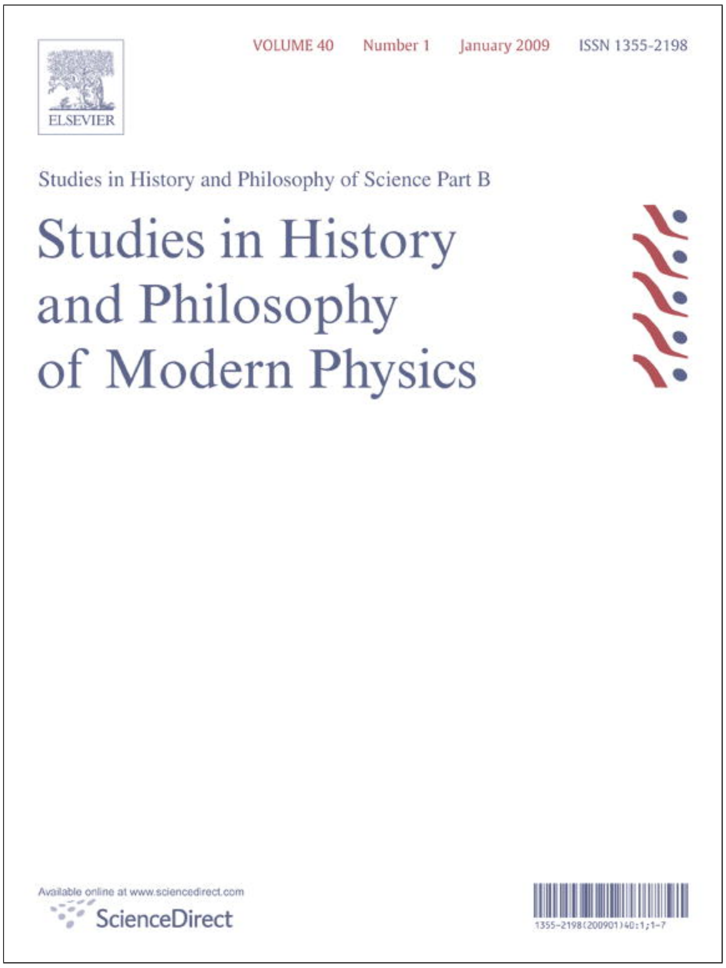
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How Weyl stumbled across electricity while pursuing mathematical justice

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ABSTRACT

It is argued that Weyl's theory of gravitation and electricity came out of 'mathematical justice': out of the *equal rights of direction and length*. Such justice was manifestly at work in the context of discovery, and is enough to derive all of source-free electromagnetism. Weyl's repeated references to coordinates and gauge are taken to express equal treatment of direction and length.

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1. Introduction

It is almost always claimed[1] that Weyl *deliberately* unified gravitation and electricity in the rectification of general relativity he attempted in 1918. In fact the unification, as Bergia (1993) and Ryckman (2005) have pointed out and a couple of passages[2] show, was the unintended outcome of *a priori*[3] prejudice.[4] But what prejudice?

The evidence as I read it suggests the theory came straight out of Weyl's sense of mathematical 'justice,' which led him to put the direction and length of a vector on an equal footing, to follow the

> *D/L rule*: whatever applies to direction (in general relativity) must also apply to length; to every basic notion concerning direction there must correspond a notion concerning length in the same way

which will be more fully expressed in the table (p. 4). Of course neither the D/L rule nor the table appears explicitly in Weyl's writings, but they are implicit, he follows both.

Levi-Civita (1917) had discovered that the parallel transport determined by Einstein's covariant derivative was not integrable—while length, far from depending on the path taken, remained unaltered. For Weyl this was unfair: both features deserved the same treatment.[5] He remedied with a connection that made *congruent* transport (of length) just as path-dependent as parallel transport. This 'total' connection restored justice through a *length connection* it included, an inexact one-form Weyl could not help identifying with the electromagnetic four-potential A, whose four-curl $F = dA$, being closed (for $dF = d^2A$ vanishes identically), provides Maxwell's two homogeneous equations. Source-free electromagnetism (up to Hodge duality at any rate) thus came, quite unexpectedly, out of Weyl's surprising sense of mathematical justice.

Admittedly there were also intimations,[6] from the beginning, prefiguring another prejudice, a 'telescepticism' (Section 3.3)

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[1] Certainly by Folland (1970), Trautman (1982), Yang (1985), Perlick (1991), Vizgin (1994) and others.

[2] Weyl (1918a, pp. 148–149) and Weyl (1918b): "And you must not think I was led to introduce the linear differential form $d\varphi$ alongside the quadratic one, from physics; rather, I wanted to get rid of this 'imbalance,' which I had always found an eyesore, once and for all, and then noticed to my own amazement: it looks as though it explains electricity." (My translation.)

[3] As opposed to 'experimentally founded' or even 'empirically justified' (with respect to the past—to what *was* the past, back then; *a posteriori* justification, once the sequel is known, is of course another matter). *A priori* considerations can be æsthetic or mathematical, for instance.

[4] I say 'prejudice'—and not 'principle' or 'assumption,' for instance—to emphasize the unexpected, gratuitous, almost unaccountable character of the considerations.

[5] Weyl (1918a, p. 148, last sentence).

[6] The phrase between dashes, and the text after it in italics, on p. 148 of Weyl (1918a). Further adumbrations (including the title itself) can be found in Weyl (1918c), which came out about half a year after the communication of Weyl (1918a).

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opposed to distant comparisons. Other texts[7] from the same period are in a similar spirit, delineating a rather murky phenomenological background broadly consistent with this 'tele-scepticism.' Such a background, which Ryckman rightly traces back to Husserl, gives substance and context to the intimations and helps situate them within a developing 'infinitesimal' agenda. But this conceptual framework, as it appears around 1918 at any rate, is logically insufficient to bring together gravitation and electricity. The later texts cited in footnote 25 contribute much clarity and definiteness, which could then be ascribed to the whole agenda *a posteriori*. My interpretation (Section 3.1) of an insistent contraposition of coordinates and gauge has led me to stress the role of 'mathematical justice' instead, whose compelling logical sufficiency spares one the ambiguities and chronological uncertainties of a more ambitious (and ideologically richer) kind of reconstruction. I have little or nothing to add to Ryckman's interpretation, which retains its validity and great interest. But since a viable alternative can only enrich our understanding of Weyl's theory, especially of its origins, I will contend that its derivation *can do without Husserl*; that what was really at work in the spring of 1918, what effectively gave rise to the theory, was the *equal rights of direction and length*, rather than the telescepticism considered in Section 3.3.[8] As the textual evidence underdetermines its interpretation, why not explore the available 'freedom' and offer a new reconstruction consistent with that evidence. The freedom need not be entirely even and uniform, without relief or texture; it could be varied, with 'accidents' of all sorts; regions of it may be favoured by the familiar criteria of simplicity, elegance, economy, expediency and so forth. Needless to say I claim to have found a distinguished 'sweet spot' within the freedom, and that my interpretation is not only *compatible* with the evidence, but even *suggested* by it—especially by Weyl's harping on coordinates vs. gauge, the way I read it at any rate ...

My position is contained in the following claims (which have much in common):

1. *The D/L rule, fully outlined in the table* (p. 4), *effectively produces Weyl's theory.*
2. *The table—its right column at any rate—does not come straight out of a few* '*infinitesimal*' *hints from* 1918. It may be possible to force it out, but not without considerable logical violence.
3. *The infinitesimal agenda, in the vague and embryonic form it has in* 1918, *is insufficient on its own to produce Weyl's theory.*
4. However relevant Husserl may be to the whole story, as it extends over the decades, *Weyl's theory can do without Husserl, who is not indispensable for its derivation*. The theory can be accounted for without any reference to him.

2. Background: Einstein, Levi-Civita

We can begin with aspects of Einstein's theory of gravitation, since Weyl's theory grew out of it. What interests us above all is affine structure, given by the Christoffel symbols Γ^a_{bc}. Through the geodesic equation

$$\frac{\mathrm{d}^2x^a}{\mathrm{d}s^2} + \Gamma^a_{\mathrm{bc}}\frac{\mathrm{d}x^b}{\mathrm{d}s}\frac{\mathrm{d}x^c}{\mathrm{d}s} = 0 \tag{1}$$

$(a = 0, \ldots, 3)$ and the worldlines satisfying it, the Christoffel symbols provide a notion of (parametrized[9]) straightness, of inertial, unaccelerated motion, of free fall.

The left-hand side of (1) gives the components $\langle \mathrm{d}x^a, \nabla_{\dot\sigma}\dot\sigma\rangle$ of the covariant derivative $\nabla_{\dot\sigma}\dot\sigma$ of the vector $\dot\sigma$ with components $\mathrm{d}x^a/\mathrm{d}s = \langle \mathrm{d}x^a, \dot\sigma\rangle$, in the direction $\dot\sigma$ tangent to the worldline $\sigma : I \to M; s \mapsto \sigma(s)$ with coordinates $\sigma^a(s) = x^a(\sigma(s))$, where I is an appropriate interval and M the differential manifold representing the universe; $a = 0, \ldots, 3$. The Christoffel symbols are related to ∇ by

$$\Gamma^a_{\mathrm{bc}} = \langle \mathrm{d}x^a, \nabla_{\partial_b}\partial_c\rangle, \tag{2}$$

where the basis vectors $\partial_a = \partial_{x^a} = \partial/\partial x^a$ are tangent to the coordinate lines of the system x^a.

Einstein only appears to have explored the *infinitesimal* behaviour of the parallel transport determined by his covariant derivative. It was Levi-Civita who first understood that if ∇g vanishes, as in Einstein's theory, the direction of the vector $V_s \in T_{\sigma(s)}M$ transported according to $\nabla_{\dot\sigma}V_s = 0$ depends[10] on the path σ taken—whereas the squared length $l_s = g(V_s, V_s)$ remains constant along σ, for

$$\nabla g = 0 = \nabla_{\dot\sigma}V_s \tag{3}$$

means that $\mathrm{d}l_s/\mathrm{d}s = \nabla_{\dot\sigma}l_s$ vanishes.

3. The emergence of Weyl's theory

3.1. The equal rights of direction and length

Weyl felt that as parallel transport depended on the path taken, congruent transport ought to as well. Indeed his generalization of Einstein's theory appears to have been almost entirely determined by the intention of putting direction and length on an equal footing. The following table[11]—parts of which may for the time being be more intelligible than others—outlines Weyl's programme.

DIRECTION		**LENGTH**
coordinates (up to gauge)		gauge
gravitation		electricity
parallel	**transport**	congruent
Γ^a_{bc}(Levi-Civita)	**connection**	A
$\delta V^a = -\Gamma^a_{\mathrm{bc}}X^bV^c$		$\delta l = -\langle\alpha, X\rangle = -A_bX^bl$
R^a_{bcd}(of Γ^a_{bc})	**curvature**	$F = dA$
coordinates y^a(at P) : $\Gamma^a_{\mathrm{bc}} = 0$	**geodesic**	gauge(at P) : $A' = A + d\lambda = 0$
$\ddot{x}^a + \Gamma^a_{\mathrm{bc}}\dot{x}^b\dot{x}^c \mapsto \ddot{y}^a$	**equivalence principle**	$\alpha = -lA \mapsto \alpha' = 0$

A few words about 'coordinates (up to gauge).' The parallel between coordinates and gauge, which Weyl draws[12] over and over, can be seen as a parallel between direction and length. For surely Weyl does not mean 'coordinates *including gauge*—as

[7] The first paragraph of Section 3, Weyl (1917, p. 125); the introduction to Weyl (1918d); and Weyl (1918e). All three were brought to my attention by a referee.

[8] The two prejudices are in fact independent, however connected they may be in Weyl's theory; see Section 3.4.

[9] For (1) determines an equivalence class $[s]$ of affine parameters, each parameter of which gives the proper time of a regular clock, with its own zero and unit of time. The parameters belonging to $[s]$ are related by affine transformations $s \mapsto \upsilon s + \zeta$, where the constants υ and ζ give the unit and zero. The constant υ is typically chosen so that $g(\partial_0, \partial_0) = 1$.

[10] Levi-Civita (1917, p. 175).

[11] Parts of it were inspired by Coleman & Korté (2001, pp. 204–205, 211–212).

[12] Weyl (1918a, p. 150), top; translation (Perrett and Jeffrey): 'For the purpose of analytical presentation we have firstly to choose a definite system of co-ordinates, and secondly at each point P to determine the arbitrary factor of proportionality with which the $g_{\mu\nu}$ are endowed. Accordingly the formulæ which emerge must possess a double property of invariance: they must be invariant with respect to any continuous transformations of co-ordinates, and they must remain unaltered if $\lambda g_{\mu\nu}$, where λ is an arbitrary continuous function of position, is substituted for the $g_{\mu\nu}$.' Three other passages are similar: Weyl (1918c) p. 396 ("Zum [...] gewonnen"), p. 398 ("In [...] ("Maßstab-Invarianz")"); and the first paragraph of Weyl (1919).

opposed to gauge,' which would be redundant.[13] And up to gauge, coordinates provide no more than direction: The coordinates x^a assign to each event $P \in M$ a basis $\partial_a \in T_P M$, and a dual basis

$$\mathrm{d}x^a = \partial_a^\flat = g(\partial_a, \cdot) \in T_P^* M \tag{4}$$

giving the components $V^a = \langle \mathrm{d}x^a, V \rangle$ of any vector $V \in T_P M$; $a = 0, \ldots, 3$. The recalibration[14] $g \mapsto \mathrm{e}^{2\lambda} g$ induces a transformation $V \mapsto \mathrm{e}^{\lambda} V$, or $V^a \mapsto \mathrm{e}^{\lambda} V^a$, through

$$\mathrm{e}^{2\lambda} g(V, V) = g(\partial_a, \partial_b)\mathrm{e}^{\lambda} V^a \mathrm{e}^{\lambda} V^b. \tag{5}$$

Direction, given by the ratios

$$\mathrm{e}^{\lambda} V^0 : \mathrm{e}^{\lambda} V^1 : \mathrm{e}^{\lambda} V^2 : \mathrm{e}^{\lambda} V^3 = V^0 : V^1 : V^2 : V^3, \tag{6}$$

remains unaffected.

Weyl clearly distinguishes between a 'stretch' (like a *stretch* of road) and its numerical length, determined by the gauge chosen. Just as a direction $[\mathrm{e}^{\lambda} V]_{(\text{all } \lambda)}$ is 'expanded' with respect to a coordinate system, which provides its numerical representation (the ratios $V^0 : \cdots : V^3$), a stretch gets 'expanded' in a gauge, which likewise gives a numerical representation, the (squared) length $l = \mathrm{e}^{2\lambda} g(V, V)$.

The rest of the table should in due course become clearer. Returning now to the (very similar) second and third claims made at the end of the Introduction, the most meaningful and substantial 'telesceptical' statement from 1918 seems to be

> A genuine near-geometry must, however, only know a principle of the transport of a length from one point to an infinitely neighbouring one.

Or less literally

> *T*: A genuine infinitesimal geometry must only countenance infinitesimal congruent transport.[15]

The path-dependence of congruent transport is not an immediate and obvious consequence; it may somehow emerge from the 'only,' but not without effort and interpretation. The one-form A in the fourth line of the Table would require much more effort. And the notion of electromagnetic 'free fall' contained in the last two lines seems to be logically even remoter[16] from T, as are the length difference and curvature of lines five and six. So the Table, which determines and encapsulates the theory, comes straight out of the D/L rule, but not out of the 'infinitesimal' adumbrations from 1918.

Let us now see how the inexact one-form A, which gives rise to so much of electromagnetism, emerges from the equal rights of direction and length.

3.2. *Electromagnetism from equal rights*

Weyl calls a manifold M *affinely connected* if the tangent space $T_P M$ at every point $P \in M$ is connected to all the neighbouring tangent spaces $T_{P'} M$ by a mapping

$$\begin{aligned} \Xi_X : T_P M &\to T_{P'} M \\ V_P &\mapsto V_{P'} = \Xi_X V_P \end{aligned} \tag{7}$$

linear both in the 'main' argument $V_P \in T_P M$ and in the (short[17]) directional argument $X = P' - P$ viewed (since P' is beside P) as lying in $T_P M$. Being linear, Ξ_X will be represented by a matrix:

$$\Xi^a_c = \langle \mathrm{d}x^a, \Xi_X \partial_c \rangle = \Xi^a_{bc} X^b = \langle \mathrm{d}x^a, \Xi_{\partial_b} \partial_c \rangle \langle \mathrm{d}x^b, X \rangle. \tag{8}$$

Weyl specifically refers to the components $\delta V^a = \langle \mathrm{d}x^a_{P'}, V_{P'} \rangle - \langle \mathrm{d}x^a_P, V_P \rangle$, requiring them to be linear in the components X^b and $V^c_P = \langle \mathrm{d}x^c_P, V_P \rangle$. The bilinear function

$$\Gamma^a(\{X^b\}, \{V^c\}) = \delta V^a \tag{9}$$

will be a matrix, represented by Γ^a_{bc}; the difference δV^a is therefore $-\Gamma^a_{bc} X^b V^c$ (where the minus sign is conventional).

With respect to the *geodesic* coordinates y^a which make

$$\Gamma^a_c = \Gamma^a_{bc} X^b = \langle dy^a, \nabla_X \partial_{y^c} \rangle \tag{10}$$

and δV^a vanish, leaving the components V^a unchanged, Ξ^a_c becomes the identity matrix

$$\delta^a_c = \langle \mathrm{d}y^a, \Xi_X \partial_{y^c} \rangle \leftrightarrow \mathrm{diag}(1,1,1,1). \tag{11}$$

Physically this has to do with the equivalence principle, according to which a gravitational field Γ^a_{bc} can always be eliminated or generated at P by an appropriate choice of coordinates.

With equal rights in mind Weyl turns to length, using the very same scheme. To clarify his procedure we can take just a single component of the difference $\{\delta V^0, \ldots, \delta V^3\}$, calling it δl (this will be the 'squared-length-difference scalar').[18] The upper index of Γ^a_{bc} accordingly disappears, leaving[19] $\delta l = \Gamma_{bc} X^b V^c$. If we now take a single component of the main argument $\{V^0, \ldots, V^3\}$, calling it l (this will be the squared length), the second index of Γ_{bc} disappears as well, and we are left with $\delta l = \Gamma_b X^b l$, where $\Gamma_b = \langle A, \partial_b \rangle$ are the components of a one-form,[20] denoted A with electricity in mind.

But this is not really Weyl's argument, which is more accurately rendered as follows. The object A generating the squared-length-difference scalar δl has to be linear in the squared length l and the direction X. A linear function $A(l, X) = \delta l$ of a scalar l and vector X yielding a scalar δl will be a one-form[21]:

$$\delta l = -\langle \alpha, X \rangle = -\langle \alpha, \partial_b \rangle \langle \mathrm{d}x^b, X \rangle = -\alpha_b X^b = -\langle A, X \rangle l = -\langle A, \partial_b \rangle \langle \mathrm{d}x^b, X \rangle l = -A_b X^b l, \tag{12}$$

where α is the squared-length-difference one-form. An exact one-form $A = \mathrm{d}\mu$ would make congruent transfer integrable, removing

[13] *Cf.* Weyl (1918a, p. 149), last two sentences; translation (Perrett and Jeffrey): 'We take it, however, that this form is determined only so far as to a positive factor of proportionality, which remains arbitrary. If the manifold of points of space is represented by co-ordinates x_μ, the $g_{\mu\nu}$ are determined by the metrical properties at the point P only to the extent of their proportionality.'

[14] The convenient 'exponential' recalibration is not used by Weyl.

[15] My translations of 'Eine Wahrhafte Nahe-Geometrie darf jedoch nur ein Prinzip der Übertragung einer Länge von einem Punkt zu einem unendlich benachbarten kennen [...]' Weyl (1918a, p. 148).

[16] I am deliberately conflating—when I say 'remoter' or 'comes straight out of,' for instance—two kinds of 'logical distance' which can be roughly illustrated as follows: 1. a hard theorem is 'farther' from the assumptions than an easy one; 2. a theorem requiring assumptions a, b and c is 'farther' from a than a theorem requiring only a and b.

[17] The necessary shortness of X seems inconsistent with linearity, which would 'connect' P with the entire tangent space $T_P M$ and not just with the small neighbourhood 'covering' M. In this context it may be best to view the linearity in the directional argument as being appropriately restricted (of course the length of X does not matter in differentiation, in which limits are taken).

[18] Weyl appears to use d and δ interchangeably, and d in a way—see footnote 21—that is unusual not only today, but even then. He does not distinguish between the *scalar* representing the difference in squared length, and the corresponding *one-form* (as we would call it); but the distinction nonetheless seems useful.

[19] We can perhaps think of the hybrid, intermediate connection Γ_{bc} as being something like $\langle A, \nabla_{\partial_b} \partial_c \rangle$.

[20] One may wonder how the tensor A can be the counterpart of the connection Γ^a_{bc}, which is not a tensor. The components $A_a = \langle A, \partial_a \rangle = \Gamma_a$ only transform 'tensorially' with respect to coordinate transformations $A_a \mapsto \bar{A}_b = A_a \langle d\bar{x}^b, \partial_{x^a} \rangle$, however; with respect to recalibration $A_a \mapsto A'_a = A_a + \partial_a \lambda$ the components A_a transform differently and can be locally cancelled, for instance.

[21] Weyl in fact writes $\mathrm{d}l = -l\,\mathrm{d}\varphi$, where I write $\alpha = -lA$. The misleading d's cannot be understood globally—or even locally, in the theory of gravitation and electricity, in which $F = \mathrm{d}^2\varphi$ will be the Faraday two-form: where $\mathrm{d}\varphi$ is closed, in other words the differential (even only locally) of a function φ, there would be no electromagnetism.

the dependence of the recalibration

$$e^{\int_\gamma A} = e^{\int d\mu} = e^{\Delta\mu} \tag{13}$$

on the path $\gamma : [0,1] \to M$, where $\Delta\mu = \mu_1 - \mu_0$ is the difference between the values $\mu_1 = \mu(P_1)$ and $\mu_0 = \mu(P_0)$ of μ at $P_1 = \gamma(1)$ and $P_0 = \gamma(0)$. Mathematical justice therefore demands that A be inexact; so the curl $F = dA$ cannot vanish identically.

Confirmation that A has to be one-form, possibly inexact, is provided by Weyl's requirement that the squared-length-difference one-form $\alpha = -lA$ be eliminable at any point P by recalibration. As l is given (and does not vanish), this amounts to $A + d\lambda = 0$ at P, where the gauge λ is *geodesic*.[22] Since $d\lambda$ is a one-form, A must be one too. Though $d\lambda$ is exact, Weyl only asks that it cancel A *at P*—so A need not even be *closed*, or 'locally exact.'

With $F = dA$ and its consequence $dF = 0$ before him Weyl could not help seeing the electromagnetic four-potential A, the Faraday two-form $F = dA$ (which vanishes wherever A is closed) and Maxwell's two homogeneous equations,[23] expressed by $dF = 0$—not to mention an electromagnetic 'equivalence principle'[24] according to which the squared-length-difference scalar δl and one-form α, as well as the electromagnetic four-potential A, can be eliminated or produced at a point by an appropriate gauge function λ.

In coordinates

$$F_{ab} = F(\partial_a, \partial_b) = \partial_a A_b - \partial_b A_a \leftrightarrow \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}, \tag{14}$$

where E_x, E_y, E_z are the components of the electric field and B_x, B_y, B_z those of the magnetic field. Or $F = F_{ab} dx^a \wedge dx^b/2$. The vanishing three-form

$$dF = \tfrac{1}{2} dF_{bc} \wedge dx^b \wedge dx^c = \tfrac{1}{6} \partial_a F_{bc} dx^a \wedge dx^b \wedge dx^c \tag{15}$$

has components $dF(\partial_a, \partial_b, \partial_c) = \partial_a F_{bc} + \partial_b F_{ca} + \partial_c F_{ab}$.

Maxwell's other two equations are obtained, in 'source-free' form, by setting d^*F equal to zero, where $^*F$ is the Hodge dual of the Faraday two-form, with coordinates

$$(^*F)_{ab} = (^*F)(\partial_a, \partial_b) \leftrightarrow \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & E_z & -E_y \\ -B_y & -E_z & 0 & E_x \\ -B_z & E_y & -E_x & 0 \end{pmatrix}. \tag{16}$$

Electromagnetism thus emerged, altogether unexpectedly, from the equal rights of direction and length.

3.3. The illegitimacy of distant comparisons

Weyl has another *a priori* prejudice, rooted, as Ryckman (2005) has cogently argued, in Husserl's transcendental phenomenology. It is expressed in two similar passages,[25] which roughly say: As the curvature $R(P)$ is subtle and hard to perceive directly, a 'cognizing ego' at the 'ego centre' $P \in M$ takes itself to be immersed in the 'psychologically privileged' tangent space $T_P M$. The universe M resembles $T_P M$ in the immediate vicinity $\mathfrak{U}$ of P, where they practically coincide, and 'cover' one another. Beyond $\mathfrak{U}$ the relation between M and the 'intuitive' space $T_P M$ grows looser, as the universe goes its own way, bending as the energy-momentum tensor T varies.

Ryckman writes (p. 148) that

> Weyl restricted the homogeneous space of phenomenological intuition, the locus of phenomenological *Evidenz*, to what is given at, or neighboring, the cognizing ego [...]. But in any case, by delimiting what Husserl termed 'the sharply illuminated circle of perfect givenness,' the domain of 'eidetic vision,' to the infinitely small homogeneous space of intuition surrounding the 'ego-center' [...].

This restriction or delimitation can be understood in two ways: directly, in terms of the limitations of our senses, and of an accordingly circumscribed domain of sensory access, of 'eidetic vision'; or more mathematically, as follows: The cognizing ego attaches a kind of intuitive 'certainty' to all of $T_P M$, which, being flat and homogeneous,[26] can be captured or 'understood' in its entirety once any little piece is. The universe shares that certainty as long as it resembles $T_P M$, and hence only in $\mathfrak{U}$, outside of which it is subject to all sorts of unforeseeable variations.

Integrable congruent propagation had to be rejected as allowing the certain comparison of lengths well beyond $\mathfrak{U}$, indeed at any distance, without the welcome ambiguities related to the path followed. Returning to Ryckman (p. 149):

> Guided by the phenomenological methods of 'eidetic insight' and 'eidetic analysis', the epistemologically privileged purely infinitesimal comparison relations of *parallel transport* of a vector, and the *congruent displacement* of vector magnitude, will be the foundation stones of Weyl's reconstruction. The task of comprehending 'the sense and justification' of the mathematical structures of classical field theory is accordingly to be addressed through a construction or *constitution* of the latter within a world geometry entirely built up from these basic geometrical relations immediately evident within a purely infinitesimal space of intuition. A wholly *epistemological* project, it nonetheless coincides with the explicitly *metaphysical* aspirations of Leibniz and Riemann to 'understand the world from its behaviour in the infinitesimally small.'

3.4. The two prejudices

Removed from the context of Weyl's theory, the two prejudices are entirely distinct. While one is markedly infinitesimal, the other—'mathematical justice'—has nothing (necessarily) infinitesimal about it: in a spirit of equal rights one could require, for instance, both the directions and lengths of the vectors in some set to have the same kind of distribution—uniform, say, or Gaussian—around a given vector. Nothing infinitesimal about that.

An abundant insistence in the early going on the equal rights of direction and length, together with the absence, back then, of any explicit, articulated expression of the telescepticism of Section 3.3, suggests the following account. First there was mathematical justice, which, far from being at odds with Weyl's nascent infinitesimal agenda, supported it, perhaps even suggesting aspects. In due course Weyl's 'purely infinitesimal geometry'

[22] By analogy one might even call it 'inertial' or 'unaccelerated.'

[23] In full, $\nabla \cdot B = 0$ and $\nabla \times E + \partial B/\partial t = 0$.

[24] Lyre (2004) speaks of a *generalized equivalence principle*.

[25] Weyl (1927, p. 98) ('Erkennt [...] Fläche.') translation (Helmer): 'In addition to the physical space one may acknowledge the existence of a *space of intuition* and maintain that its metrical structure of necessity satisfies Euclidean geometry. This view does not contradict physics, in so far as physics adheres to the Euclidean quality of the infinitely small neighborhood of a point O (at which the ego happens to be at the moment). [...] But then it must be admitted that the relation of the intuitive to the physical space becomes the vaguer the farther one departs from the ego center. The intuitive space may be likened to a tangent plane touching a curved surface (the physical space) at a point O; in the immediate vicinity of O the two coincide, but the larger the distance from O the more arbitrary will the one-to-one correspondence between plane and surface become that one tries to establish by continuing the relation of coincidence near O.' The passage ('Die [...] Ebene') on p. 52 (second column, towards the bottom) of Weyl (1931) is similar.

[26] Curvature (which vanishes identically) and the metric are constant.

acquired more explicit transcendental-phenomenological grounding (footnote 25), which can in hindsight make the apparently gratuitous early insistence on equal rights somewhat less surprising.

4. Compensating transformations

We have seen how Weyl's theory, building on general relativity, came out of the inexact one-form A—whose transformations

$$A \mapsto A' = A + \mathrm{d}\mu \tag{17}$$

are counterbalanced in the theory by

$$g \mapsto g' = \mathrm{e}^{\mu} g, \tag{18}$$

leaving length unaltered. Such compensation is fundamental enough to be worth looking at briefly.

Freedom to transform A according to (17) is left by the length curvature $F = \mathrm{d}A$, which is indifferent to an exact term $\mathrm{d}\mu$, as

$$F = \mathrm{d}A' = \mathrm{d}A + \mathrm{d}^2\mu = \mathrm{d}A. \tag{19}$$

But (17) does change length. Transporting the vector X_0 from point P_0 with value $\mu_0 = \mu(P_0)$ to point P_1 with value $\mu_1 = \mu(P_1)$, the final squared length $g_1(X_1, X_1)$ acquires the additional (integrable) factor $\mathrm{e}^{\Delta\mu}$, where $\Delta\mu = \mu_1 - \mu_0$. For μ recalibrates, along a curve γ, according to

$$\mathrm{e}^{\int_\gamma A} \mapsto \mathrm{e}^{\int_\gamma A'} = \mathrm{e}^{\int_\gamma A} \mathrm{e}^{\Delta\mu} \neq \mathrm{e}^{\int_\gamma A}, \tag{20}$$

and therefore

$$g_1(X_1, X_1) = \mathrm{e}^{\int_\gamma A} g_0(X_0, X_0) \neq \mathrm{e}^{\int_\gamma A'} g_0(X_0, X_0). \tag{21}$$

But the conformal transformation (18) compensates, leaving length unchanged:

$$g'_1(X_1, X_1) = \mathrm{e}^{\mu_1} g_1(X_1, X_1) = \mathrm{e}^{\int_\gamma A'} g'_0(X_0, X_0) = \mathrm{e}^{\int_\gamma A} \mathrm{e}^{\Delta\mu} \mathrm{e}^{\mu_0} g_0(X_0, X_0). \tag{22}$$

The exponents cancel, yielding the original dilation

$$g_1(X_1, X_1) = \mathrm{e}^{\int_\gamma A} g_0(X_0, X_0). \tag{23}$$

The metric g is *compatible* with the covariant derivative ∇ if ∇g vanishes, in which case the straightest worldlines (satisfying $\nabla_{\dot\sigma}\dot\sigma = 0$) will also be stationary, satisfying

$$\delta \int \sqrt{g(\dot\sigma, \dot\sigma)}\, \mathrm{d}s = \delta \int \mathrm{d}s = 0 \tag{24}$$

too. The covariant derivative of the recalibrated metric g' only vanishes if μ is a constant (for then $\mathrm{d}\mu$ vanishes); otherwise

$$\nabla g' = \mathrm{d}\mu \otimes g', \tag{25}$$

which combines (17) and (18), to express the weaker *Weyl compatibility*.

5. Einstein's objection

Out of a sense of mathematical justice, then, Weyl made congruent displacement just as path dependent as parallel transport. But experience, objected Einstein, is unfair, showing congruent displacement to be integrable. In a letter to Weyl dated April 15th (1918) he argued that clocks running at the same rate at one point will *continue* to run at the same rate at another, however, they get there—whatever the requirements of mathematical justice. Four days later he reformulated the objection in terms of the 'proper frequencies' of atoms (rather than genuine macroscopic clocks) 'of the same sort': if such frequencies depended on the path followed, and hence on the different (electromagnetic) vicissitudes of the atoms, the chemical elements they would make up if brought together would not have the clean spectral lines one sees.

But even if experience shows congruent displacement to be integrable, it would be wrong to conclude that the equal rights of direction and length led nowhere; for the structure that came out of Weyl's surprising sense of mathematical justice would survive in our standard gauge theories, whose accuracy is less doubtful.

6. Final remarks

There are various levels of 'experience,' ranging from the most concrete to the most abstract: from the most obvious experimental level, having to do with the results of particular experiments, to principles, perhaps even instincts, distilled from a lifetime of experience. One such principle could be Einstein's 'I am convinced that God does not play dice,' to which, having—we may conjecture—noticed that the causal regularities behind apparent randomness eventually tend to emerge, he may ultimately have been led by experience: by his own direct experience, together with his general knowledge of science and the world. One would nonetheless hesitate to view so general and abstract a principle as being *a posteriori*, empirical. It is clearly not *a posteriori* with respect to any particular experiment; only, if at all, with respect to a very loose, general and subjective kind of ongoing experience, capable of being interpreted in very different ways.

An unexpected empirical fertility of apparently *a priori* and unempirical prejudice can sometimes be accounted for in terms of a derivation, however indirect, from experience: by attributing remote empirical roots to considerations which at first seem to have nothing at all to do with experience. The world can admittedly be experienced in very different ways, some much less obvious and straightforward than others; but here we have a prejudice which—however subtle and developed one's faculties for interpreting experience—seems to be completely unempirical. Perhaps the empirical shortcomings of the theory are best blamed, then, on the totally unempirical nature of the prejudice from which it stemmed.

Or is it so completely unempirical? As mathematical justice is at issue, the principle of sufficient reason can come to mind: if there is an imbalance, an unexpected difference, there had better be a reason for it—failing which, balance, or rather justice should prevail. Even Einstein's dice may come to mind: If a situation of apparent balance, such as

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|\alpha\rangle + |\beta\rangle), \tag{26}$$

gives rise to an imbalance (as it must, if a measurement is made), say the eigenvalue $+1$ of the operator $A = |\alpha\rangle\langle\alpha| - |\beta\rangle\langle\beta|$, there ought to be a reason: a circumstance unrepresented in (26) which favours $|\alpha\rangle$. For God does not play dice: symmetry-breaking is never *entirely* spontaneous. But the 'balance' before the disruption is not always so easily seen; what tells us in general which objects or entities are to be put on an equal footing, for imbalances to be visible? *Judgment*, surely; a judgment somehow *founded in experience*, which assesses the relevant peculiarities of the context and determines accordingly. And here Weyl's judgment and sense of balance accord the same status to direction and length—for he sees nothing to justify a preference, an injustice.

Can the success of modern gauge theories really be attributed to Weyl's sense of mathematical justice? Or is the connection between those theories and the equal rights of direction and length too tenuous to be worth speaking of? The lineage is unmistakable, and can be traced through Yang and Mills (1954) and Weyl (1929), back to 1918; the scheme of compensation outlined in Section 4 survives in today's theories, and is central to their success; but the mathematical justice that produced it no longer has any role. Gauge theory may have outlived the equal treatment of length and direction, but not the abstract structure that came out of it.

Whatever the relationship between mathematical justice and experience, we have a surprising example of how directly an elaborate theory can emerge from simple *a priori* prejudice. The prejudice seems gratuitous in the context of discovery, and only acquires justification and phenomenological grounding years later, in an explicit, articulated 'telescepticism' which provides epistemology and motivation.

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