqrt{c_2}|}\right)\right)}. \end{aligned} \quad (4.2.31)$$

Thus, a formula for a 2-soliton of (mKdV) can be parametrized by shape parameters $c_1, c_2 > 0$ such that $c_1 \neq c_2$ and translation parameters $x_1, x_2 \in \mathbb{R}$ and can be given by the following expression :

$$r(t, x) := -2\sqrt{2} \frac{\partial}{\partial x} \arctan \left[\frac{\sqrt{c_1} + \sqrt{c_2}}{|\sqrt{c_1} - \sqrt{c_2}|} \frac{\cosh \left(\frac{\sqrt{c_1}c_1 - \sqrt{c_2}c_2}{2} t - \frac{\sqrt{c_1} - \sqrt{c_2}}{2} x + x_1 \right)}{\sinh \left(-\frac{\sqrt{c_1}c_1 + \sqrt{c_2}c_2}{2} t + \frac{\sqrt{c_1} + \sqrt{c_2}}{2} x + x_2 \right)} \right]. \quad (4.2.32)$$

To obtain a 2-antisoliton, one can take the opposite of the formula given above.

And, if we want a formula for a couple soliton-antisoliton (with distinct velocities), we need to make the computations above supposing that \mathcal{B}_1 and \mathcal{B}_2 have opposite signs (as we will see in the following). So, a couple soliton-antisoliton, where the soliton has the shape parameter $c_1 > 0$ and the antisoliton has the shape parameter $c_2 > 0$, with $c_1 \neq c_2$, parametrized by translation parameters $x_1, x_2 \in \mathbb{R}$ is given by the following expression :

$$r(t, x) := -2\sqrt{2} \frac{\partial}{\partial x} \arctan \left[\frac{\sqrt{c_1} + \sqrt{c_2}}{|\sqrt{c_1} - \sqrt{c_2}|} \frac{\sinh \left(\frac{\sqrt{c_1}c_1 - \sqrt{c_2}c_2}{2} t - \frac{\sqrt{c_1} - \sqrt{c_2}}{2} x + x_1 \right)}{\cosh \left(-\frac{\sqrt{c_1}c_1 + \sqrt{c_2}c_2}{2} t + \frac{\sqrt{c_1} + \sqrt{c_2}}{2} x + x_2 \right)} \right]. \quad (4.2.33)$$

We can observe that when c_1 tends to c_2 , we are able to find the formula for dipoles given in 1.4.6.

4.2.2 Pointwise convergence

Determination of $\mathcal{A}_n, \mathcal{B}_n$ in terms of translation parameters when $t \rightarrow +\infty$

First, we study the pointwise convergence of p when $t \rightarrow +\infty$.

We observe that $\det(I + V)$ is a linear combination of exponentials of the following type

$$\sum_i w_i \exp(u_i t + v_i x), \quad (4.2.34)$$

where $u_i, v_i, w_i \in \mathbb{C}$.

So, $\Im \det(I + V)$ and $\Re \det(I + V)$ are linear combinations of the following type

$$\sum_i w_i \sin(\nu_i t + \omega_i x + \phi_i) \exp(u_i t + v_i x), \quad (4.2.35)$$

where $u_i, v_i, w_i, \nu_i, \omega_i, \phi_i \in \mathbb{R}$.

We can have a first idea of the form of the expression of p by writing :

$$p(t, x) = 2\sqrt{2} \frac{\left[\frac{\partial}{\partial x} \Im \det(I + V) \right] \Re \det(I + V) - \Im \det(I + V) \left[\frac{\partial}{\partial x} \Re \det(I + V) \right]}{(\Re \det(I + V))^2 + (\Im \det(I + V))^2}. \quad (4.2.36)$$

In any case, the biggest part of the work is in the study of $\det(I + V)$.

The first goal is to find how to choose the parameters $\mathcal{A}_n, \mathcal{B}_n$ so that p converges to P when $t \rightarrow +\infty$ (with the translation parameters given by P).

For this, one needs to study the pointwise limit of

$$p(t, x + v_j t) \quad (4.2.37)$$

when $t \rightarrow +\infty$. And the goal is to find the translation parameters of P_j . (By this manner, we make the velocity of P_j equal to zero, and so P_j becomes centered in a fixed point when $t \rightarrow +\infty$, which is easily detectable by a pointwise convergence.)

We have

$$\begin{aligned}
& \det[I + V](t, x + v_j t) \\
&= \sum_{E \subset \llbracket 1, N \rrbracket} i^{|E|} \prod_{n \in E} (\mathcal{A}_n + i\mathcal{B}_n) \exp\left(8i \sum_{n \in E} \zeta_n^3 t\right) \exp\left(2i \sum_{n \in E} \zeta_n x\right) \\
&\quad \cdot \exp\left(2i \sum_{n \in E} \zeta_n v_j t\right) \sum_{\sigma \in \mathfrak{S}(E)} \varepsilon(\sigma) \prod_{n \in E} \frac{1}{\zeta_n + \zeta_{\sigma(n)}} \\
&= \sum_{E \subset \llbracket 1, N \rrbracket} i^{|E|} \prod_{n \in E} (\mathcal{A}_n + i\mathcal{B}_n) \exp\left(2i \sum_{n \in E} \zeta_n (4\zeta_n^2 + v_j) t\right) \exp\left(2i \sum_{n \in E} \zeta_n x\right) \sum_{\sigma \in \mathfrak{S}(E)} \varepsilon(\sigma) \prod_{n \in E} \frac{1}{\zeta_n + \zeta_{\sigma(n)}} \\
&= \sum_{E \subset \llbracket 1, N \rrbracket} i^{|E|} \prod_{n \in E} (\mathcal{A}_n + i\mathcal{B}_n) \exp\left(\sum_{n=1}^K (\mathbf{1}_{2n-1 \in E} + \mathbf{1}_{2n \in E}) \beta_n (-\gamma_n - v_j) t\right) \\
&\quad \exp\left(\sum_{n=1}^L \mathbf{1}_{2K+n \in E} \sqrt{c_n} (c_n - v_j) t\right) \exp\left(i \sum_{n=1}^K (\mathbf{1}_{2n-1 \in E} - \mathbf{1}_{2n \in E}) \alpha_n (\delta_n + v_j) t\right) \exp\left(-\sum_{n=1}^L \mathbf{1}_{2K+n \in E} \sqrt{c_n} x\right) \\
&\quad \exp\left(-\sum_{n=1}^K (\mathbf{1}_{2n-1 \in E} + \mathbf{1}_{2n \in E}) \beta_n x\right) \exp\left(i \sum_{n=1}^K (\mathbf{1}_{2n-1 \in E} - \mathbf{1}_{2n \in E}) \alpha_n x\right) \sum_{\sigma \in \mathfrak{S}(E)} \varepsilon(\sigma) \prod_{n \in E} \frac{1}{\zeta_n + \zeta_{\sigma(n)}},
\end{aligned} \tag{4.2.38}$$

where

$$\sum_{\sigma \in \mathfrak{S}(E)} \varepsilon(\sigma) \prod_{n \in E} \frac{1}{\zeta_n + \zeta_{\sigma(n)}} \tag{4.2.39}$$

corresponds to a Cauchy determinant. We have an explicit formula to compute it.

Let us start by detecting the translation parameters of $P_j = B_k$ from the parameters given by the formula.

We are concerned by the pointwise convergence of

$$p(t, x - \gamma_k t). \tag{4.2.40}$$

When $t \rightarrow +\infty$, in the expression of $\det(I + V)$, we can keep the terms corresponding to E such that

$$\sum_{n=1}^K (\mathbf{1}_{2n-1 \in E} + \mathbf{1}_{2n \in E}) \beta_n (-\gamma_n + \gamma_k) + \sum_{n=1}^L \mathbf{1}_{2K+n \in E} \sqrt{c_n} (c_n + \gamma_k) \tag{4.2.41}$$

is maximal, the other terms vanish when $t \rightarrow +\infty$. $-\gamma_n + \gamma_k$ corresponds to the velocity of B_n from which we have removed the velocity of B_k , and $c_n + \gamma_k$ corresponds to the velocity of R_n from which we have removed the velocity of B_k . So, we need to take the sets E that correspond to all the objects that have larger velocities than B_k , eventually including B_k . Let l be the largest index such that $c_l < -\gamma_k$ (in the case when there is no such index, we set $l = 0$). In this case, the appropriate sets E are $E_1 = \llbracket 2k+1, 2K \rrbracket \cup \llbracket 2K+l+1, N \rrbracket$, $E_2 = \llbracket 2k, 2K \rrbracket \cup \llbracket 2K+l+1, N \rrbracket$, $E_3 = \{2k-1\} \cup \llbracket 2k+1, 2K \rrbracket \cup \llbracket 2K+l+1, N \rrbracket$ and $E_4 = \llbracket 2k-1, 2K \rrbracket \cup \llbracket 2K+l+1, N \rrbracket$.

We take as **convention** that if $a > b$, the set $\llbracket a, b \rrbracket$ corresponds to the empty set. We will use the same convention for sums and products.

Let us note

$$\begin{aligned} C &:= (-1)^{K-k} i^{L-l} \prod_{n=2k+1}^{2K} (\mathcal{A}_n + i\mathcal{B}_n) \prod_{n=2K+l+1}^N (\mathcal{A}_n + i\mathcal{B}_n) \\ &= (-1)^{L-l} \prod_{n=k+1}^K (\mathcal{A}_{2k}^2 + \mathcal{B}_{2k}^2) \prod_{n=l+1}^L \mathcal{B}_{2K+n} \in \mathbb{R}, \end{aligned} \quad (4.2.42)$$

which is a translation parameter that is determined by objects that are faster than the considered breather. It is given by the first factors of the terms that compose the sum $\det(I + V)$.

Let us also denote the Cauchy determinants that will appear (and that depend only on the shape parameters of the objects) in the following way :

$$\begin{aligned} \mathcal{D}_m &:= \sum_{\sigma \in \mathfrak{S}(E_m)} \varepsilon(\sigma) \prod_{n \in E_m} \frac{1}{\zeta_n + \zeta_{\sigma(n)}} \\ &= \frac{\prod_{i < j; (i,j) \in E_m^2} (\zeta_j - \zeta_i)^2}{\prod_{(i,j) \in E_m^2} (\zeta_j + \zeta_i)}, \end{aligned} \quad (4.2.43)$$

for $m = 1, 2, 3, 4$. We usefully remark that

$$\mathcal{D}_1, \mathcal{D}_4 \in \mathbb{R} \quad \text{if } L - l \text{ is even,} \quad (4.2.44)$$

$$\mathcal{D}_1, \mathcal{D}_4 \in i\mathbb{R} \quad \text{if } L - l \text{ is odd.} \quad (4.2.45)$$

and

$$\mathcal{D}_2 + \overline{\mathcal{D}_3} = 0 \quad \text{if } L - l \text{ is even,} \quad (4.2.46)$$

$$\mathcal{D}_2 - \overline{\mathcal{D}_3} = 0 \quad \text{if } L - l \text{ is odd.} \quad (4.2.47)$$

Let us also discuss the sign of \mathcal{D}_1 or \mathcal{D}_4 :

$$\mathcal{D}_1, \mathcal{D}_4 \in \mathbb{R}_+^* \quad \text{if } L - l \equiv 0[4], \quad (4.2.48)$$

$$\mathcal{D}_1, \mathcal{D}_4 \in -i\mathbb{R}_+^* \quad \text{if } L - l \equiv 1[4], \quad (4.2.49)$$

$$\mathcal{D}_1, \mathcal{D}_4 \in -\mathbb{R}_+^* \quad \text{if } L - l \equiv 2[4], \quad (4.2.50)$$

$$\mathcal{D}_1, \mathcal{D}_4 \in i\mathbb{R}_+^* \quad \text{if } L - l \equiv 3[4]. \quad (4.2.51)$$

To study the pointwise convergence of p , we can leave out the terms corresponding to other sets E , which are negligible. So, up to negligible terms, when $t \rightarrow +\infty$,

$$\begin{aligned} \det[I + V](t, x - \gamma_k t) &\sim \mathcal{D}_1 \\ &\quad + i(\mathcal{A}_{2k} + i\mathcal{B}_{2k}) \exp(-i\alpha_k(\delta_k - \gamma_k)t) \exp(-\beta_k x) \exp(-i\alpha_k x) \mathcal{D}_2 \end{aligned}$$

$$\begin{aligned}
& -i(\mathcal{A}_{2k} - i\mathcal{B}_{2k}) \exp(i\alpha_k(\delta_k - \gamma_k)t) \exp(-\beta_k x) \exp(i\alpha_k x) \mathcal{D}_3 \\
& + (\mathcal{A}_{2k}^2 + \mathcal{B}_{2k}^2) \exp(-2\beta_k x) \mathcal{D}_4
\end{aligned} \tag{4.2.52}$$

multiplied by

$$C \exp\left(2 \sum_{n=k+1}^K \beta_n (-\gamma_n + \gamma_k) t\right) \exp\left(\sum_{n=l+1}^L \sqrt{c_n} (c_n + \gamma_k) t\right) \exp\left(-2 \sum_{n=k+1}^K \beta_n x\right) \exp\left(-\sum_{n=l+1}^L \sqrt{c_n} x\right) \in \mathbb{R}. \tag{4.2.53}$$

We place ourselves in the case when $L - l$ is even and divisible by 4.

We remark that the sum of the first and the fourth line of the sum (4.2.52) is real, and that the sum of the second and the third line is imaginary.

So,

$$\frac{\Im \det[I + V]}{\Re \det[I + V]}(t, x - \gamma_k t) \sim \frac{2\Re[(\mathcal{A}_{2k} + i\mathcal{B}_{2k}) \exp(-i\alpha_k(\delta_k - \gamma_k)t) \exp(-\beta_k x) \exp(-i\alpha_k x) \mathcal{D}_2]}{\mathcal{D}_1 + (\mathcal{A}_{2k}^2 + \mathcal{B}_{2k}^2) \exp(-2\beta_k x) \mathcal{D}_4}, \tag{4.2.54}$$

when $t \rightarrow +\infty$, the other terms being exponentially negligible.

It leaves now only to give to fraction that has just appeared the following form :

$$\frac{\beta_k \sin(\alpha_k(x + (\delta_k - \gamma_k)t + x_{1,k}))}{\alpha_k \cosh(\beta_k(x + x_{2,k}))}. \tag{4.2.55}$$

Let us start by rewriting the numerator :

$$2\sqrt{\mathcal{A}_{2k}^2 + \mathcal{B}_{2k}^2} \exp(-\beta_k x) |\mathcal{D}_2| \cos(\alpha_k(x + (\delta_k - \gamma_k)t) - \arg(\mathcal{D}_2) - \arg(\mathcal{A}_{2k} + i\mathcal{B}_{2k})), \tag{4.2.56}$$

which is the same as :

$$2\sqrt{\mathcal{A}_{2k}^2 + \mathcal{B}_{2k}^2} \exp(-\beta_k x) |\mathcal{D}_2| \sin\left(\alpha_k(x + (\delta_k - \gamma_k)t) - \arg(\mathcal{D}_2) - \arg(\mathcal{A}_{2k} + i\mathcal{B}_{2k}) + \frac{\pi}{2}\right). \tag{4.2.57}$$

So, $\frac{\Im \det[I+V]}{\Re \det[I+V]}(t, x - \gamma_k t)$ is equivalent when $t \rightarrow +\infty$ to :

$$\frac{\beta_k \sin(\alpha_k(x + (\delta_k - \gamma_k)t) - \arg(\mathcal{D}_2) - \arg(\mathcal{A}_{2k} + i\mathcal{B}_{2k}) + \frac{\pi}{2})}{\alpha_k \frac{\beta_k}{2\sqrt{\mathcal{A}_{2k}^2 + \mathcal{B}_{2k}^2} |\mathcal{D}_2|} (\mathcal{D}_1 \exp(\beta_k x) + (\mathcal{A}_{2k}^2 + \mathcal{B}_{2k}^2) \exp(-\beta_k x) \mathcal{D}_4)}. \tag{4.2.58}$$

In order to present the denominator as a hyperbolic cosine, we need to check that :

$$\frac{\beta_k^2}{\alpha_k^2} \frac{1}{|\mathcal{D}_2|^2} \frac{1}{\mathcal{A}_{2k}^2 + \mathcal{B}_{2k}^2} \mathcal{D}_1 \mathcal{D}_4 (\mathcal{A}_{2k}^2 + \mathcal{B}_{2k}^2) = 1. \tag{4.2.59}$$

We have that :

$$\begin{aligned}
& \frac{\beta_k^2}{\alpha_k^2} \frac{1}{|\mathcal{D}_2|^2} \frac{1}{\mathcal{A}_{2k}^2 + \mathcal{B}_{2k}^2} \mathcal{D}_1 \mathcal{D}_4 (\mathcal{A}_{2k}^2 + \mathcal{B}_{2k}^2) \\
& = \frac{\beta_k^2}{\alpha_k^2} \frac{\mathcal{D}_1 \mathcal{D}_4}{|\mathcal{D}_2|^2}
\end{aligned}$$

$$\begin{aligned}
&= \frac{\prod_{n=k+1}^K \left[\left(\frac{\alpha_k - \alpha_n}{2} + i \frac{\beta_k - \beta_n}{2} \right)^2 \left(\frac{\alpha_k + \alpha_n}{2} + i \frac{\beta_k - \beta_n}{2} \right)^2 \left(\frac{-\alpha_k - \alpha_n}{2} + i \frac{\beta_k - \beta_n}{2} \right)^2 \left(\frac{-\alpha_k + \alpha_n}{2} + i \frac{\beta_k - \beta_n}{2} \right)^2 \right]}{\prod_{n=k+1}^K \left[\left(\frac{\alpha_k + \alpha_n}{2} + i \frac{\beta_k + \beta_n}{2} \right)^2 \left(\frac{\alpha_k - \alpha_n}{2} + i \frac{\beta_k + \beta_n}{2} \right)^2 \left(\frac{-\alpha_k + \alpha_n}{2} + i \frac{\beta_k + \beta_n}{2} \right)^2 \left(\frac{-\alpha_k - \alpha_n}{2} + i \frac{\beta_k + \beta_n}{2} \right)^2 \right]} \\
&\cdot \frac{\beta_k^2 \prod_{n=l+1}^L \left[\left(\frac{\alpha_k}{2} + i \frac{\beta_k - \sqrt{c_n}}{2} \right)^2 \left(-\frac{\alpha_k}{2} + i \frac{\beta_k - \sqrt{c_n}}{2} \right)^2 \right]}{\alpha_k^2 \prod_{n=l+1}^L \left[\left(\frac{\alpha_k}{2} + i \frac{\beta_k + \sqrt{c_n}}{2} \right)^2 \left(-\frac{\alpha_k}{2} + i \frac{\beta_k + \sqrt{c_n}}{2} \right)^2 \right]} \frac{\alpha_k^2}{\beta_k^2 (\alpha_k^2 + \beta_k^2)} \\
&\cdot \frac{(\alpha_k^2 + \beta_k^2) \prod_{n=k+1}^K \left[\left(\left(\frac{\alpha_k + \alpha_n}{2} \right)^2 + \left(\frac{\beta_k + \beta_n}{2} \right)^2 \right) \left(\left(\frac{\alpha_k - \alpha_n}{2} \right)^2 + \left(\frac{\beta_k + \beta_n}{2} \right)^2 \right) \right] \prod_{n=l+1}^L \left(\left(\frac{\alpha_k}{2} \right)^2 + \left(\frac{\beta_k + \sqrt{c_n}}{2} \right)^2 \right)^2}{\prod_{n=k+1}^K \left[\left(\left(\frac{\alpha_k - \alpha_n}{2} \right)^2 + \left(\frac{\beta_k - \beta_n}{2} \right)^2 \right) \left(\left(\frac{\alpha_k + \alpha_n}{2} \right)^2 + \left(\frac{\beta_k - \beta_n}{2} \right)^2 \right) \right] \prod_{n=l+1}^L \left(\left(\frac{\alpha_k}{2} \right)^2 + \left(\frac{\beta_k - \sqrt{c_n}}{2} \right)^2 \right)^2} \\
&= 1.
\end{aligned} \tag{4.2.60}$$

So, $\frac{\Im \det[I+V]}{\Re \det[I+V]}(t, x - \gamma_k t)$ is equivalent when $t \rightarrow +\infty$ to :

$$\frac{\beta_k \sin \left(\alpha_k (x + (\delta_k - \gamma_k) t) - \arg(\mathcal{D}_2) - \arg(\mathcal{A}_{2k} + i\mathcal{B}_{2k}) + \frac{\pi}{2} \right)}{\alpha_k \cosh \left(\beta_k x + \ln \left(\frac{\mathcal{D}_1}{\sqrt{\mathcal{A}_{2k}^2 + \mathcal{B}_{2k}^2} |\mathcal{D}_2|} \frac{\beta_k}{\alpha_k} \right) \right)}. \tag{4.2.61}$$

So,

$$x_{1,k} = -\frac{1}{\alpha_k} \left(\arg(\mathcal{D}_2) + \arg(\mathcal{A}_{2k} + i\mathcal{B}_{2k}) - \frac{\pi}{2} \right), \tag{4.2.62}$$

and

$$x_{2,k} = \frac{1}{\beta_k} \ln \left(\frac{\mathcal{D}_1}{\sqrt{\mathcal{A}_{2k}^2 + \mathcal{B}_{2k}^2} |\mathcal{D}_2|} \frac{\beta_k}{\alpha_k} \right). \tag{4.2.63}$$

Thus, there is a one-to-one correspondence between the translation parameters of the breather and $\mathcal{A}_{2k} + i\mathcal{B}_{2k} \in \mathbb{C}^*$.

In the case when $L - l$ is even without being divisible by 4, things happen in a nearly same way, with the difference that we obtain the opposite. More precisely, $\frac{\Im \det[I+V]}{\Re \det[I+V]}(t, x - \gamma_k t)$ is equivalent when $t \rightarrow +\infty$ to :

$$-\frac{\beta_k \sin \left(\alpha_k (x + (\delta_k - \gamma_k) t) - \arg(\mathcal{D}_2) - \arg(\mathcal{A}_{2k} + i\mathcal{B}_{2k}) + \frac{\pi}{2} \right)}{\alpha_k \cosh \left(\beta_k x + \ln \left(\frac{-\mathcal{D}_1}{\sqrt{\mathcal{A}_{2k}^2 + \mathcal{B}_{2k}^2} |\mathcal{D}_2|} \frac{\beta_k}{\alpha_k} \right) \right)}. \tag{4.2.64}$$

In this case,

$$x_{1,k} = -\frac{1}{\alpha_k} \left(\arg(\mathcal{D}_2) + \arg(\mathcal{A}_{2k} + i\mathcal{B}_{2k}) + \frac{\pi}{2} \right), \tag{4.2.65}$$

and

$$x_{2,k} = \frac{1}{\beta_k} \ln \left(\frac{-\mathcal{D}_1}{\sqrt{\mathcal{A}_{2k}^2 + \mathcal{B}_{2k}^2} |\mathcal{D}_2|} \frac{\beta_k}{\alpha_k} \right). \tag{4.2.66}$$

We have still the one-to-one correspondence between the breather's translation parameters and \mathbb{C}^* .

In the case when $L - l$ is odd, the computations are nearly the same. To be complete, let us write these computations.

In this case, in (4.2.52), the sum of the first and the fourth line is imaginary, and the sum of the second and the third line is real.

So,

$$\frac{\Im \det [I + V]}{\Re \det [I + V]}(t, x - \gamma_k t) \sim \frac{(-i\mathcal{D}_1) + (\mathcal{A}_{2k}^2 + \mathcal{B}_{2k}^2) \exp(-2\beta_k x) (-i\mathcal{D}_4)}{2\Re[(\mathcal{A}_{2k} + i\mathcal{B}_{2k}) \exp(-i\alpha_k(\delta_k - \gamma_k)t) \exp(-\beta_k x) \exp(-i\alpha_k x) (i\mathcal{D}_2)]}, \quad (4.2.67)$$

when $t \rightarrow +\infty$.

The denominator can be rewritten in the following way :

$$2\sqrt{\mathcal{A}_{2k}^2 + \mathcal{B}_{2k}^2} \exp(-\beta_k x) |\mathcal{D}_2| \cos\left(\alpha_k((\delta_k - \gamma_k)t + x) - \arg(\mathcal{A}_{2k} + i\mathcal{B}_{2k}) - \arg(\mathcal{D}_2) - \frac{\pi}{2}\right), \quad (4.2.68)$$

that is equal to :

$$2\sqrt{\mathcal{A}_{2k}^2 + \mathcal{B}_{2k}^2} \exp(-\beta_k x) |\mathcal{D}_2| \sin\left(\alpha_k((\delta_k - \gamma_k)t + x) - \arg(\mathcal{A}_{2k} + i\mathcal{B}_{2k}) - \arg(\mathcal{D}_2)\right). \quad (4.2.69)$$

We see that we obtain the inverse of the fraction that we have obtained in the first case. However, we recall that with a definition of arctan adapted to our situation, that can be done by performing a shift of π when we pass from $\arctan(+\infty)$ to $\arctan(-\infty)$, we have

$$\arctan(y) + \arctan\left(\frac{1}{y}\right) = cst. \quad (4.2.70)$$

Because we have to deal with a derivative in the formula for p , we can replace the fraction that we obtain by the opposite of its inverse, and so we can considerate the fraction

$$-\frac{\Re \det [I + V]}{\Im \det [I + V]}(t, x - \gamma_k t) \quad (4.2.71)$$

that gives the same result after differentiation of its arctan.

So, $-\frac{\Re \det [I + V]}{\Im \det [I + V]}(t, x - \gamma_k t)$ is equivalent, when $t \rightarrow +\infty$, to :

$$\frac{\beta_k}{\alpha_k} \frac{\sin(\alpha_k((\delta_k - \gamma_k)t + x) - \arg(\mathcal{A}_{2k} + i\mathcal{B}_{2k}) - \arg(\mathcal{D}_2))}{\frac{\beta_k}{\alpha_k} \frac{1}{2\sqrt{\mathcal{A}_{2k}^2 + \mathcal{B}_{2k}^2} |\mathcal{D}_2|} (i\mathcal{D}_1 \exp(\beta_k x) + i\mathcal{D}_4 (\mathcal{A}_{2k}^2 + \mathcal{B}_{2k}^2) \exp(-\beta_k x))}. \quad (4.2.72)$$

There is nothing new to do to check that

$$\frac{\beta_k^2}{\alpha_k^2} \frac{1}{|\mathcal{D}_2|^2} \frac{1}{\mathcal{A}_{2k}^2 + \mathcal{B}_{2k}^2} (i\mathcal{D}_1) (i\mathcal{D}_4) (\mathcal{A}_{2k}^2 + \mathcal{B}_{2k}^2) = 1, \quad (4.2.73)$$

paying attention to the fact that \mathcal{D}_1^2 is a real *negative* number in this case (in the first case, it was a real positive number).

If $L - l$ is congruent to 1 modulo 4, then $i\mathcal{D}_1$ and $i\mathcal{D}_4$ are positive real numbers. In this case,

$$x_{1,k} = -\frac{1}{\alpha_k} (\arg(\mathcal{A}_{2k} + i\mathcal{B}_{2k}) + \arg(\mathcal{D}_2)), \quad (4.2.74)$$

and

$$x_{2,k} = \frac{1}{\beta_k} \ln \left(\frac{i\mathcal{D}_1}{\sqrt{\mathcal{A}_{2k}^2 + \mathcal{B}_{2k}^2} |\mathcal{D}_2|} \frac{\beta_k}{\alpha_k} \right). \quad (4.2.75)$$

And there is again a one-to-one correspondence between the breather's parameters in $\mathbb{R}/(2\pi\alpha_k\mathbb{Z}) \times \mathbb{R}$ and $\mathcal{A}_{2k} + i\mathcal{B}_{2k} \in \mathbb{C}^*$.

If $L - l$ is congruent to 3 modulo 4, then $i\mathcal{D}_1$ et $i\mathcal{D}_4$ are negative real numbers. In this case,

$$x_{1,k} = -\frac{1}{\alpha_k} (\arg(\mathcal{A}_{2k} + i\mathcal{B}_{2k}) + \arg(\mathcal{D}_2) + \pi), \quad (4.2.76)$$

and

$$x_{2,k} = \frac{1}{\beta_k} \ln \left(\frac{-i\mathcal{D}_1}{\sqrt{\mathcal{A}_{2k}^2 + \mathcal{B}_{2k}^2} |\mathcal{D}_2|} \frac{\beta_k}{\alpha_k} \right). \quad (4.2.77)$$

Conclusion : We have studied the pointwise convergence of $p(t, x - \gamma_k t)$ when $t \rightarrow +\infty$. This study allows us to express translation parameters of the breather B_k in terms of $\mathcal{A}_{2k} + i\mathcal{B}_{2k}$, and that will be useful to study the asymptotic behavior of p more precisely in the following.

We remark that we can express the translation parameters without distinguishing the 4 cases considered above. They can be expressed in a simplified manner in the following way :

$$x_{1,k} = \frac{1}{\alpha_k} \arg \left(\frac{i\mathcal{D}_1}{(\mathcal{A}_{2k} + i\mathcal{B}_{2k}) \mathcal{D}_2} \frac{\beta_k}{\alpha_k} \right), \quad (4.2.78)$$

and

$$x_{2,k} = \frac{1}{\beta_k} \ln \left(\left| \frac{i\mathcal{D}_1}{(\mathcal{A}_{2k} + i\mathcal{B}_{2k}) \mathcal{D}_2} \frac{\beta_k}{\alpha_k} \right| \right). \quad (4.2.79)$$

Let us now detect the translation parameters of $P_j = R_l$ in terms of \mathcal{B}_{2K+l} .

We are looking for the pointwise convergence of

$$p(t, x + c_l t), \quad (4.2.80)$$

when $t \rightarrow +\infty$.

Let k be the largest index such that $-\gamma_k < c_l$ (in the case when such indexes do not exist, we take $k = 0$). Here, the sets E that are appropriate are $E_1 = \llbracket 2k + 1, 2K \rrbracket \cup \llbracket 2K + l + 1, N \rrbracket$ and $E_2 = \llbracket 2k + 1, 2K \rrbracket \cup \llbracket 2K + l, N \rrbracket$.

We set

$$C := (-1)^{L-l} \prod_{n=k+1}^K (\mathcal{A}_{2n}^2 + \mathcal{B}_{2n}^2) \prod_{n=l+1}^L \mathcal{B}_{2K+n}. \quad (4.2.81)$$

C does not depend on the translation parameter \mathcal{B}_{2K+l} associated to the soliton R_l .

We also define Cauchy determinants :

$$\mathcal{D}_m := \sum_{\sigma \in \mathfrak{S}(E_m)} \varepsilon(\sigma) \prod_{n \in E_m} \frac{1}{\zeta_n + \zeta_{\sigma(n)}}$$

$$= \frac{\prod_{i < j; (i,j) \in E_m^2} (\zeta_j - \zeta_i)^2}{\prod_{(i,j) \in E_m^2} (\zeta_j + \zeta_i)}, \quad (4.2.82)$$

for $m = 1, 2$. We usefully remark that

$$\mathcal{D}_1 \in \mathbb{R}_+^*, \mathcal{D}_2 \in -i\mathbb{R}_+^* \quad \text{if } L - l \equiv 0[4], \quad (4.2.83)$$

$$\mathcal{D}_1 \in -i\mathbb{R}_+^*, \mathcal{D}_2 \in -\mathbb{R}_+^* \quad \text{if } L - l \equiv 1[4], \quad (4.2.84)$$

$$\mathcal{D}_1 \in -\mathbb{R}_+^*, \mathcal{D}_2 \in i\mathbb{R}_+^* \quad \text{if } L - l \equiv 2[4], \quad (4.2.85)$$

$$\mathcal{D}_1 \in i\mathbb{R}_+^*, \mathcal{D}_2 \in \mathbb{R}_+^* \quad \text{if } L - l \equiv 3[4]. \quad (4.2.86)$$

This will allow us to distinguish cases in function of the congruence of $L - l$ modulo 4.

We note that in any case (independently on the congruence of $L - l$ modulo 4),

$$\frac{\mathcal{D}_1}{i\mathcal{D}_2} > 0, \quad (4.2.87)$$

$$\frac{i\mathcal{D}_1}{\mathcal{D}_2} < 0. \quad (4.2.88)$$

So, when $t \rightarrow +\infty$,

$$\det[I + V](t, x + c_l t) \sim \mathcal{D}_1 - \mathcal{B}_{2K+l} \exp(-\sqrt{c_l} x) \mathcal{D}_2 \quad (4.2.89)$$

multiplied by

$$C \exp\left(2 \sum_{n=k+1}^K \beta_n (-\gamma_n - c_l) t\right) \exp\left(\sum_{n=l+1}^L \sqrt{c_n} (c_n - c_l) t\right) \exp\left(-2 \sum_{n=k+1}^K \beta_n x\right) \exp\left(-\sum_{n=l+1}^L \sqrt{c_n} x\right) \in \mathbb{R}, \quad (4.2.90)$$

the other terms being exponentially negligible in time.

The goal is now to present $\frac{\Im \det(I+V)}{\Re \det(I+V)}(t, x + c_l t)$ or $-\frac{\Re \det(I+V)}{\Im \det(I+V)}(t, x + c_l t)$ under the following form :

$$\pm \exp(\sqrt{c_l} (x - x_{0,l})), \quad (4.2.91)$$

and to determine $x_{0,l}$ in terms of \mathcal{B}_{2K+l} (with a dependence on c_l and the parameters of the other objects), where if the sign in front of the exponential is positive, we have a soliton, and if it is negative, we have an antisoliton.

First case : $L - l$ is divisible by 2

Then,

$$-\frac{\Re \det(I + V)}{\Im \det(I + V)}(t, x + c_l t) \sim -\frac{\mathcal{D}_1 \exp(\sqrt{c_l} x)}{\mathcal{B}_{2K+l} (i\mathcal{D}_2)}. \quad (4.2.92)$$

So, if $\mathcal{B}_{2K+l} > 0$, we detect an antisoliton when $t \rightarrow +\infty$; if $\mathcal{B}_{2K+l} < 0$, we detect a soliton when $t \rightarrow +\infty$. From this, we deduce that

$$-\frac{\Re \det(I + V)}{\Im \det(I + V)}(t, x + c_l t) \sim \pm \exp\left(\sqrt{c_l} x + \ln\left(\frac{\mathcal{D}_1}{i\mathcal{D}_2} \frac{1}{|\mathcal{B}_{2K+l}|}\right)\right). \quad (4.2.93)$$

Thus,

$$x_{0,l} = -\frac{1}{\sqrt{c_l}} \ln \left(\frac{\mathcal{D}_1}{i\mathcal{D}_2} \frac{1}{|\mathcal{B}_{2K+l}|} \right). \quad (4.2.94)$$

In particular, there is a one-to-one correspondence between $x_{0,l} \in \mathbb{R}$ and $|\mathcal{B}_{2K+l}| \in \mathbb{R}_+^*$.

Second case : $L - l$ is odd

Then,

$$\begin{aligned} \frac{\Im \det(I + V)}{\Re \det(I + V)}(t, x + c_l t) &\sim \frac{(-i\mathcal{D}_1)}{-\mathcal{B}_{2K+l} \exp(-\sqrt{c_l}x) \mathcal{D}_2} \\ &\sim \frac{i\mathcal{D}_1}{\mathcal{B}_{2K+l} \mathcal{D}_2} \exp(\sqrt{c_l}x). \end{aligned} \quad (4.2.95)$$

We obtain the same conclusion : if $\mathcal{B}_{2K+l} > 0$, we detect an antisoliton, and if $\mathcal{B}_{2K+l} < 0$, we detect a soliton. So, in this case,

$$x_{0,l} = -\frac{1}{\sqrt{c_l}} \ln \left(\left| \frac{\mathcal{D}_1}{\mathcal{D}_2 \mathcal{B}_{2K+l}} \right| \right). \quad (4.2.96)$$

We remark that this translation parameter, expressed as above, is valid for the both cases.

Conclusion : We have studied the pointwise convergence of $p(t, x + c_l t)$ when $t \rightarrow +\infty$. This study permits us to express translation parameters of the soliton R_l in terms of \mathcal{B}_{2K+l} , and that will be useful to study the asymptotic behavior of p more precisely in the following.

To answer completely to the question about the shift experienced by a soliton or a breather after collision with all of the other objects, we need to compare the translation parameters in $+\infty$ with the translation parameters in $-\infty$.

Determination of $\mathcal{A}_n, \mathcal{B}_n$ in terms of translation parameters when $t \rightarrow -\infty$

To be complete, let us also study the pointwise convergence of p when $t \rightarrow -\infty$.

We will do the same reasoning as when $t \rightarrow +\infty$ insisting on the differences. In particular, we will study the pointwise limit of $p(t, x + v_j t)$ when $t \rightarrow -\infty$.

Let us start by doing the study for $P_j = B_k$.

So, we study the pointwise convergence of $p(t, x - \gamma_k t)$. This time, we want to keep the terms corresponding to E such that

$$\sum_{n=1}^K (\mathbf{1}_{2n-1 \in E} + \mathbf{1}_{2n \in E}) \beta_n (-\gamma_n + \gamma_k) + \sum_{n=1}^L \mathbf{1}_{2K+n \in E} \sqrt{c_n} (c_n + \gamma_k) \quad (4.2.97)$$

is minimal. The other terms will be exponentially negligible in time. Let l be the largest index such that $c_l < -\gamma_k$ (in the case when such indexes do not exist, we set $l = 0$). The appropriate sets E are : $E_1 := \llbracket 1, 2k-2 \rrbracket \cup \llbracket 2K+1, 2K+l \rrbracket$, $E_2 := \llbracket 1, 2k-2 \rrbracket \cup \{2k\} \cup \llbracket 2K+1, 2K+l \rrbracket$, $E_3 := \llbracket 1, 2k-1 \rrbracket \cup \llbracket 2K+1, 2K+l \rrbracket$ and $E_4 := \llbracket 1, 2k \rrbracket \cup \llbracket 2K+1, 2K+l \rrbracket$.

We note

$$C := (-1)^l \prod_{n=1}^{k-1} (\mathcal{A}_{2n}^2 + \mathcal{B}_{2n}^2) \prod_{n=1}^l \mathcal{B}_{2K+n}. \quad (4.2.98)$$

We also define Cauchy determinants

$$\mathcal{D}_m := \frac{\prod_{i < j; (i,j) \in E_m^2} (\zeta_j - \zeta_i)^2}{\prod_{(i,j) \in E_m^2} (\zeta_j + \zeta_i)}, \quad (4.2.99)$$

for $m = 1, 2, 3, 4$. In a similar manner, we remark that

$$\mathcal{D}_1, \mathcal{D}_4 \in \mathbb{R}_+^* \quad \text{if } l \equiv 0[4], \quad (4.2.100)$$

$$\mathcal{D}_1, \mathcal{D}_4 \in -i\mathbb{R}_+^* \quad \text{if } l \equiv 1[4], \quad (4.2.101)$$

$$\mathcal{D}_1, \mathcal{D}_4 \in -\mathbb{R}_+^* \quad \text{if } l \equiv 2[4], \quad (4.2.102)$$

$$\mathcal{D}_1, \mathcal{D}_4 \in i\mathbb{R}_+^* \quad \text{if } l \equiv 3[4]. \quad (4.2.103)$$

And also,

$$\mathcal{D}_2 + \overline{\mathcal{D}_3} = 0 \quad \text{if } l \text{ is even}, \quad (4.2.104)$$

$$\mathcal{D}_2 - \overline{\mathcal{D}_3} = 0 \quad \text{if } l \text{ is odd}. \quad (4.2.105)$$

So, up to negligible terms, when $t \rightarrow -\infty$,

$$\begin{aligned} \det [I + V] (t, x - \gamma_k t) &\sim \mathcal{D}_1 \\ &+ i (\mathcal{A}_{2k} + i\mathcal{B}_{2k}) \exp(-i\alpha_k (\delta_k - \gamma_k) t) \exp(-\beta_k x) \exp(-i\alpha_k x) \mathcal{D}_2 \\ &+ i (\mathcal{A}_{2k} - i\mathcal{B}_{2k}) \exp(i\alpha_k (\delta_k - \gamma_k) t) \exp(-\beta_k x) \exp(i\alpha_k x) (-\mathcal{D}_3) \\ &+ (\mathcal{A}_{2k}^2 + \mathcal{B}_{2k}^2) \exp(-2\beta_k x) \mathcal{D}_4 \end{aligned} \quad (4.2.106)$$

multiplied by

$$C \exp \left(2 \sum_{n=1}^{k-1} \beta_n (-\gamma_n + \gamma_k) t \right) \exp \left(\sum_{n=1}^l \sqrt{c_n} (c_n + \gamma_k) t \right) \exp \left(-2 \sum_{n=1}^{k-1} \beta_n x \right) \exp \left(- \sum_{n=1}^l \sqrt{c_n} x \right) \in \mathbb{R}. \quad (4.2.107)$$

We remark that, apart from the fact that definitions of \mathcal{D}_m are not the same, we obtain exactly the same expression as when $t \rightarrow +\infty$. And so, exactly the same computations can be done. Thus, we can write the same conclusion.

So,

$$x_{1,k} = \frac{1}{\alpha_k} \arg \left(\frac{i\mathcal{D}_1}{(\mathcal{A}_{2k} + i\mathcal{B}_{2k}) \mathcal{D}_2} \frac{\beta_k}{\alpha_k} \right), \quad (4.2.108)$$

and

$$x_{2,k} = \frac{1}{\beta_k} \ln \left(\left| \frac{i\mathcal{D}_1}{(\mathcal{A}_{2k} + i\mathcal{B}_{2k}) \mathcal{D}_2} \frac{\beta_k}{\alpha_k} \right| \right). \quad (4.2.109)$$

And for $P_j = R_l$:

Let k be the largest index such that $-\gamma_k < c_l$ (in the case when such indexes do not exist, we take $k = 0$). So, the appropriate sets E are : $E_1 := \llbracket 1, 2k \rrbracket \cup \llbracket 2K + 1, 2K + l - 1 \rrbracket$ and $E_2 := \llbracket 1, 2k \rrbracket \cup \llbracket 2K + 1, 2K + l \rrbracket$.

We denote

$$C := (-1)^{l-1} \prod_{n=1}^k (\mathcal{A}_{2n}^2 + \mathcal{B}_{2n}^2) \prod_{n=1}^{l-1} \mathcal{B}_{2K+n}. \quad (4.2.110)$$

We also define two Cauchy determinants

$$\mathcal{D}_m := \frac{\prod_{i < j; (i,j) \in E_m^2} (\zeta_j - \zeta_i)^2}{\prod_{(i,j) \in E_m^2} (\zeta_j + \zeta_i)}, \quad (4.2.111)$$

for $m = 1, 2$. Like before, we remark that

$$\mathcal{D}_1 \in \mathbb{R}_+^*, \mathcal{D}_2 \in -i\mathbb{R}_+^* \quad \text{if } l-1 \equiv 0[4], \quad (4.2.112)$$

$$\mathcal{D}_1 \in -i\mathbb{R}_+^*, \mathcal{D}_2 \in -\mathbb{R}_+^* \quad \text{if } l-1 \equiv 1[4], \quad (4.2.113)$$

$$\mathcal{D}_1 \in -\mathbb{R}_+^*, \mathcal{D}_2 \in i\mathbb{R}_+^* \quad \text{if } l-1 \equiv 2[4], \quad (4.2.114)$$

$$\mathcal{D}_1 \in i\mathbb{R}_+^*, \mathcal{D}_2 \in \mathbb{R}_+^* \quad \text{if } l-1 \equiv 3[4]. \quad (4.2.115)$$

Here again, the reasoning is similar to the case $t \rightarrow +\infty$ and we obtain

$$x_{0,l} = -\frac{1}{\sqrt{c_l}} \ln \left(\left| \frac{\mathcal{D}_1}{\mathcal{D}_2 \mathcal{B}_{2K+l}} \right| \right), \quad (4.2.116)$$

and if $\mathcal{B}_{2K+l} > 0$, R_l is an antisoliton; if $\mathcal{B}_{2K+l} < 0$, R_l is a soliton. **We note that, as expected, a soliton does not change its sign between $t \rightarrow -\infty$ and $t \rightarrow +\infty$.**

What happens with the derivatives of p ?

The pointwise convergences that we have showed for p are also valid for spatial derivatives of p , because spatial derivatives of exponentially negligible in time terms stay exponentially negligible in time.

Determination of the shift after all the collisions

We can now compute the shift experienced by an object after collision with the other objects.

From now on, we add $+\infty$ or $-\infty$ in superscript to distinguish quantities computed for $t \rightarrow +\infty$ and for $t \rightarrow -\infty$.

In the case of the soliton R_l , with the associated parameter k (the parameter that corresponds to the fastest breather that is slower than R_l),

$$\begin{aligned} x_{0,l}^{+\infty} - x_{0,l}^{-\infty} &= -\frac{1}{\sqrt{c_l}} \ln \left(\left| \frac{\mathcal{D}_1^{+\infty} \mathcal{D}_2^{-\infty}}{\mathcal{D}_1^{-\infty} \mathcal{D}_2^{+\infty}} \right| \right) \\ &= -\frac{1}{\sqrt{c_l}} \ln \left(\frac{\prod_{n=1}^k \left(\frac{\alpha_n^2}{4} + \frac{(\beta_n - \sqrt{c_l})^2}{4} \right)^2 \prod_{n=1}^{l-1} \left(\frac{\sqrt{c_n}}{2} - \frac{\sqrt{c_l}}{2} \right)^2}{\sqrt{c_l} \prod_{n=1}^k \left(\frac{\alpha_n^2}{4} + \frac{(\beta_n + \sqrt{c_l})^2}{4} \right)^2 \prod_{n=1}^{l-1} \left(\frac{\sqrt{c_n}}{2} + \frac{\sqrt{c_l}}{2} \right)^2} \right) \\ &\quad - \frac{1}{\sqrt{c_l}} \ln \left(\frac{\sqrt{c_l} \prod_{n=k+1}^K \left(\frac{\alpha_n^2}{4} + \frac{(\beta_n + \sqrt{c_l})^2}{4} \right)^2 \prod_{n=l+1}^L \left(\frac{\sqrt{c_n}}{2} + \frac{\sqrt{c_l}}{2} \right)^2}{\prod_{n=k+1}^K \left(\frac{\alpha_n^2}{4} + \frac{(\beta_n - \sqrt{c_l})^2}{4} \right)^2 \prod_{n=l+1}^L \left(\frac{\sqrt{c_n}}{2} - \frac{\sqrt{c_l}}{2} \right)^2} \right) \end{aligned}$$

$$= -\frac{2}{\sqrt{c_l}} \left[\ln \left(\prod_{n=1}^k \frac{\alpha_n^2 + (\beta_n - \sqrt{c_l})^2}{\alpha_n^2 + (\beta_n + \sqrt{c_l})^2} \prod_{n=1}^{l-1} \frac{\sqrt{c_l} - \sqrt{c_n}}{\sqrt{c_n} + \sqrt{c_l}} \right) - \ln \left(\prod_{n=k+1}^K \frac{\alpha_n^2 + (\beta_n - \sqrt{c_l})^2}{\alpha_n^2 + (\beta_n + \sqrt{c_l})^2} \prod_{n=l+1}^L \frac{\sqrt{c_n} - \sqrt{c_l}}{\sqrt{c_n} + \sqrt{c_l}} \right) \right]. \quad (4.2.117)$$

In the case of the breather B_k , with the associated parameter l (the parameter that corresponds to the fastest soliton that is slower than B_k),

$$\begin{aligned} x_{2,k}^{+\infty} - x_{2,k}^{-\infty} &= \frac{1}{\beta_k} \ln \left(\left| \frac{\mathcal{D}_1^{+\infty} \mathcal{D}_2^{-\infty}}{\mathcal{D}_1^{-\infty} \mathcal{D}_2^{+\infty}} \right| \right) \\ &= \frac{1}{\beta_k} \ln \left(\prod_{n=1}^{k-1} \frac{((\alpha_k - \alpha_n)^2 + (\beta_k - \beta_n)^2)}{((\alpha_k + \alpha_n)^2 + (\beta_k + \beta_n)^2)} \frac{((\alpha_k + \alpha_n)^2 + (\beta_k - \beta_n)^2)}{((\alpha_k - \alpha_n)^2 + (\beta_k + \beta_n)^2)} \prod_{n=1}^l \frac{\alpha_k^2 + (\beta_k - \sqrt{c_l})^2}{\alpha_k^2 + (\beta_k + \sqrt{c_l})^2} \right) \\ &\quad - \frac{1}{\beta_k} \ln \left(\prod_{n=k+1}^K \frac{((\alpha_k - \alpha_n)^2 + (\beta_k - \beta_n)^2)}{((\alpha_k + \alpha_n)^2 + (\beta_k + \beta_n)^2)} \frac{((\alpha_k + \alpha_n)^2 + (\beta_k - \beta_n)^2)}{((\alpha_k - \alpha_n)^2 + (\beta_k + \beta_n)^2)} \prod_{n=l+1}^L \frac{\alpha_k^2 + (\beta_k - \sqrt{c_l})^2}{\alpha_k^2 + (\beta_k + \sqrt{c_l})^2} \right). \end{aligned} \quad (4.2.118)$$

And,

$$\begin{aligned} x_{1,k}^{+\infty} - x_{1,k}^{-\infty} &= \frac{1}{\alpha_k} \arg \left(\frac{\mathcal{D}_1^{+\infty} \mathcal{D}_2^{-\infty}}{\mathcal{D}_1^{-\infty} \mathcal{D}_2^{+\infty}} \right) \\ &= \frac{2}{\alpha_k} \arg \left(\prod_{n=1}^{k-1} \frac{(\alpha_k - \alpha_n + i(\beta_k - \beta_n))(\alpha_k + \alpha_n + i(\beta_k - \beta_n))}{(\alpha_k + \alpha_n + i(\beta_k + \beta_n))(\alpha_k - \alpha_n + i(\beta_k + \beta_n))} \prod_{n=1}^l \frac{\alpha_k + i(\beta_k - \sqrt{c_l})}{\alpha_k + i(\beta_k + \sqrt{c_l})} \right) \\ &\quad - \frac{2}{\alpha_k} \arg \left(\prod_{n=k+1}^K \frac{(\alpha_k - \alpha_n + i(\beta_k - \beta_n))(\alpha_k + \alpha_n + i(\beta_k - \beta_n))}{(\alpha_k + \alpha_n + i(\beta_k + \beta_n))(\alpha_k - \alpha_n + i(\beta_k + \beta_n))} \prod_{n=l+1}^L \frac{\alpha_k + i(\beta_k - \sqrt{c_l})}{\alpha_k + i(\beta_k + \sqrt{c_l})} \right). \end{aligned} \quad (4.2.119)$$

Remark : the parameter x_0 of a soliton corresponds to the parameter $-x_2$ of a breather and that corresponds to the *natural position* of the objet.

4.2.3 Shift after a collision between two objects

We note that the total effect of the collisions corresponds to the **sum** of the effects of collisions with each object. So, we have established that it is enough to understand the collision between two objects to understand the cumulated effect of collisions between an object and a set of objects.

The general rule governing the collision between two objects is that after collision between two objects, the slowest object experiences a shift to the left, the fastest object experiences a shift to the right.

We can deduce from this what happens after a collision between two solitons :

When there is a collision between two solitons of parameters $c_1 < c_2$, the soliton of parameter c_1 experiences a shift to the left that has for absolute value

$$\frac{2}{\sqrt{c_1}} \ln \left(\frac{\sqrt{c_2} + \sqrt{c_1}}{\sqrt{c_2} - \sqrt{c_1}} \right). \quad (4.2.120)$$

The soliton of parameter c_2 experiences a shift to the right that has for absolute value

$$\frac{2}{\sqrt{c_2}} \ln \left(\frac{\sqrt{c_2} + \sqrt{c_1}}{\sqrt{c_2} - \sqrt{c_1}} \right). \quad (4.2.121)$$

We can deduce what happens after a collision between a soliton and a breather :

When there is a collision between a soliton of parameter c and a breather of parameters α, β , the soliton experiences a shift that has for absolute value

$$\frac{2}{\sqrt{c}} \ln \left(\frac{\alpha^2 + (\beta + \sqrt{c})^2}{\alpha^2 + (\beta - \sqrt{c})^2} \right). \quad (4.2.122)$$

The breather experiences a shift that has for absolute value

$$\frac{1}{\beta} \ln \left(\frac{\alpha^2 + (\beta + \sqrt{c})^2}{\alpha^2 + (\beta - \sqrt{c})^2} \right). \quad (4.2.123)$$

Moreover, the natural phase of the breather (the parameter $-x_1$) experiences a shift of

$$\frac{2}{\alpha} \arg \left(\frac{\alpha + i(\beta + \sqrt{c})}{\alpha + i(\beta - \sqrt{c})} \right), \quad (4.2.124)$$

in the case when the breather is faster than the soliton ; in the other case, the shift is the opposite of the quantity above.

Finally, we can deduce what happens after a collision between two breathers :

When there is a collision between two breathers of parameters α_1, β_1 and α_2, β_2 , the breather of parameters α_k, β_k experiences a shift that has for absolute value

$$\frac{1}{\beta_k} \ln \left(\frac{\left((\alpha_1 + \alpha_2)^2 + (\beta_1 + \beta_2)^2 \right) \left((\alpha_1 - \alpha_2)^2 + (\beta_1 + \beta_2)^2 \right)}{\left((\alpha_1 + \alpha_2)^2 + (\beta_1 - \beta_2)^2 \right) \left((\alpha_1 - \alpha_2)^2 + (\beta_1 - \beta_2)^2 \right)} \right). \quad (4.2.125)$$

Moreover, the natural phase (the parameter $-x_1$) of the breather of parameters α_k, β_k experiences a shift of

$$\pm \frac{2}{\alpha_k} \arg \left(\frac{(\alpha_1 + \alpha_2 + i(\beta_1 + \beta_2))(\alpha_1 - \alpha_2 + i(\beta_1 + \beta_2))}{(\alpha_1 + \alpha_2 + i(\beta_1 - \beta_2))(\alpha_1 - \alpha_2 + i(\beta_1 - \beta_2))} \right), \quad (4.2.126)$$

where the sign is $+$ in the case when the breather of parameters α_k, β_k is the fastest, and the sign is $-$ in the other case.

Remark : We can explain the coefficients « 1 » and « 2 » that appear in the formulas, by seeing a soliton of parameter c as the half of a breather of parameters $0, \sqrt{c}$ (which corresponds to a couple soliton-antisoliton), to explain the effects of a collision.

4.2.4 Uniform convergence of p independent from translation parameters

We recall that the definition of $p(t, x)$ given by (4.2.26) is parametrized by K (the number of breathers), by L (the number of solitons or antisolitons), from which we set $N := 2K + L$, by $\alpha_k, \beta_k > 0$ for $1 \leq k \leq K$ (shape parameters of the breathers), by $c_l > 0$ for $1 \leq l \leq L$ (shape parameters of the solitons), by $\mathcal{A}_n, \mathcal{B}_n$ for $1 \leq n \leq N$ (translation parameters of the multi-breather that satisfy conditions (4.2.18), (4.2.19), (4.2.20) and (4.2.21)). That defines a

solution $p(t, x; \mathcal{A}_n, \mathcal{B}_n) := p(t, x)$ of (mKdV). We write explicitly the dependence on translation parameters here, and we suppose that shape parameters are always the same in our study.

We recall that parameters $\zeta_n \in \mathbb{C}$ are defined from shape parameters by (4.2.15), (4.2.16) and (4.2.17).

We would like to observe that a translation in time corresponds to translations on parameters $\mathcal{A}_n, \mathcal{B}_n$:

$$p(t, x; \mathcal{A}_n, \mathcal{B}_n) = p(0, x; \mathcal{A}_n(t), \mathcal{B}_n(t)), \quad (4.2.127)$$

where, for any $1 \leq n \leq N$, for any $t \in \mathbb{R}$,

$$\mathcal{A}_n(t) + i\mathcal{B}_n(t) := (\mathcal{A}_n + i\mathcal{B}_n) \exp(8i\zeta_n^3 t). \quad (4.2.128)$$

This is why, we may consider t as an artificial parameter, and we can consider a multi-breather as a $\mathbb{R} \rightarrow \mathbb{R}$ function that can be determined by a choice of translation parameters (the time parameter t being redundant).

We will prove a uniform version of Proposition 4.4, from which Proposition 4.4 will be a consequence.

Proposition 4.5. *Let us suppose (4.2.6) for a set α_k, β_k for $1 \leq k \leq K$ and c_l for $1 \leq l \leq L$ of shape parameters of solitons and breathers of (mKdV), and we denote $v_1 < \dots < v_J$ the associated velocities. There exists $\theta > 0$, $D_0 > 0$ and $A_s > 0$ for any $s \geq 0$ (that depend only on the shape parameters) such that the following is true. For any set of breathers B_1, \dots, B_K of shape parameters α_k, β_k (given above) and translation parameters $x_{1,k}, x_{2,k}$ and any set of solitons R_1, \dots, R_L of shape parameters c_l (given above), signs κ_l and translation parameters $x_{0,l}$, we may denote P_1, \dots, P_J the same set of objects ordered by increasing velocity (with $J = K + L$), and P their sum as usual. We denote x_j the position of P_j at $t = 0$, for any $1 \leq j \leq J$: more precisely, if $P_j = B_k$, then $x_j = -x_{2,k}$, and if $P_j = R_l$, then $x_j = x_{0,l}$.*

For $1 \leq n \leq N = 2K + L$, let $\mathcal{A}_n, \mathcal{B}_n$ be translation parameters associated to $x_{1,k}, x_{2,k}, x_{0,l}$ by (4.2.96), (4.2.78) and (4.2.79).

For any $D \geq D_0$, if for $1 \leq j \leq J - 1$,

$$x_{j+1} > x_j + D, \quad (4.2.129)$$

then, for any $s > 0$,

$$\|p(0; \mathcal{A}_n, \mathcal{B}_n) - P(0)\|_{H^s} \leq A_s \exp(-\theta D). \quad (4.2.130)$$

Proposition 4.4 is an immediate consequence of the uniform proposition above, because the minimal distance between two consecutive objects increases linearly as time evolves.

4.2.5 Proof of Proposition 4.5

As we have seen in the computations made for pointwise convergence, we point out that for any $j = 1, \dots, J$, we can write $\det(I + V)$ as terms corresponding to P_j (i.e. the sum restricted on four sets E in the case when P_j is a breather, or the sum restricted on two sets E in the case when P_j is a soliton) summed with terms corresponding to other sets E .

From (4.2.38),

$$\begin{aligned} \det[I + V](0, x) &= \sum_{E \subset \llbracket 1, N \rrbracket} i^{|E|} \prod_{n \in E} (\mathcal{A}_n + i\mathcal{B}_n) \exp \left(- \sum_{n=1}^L \mathbf{1}_{2K+n \in E} \sqrt{c_n} x \right) \\ &\quad \exp \left(- \sum_{n=1}^K (\mathbf{1}_{2n-1 \in E} + \mathbf{1}_{2n \in E}) \beta_n x \right) \exp \left(i \sum_{n=1}^K (\mathbf{1}_{2n-1 \in E} - \mathbf{1}_{2n \in E}) \alpha_n x \right) \mathcal{D}_E, \end{aligned} \quad (4.2.131)$$

where

$$\mathcal{D}_E := \sum_{\sigma \in \mathfrak{S}(E)} \varepsilon(\sigma) \prod_{n \in E} \frac{1}{\zeta_n + \zeta_{\sigma(n)}}. \quad (4.2.132)$$

From the form of (4.2.96) and (4.2.79), we see that it is useful to set the following notations : for $1 \leq k \leq K$,

$$y_k^b := \frac{1}{\beta_k} \ln(|\mathcal{A}_{2k} + i\mathcal{B}_{2k}|) = \frac{1}{\beta_k} \ln(|\mathcal{A}_{2k-1} + i\mathcal{B}_{2k-1}|), \quad (4.2.133)$$

and for $1 \leq l \leq L$,

$$y_l^s := \frac{1}{\sqrt{c_l}} \ln(|\mathcal{B}_{2K+l}|), \quad (4.2.134)$$

and y_j denotes y_k^b if $P_j = B_k$ is a breather or y_l^s if $P_j = R_l$ is a soliton.

We can consider that y_j corresponds to the position of P_j that does not take account of collisions between objects (and collisions between objects do only depend on shape parameters). In other words, the distance between y_j and x_j is bounded in terms of the shape parameters.

In this subsection, we suppose that

$$x_1 < x_2 < \dots < x_J. \quad (4.2.135)$$

In other words, we place ourselves in a situation when the objects are well-ordered according to their velocity. We set D the minimal distance between two consecutive objects. In other words,

$$D := \min_{1 \leq j \leq J-1} (x_{j+1}(0) - x_j(0)). \quad (4.2.136)$$

We can write that

$$D = \min_{1 \leq j \leq J-1} (y_{j+1} - y_j) + K, \quad (4.2.137)$$

where $K \in \mathbb{R}$ is a constant that depends only on shape parameters of the objects. We denote $D_1 := L - K$.

We can say that the assumption (4.2.135) can be reformulated as $D > 0$ or $D_1 > -K$.

Moreover, we suppose here that $D > D_0$ or $D_1 > D_0 - K$, where D_0 is constant large enough in order to be able to establish the estimate (4.2.144) below.

The goal is to bound $p - P$ only in terms of the shape parameters and D .

We denote

$$\beta := \min \left(\min_{1 \leq k \leq K} \beta_k, \min_{1 \leq l \leq L} \sqrt{c_l} \right). \quad (4.2.138)$$

We will divide \mathbb{R} in zones that correspond to each object.

Let us introduce notations for the means of the consecutive positions (we prefer to work with y_j at the place of x_j). For $j = 2, \dots, J$, let

$$m_j := \frac{y_{j-1} + y_j}{2}. \quad (4.2.139)$$

Then,

$$Z_1 := (-\infty, m_2) \quad (4.2.140)$$

is the zone that corresponds to P_1 ;

$$Z_j := (m_j, m_{j+1}) \quad (4.2.141)$$

is the zone that corresponds to P_j for $2 \leq j \leq J-1$; and

$$Z_J := (m_J, +\infty) \quad (4.2.142)$$

is the zone that corresponds to P_J .

We note that for any $1 \leq j \leq J$, if $x \in Z_j$, then for any $n \neq j$, $|x - y_n| \geq \frac{D_1}{2}$.

For any $E \subset \llbracket 1, N \rrbracket$, we note $a(E)$ the number of $n \in E$ that correspond to an antisoliton, and $s(E)$ the number of $n \in E$ that correspond to a soliton. Let us now rewrite $\det[I + V](x) := \det[I + V](0, x)$ in a more convenient way :

$$\begin{aligned} & \det[I + V](x) \\ &= \sum_{E \subset \llbracket 1, N \rrbracket} i^{|E|} i^{a(E)-s(E)} \mathcal{D}_E \exp \left(- \sum_{n=1}^L \mathbf{1}_{2K+n \in E} \sqrt{c_n} (x - y_l^s) \right) \exp \left(- \sum_{n=1}^K (\mathbf{1}_{2n-1 \in E} + \mathbf{1}_{2n \in E}) \beta_n (x - y_k^b) \right) \\ & \exp \left(i \sum_{n=1}^K \mathbf{1}_{2n-1 \in E} \alpha_n \left(x + \frac{\arg(\mathcal{A}_{2n-1} + i\mathcal{B}_{2n-1})}{\alpha_n} \right) \right) \exp \left(-i \sum_{n=1}^K \mathbf{1}_{2n \in E} \alpha_n \left(x - \frac{\arg(\mathcal{A}_{2n} + i\mathcal{B}_{2n})}{\alpha_n} \right) \right). \end{aligned} \quad (4.2.143)$$

The reason why the expression above is better is that translation parameters do not appear as coefficients anymore. That is why, any constants that will appear from now on are **independent of translation parameters**.

After that, it is easy to see that for any $j \in \llbracket 1, J \rrbracket$, for any $x \in Z_j$,

$$|p - P_j| \leq C \sum_{n \neq j} \exp(-\beta |x - y_n|), \quad (4.2.144)$$

where $C > 0$ is a suitable constant that depends only on shape parameters.

This is why, by integration, for any $1 \leq j \leq J$,

$$\int_{Z_j} |p - P_j|^2 \leq C \exp(-\beta D_1). \quad (4.2.145)$$

For the same reason, for any $1 \leq j \leq J$, for any $n \neq j$,

$$\int_{Z_j} |P_n|^2 \leq C \exp(\beta K) \exp(-\beta D_1). \quad (4.2.146)$$

By summation,

$$\|p - P\|_{L^2} \leq C \exp(-\beta D_1/2) \leq C \exp(\beta K) \exp(-\beta D/2), \quad (4.2.147)$$

where $C \exp(\beta K)$ is a constant that depends only on shape parameters.

The same can be said for any H^s norm, because we can make exactly the same reasoning for any derivative of $\det[I + V]$.

4.3 Liouville theorem in 0 for (mKdV)

In the hope to obtain a Liouville theorem at a neighborhood of a breather, we give a Liouville theorem at a neighborhood of 0. Indeed, a double Bäcklund transform can transform a problem at a neighborhood of a breather in a problem at a neighborhood of 0. However, there is still some difficulties to overcome in order to use the Bäcklund transforms that transform a breather in a complex soliton, and a complex soliton in 0.

After application of the double Bäcklund transform, the Liouville problem for a breather should be transformed in a Liouville problem in 0. Let us denote ω the velocity of the initial breather and $\sigma > 0$ the small enough parameter of exponential decrease that is associated to this breather. We denote :

$$H_\sigma^1 := \{z \in H^1(\mathbb{R}), \|z\|_{H_\sigma^1} < +\infty\}, \quad (4.3.1)$$

where for $z \in H^1$,

$$\|z\|_{H_\sigma^1}^2 := \int (|z(x)|^2 + |\partial_x z(x)|^2) e^{\sigma|x|} dx. \quad (4.3.2)$$

We prove the following theorem :

Theorem 4.6. *If $\omega > 0$, there exists $\varepsilon > 0$ such that the following is true. Let v be a solution of (mKdV) such that there exists $t \mapsto y(t) \in \mathbb{R}$ such that*

$$\forall t \in \mathbb{R}, \quad \|v(t, x + y(t))\|_{H_\sigma^1} < \varepsilon, \quad (4.3.3)$$

and

$$\forall t \in \mathbb{R}, \quad y'(t) \in [\omega - \varepsilon, \omega + \varepsilon]. \quad (4.3.4)$$

Then, $v = 0$.

Actually, we would to have this theorem without any condition on the sign of ω , but the proof we make can not be done for the general case. The proof of this theorem can be done in several steps.

Remark 4.7. In this theorem, we suppose that $v(t)$ is localized in $y(t)$. So, $v(t, x + y(t))$ is always localized in 0. σ and ω can be seen as problem data.

4.3.1 Study of a virial

The integral $\int x v(t, x + y(t))^2 dx$ is bounded above by a constant multiplied by ε^2 . So, we can write :

$$\frac{d}{dt} \int (x - y(t)) v^2 = -3 \int v_x^2 + \frac{3}{2} \int v^4 - y'(t) \int v^2 \quad (4.3.5)$$

$$= -6E[v(t)] - 2y'(t)M[v(t)]. \quad (4.3.6)$$

Because the energy and the mass of a solution are integrals of motion, we can write E at the place of $E[v(t)]$ and M at the place of $M[v(t)]$.

In the case when for any position of $y'(t)$ in the interval $[\omega - \varepsilon, \omega + \varepsilon]$, the expression $-6E - 2y'(t)M$ has the same sign, we have that $\int (x - y(t)) v^2$ could not be bounded. Thus, this case is impossible.

Thus, there exists t_n an increasing sequence such that $t_n \rightarrow +\infty$ that satisfies

$$-6E - 2y'(t_n)M = 0. \quad (4.3.7)$$

Therefore, we deduce that there exists $\omega' \in \mathbb{R}$ such that $|\omega - \omega'| < \varepsilon$ and

$$-6E - 2\omega' M = 0. \quad (4.3.8)$$

We deduce from that :

$$E = -\frac{\omega'}{3} M. \quad (4.3.9)$$

4.3.2 A bound for the \dot{H}^1 norm

The H^1 norm is not constant for a solution of (mKdV), but the energy is. We can nevertheless obtain a global bound for the H^1 norm the conservation of the energy. The energy is bounded above by the \dot{H}^1 norm. But, we would like to bound the \dot{H}^1 norm.

Let us use the *Gagliardo-Nirenberg inequality* :

$$\|v\|_{L^4} \leq C \|v\|_{L^2}^{3/4} \|v\|_{\dot{H}^1}^{1/4}. \quad (4.3.10)$$

Then :

$$E = \frac{1}{2} \|v\|_{\dot{H}^1}^2 - \frac{1}{4} \|v\|_{L^4}^4 \quad (4.3.11)$$

$$\geq \frac{1}{2} \|v\|_{\dot{H}^1}^2 - CM^{3/2} \|v\|_{\dot{H}^1}. \quad (4.3.12)$$

We would like that the largest root of the trinomial $\frac{1}{2}X^2 - CM^{3/2}X + \frac{\omega'}{3}M$ (we have rewritten the energy in terms of the mass, we can do it thanks to the virial), which is

$$CM^{3/2} + \sqrt{C^2 M^3 - \frac{2}{3}\omega' M}. \quad (4.3.13)$$

We consider the case when $\omega' \geq 0$.

In this case, we can deduce that

$$\|v\|_{\dot{H}^1} \leq CM^{3/2}. \quad (4.3.14)$$

4.3.3 Conclusion in the case when $\omega > 0$

We use the *Heisenberg inequality* :

$$\left(\int (x - y(t))^2 v^2 dx \right) \|v\|_{\dot{H}^1}^2 \geq CM^2. \quad (4.3.15)$$

Moreover, we have that

$$\left(\int (x - y(t))^2 v^2 dx \right) \|v\|_{\dot{H}^1}^2 \leq C \|v\|_{H_\sigma^1}^2 \|v\|_{\dot{H}^1}^2 \quad (4.3.16)$$

$$\leq C\varepsilon^2 M^3. \quad (4.3.17)$$

So,

$$M^2 \leq C\varepsilon^2 M^3, \quad (4.3.18)$$

then, assuming (by contradiction) that $M > 0$,

$$1 \leq C\varepsilon^2 M, \tag{4.3.19}$$

where C is a constant that do not depend on ε . Moreover, M is bounded above by ε^2 . Thus, by choosing ε small enough, we have a contradiction and we deduce that $v = 0$.

Bibliographie

- [1] M. J. ABLOWITZ, D. J. KAUP, A. C. NEWELL, AND H. SEGUR, *Nonlinear-evolution equations of physical significance*, Phys. Rev. Lett., 31 (1973), pp. 125–127.
- [2] ———, *The inverse scattering transform-Fourier analysis for nonlinear problems*, Studies in Appl. Math., 53 (1974), pp. 249–315.
- [3] J. H. ADLAM AND J. E. ALLEN, *The structure of strong collision-free hydromagnetic waves*, Phil. Mag., 3 (1958), pp. 448–455.
- [4] G. B. AIRY, *Tides and waves*, Encyclopaedia Metropolitana. Mixed Sciences, 3 (1841), pp. 1817–1845.
- [5] M. A. ALEJO, *Focusing mKdV breather solutions with nonvanishing boundary condition by the inverse scattering method*, J. Nonlinear Math. Phys., 19 (2012), pp. 1250009, 17.
- [6] ———, *Nonlinear stability of Gardner breathers*, J. Differential Equations, 264 (2018), pp. 1192–1230.
- [7] M. A. ALEJO AND C. MUÑOZ, *On the nonlinear stability of mKdV breathers*, J. Phys. A, 45 (2012), pp. 432001, 7.
- [8] ———, *Nonlinear stability of MKdV breathers*, Comm. Math. Phys., 324 (2013), pp. 233–262.
- [9] ———, *Dynamics of complex-valued modified KdV solitons with applications to the stability of breathers*, Anal. PDE, 8 (2015), pp. 629–674.
- [10] M. A. ALEJO, C. MUÑOZ, AND J. M. PALACIOS, *On the variational structure of breather solutions I : Sine-Gordon equation*, J. Math. Anal. Appl., 453 (2017), pp. 1111–1138.
- [11] ———, *On the variational structure of breather solutions II : Periodic mKdV equation*, Electron. J. Differential Equations, (2017), pp. Paper No. 56, 26.
- [12] H. BAZIN, *Recherches expérimentales sur la propagation des ondes*, Mém. présentés par divers Savants à L’Acad. Sci. Inst. France, 19 (1865), pp. 495–644.
- [13] T. B. BENJAMIN, *The stability of solitary waves*, Proc. Roy. Soc. London Ser. A, 328 (1972), pp. 153–183.
- [14] H. BERESTYCKI AND P.-L. LIONS, *Nonlinear scalar field equations. I. Existence of a ground state*, Arch. Rational Mech. Anal., 82 (1983), pp. 313–345.
- [15] B. BIRNIR, G. PONCE, AND N. SVANSTEDT, *The local ill-posedness of the modified KdV equation*, Ann. Inst. H. Poincaré C Anal. Non Linéaire, 13 (1996), pp. 529–535.
- [16] J. L. BONA, P. E. SOUGANIDIS, AND W. A. STRAUSS, *Stability and instability of solitary waves of Korteweg-de Vries type*, Proc. Roy. Soc. London Ser. A, 411 (1987), pp. 395–412.
- [17] J. BOUSSINESQ, *Théorie de l’intumescence liquide appelée onde solitaire ou de translation se propageant dans un canal rectangulaire*, CR Acad. Sci. Paris, 72 (1871), pp. 755–759.
- [18] ———, *Essai sur la théorie des eaux courantes*, Mém. présentés par divers Savants à L’Acad. Sci. Inst. France (séries 2), 23 (1877), pp. 1–680.

- [19] T. CAZENAVE AND P.-L. LIONS, *Orbital stability of standing waves for some nonlinear Schrödinger equations*, Comm. Math. Phys., 85 (1982), pp. 549–561.
- [20] N. CHEEMAA, A. R. SEADAWY, T. G. SUGATI, AND D. BALEANU, *Study of the dynamical nonlinear modified korteweg-de vries equation arising in plasma physics and its analytical wave solutions*, Results in Physics, 19 (2020), p. 103480.
- [21] G. CHEN AND J. LIU, *Soliton resolution for the focusing modified KdV equation*, Ann. Inst. H. Poincaré C Anal. Non Linéaire, 38 (2021), pp. 2005–2071.
- [22] S. N. CHOW AND J. K. HALE, *Methods of bifurcation theory*, vol. 251 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], Springer-Verlag, New York-Berlin, 1982.
- [23] J. COLLIANDER, M. KEEL, G. STAFFILANI, H. TAKAOKA, AND T. TAO, *Sharp global well-posedness for KdV and modified KdV on \mathbb{R} and \mathbb{T}* , J. Amer. Math. Soc., 16 (2003), pp. 705–749.
- [24] V. COMBET, *Multi-soliton solutions for the supercritical gKdV equations*, Comm. Partial Differential Equations, 36 (2011), pp. 380–419.
- [25] S. CORREIA AND R. CÔTE, *Perturbation at blow-up time of self-similar solutions for the modified Korteweg-de vries equation*, 2022.
- [26] S. CORREIA, R. CÔTE, AND L. VEGA, *Asymptotics in Fourier space of self-similar solutions to the modified Korteweg-de Vries equation*, J. Math. Pures Appl. (9), 137 (2020), pp. 101–142.
- [27] ———, *Self-similar dynamics for the modified Korteweg-de Vries equation*, Int. Math. Res. Not. IMRN, (2021), pp. 9958–10013.
- [28] R. CÔTE AND X. FRIEDERICH, *On smoothness and uniqueness of multi-solitons of the non-linear Schrödinger equations*, Comm. Partial Differential Equations, 46 (2021), pp. 2325–2385.
- [29] R. CÔTE AND Y. MARTEL, *Multi-travelling waves for the nonlinear Klein-Gordon equation*, Trans. Amer. Math. Soc., 370 (2018), pp. 7461–7487.
- [30] R. CÔTE, Y. MARTEL, AND F. MERLE, *Construction of multi-soliton solutions for the L^2 -supercritical gKdV and NLS equations*, Rev. Mat. Iberoam., 27 (2011), pp. 273–302.
- [31] R. CÔTE AND C. MUÑOZ, *Multi-solitons for nonlinear Klein-Gordon equations*, Forum Math. Sigma, 2 (2014), pp. Paper No. e15, 38.
- [32] R. CÔTE, C. MUÑOZ, D. PILOD, AND G. SIMPSON, *Asymptotic stability of high-dimensional Zakharov-Kuznetsov solitons*, Arch. Ration. Mech. Anal., 220 (2016), pp. 639–710.
- [33] P. DEIFT AND X. ZHOU, *A steepest descent method for oscillatory Riemann-Hilbert problems. Asymptotics for the MKdV equation*, Ann. of Math. (2), 137 (1993), pp. 295–368.
- [34] K. EL DIKA, *Asymptotic stability of solitary waves for the Benjamin-Bona-Mahony equation*, Discrete Contin. Dyn. Syst., 13 (2005), pp. 583–622.
- [35] K. EL DIKA AND L. MOLINET, *Stability of multipeakons*, Ann. Inst. H. Poincaré C Anal. Non Linéaire, 26 (2009), pp. 1517–1532.
- [36] K. O. FRIEDRICHS AND D. H. HYERS, *The existence of solitary waves*, Comm. Pure Appl. Math., 7 (1954), pp. 517–550.
- [37] C. S. GARDNER, *Korteweg-de Vries equation and generalizations. IV. The Korteweg-de Vries equation as a Hamiltonian system*, J. Mathematical Phys., 12 (1971), pp. 1548–1551.
- [38] C. S. GARDNER, J. M. GREENE, M. D. KRUSKAL, AND R. M. MIURA, *Method for solving the korteweg-devries equation*, Phys. Rev. Lett., 19 (1967), pp. 1095–1097.

- [39] —, *Korteweg-de Vries equation and generalization. VI. Methods for exact solution*, Comm. Pure Appl. Math., 27 (1974), pp. 97–133.
- [40] C. S. GARDNER AND G. K. MORIKAWA, *Similarity in the asymptotic behavior of collision-free hydromagnetic waves and water waves*, New York University, Courant Institute Math. Sci. Res. Rep. NYO-9082, (1960).
- [41] P. GERMAIN, F. PUSATERI, AND F. ROUSSET, *Asymptotic stability of solitons for mKdV*, Adv. Math., 299 (2016), pp. 272–330.
- [42] C. GORRIA, M. A. ALEJO, AND L. VEGA, *Discrete conservation laws and the convergence of long time simulations of the mkdv equation*, J. Comput. Phys., 235 (2013), pp. 274–285.
- [43] L. GREENBERG, *An oscillation method for fourth-order, selfadjoint, two-point boundary value problems with nonlinear eigenvalues*, SIAM J. Math. Anal., 22 (1991), pp. 1021–1042.
- [44] M. GRILLAKIS, J. SHATAH, AND W. STRAUSS, *Stability theory of solitary waves in the presence of symmetry. II*, J. Funct. Anal., 94 (1990), pp. 308–348.
- [45] B. HARROP-GRIFFITHS, *Long time behavior of solutions to the mKdV*, Comm. Partial Differential Equations, 41 (2016), pp. 282–317.
- [46] B. HARROP-GRIFFITHS, R. KILLIP, AND M. VISAN, *Sharp well-posedness for the cubic NLS and mKdV in $H^s(\mathbb{R})$* , 2020.
- [47] N. HAYASHI AND P. NAUMKIN, *On the modified Korteweg-de Vries equation*, Math. Phys. Anal. Geom., 4 (2001), pp. 197–227.
- [48] N. HAYASHI AND P. I. NAUMKIN, *Large time asymptotics of solutions to the generalized Korteweg-de Vries equation*, J. Funct. Anal., 159 (1998), pp. 110–136.
- [49] —, *Large time behavior of solutions for the modified Korteweg-de Vries equation*, Internat. Math. Res. Notices, (1999), pp. 395–418.
- [50] M. A. HELAL, *A Chebyshev spectral method for solving Korteweg-de Vries equation with hydrodynamical application*, Chaos Solitons Fractals, 12 (2001), pp. 943–950.
- [51] R. HIROTA, *Exact solution of the modified korteweg-de vries equation for multiple collisions of solitons*, Journal of the Physical Society of Japan, 33 (1972), pp. 1456–1458.
- [52] J. HOLMER, G. PERELMAN, AND M. ZWORSKI, *Effective dynamics of double solitons for perturbed mKdV*, Comm. Math. Phys., 305 (2011), pp. 363–425.
- [53] B. B. KADOMTSEV AND V. I. PETVIASHVILI, *On the Stability of Solitary Waves in Weakly Dispersing Media*, Soviet Physics Doklady, 15 (1970), pp. 539–541.
- [54] T. KAKUTANI AND H. ONO, *Weak non-linear hydromagnetic waves in a cold collision-free plasma*, Journal of the Physical Society of Japan, 26 (1969), pp. 1305–1318.
- [55] Y. KAMETAKA, *Korteweg-de Vries equation. IV. Simplest generalization*, Proc. Japan Acad., 45 (1969), pp. 661–665.
- [56] T. KAPPELER AND P. TOPALOV, *Global wellposedness of KdV in $H^{-1}(\mathbb{T}, \mathbb{R})$* , Duke Math. J., 135 (2006), pp. 327–360.
- [57] C. E. KENIG AND Y. MARTEL, *Asymptotic stability of solitons for the Benjamin-Ono equation*, Rev. Mat. Iberoam., 25 (2009), pp. 909–970.
- [58] C. E. KENIG, G. PONCE, AND L. VEGA, *Well-posedness and scattering results for the generalized Korteweg-de Vries equation via the contraction principle*, Comm. Pure Appl. Math., 46 (1993), pp. 527–620.
- [59] —, *On the concentration of blow up solutions for the generalized KdV equation critical in L^2* , in Nonlinear wave equations (Providence, RI, 1998), vol. 263 of Contemp. Math., Amer. Math. Soc., Providence, RI, 2000, pp. 131–156.

- [60] ———, *On the ill-posedness of some canonical dispersive equations*, Duke Math. J., 106 (2001), pp. 617–633.
- [61] R. KILLIP AND M. VIŞAN, *KdV is well-posed in H^{-1}* , Ann. of Math. (2), 190 (2019), pp. 249–305.
- [62] D. J. KORTEWEG AND G. DE VRIES, *On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves*, Philos. Mag. (5), 39 (1895), pp. 422–443.
- [63] M. D. KRUSKAL, R. M. MIURA, C. S. GARDNER, AND N. J. ZABUSKY, *Korteweg-de Vries equation and generalizations. V. Uniqueness and nonexistence of polynomial conservation laws*, J. Mathematical Phys., 11 (1970), pp. 952–960.
- [64] G. L. LAMB, JR., *Elements of soliton theory*, Pure and Applied Mathematics, John Wiley & Sons, Inc., New York, 1980.
- [65] K. G. LAMB, O. POLUKHINA, T. TALIPOVA, E. PELINOVSKY, W. XIAO, AND A. KURKIN, *Breather generation in fully nonlinear models of a stratified fluid*, Phys. Rev. E (3), 75 (2007), pp. 046306, 6.
- [66] M. A. LAVRENT'EV, *On the theory of longwaves*, Transactions of the Institute of Mathematics, Academy of Sciences of the Ukrainian SSR, 8 (1947), pp. 13–69.
- [67] P. D. LAX, *Integrals of nonlinear equations of evolution and solitary waves*, Comm. Pure Appl. Math., 21 (1968), pp. 467–490.
- [68] S. LE COZ AND T.-P. TSAI, *Finite and infinite soliton and kink-soliton trains of nonlinear Schrödinger equations*, in Proceedings of the Sixth International Congress of Chinese Mathematicians. Vol. I, vol. 36 of Adv. Lect. Math. (ALM), Int. Press, Somerville, MA, 2017, pp. 43–56.
- [69] S. LE COZ AND Z. WANG, *Stability of the multi-solitons of the modified Korteweg-de Vries equation*, Nonlinearity, 34 (2021), pp. 7109–7143.
- [70] J. H. MADDOCKS AND R. L. SACHS, *On the stability of KdV multi-solitons*, Comm. Pure Appl. Math., 46 (1993), pp. 867–901.
- [71] Y. MARTEL, *Asymptotic N -soliton-like solutions of the subcritical and critical generalized Korteweg-de Vries equations*, Amer. J. Math., 127 (2005), pp. 1103–1140.
- [72] ———, *Linear problems related to asymptotic stability of solitons of the generalized KdV equations*, SIAM J. Math. Anal., 38 (2006), pp. 759–781.
- [73] Y. MARTEL AND F. MERLE, *Asymptotic stability of solitons for subcritical generalized KdV equations*, Arch. Ration. Mech. Anal., 157 (2001), pp. 219–254.
- [74] Y. MARTEL AND F. MERLE, *Instability of solitons for the critical generalized Korteweg-de Vries equation*, Geom. Funct. Anal., 11 (2001), pp. 74–123.
- [75] Y. MARTEL AND F. MERLE, *Blow up in finite time and dynamics of blow up solutions for the L^2 -critical generalized KdV equation*, J. Amer. Math. Soc., 15 (2002), pp. 617–664.
- [76] ———, *Nonexistence of blow-up solution with minimal L^2 -mass for the critical gKdV equation*, Duke Math. J., 115 (2002), pp. 385–408.
- [77] ———, *Stability of blow-up profile and lower bounds for blow-up rate for the critical generalized KdV equation*, Ann. of Math. (2), 155 (2002), pp. 235–280.
- [78] ———, *Asymptotic stability of solitons of the subcritical gKdV equations revisited*, Nonlinearity, 18 (2005), pp. 55–80.
- [79] ———, *Multi solitary waves for nonlinear Schrödinger equations*, Ann. Inst. H. Poincaré C Anal. Non Linéaire, 23 (2006), pp. 849–864.
- [80] ———, *Asymptotic stability of solitons of the gKdV equations with general nonlinearity*, Math. Ann., 341 (2008), pp. 391–427.

- [81] ———, *Description of two soliton collision for the quartic gKdV equation*, Ann. of Math. (2), 174 (2011), pp. 757–857.
- [82] ———, *On the nonexistence of pure multi-solitons for the quartic gKdV equation*, Int. Math. Res. Not. IMRN, (2015), pp. 688–739.
- [83] Y. MARTEL, F. MERLE, AND P. RAPHAËL, *Blow up and near soliton dynamics for the L^2 critical gKdV equation*, in Séminaire Laurent Schwartz—Équations aux dérivées partielles et applications. Année 2011–2012, Sémin. Équ. Dériv. Partielles, École Polytech., Palaiseau, 2013, pp. Exp. No. XXXVII, 14.
- [84] ———, *Blow up for the critical generalized Korteweg–de Vries equation. I : Dynamics near the soliton*, Acta Math., 212 (2014), pp. 59–140.
- [85] ———, *Blow up for the critical gKdV equation. II : Minimal mass dynamics*, J. Eur. Math. Soc. (JEMS), 17 (2015), pp. 1855–1925.
- [86] ———, *Blow up for the critical gKdV equation III : exotic regimes*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5), 14 (2015), pp. 575–631.
- [87] Y. MARTEL, F. MERLE, AND T.-P. TSAI, *Stability and asymptotic stability in the energy space of the sum of N solitons for subcritical gKdV equations*, Comm. Math. Phys., 231 (2002), pp. 347–373.
- [88] F. MERLE, *Construction of solutions with exactly k blow-up points for the Schrödinger equation with critical nonlinearity*, Comm. Math. Phys., 129 (1990), pp. 223–240.
- [89] M. MING, F. ROUSSET, AND N. TZVETKOV, *Multi-solitons and related solutions for the water-waves system*, SIAM J. Math. Anal., 47 (2015), pp. 897–954.
- [90] R. M. MIURA, *Korteweg-de Vries equation and generalizations. I. A remarkable explicit nonlinear transformation*, J. Mathematical Phys., 9 (1968), pp. 1202–1204.
- [91] ———, *The Korteweg-de Vries equation : a survey of results*, SIAM Rev., 18 (1976), pp. 412–459.
- [92] R. M. MIURA, C. S. GARDNER, AND M. D. KRUSKAL, *Korteweg-de Vries equation and generalizations. II. Existence of conservation laws and constants of motion*, J. Mathematical Phys., 9 (1968), pp. 1204–1209.
- [93] C. MUÑOZ, *On the inelastic two-soliton collision for gKdV equations with general nonlinearity*, Int. Math. Res. Not. IMRN, (2010), pp. 1624–1719.
- [94] C. MUÑOZ AND G. PONCE, *Breathers and the dynamics of solutions in KdV type equations*, Comm. Math. Phys., 367 (2019), pp. 581–598.
- [95] T. V. NGUYEN, *Strongly interacting multi-solitons with logarithmic relative distance for the gKdV equation*, Nonlinearity, 30 (2017), pp. 4614–4648.
- [96] ———, *Existence of multi-solitary waves with logarithmic relative distances for the NLS equation*, C. R. Math. Acad. Sci. Paris, 357 (2019), pp. 13–58.
- [97] T. PERELMAN, A. FRIDMAN, AND M. EL’YASHEVICH, *A modified korteweg-de vries equation in electrodynamics*, Sov. Phys. JETP, 39 (1974), pp. 643–646.
- [98] V. I. PETVIASHVILI AND V. V. IANKOV, *Solitons and turbulence*, Moscow Energoizdat, (1985), pp. 3–55.
- [99] L. RAILEYGH, *On waves*, Phil. Mag., 1 (1876), pp. 257–259.
- [100] F. ROUSSET AND N. TZVETKOV, *Stability and instability of the KdV solitary wave under the KP-I flow*, Comm. Math. Phys., 313 (2012), pp. 155–173.
- [101] J. S. RUSSELL, *Report of the committee on waves*, vol. 417496, Report of the 7th meeting of the British Association for the Advancement of Science, John Murray, London, 1838.
- [102] ———, *Report on waves*, Report for the fourteenth meeting of the British Association for the Advancement of Science, York, sept 1844.

- [103] ———, *The great solitary wave of the first order, or the wave of translation*, Transactions of the Royal Society of Edinburgh, (1849).
- [104] ———, *The wave of translation in the oceans of water, air, and ether*, Nature, 32 (1885), pp. 546–547.
- [105] R. Z. SAGDEEV, *Plasma physics and the problem of controlled thermonuclear reactions*, vol. 4, Pergamon, Oxford, 1960.
- [106] H. SCHAMEL, *A modified korteweg-de vries equation for ion acoustic waves due to resonant electrons*, Journal of Plasma Physics, 9 (1973), pp. 377–387.
- [107] P. C. SCHUUR, *Asymptotic analysis of soliton problems*, vol. 1232 of Lecture Notes in Mathematics, Springer-Verlag, Berlin, 1986. An inverse scattering approach.
- [108] A. SEMENOV, *On the uniqueness of multi-breathers of the modified korteweg-de vries equation*, accepted for publication in the Revista Matemática Iberoamericana, (2021).
- [109] ———, *Orbital stability of a sum of solitons and breathers of the modified korteweg-de vries equation*, 2021.
- [110] G. G. STOKES, *On the theory of oscillatory waves*, Transactions of the Cambridge Philosophical Society, 8 (1847), pp. 197–237.
- [111] S. TANAKA, *Some remarks on the modified Korteweg-de Vries equations*, Publ. Res. Inst. Math. Sci., 8 (1972/73), pp. 429–437.
- [112] T. TAO, *Nonlinear dispersive equations*, vol. 106 of CBMS Regional Conference Series in Mathematics, Published for the Conference Board of the Mathematical Sciences, Washington, DC ; by the American Mathematical Society, Providence, RI, 2006. Local and global analysis.
- [113] ———, *Scattering for the quartic generalised Korteweg-de Vries equation*, J. Differential Equations, 232 (2007), pp. 623–651.
- [114] F. D. TAPPERT AND C. M. VARMA, *Asymptotic theory of self-trapping of heat pulses in solids*, Phys. Rev. Lett., 25 (1970), pp. 1108–1111.
- [115] F. VALET, *Asymptotic K -soliton-like solutions of the Zakharov-Kuznetsov type equations*, Trans. Amer. Math. Soc., 374 (2021), pp. 3177–3213.
- [116] M. WADATI, *The exact solution of the modified korteweg-de vries equation*, Journal of the Physical Society of Japan, 32 (1972), pp. 1681–1681.
- [117] ———, *The modified korteweg-de vries equation*, Journal of the Physical Society of Japan, 34 (1973), pp. 1289–1296.
- [118] M. WADATI AND K. OHKUMA, *Multiple-pole solutions of the modified korteweg-de vries equation*, Journal of the Physical Society of Japan, 51 (1982), pp. 2029–2035.
- [119] M. WADATI AND M. TODA, *The exact n -soliton solution of the korteweg-de vries equation*, Journal of the Physical Society of Japan, 32 (1972), pp. 1403–1411.
- [120] M. I. WEINSTEIN, *Modulational stability of ground states of nonlinear Schrödinger equations*, SIAM J. Math. Anal., 16 (1985), pp. 472–491.
- [121] ———, *Lyapunov stability of ground states of nonlinear dispersive evolution equations*, Comm. Pure Appl. Math., 39 (1986), pp. 51–67.
- [122] C. WEXLER AND A. T. DORSEY, *Contour dynamics, waves, and solitons in the quantum hall effect*, Phys. Rev. B, 60 (1999), pp. 10971–10983.
- [123] G. B. WHITHAM, *Non-linear dispersive waves*, Proc. Roy. Soc. London Ser. A, 283 (1965), pp. 238–261.
- [124] N. J. ZABUSKY, *A synergetic approach to problem of nonlinear dispersive wave propagation and interaction*, Proceedings of the Symposium Nonlinear Partial Differential Equation, Academic Press, New York, (1967), pp. 223–258.

- [125] N. J. ZABUSKY AND M. D. KRUSKAL, *Interaction of "solitons" in a collisionless plasma and the recurrence of initial states*, Phys. Rev. Lett., 15 (1965), pp. 240–243.
- [126] V. E. ZAHAROV AND L. D. FADDEEV, *The Korteweg-de Vries equation is a fully integrable Hamiltonian system*, Funkcional. Anal. i Priložen., 5 (1971), pp. 18–27.