The two revolutions of 1917 and their philosophical roots

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Abstract

Revolutions are often rooted in philosophy, 1917 being no exception. I consider two (topological) revolutions, both with philosophical roots. One of them shook the world (*die Welt*)—which became cylindrical in adherence to a 'Machian' principle proposed by Einstein. And his methodological insistence on coordinate-independence in analytical micromechanics gave rise to highly invariant quantum conditions on a torus.

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Introduction

> It is a philosophy manifested over his entire career, consisting in the advance or critique of theory through the use, and presumed general validity, of certain principles; physical, formal, methodological, and metaphysical. [...] it is a "philosophy of principles" guiding theoretical practice and shaping philosophical pronouncements rather than the other way around [...].
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> Historians of physics have remarked that Einstein's strivings and accomplishments in physical theory have a distinctive character: they reveal the quest of the *natural philosopher* [...].[1]

In three texts from 1917(a,b,c), Einstein's well-known methodological idiosyncrasies[2]—which sometimes amount almost to a contempt for experience—are manifest in two fields: cosmology and analytical micromechanics. The point here is that in both fields Einstein applies philosophical principles that somehow lead to *topological* revolutions; the cylinder being clearly revolutionary, the torus even more. Experience as usual seems rather remote and marginal, its place taken by philosophy, abstract methodology.

1 In cosmology Einstein applies a principle (**MP**) of markedly Machian character: inertia is to be fixed by matter (§1). But the asymptotic decay of inertia, in the mathematical formulation chosen, is inconsistent with the observed slowness of celestial bodies. Or *curvature* can diminish far from matter instead; but such an assumption, which would indeed fix inertia, seems unsatisfactorily un-Machian. Since asymptotic decay—of inertia or curvature—at spatial infinity is so troublesome, Einstein proposes a revolutionary topology with only temporal infinities, that promises to be more Machian (§1.2).

The continuity (or even constancy) of the cosmological matter density ρ presupposes a scale so large that celestial bodies lose their individuality and just 'disappear into the global cosmic smear.' One isn't supposed to look closely enough to see single stars or planets—whose detailed configuration would require a completely different kind of description. One of Einstein's many methodological peculiarities, or even novelties, is to take the corpuscular nature of celestial matter seriously enough to treat it statistical-mechanically, while maintaining the large-scale smoothness of ρ. Using both descriptions at the same time is asking for trouble, and indeed gives rise to the contradictions we'll soon see (§1), which Einstein resolves with a new topology engendered by the cosmological constant.

2 In the last phase of the old quantum theory, say from 1915, Schwarzschild, Epstein and others apply the techniques of celestial mechanics to microphysics; Sommerfeld generalises the Bohr-Sommerfeld quantum condition (§2.1) for one freedom degree to N conditions for as many coordinates (§2.2.1). In 1917 Einstein's head has been brimming with tensors and covariance principles for years, he cannot but object to a quantum rule referring to particular coordinates—and accordingly proposes a more invariant rule (§2.2.3), with N conditions for the N homotopy classes of an N-torus (§2.3).

So we have two cases, a few months apart, illustrating Einstein's peculiar methodology that takes him, as it often does, from philosophy to radically new physics—topologically new here. Both cases involve celestial mechanics, of sorts; §2 is about

[1] Ryckman (2017) pp. 4-5

[2] See Ryckman (2017) for instance, and the literature cited therein.

the sophisticated 'celestial micromechanics' that grew out of Bohr's atom; Einstein's is Lagrangian and only apparently Hamiltonian (§2.6), whereas the theory (§2.4) proposed by Schwarzschild, Sommerfeld and Epstein (SSE) is genuinely so.

1 The cylindrical revolution

Einstein (1917a, p. 145) proposes the following 'Machian' principle **MP**:

> In einer konsequenten Relativitätstheorie kann es keine Trägheit g e - g e n ü b e r d e m „R a u m e" geben, sondern nur eine Trägheit der Massen g e g e n e i n a n d e r. Wenn ich daher eine Masse von allen anderen Massen der Welt räumlich genügend entferne, so muß ihre Trägheit zu Null herabsinken.'[3]

So there's no inertia with respect to space, only between masses; far from other bodies, a mass loses its inertia. But before turning to general relativity he considers Newtonian theory, represented by Poisson's equation[4]

$$\Delta\phi = \rho, \tag{1}$$

which corresponds to the relativistic equations[5] (without the cosmological constant) we now call 'Einstein's'; ρ being the matter density, ϕ the gravitational potential, and $\Delta := \nabla \cdot \nabla = \mathrm{div}\,\mathrm{grad}$ (but differential forms are more convenient and will be used below). By generating contradictions, considered in §1.1 below, Einstein wants to show that *Newtonian gravity doesn't work* and should be corrected; or rather on its own, without statistics, it may make sense, but *statistical mechanics is incompatible with Newton-Poisson gravity*.[6] For it to make 'statistical sense,' (1) has to be modified; Einstein will propose the altered equation[7]

$$\Delta\phi - \lambda\phi = \rho \tag{2}$$

whose λ corresponds to the cosmological constant he'll introduce in general relativity[8] (§1.2 below). And even there he'll see a *logical* problem, of consistency: the $\mathbb{R}^4$ topology is *inconsistent* with certain statistical considerations required by cosmology; consistency is restored in the cylindrical topology produced by λ.[9]

[3] *Cf.* Einstein (1916a): "In der Gravitation suche ich nun nach den Grenzbedingungen im Unendlichen; es ist doch interessant, sich zu überlegen, inwiefern es eine *endliche* Welt gibt, d. h. eine Welt von natürlicher gemessenen endlicher Ausdehnung, in der wirklich alle Trägheit relativ ist."

[4] Einstein includes the factor $4\pi K$ which we can ignore, with an appropriate choice of units.

[5] Einstein (1917a) p. 144: "die von mir bisher vertretenen Feldgleichungen der Gravitation [. . .]."

[6] Einstein (1917a) p. 143: "Wendet man das BOLTZMANNsche Verteilungsgesetz für Gasmoleküle auf die Sterne an, indem man das Sternsystem mit einem Gase von stationärer Wärmebewegung vergleicht, so volgt, daß das NEWTONsche Sternsystem überhaupt nicht existieren kann." P. 144: "Der bei der NEWTONschen Theorie konstatierte Konflikt mit der statistischen Mechanik [. . .]."

[7] Einstein (1917a) p. 143: "Diese Schwierigkeiten lassen sich auf dem Boden der NEWTONschen Theorie [(1)] wohl kaum überwinden. Man kann sich die Frage vorlegen, ob sich dieselben durch eine Modifikation [(2)] der NEWTONschen Theorie beseitigen lassen."

[8] Einstein (1917a) p. 148: "allerdings bedarf es für die Durchführung dieses Gedankens einer verallgemeinernden Modifikation der Feldgleichungen der Gravitation." P. 151: "Das Gleichungssystem [Einstein's equations without λ] erlaubt jedoch eine naheliegende, mit dem Relativitätspostulat vereinbare Erweiterung, welche der durch Gleichung (2) gegebenen Erweiterung der POISSONschen Gleichung vollkommen analog ist."

[9] Einstein (1917a) p. 152: "Jedenfalls ist diese Auffassung logisch Widerspruchsfrei [. . .]. Um zu dieser widerspruchsfreien Auffassung zu gelangen [. . .]."

1.1 Newton-Poisson gravity

Even if Einstein writes (1), it isn't pointless to begin—logically just upstream—with the weaker Gauss-Maxwell condition

$$dE = \rho, \tag{3}$$

where the matter density ρ can be viewed as a three-form and the gravitational field E as a two-form; its divergence dE will then be a three-form. **MP** isn't satisfied by (3), since matter ρ only determines the flux

$$\mathbf{F} := \iint_{\partial\zeta} E = \iiint_{\zeta} dE = \iiint_{\zeta} \rho$$

through an enclosing membrane $\partial\zeta$, but not the field E itself, whose lines can be freely deformed without affecting $\mathbf{F}$. Since d^2 (or $\nabla \cdot \nabla\times = \operatorname{div}\operatorname{curl}$) vanishes, the divergence $dE = d\bar{E}$ is unaffected by the deformation

$$E \mapsto \bar{E} := E + d\xi$$

induced by the exact term $d\xi$ (whatever the form of the two-form E).

To get from (3) to (1) one has to assume that force is conservative, derived from a potential, in other words

$$E = *d\phi, \tag{4}$$

where ϕ is a zero-form and the star denotes Hodge duality. With differential forms Poisson's equation (1) becomes $\underline{\Delta}\phi = \rho$, where each of the three operations represented by $\underline{\Delta} := d{*}d$ increases ϕ's degree by one, ultimately turning a zero-form into a three-form.

To fix E by establishing that

$$d\xi \text{ vanishes} \tag{5}$$

Einstein needs both (4) and the asymptotic constancy[10] **AC** of ϕ—which is by no means enough on its own, since it can be applied beyond a sphere σ_r of radius[11] r. Without (4), the un-Machian one-form ξ could freely deform the field in σ_r but vanish before r, leaving **AC** to fix E beyond.

Once (4) has been assumed on physical grounds, it seems just as reasonable to assume the exactness of

$$*\bar{E} = d\phi + *d\xi$$

as well. The resulting exactness of $*d\xi =: d\gamma$ then allows us to write

$$\bar{E} = *(d\phi + d\gamma) \tag{6}$$

and indeed

$$d(*d\phi + *d\gamma) = d{*}d\phi = \rho,$$

so that

$$d{*}d\gamma \text{ vanishes;} \tag{7}$$

[10] Einstein (1917a) p. 142: "im räumlich Unendlichen das Potential ϕ einem festen Grenzwerte zustrebt." See also footnote 19 below.

[11] The centre **O** (*Mittelpunkt*) from which the radius is taken will be introduced below, see footnote 14.

(4) & (6) $\Rightarrow$ (7). But the harmonic zero-form γ cannot be asymptotically constant without being globally so;

(7) & **AC** $\Rightarrow$ (5).[12]

So far the problem is that matter needs a lot of help—(4) & **AC**—to fix E. By using Poisson's equation (rather than the slightly weaker (3)) Einstein assumes (4) tacitly, without question, but clearly finds **AC** disturbingly un-Machian. **MP** has led to **AC**, which will in turn impose on ρ its own (vanishing) asymptotic constancy.[13] At a large enough scale ρ looks spherically symmetrical,[14] around a centre **0**, from which one can consider radial distances r; (1) & **AC** imply that ρ has to vanish so fast—faster than[15] $1/r^2$—that Einstein seems to view matter as 'practically circumscribed'; he speaks of a "finite universe"[16]

FU: *ρ diminishes faster than $1/r^2$*,

which he will eventually violate using the (celestial) statistical mechanics he proposes.

Statistical mechanics? From afar, the matter distribution ρ looks smooth, but closer inspection reveals the individual bodies of a celestial *gas*.[17] Its constituents interact in a statistical mechanics which may concentrate enough of the available energy into a single body[18] to dispatch it irretrievably to spatial infinity,[19] eventually producing *Verödung*; the large numbers involved can eventually 'realise' or 'actualise' apparently unlikely possibilities. Einstein develops the contradictions that somehow lead to λ using two conflicting tendencies, which we can call *centripetal* and *centrifugal*. The first is purely gravitational: increasing the potential difference would increase the centripetal force $E_1 = d\phi$ which compresses the celestial gas towards **0**—and may even 'seal it off from infinity' at some 'barrier-radius.'[20] Even if Einstein only speaks of speed and never of anything like heat, we can view the centrifugal tendency as 'thermodynamic':

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[12]Jean-Philippe Nicolas suggested the proof strategy; I am also indebted to Vieri Benci, Julien Bernard and Eric Gourgoulhon, for similar strategies.

[13]Einstein (1917a) p. 142: "Es ist wohlbekannt, daß die NEWTONsche Grenzbedingung des konstanten Limes für ϕ im räumlich Unendlichen zu der Auffassung hinführt, daß die Dichte der Materie im Unendlichen zu null wird." But the modified equation (2) will allow ρ to remain constant; Einstein (1917a) p. 144: "Ein Abnehmen der Dichte im räumlich Unendlichen müßte nicht angenommen werden, sondern es wäre sowohl das mittlere Potential als auch die mittlere Dichte bis ins Unendliche konstant."

[14]Einstein (1917a) p. 142: "Wir denken uns nämlich, es lasse sich ein Ort im Weltraum finden, um den herum das Gravitationsfeld der Materie, im großen betrachtet, Kugelsymmetrie besitzt (Mittelpunkt)."

[15]Einstein (1917a) pp. 142-3: "Dann folgt aus der POISSONschen Gleichung, daß die mittlere Dichte ρ rascher als $\frac{1}{r^2}$ mit wachsender Entfernung r vom Mittelpunkt zu null herabsinken muß, damit ϕ im Unendlichen einem Limes zustrebe."

[16]*Cf.* Einstein (1916a): "In der Gravitation suche ich nun nach den Grenzbedingungen im Unendlichen; es ist doch interessant, sich zu überlegen, inwiefern es eine *endliche* Welt gibt, d. h. eine Welt von natürlicher gemessenen endlicher Ausdehnung, in der wirklich alle Trägheit relativ ist." Einstein (1917a) p. 143: "In diesem Sinne ist also die Welt nach Newton endlich, wenn sie auch unendlich große Gesamtmasse besitzen kann."

[17]Einstein (1917a) p. 143, footnote: "ρ is die Mittlere Dichte der Materie, gebildet für einen Raum, der groß ist gegenüber der Distanz benachbarter Fixsterne, aber klein gegenüber den Abmessungen des ganzen Sternsystems." P. 148: "Wenn es uns aber nur auf die Struktur im großen ankommt, dürfen wir uns die Materie als über ungeheure Räume gleichmäßig ausgebreitet vorstellen, so daß deren Verteilungsdichte eine ungeheuer langsam veränderliche Funktion wird."

[18]Einstein (1917a) p. 143: "Dieser Fall muß nach der statistischen Mechanik solange immer wieder eintreten, als die gesamte Energie des Sternsystems genügend groß ist, um — auf einen einzigen Himmelskörper übertragen — diesem die Reise ins Unendliche zu gestatten, von welcher er nie wieder zurückkehren kann."

[19]Einstein (1917a) p. 143: "Denn aus der Voraussetzung [**AC**] eines endlichen Limes für ϕ im räumlich Unendlichen folgt, daß ein mit endlicher kinetischer Energie begabter Himmelskörper das räumlich Unendliche unter Überwindung der NEWTONschen Anziehungskräfte erreichen kann."

[20]Einstein (1917a) p. 143: "Man könnte dieser eigentümlichen Schwierigkeit durch die Annahme zu entrinnen versuchen, daß jenes Grenzpotential im Unendlichen einen sehr hohen Wert habe."

increasing the potential differences would also increase the thermal agitation—the average kinetic energy—and hence the outward pressure of the gas. The **FU** rate $1/r^2$ is at the heart of Einstein's contradictory scheme, in which the centrifugal pressure **P** generated by the potential differences exceeds $1/r^2$ by overcoming the centripetal tendency, also due to the potential differences. If one could produce a large potential difference without also increasing the kinetic energy and hence centrifugal pressure, *Verödung* from hemorrhage to spatial infinity (¬ **FU**) could be prevented; but the impermeability of the potential barrier depends on the very quantities that would make it more permeable.[21] We appear to have

$$\mathbf{MP} \Rightarrow \mathbf{AC} \Rightarrow \mathbf{FU} \Rightarrow \neg\, \mathbf{P},$$

and hence $\mathbf{P} \Rightarrow \neg\, \mathbf{MP}$.

Even if Einstein envisages *infinite*[22] mass

$$\infty = \int \rho, \tag{8}$$

we can first consider a finite mass

$$\mu := \int_{\mathbf{0}}^{r_\mu} \rho_f$$

enclosed in a radius r_μ, where the 'finite' density ρ_f has compact support. By (1), increasing μ would increase the potential differences,[23] both small $d\phi$ and large

$$\mathit{\Delta}\phi := \phi_\infty - \phi_{\mathbf{0}},$$

which would have to dominate the kinetic energy of the celestial gas to check *Verödung*.[24] The merely finite mass μ simplifies our logic: rather than the **FU** rate $1/r^2$ we just have the clean distinction between *masses at a finite distance from* **0** and *masses lost to spatial infinity*. Einstein doesn't provide enough mathematical detail for genuine rigor or calculation, but there may be cases in which *any* total mass μ would produce enough kinetic energy, temperature and hence centrifugal pressure to overcome the centripetal tendency and allow *Verödung*.

Since Einstein's infinite mass cannot be *entirely* enclosed within a radius r_μ, the best one can do—to obtain the 'approximate finiteness' of the **FU**—is enclose *almost all* of it, say nine tenths of it,[25] in a radius $\bar{r}_9$:

$$\bar{\mu}_9 = \int_{\mathbf{0}}^{\bar{r}_9} \rho.$$

Since **FU** is deduced from (1) & **AC** and not independently postulated, violation of the corresponding bound $1/r^2$ by hemorrhage to spatial infinity undermines the whole Newtonian scheme.[26] The law relating the distribution ρ of the celestial gas to the

[21] Einstein (1917a) p. 143: "Dies wäre ein gangbarer Weg, wenn nicht der Verlauf des Gravitationspotentials durch die Himmelskörper selbst bedingt sein müßte."

[22] Einstein (1917a) p. 143: "die Welt nach NEWTON [. . .] unendlich große Gesamtmasse besitzen kann."

[23] Einstein (1917a) p. 143: "der Verlauf des Gravitationspotentials durch die Himmelskörper selbst bedingt sein müßte."

[24] See footnote 18 above.

[25] *Cf.* Einstein (1921): "$2a$ is der Radius, in dem die Dichte auf etwa 2% der zentralen Dichte gesunken ist."

[26] See footnote 6 above.

centripetal tendency is essentially (1); what Einstein doesn't specify here is the law relating ρ to temperature and hence **P**. If provided, such a law may already suffice to produce the contradiction(s) needed to justify the Newtonian cosmological constant λ, which will lead to the corresponding relativistic constant.

This brings us to the role played by the *slowness* Einstein mentions once[27] in this Newtonian section (§ I. Die NEWTONsche Theorie). He gives the impression that his cosmology, however philosophical and unempirical, does need one *Tatsache*: slow celestial bodies. In §1.2 below we'll see that Einstein's *relativistic* cosmo-logic does indeed seem to require such a *Tatsache*; but in this Newtonian context one can wonder whether more mathematical detail—differential equations—wouldn't suffice to generate the contradiction(s) Einstein is after without the need for anything as unseemly as an observed fact.

So how does slowness **S** contribute to Einstein's contradictory Newtonian cosmologic? This seems to be the argument: if $\mathit{\Delta}\phi$ were large enough to prevent *Verödung* and preserve **FU** by sealing it off with a really impenetrable potential barrier, $d\phi$ would also be large, as would forces, accelerations and indeed *speeds*. But since such speeds are not observed, accelerations and forces must be small, as must $d\phi$ and $\mathit{\Delta}\phi$, thus undermining the very impermeability of the potential barrier that would otherwise preserve **FU**. We appear to have

$$\mathbf{S} \Rightarrow \neg\,\mathbf{FU} \Rightarrow \neg\,\mathbf{MP}.$$

Increasing $\bar{\mu}_9$ would increase both of the conflicting tendencies, centripetal and centrifugal; but without knowing exactly how **P** is related to ρ, one can only guess whether such pressure would, by overcoming gravity, produce the hemorrhage needed for the contradiction(s) Einstein wants; so he seems to treat speed as an independent quantity, to be observed[28] and not deduced (from ρ, for instance).

To make sense of the somewhat cryptic sentences

> Denn der endlichen Potentialdifferenz zwischen dem Mittelpunkt und dem räumlich Unendlichen entspricht ein endliches Verhältnis der Dichten. Ein Verschwinden der Dichte im Unendlichen zieht also ein Verschwinden der Dichte im Mittelpunkt an sich.[29]

we can begin with no matter at all, $\rho = 0$, which makes $\underline{\Delta}\phi$, $d\phi$ and $\mathit{\Delta}\phi$ vanish too; ϕ is constant. Now increase ρ (say with spherical symmetry): $\mathit{\Delta}\phi$ increases accordingly, and with it the number and concentration of the gravitational force lines, and indeed the divergence $dE = d{*}E_1$ and gravitational flux **F** through an enclosing membrane. The *Potentialdifferenz* $\mathit{\Delta}\phi$ becomes *endlich* in both senses of the word (neither infinitely small nor infinite), whereas $\rho_\infty/\rho_\mathbf{0}$ can vanish forever for a 'nongaseous' ρ: a single body—or even one body in a stable Kepler orbit around another—would have no reason to budge, and escape towards infinity. But the expansion of a *gaseous* ρ prevents $\rho_\infty/\rho_\mathbf{0}$ from vanishing, thus rendering it *endlich*—not infinitely small. An infinite $\mathit{\Delta}\phi$ would

[27]Einstein (1917a) p. 143: "In Wahrheit werden wir mit Notwendigkeit zu der Auffassung gedrängt, daß das Auftreten bedeutender Potentialdifferenzen des Gravitationsfeldes mit den Tatsachen im Widerspruch ist. Dieselben müssen vielmehr von so geringer Größenordnung sein, daß die durch sie erzeugbaren Sterngeschwindigkeiten die tatsächlich beobachteten nicht übersteigen."

[28]Einstein (1917a) p. 143: "In Wahrheit werden wir mit Notwendigkeit zu der Auffassung gedrängt, daß das Auftreten bedeutender Potentialdifferenzen des Gravitationsfeldes mit den Tatsachen im Widerspruch ist. Dieselben müssen vielmehr von so geringen Größenordnung sein, daß sie durch sie erzeugbaren Sterngeschwindigkeiten die tatsächlich beobachteten nicht übersteigen."

[29]Einstein (1917a) p. 143

be needed to preserve Einstein's **FU** (deduced from **AC**) by checking the expansion of the celestial gas enough to keep $\rho_\infty/\rho_\mathbf{0}$ infinitely small. He may have in mind something like an initially uniform ρ, with unit *Verhältnis der Dichten* $1 = \rho_\infty/\rho_\mathbf{0}$ and an infinite *Potentialdifferenz*

$$\Delta\phi = \infty = \int \rho,$$

which would then make ρ_∞ vanish and $\rho_\infty/\rho_\mathbf{0}$ infinitely small by compressing the celestial gas towards **0**.

Einstein objects that this statistical Newtonian cosmology makes no sense and has to be corrected to restore consistency.[30]

With (1), a constant potential makes ρ vanish. But the second term $-\lambda\phi$ in (2) allows the first term $\Delta\phi$ to vanish without matter doing so too. Einstein considers two very different scales; at the larger one $\mathcal{L}$, matter[31] and the potential can both be constant without vanishing:

$$\phi_\mathcal{L} = \text{const.} = -\frac{\rho_\mathcal{L}}{\lambda}. \tag{9}$$

At the smaller scale $\mathcal{S}$, inhomogeneities in the matter distribution appear, and[32]

$$(2) \to (1) \quad \text{wherever} \quad \frac{\lambda}{\rho_\mathcal{S}} \to 0.$$

The cosmological factor λ brings us to general relativity, and similar issues.

1.2 General relativity

To convert the four-velocity $V^\mu = dx_\mu/ds$ into the energy-momentum

$$P_\mu = m\sqrt{-\mathbf{g}}\, g_{\mu\nu}\frac{dx_\mu}{ds}$$

needed to characterize the "inertia" which by **MP** has to decay far from matter, Einstein lowers the index μ using the metric $g_{\mu\nu}$; where[33] $x_0, \ldots, x_3$ are space-time coordinates and s is a parameter along the world-line (with tangent V) described by a body of mass m. His whole argument turns on the metric,[34] whose determinant $\mathbf{g}$ has to satisfy $\sqrt{-\mathbf{g}} = 1$. Spatial isotropy allows him to write the metric as $\operatorname{diag}(B, -A, -A, -A)$, which yields the condition

$$\sqrt{A^3B} = \sqrt{-\mathbf{g}} = 1.$$

Simplifying assumptions then lead to the rest (and potential) energy $m\sqrt{B}$ and momentum factor $mA/\sqrt{B}$. Since Einstein views m as a constant, unrelated to position,[35] **MP** requires that far from matter $A \to 0$ and[36] $B \to \infty$.

[30] Einstein (1917a) p. 143: "Man kann sich die Frage vorlegen, ob sich dieselben durch eine Modifikation der NEWTONschen Theorie beseitigen lassen."

[31] *Cf.* Einstein (1917a) p. 149: "Wenn wir aber die Welt als räumlich in sich geschlossen annehmen, so liegt die Hypothese nahe, daß ρ unabhängig vom Orte sei [. . .]."

[32] Einstein (1917a) p. 144: "Denkt man sich [. . .] die Materie örtlich ungleichmäßig verteilt, so wird sich über den konstanten ϕ-Wert [$\phi_\mathcal{L}$] der Gleichung (3) [(9) above] ein zusätzliches ϕ [$\phi_\mathcal{S}$] überlagern, welches in der Nähe dichterer Massen einem NEWTONschen Felde um so ähnlicher ist, je kleiner λ_ϕ gegenüber $4\pi K\rho$ ist."

[33] Einstein's time component is the fourth, here it is the first (or rather zeroth).

[34] Einstein (1917a) p. 147: "Die Trägheit [. . .] ist nämlich von den $g_{\mu\nu}$ abhängig [. . .]."

[35] Einstein (1917a) p. 145: "Da m eine dem Massenpunkt unabhängig von seiner Lage eigentümliche Konstante ist [. . .]."

[36] Einstein (1917a) p. 146: "Ein solches Ausarten der Koeffizienten $g_{\mu\nu}$ scheint also durch das Postulat [**MP**] von der Relativität aller Trägheit gefordert zu werden."

What seems to trouble Einstein is that[37] $m\sqrt{B} \to \infty$ would lead to various large quantities: potential differences, forces, accelerations, and indeed speeds—which are not seen. That's the contradiction he ultimately produces: **MP** would lead to speeds never in fact observed.[38] The argument is again markedly statistical inasmuch as statistical mechanics gets used to *realize* possibilities; a *single* body in a circumscribed orbit—a relativistic *Rosettenbahn*—with an equally circumscribed range of speeds would have no reason to leave that range and move faster than observed celestial bodies: the ('purely statistical') possibility of large accelerations and speeds would never be actualised.[39]

Einstein wants to transform away[40] all but the time-time component $T^{00} = \rho$ of the *Energietensor* T; annihilation of the time-space components (momenta) T^{01}, T^{02}, T^{03} presupposes $\mathfrak{k} \ll \mathfrak{K}$ (the cone $\mathfrak{k}$ containing the stellar velocities has to be much smaller than the light cone $\mathfrak{K}$); which in turn requires stellar slowness. Even if coordinate transformations 'displace' the stellar cone $\mathfrak{k}$ within the light cone $\mathfrak{K}$, they have little effect on the size relation $\mathfrak{k} \ll \mathfrak{K}$.

Having concluded that **MP**, through its consequence $m\sqrt{B} \to \infty$, is incompatible with slowness, Einstein explores another two possibilities (before concluding that the $\mathbb{R}^4$ topology is itself to blame, being fundamentally incompatible with **MP**):

(*a*) $g \to \eta$: at spatial infinity the metric approaches its diagonal Minkowski form η

(*b*) *no condition*: 'anything goes.'

The first condition (*a*) resembles[41] what we now call *asymptotic flatness*: what degenerates far from matter is not the metric itself but its variation, its derivatives, in other words curvature. Since Einstein views the metric as the gravitational potential, which therefore corresponds to ϕ, he'll be no happier with asymptotic flatness than he was with the asymptotic constancy **AC** of Newton-Poisson theory. He mentions two problems in particular: *Erstens*, (*a*) would impose a given coordinate system.[42] *Zweitens*, (*a*) wouldn't square with **MP**. The metric far from the origin in a universe containing only a single mass (at the origin) would differ very little from the metric far from the origin in a flat universe; inertia—the metric—would therefore be influenced but not fixed by matter.[43] And a single body in an otherwise empty universe would have inertia (already questionable in itself)—which would differ very little from its value in a universe containing many other bodies.

[37] Einstein (1917a) p. 146: "Diese Forderung [**MP**] bringt es auch mit sich, daß die Potentielle Energie $m\sqrt{B}$ des Punktes im Unendlichen unendlich groß wird."

[38] Einstein (1917a) p. 146: "Bei geeignet gewählten Koordinatensystem sind die Sterngeschwindigkeiten sehr klein gegenüber der Lichtgeschwindigkeit." P. 148: "Das Wichtigste, was wir über die Verteilung der Materie aus der Erfahrung wissen, ist dies, daß die Relativgeschwindigkeiten der Sterne sehr klein sind gegenüber der Lichtgeschwindigkeit."

[39] *Cf.* Einstein (1917a) p. 146: "Die Tatsache der geringen Sterngeschwindigkeiten läßt den Schluß zu, daß nirgends, wo es Fixsterne gibt, das Gravitationspotential (in unserem Falle $\sqrt{B}$) erheblich größer sein kann als bei uns; es folgt dies aus statistischen Überlegungen, genau wie im Falle der NEWTONschen Theorie."

[40] Einstein (1917a) p. 148: "Ich glaube deshalb, daß wir fürs erste folgende approximierende Annahme unserer Betrachtung zugrunde legen dürfen: Es gibt ein Koordinatensystem, relativ zu welchem die Materie als dauernd ruhend angesehen werden darf."

[41] Since Kretschmann (1917) we've known that $g = \eta$ is only a sufficient—by no means a necessary—condition for flatness, which is compatible with a broad class of curved coordinates.

[42] But see footnote 41 above.

[43] Einstein (1917a) p. 147: "Somit würde die Trägheit durch die (im Endlichen vorhandene) Materie zwar b e e i n f l u ß t aber nicht b e d i n g t." *Cf.* p. 148: "Der metrische Charakter (Krümmung) des vierdimensionalen raumzeitlichen Kontinuums wird nach der allgemeinen Relativitätstheorie in jedem Punkte durch die daselbst befindliche Materie und deren Zustand bestimmt."

And Einstein dismisses (*b*) as an ignominious capitulation, a premature and ultimately unnecessary surrender. Unnecessary because a further—topologically revolutionary—possibility remains …

Since the trouble is at spatial infinity, he does away with it,[44] using the cosmological constant λ to produce a cylindrical universe whose only infinities are temporal, not spatial; in the hope that the new cylindrical topology will be more Machian than $\mathbb{R}^4$.[45] The whole operation—amputation—is highly *a priori*, even philosophical; empirically, Einstein only needs slowness.

The perfectly vertical cylinder (as opposed to De Sitter's topologically equivalent expanding hyperboloid, for instance) is obtained by balancing the attraction of matter ρ with the repulsion of λ. The scheme has its drawbacks: the implausible balancing of λ and ρ requires an explanation, and the very verticality of the cylinder singles out (unlike De Sitter's more invariant hyperboloid) a relativistically problematic perpendicular foliation, a privileged class of 'times.'

Returning to **MP**, the condition "Wenn ich daher [eine Masse von allen anderen Massen der Welt räumlich genügend entferne], so muß {ihre Trägheit zu Null herabsinken}" can be simplified as follows. The part [eine Masse … entferne] in square brackets can be abbreviated as [$\mathfrak{A}$], the part {ihre … herabsinken} in curly brackets as $\{\mathfrak{B}\}$. One way of satisfying Einstein's Machian requirement $[\mathfrak{A}] \Rightarrow \{\mathfrak{B}\}$ is to arrange for the satisfaction of $\{\mathfrak{B}\}$; another is to act preemptively, preventing the satisfaction of [$\mathfrak{A}$] in the first place. Einstein appears to adopt this latter course, both in Newton-Poisson gravity and in general relativity. So we seem to have an Einstein-Mach agenda that amounts to this: 'Since we want matter to fix inertia—and spatial infinity far from matter is troublesome—matter never has to be too far away; one therefore has to arrange for matter to be present everywhere to help fix inertia everywhere.'

The very last paragraph (p. 152) spells out Einstein's cosmo-logic pretty concisely: Curvature varies locally, but at large scale space becomes very nearly spherical. The proposed scheme is *logically consistent* and the most natural from the point of view of general relativity—*empirical adequacy is another matter*. Consistency is obtained by an extension (λ) of the gravitational equations *not required by the facts*. Positive spatial curvature can be obtained without the cosmological term; the latter is only needed for the quasi-static (large-scale) distribution ρ of matter given by stellar slowness: $\mathfrak{k} \ll \mathfrak{K}$ allows the reduction (diagonalisation) of T to the mere time-time component ρ exactly balanced by λ, yielding the perfect cylinder without the spatial infinities that make general relativity incompatible with statistical mechanics.

All this is reminiscent of Einstein's strategy in his famous 'photon' paper of 1905: Ordinary 'macroscopic' electromagnetism, with its smooth electromagnetic fields, accounts for most optical phenomena fairly well—as long as one doesn't look too closely (at the details of light's generation or transformation). If one does, the resulting statistics could lead to *Widersprüchen*.[46] All the bewildering peculiarities of complemen-

[44]Einstein (1917a) p. 144: "Es ergibt sich dann schließlich, daß Grenzbedingungen im räumlich Unendlichen überhaupt entfallen, da das Weltkontinuum bezüglich seiner räumlichen Erstreckungen als ein in sich geschlossenes von endlichem, räumlichem (dreidimensionalem) Volumen aufzufassen ist."

[45]*Cf.* Einstein (1917b): "Ich aber sehe die Sache als eine notwendige Ergänzung an, ohne welche weder die Trägheit noch die Geometrie wahrhaft relativ ist. Wer es aber nicht als störend empfindet, wenn die Existenz eines $g_{\mu\nu}$-Feldes ohne felderzeugende Materie aus der Theorie sich als möglich ergibt, und wenn eine einzige (allein in der Welt vorhandenen gedachte) Masse Trägheit besitzen kann, der ist von der Notwendigkeit des neuen Schrittes nicht zu überzeugen."

[46]Einstein (1905) p. 133: "die mit kontinuierlichen Raumfunktionen operierende Theorie des Lichtes zu Widersprüchen mit der Erfahrung führt, wenn man sie auf die Erscheinungen der Lichterzeugung und Lichtverwandlung anwendet."

tarity, wave-particle duality, and uncertainty relations are of course decades away; but Einstein is already trying to work out how to relate individual bodies to their smooth, large-scale fields. Twelve years later, in *Kosmologische Betrachtungen*, he's struggling with a similar tension: between the (statistics of the) individual bodies one might accordingly call 'cosmic quanta' and their smooth, large-scale fields.

Let us now turn to the other topological revolution of 1917, in analytical micromechanics.

2 The toroidal revolution

2.1 One freedom degree

To avoid going back to Democritus—ἀνάγκη στῆναι—we can pick up the story when micromechanics begins to look roughly 'celestial'; Einstein's point of departure[47] is Sommerfeld's version

$$\oint pdq = nh \tag{10}$$

of Bohr's rule for one freedom degree q, where n is an integer, h Planck's constant and pdq can be anachronistically reinterpreted as a one-form $\underline{\alpha} := pdq + 0dp$ on the tangent bundle $\mathbf{T}\Gamma = \mathbf{TT}^*\mathcal{Q}$; the configuration space $\mathcal{Q}$ here being the real line $\mathbb{R}$, with cotangent bundle $\Gamma = \mathbf{T}^*\mathbb{R}$. For the time being one can think of the simplest possible configuration: an electron describing circular orbits (given by values of n) around a nucleus. What Einstein doesn't point out is the *canonical*[48] invariance of the integral[49]

$$\oint_{\partial\zeta} \underline{\alpha} = \oint_{\partial\zeta} \underline{\bar{\alpha}} = \iint_{\zeta} \omega,$$

where

$$\underline{\alpha} \mapsto \underline{\bar{\alpha}} := \underline{\alpha} + dF$$

are two primitives $\in [\underline{\alpha}] = d^{-1}\omega$ of the symplectic two-form

$$\omega = d\underline{\alpha} = d\underline{\bar{\alpha}},$$

$\partial\zeta$ is the boundary[50] of the region ζ, and the generating function F could (for instance) tilt $\mathcal{Q}$ in the plane Γ. The invariance he sees[51] is just the *Koordinateninvarianz* $pdq = PdQ$ of the integrand: the configuration space $\mathcal{Q}$ exceptionally stays put (since dF, which could tilt it in Γ, vanishes) and can only be transformed *internally*, by a global or even local recalibration, which amounts to a diffeomorphism $\mathcal{Q} \to \mathcal{Q}$. In mechanics, the mere coordinate transformation $q \mapsto Q(q)$ is known as a *point* transformation; in Lagrangian mechanics it trivially induces the transformation

$$\dot{q} := \frac{dq}{dt} \mapsto \dot{Q}(Q(q))$$

[47] Einstein (1917c) p. 82: "B i s h e r i g e F o r m u l i e r u n g. Es unterliegt wohl keinem Zweifel mehr, daß für periodische mechanische Systeme von einem Freiheitsgrad die Quantenbedingung [. . .]."

[48] *Cf.* Graffi (2005).

[49] *Cf.* Sommerfeld (1915) p. 427, Sommerfeld (1916) pp. 5-6, Sommerfeld (1951) pp. 85ff.

[50] In §2.2.3 below this will become a loop.

[51] Einstein (1917c) p. 84: "*pdq* bei Systemen von einem Freiheitsgrad eine Invariante, d. h. von der Wahl der Koordinate *q* unabhängig ist [. . .]"

in each tangent space $\mathbf{T}_q\mathcal{Q}$, and hence on the tangent bundle $\mathbf{T}\mathcal{Q}$. In Hamiltonian mechanics the degenerate transformation $q \mapsto Q(q)$, which depends on q alone (and not on p too), can be just as trivially *extended*

$$p = \dot{q}^\flat = \frac{\partial \mathscr{L}}{\partial \dot{q}} \mapsto P(Q(q)) = \dot{Q}^\flat(Q(q)) = \frac{\partial \mathscr{L}}{\partial \dot{Q}}(Q(q)) \in \mathbf{T}^*_q\mathcal{Q}$$

to the cotangent bundle $\mathbf{T}^*\mathcal{Q}$ by the duality $p = \dot{q}^\flat$, where $\mathscr{L}$ is the Lagrangian and everything depends on $Q(q)$. Such *extended point* transformations are of course much less general than the *canonical* transformations

$$(q, p) \mapsto (Q(q, p), P(q, p))$$

of Hamiltonian mechanics, where both new canonical variables depend on both of the old ones; limiting Hamiltonian mechanics to extended point transformations amounts to doing Lagrangian mechanics in Hamiltonian disguise. Einstein's 'Hamiltonian' *Koordinatentransformation* is an extended point transformation and hence only a degenerate canonical transformation.[52]

2.2 More freedom degrees

2.2.1 Sommerfeld's rule

Since most systems have more than a single freedom degree q, the need was soon felt to generalise (10); supplying an index $k = 1, \ldots, N$, Sommerfeld[53] very naturally proposed:

$$J_k = \oint p_k dq_k = n_k h. \tag{11}$$

Each integrand $p_k dq_k$ is *Koordinateninvariant* (whereas each integral J_k, by acquiring the 'canonical' gauge freedom dF_k, would even be *canonically* invariant) on each two-dimensional symplectic manifold Γ_k satisfying

$$\mathbf{T}^*_{z_k}\Gamma_k = \mathrm{span}\{dq_k, dp_k\},$$

where $z_k = (q_k, p_k)$. But the (local) decomposition $\Gamma = \Gamma_1 \times \cdots \times \Gamma_N$ will only be preserved by unentangled canonical transformations $\hat{F}_k : \Gamma_k \to \Gamma_k$ generated by a function of the form $F = F_1 + \cdots + F_N$, where the hat indicates generation of the canonical transformation $\hat{F} : \Gamma \to \Gamma$ by F. Such separation into two-dimensional submanifolds is indicted as the 'coordinate-dependence' of (11); *general* canonical transformations on the $2N$-dimensional phase space Γ would of course entangle the initial two-dimensional manifolds Γ_k.

Here we already see the modern but not entirely anachronistic distinction between local and global. Einstein clearly thinks globally[54] and considers *global* separability (perhaps even integrability). Sommerfeld *&al.* appear to think more locally; their separability concerns no more than a *period* or *libration*.[55]

[52] Einstein (1916b): "Wann Sie sich die Mühe geben wollen, mir auch noch die kanonischen Transformationen darzulegen, werden Sie einen dankbaren und gewissenhaften Zuhörer finden." And the evidence indicates that he still hadn't worked out what they were a few months later, in Einstein (1917c,d,e).

[53] Sommerfeld (1915) p. 429, Sommerfeld (1916) p. 9; see also Sommerfeld (1951) p. 90.

[54] Einstein (1917c) p. 89: "so wird die Bahnkurve im Laufe der Zeit unendlich oft beliebig nahe an ihm vorbeikommen [. . .]." P. 91: "Wenn nun die Bahn nach einer gewissen (sehr großen) Zeit [. . .]." P. 92: "d i e u n e n d l i c h f o r t g e s e t z t e Bewegung."

[55] But see Epstein (1916a) p. 491, Epstein (1916b) pp. 820-2.

2.2.2 Separability

Before we reach Einstein's more invariant rule, a few words about the (non)separability it relies on. In Hamilton-Jacobi theory, the momentum canonically conjugate to the coordinate q_k is given by $p_k = \partial S/\partial q_k$, where S satisfies the Hamilton-Jacobi equation

$$\mathscr{H}\left(q_k, \frac{\partial S}{\partial q_k}, t\right) = -\frac{\partial S}{\partial t}.$$

Einstein attributes[56] the separability condition[57]

$$S = \sum_{k=1}^{N} S_k(q_k) \tag{12}$$

to Epstein;[58] since p_k depends only on its own coordinate q_k, the above partial derivatives become ordinary: $p_k = dS_k/dq_k$. Like its modern counterpart,[59] complete integrability,[60] such separability can be local or global. An unperturbed Kepler motion is globally but *ambiguously* separable—with respect to a 'whole line' (essentially their common principal axis) of coordinate systems: elliptical (finite distance between the *foci* of the coordinate system), parabolic (infinite distance), polar (vanishing distance). The perturbations mentioned in §2.5 below produce a *Rosettenbahn* whose self-intersections undermine the global[61] separability considered by Einstein (rendering it merely local), and reduce or even eliminate the coordinate ambiguities that trouble SSE.

2.2.3 Einstein's rule

After years of general covariance, invariance and tensors, Einstein was shocked by the coordinate-dependence of Sommerfeld's rule (11). It must have troubled him philosophically, æsthetically, methodologically; how can the laws of nature depend on something as contingent and subjective as a choice[62] of coordinates? He accordingly proposed the more invariant condition

$$\oint_{L_k} \left(\sum_{m=1}^{N} p_m dq_m \right) = n_k h, \tag{13}$$

[56]Einstein (1917c) p. 83: "Nach EPSTEIN hat man nun die Koordinaten q_i so zu wählen, daß ein vollständiges Integral von 5a) von der Form

$$J^* = \sum_i J_i(q_i)$$

existiert, wobei J_i von abhängt, von den übrigen q aber unabhängig ist. " Einstein (1917d): "Der Epstein'sche SpezialFall ist einfach der, dass jedes p_ν nur von dem zugehörigen q_ν abhängt."

[57]For modern treatments see Arnol'd (1988) pp. 265ff, Landau & Lifschitz (1990) pp. 184ff, Fasano & Marmi (2006) pp. 421ff.

[58]But see Charlier (1902) pp. 77ff and Levi-Civita (1904).

[59]Gutzwiller (1990) p. 31: "The adjectives 'integrable' and 'separable' designate systems that behave essentially in the same way. The small difference in meaning between these two words will be explained at the end of the chapter." *Cf.* Kozlov (1996) p. 94: "Note that if arbitrary canonical transformations are acceptable, then any completely integrable system is solvable by separation of variables. To see this it is sufficient to pass to action-angle variables. In such a general formulation, the existence problem of separable canonical variables is equivalent to that of a complete set of involutive integrals."

[60]See Arnol'd (1988) pp. 274ff, Fasano & Marmi (2006) pp. 439ff, Spivak (2010) pp. 614ff.

[61]See footnote 54 above.

[62]Einstein (1917c) p. 84: "welche wohl an sich mit dem Quantenproblem nichts zu schaffen hat."

where L_k is a class of homotopically equivalent loops,[63] $k = 1, \ldots, N$ and N is still the number of freedom degrees. The integral is affected neither by loop deformations within each homotopy class L_k, nor by *Koordinatentransformationen*

$$(q_1, \ldots, q_N) \mapsto (Q_1, \ldots, Q_N)$$

automatically balanced by

$$(p_1, \ldots, p_N) = (\dot{q}_1^\flat, \ldots, \dot{q}_N^\flat) \mapsto (P_1, \ldots, P_N) = (\dot{Q}_1^\flat, \ldots, \dot{Q}_N^\flat)$$

in each cotangent space. Einstein views

$$\alpha = \sum_{k=1}^{N} p_k dq_k \tag{14}$$

as a covariant vector[64] field on the tangent bundle $\mathbf{T}\mathcal{Q}$, which can be transformed accordingly;[65] at $\mathbf{q} \in \mathcal{Q}$ we have the covector

$$\alpha_{\mathbf{q}} : \mathbf{T}_{\mathbf{q}}\mathcal{Q} \to \mathbb{R}.$$

If it were anachronistically reinterpreted as a one-form

$$\underline{\alpha} = \sum_{k=1}^{N} p_k dq_k + 0 dp_k$$

with values $\underline{\alpha}_{\mathbf{z}} : \mathbf{T}_{\mathbf{z}}\Gamma \to \mathbb{R}$ on the tangent bundle $\mathbf{T}\Gamma$ of the $2N$-dimensional *phase* space $\Gamma = \mathbf{T}^*\mathcal{Q}$, the integral

$$\oint_{L_k} \underline{\alpha} = \oint_{L_k} (\underline{\alpha} + dF)$$

would also be indifferent to the exact term dF representing the added generality of Hamiltonian (with respect to Lagrangian) mechanics.

But such loops and homotopy classes $\mathbf{L} := \{L_1, \ldots, L_N\}$ have topological significance[66] ...

2.3 Riemannisation

Einstein begins with a plane motion confined to an annulus delimited by concentric circles; we can imagine a Kepler motion perturbed into a self-intersecting hypotrochoid, or *Rosettenbahn*. Einstein classifies motions according to the number of momenta assigned to an *Element* $d\tau$ (but see Appendix below):

[63] Einstein (1917c) p. 85: "Es wird aber eine endliche Anzahl von geschlossenen Linien im q-Raume geben, auf welche sich alle geschlossenen Linien in demselben durch stetige Änderung reduzieren lassen." P. 90: "Alle anderen geschlossenen Kurven lassen sich durch stetige Änderung in der Doppelfläche entweder auf einen Punkt zusammenziehen oder in einen oder mehrere Umläufe der Typen L_1 und L_2 überführen." See also Gutzwiller (1990) p. 209.

[64] Einstein (1917c) p. 84: "Geometrisch gesprochen kann man dann p_i als einen Vektor („kovarianten" Charakters) in dem l-dimensionalen Raume der q_i betrachten."

[65] Einstein (1917c) p. 84: "Invariant is nur die über alle l Freiheitsgrade erstreckte Summe $\sum_i p_i dq_i$." Einstein (1917d): "Nun betrachten wir die Summe $d\sigma = \sum_\nu p_\nu dq_\nu$, gebildet für ein beliebiges Linienelement des q_ν-Raumes. Diese ist invariant gegenüber Koordinatentransformationen [...]."

[66] Einstein (1917d): "Dieses verschwindet, wenn die Kurve stetig in einen Punkt zusammegezogen werden kann, was aber wegen der „Riemannisierung" keineswegs für alle Kurven der Fall ist."

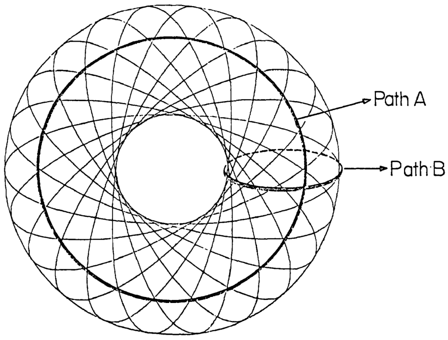

Figure 1: self-intersecting but closed hypotrochoid (case M), with two deformable loops, A and B, taken from the two homotopy classes (Gutzwiller (1990))

1 : The unperturbed Kepler motion assigns the same momentum $\alpha(\mathbf{q})$ to $d\tau \ni \mathbf{q}$ every orbit.

M: A perturbation turns the Kepler motion into a self-intersecting but still closed, periodic hypotrochoid which assigns the same finite sequence $\alpha_1, \ldots, \alpha_M$ of momenta to $d\tau$ every orbit (Figure 1 above).

∞: A perturbation turns the Kepler motion into an ergodic hypotrochoid which eventually fills the torus, assigning to $d\tau$ infinitely many momenta $\alpha_1, \ldots \alpha_\infty$.

where of course $1 < M < \infty$. Again, Einstein needs the second case, M: a self-intersecting, closed (periodic, nonergodic) hypotrochoid. To resolve the troublesome (but finite) self-intersections,[67] Einstein superposes a second annulus on the first, identifying the delimiting circles and stipulating that whenever the motion reaches one of them it goes over to the other annulus.[68] A two-torus results, whose topology is captured by two homotopy classes[69] L_1 and L_2 (one poloidal, the other toroidal—Paths A & B in Figure 1 above), with quantum conditions[70]

$$\oint_{L_1} \alpha = n_1 h \quad \text{and} \quad \oint_{L_2} \alpha = n_2 h$$

for the dynamical one-form α defined on the torus.

Einstein needs a *closed*[71] one-form α to avoid having a *third*[72] quantum condi-

[67]Einstein (1917c) p. 89: "Auf dieser Doppelfläche interpretiert sind die p_ν [...] e i n d e u t i g e Funktionen der q_ν; hierin liegt ihr Wert." Einstein (1917d): "In einem „Riemannisch geblätterten" q_ν-Raum können die p_ν als *eindeutige* und überall stetige Funktionen der q_ν interpretiert werden."

[68]Einstein (1917c) p. 89: "In den beiden Kreislinien denken wir uns die beiden Blätter miteinander verbunden, da die Bahn jeweilen von einem Kreisblatt in das andere übergehen muß, wenn die Bahnkurve einen der beiden Grenzkreise berührt." *Cf.* Epstein (1916b) p. 821.

[69]Einstein (1917c) pp. 89-90: "Auf dieser Doppelfläche gibt es offenbar zwei Typen geschlossener Kurven, welche sich durch stetige Änderung weder auf einen Punkt zusammenziehen, noch aufeinander zurückführen lassen. [...] Der Quantensatz 11) wäre hier auf die beiden Linientypen L_1 und L_2 anzuwenden."

[70]Einstein (1917c) p. 85: "In diesem Sinne kann man dann eine endliche Zahl von Bedingungen [...] als Quantenbedingungen vorschreiben." Einstein (1917d): "Die Quantenregel verlangt nun ganz einfach, dass für *jede* geschlossene Kurve $\int \sum p_\nu dq_\nu = nh$ sein soll."

[71]Einstein (1917c) p. 84: "Ist aber der Vector p_i von einem Potential J^* ableitbar [...] so hat das Integral 9) für alle geschlossenen Kurven, welche stetig ineinander übergeführt werden können, denselben Wert." Einstein (1917d): "und ausserdem *ein vollständiges Differential.*"

[72]P. 90: "Der Quantensatz 11) soll sich auf alle Linien beziehen, die im rationellen Koordinatenraume geschlossene sind."

tion on a two-torus (he wants N homotopy classes, N quantum conditions—and N integrals—for N freedom degrees). His two-torus only has *two* quantum conditions because the integral

$$J_0 = \oint_{L_0} \alpha = n_0 h$$

of the zeroth homotopy class L_0 vanishes—but only because α is closed. If the curl $d\alpha$ didn't vanish, J_0 wouldn't either.

2.4 Two programmes

Einstein (1917c) mentions Schwarzschild as well as Sommerfeld and his Russian student Epstein—to whom we can attribute a kind of 'rival' programme (SSE) that's well worth looking at. Whereas Einstein's theory, with its extended point transformations, is only nominally Hamiltonian, theirs is genuinely so; indeed they use action-angle variables,[73] which are so well adapted to the motion that the actions $\mathbf{J} := \{J_1, \ldots, J_N\}$ are conserved and Hamilton's equations assume the conveniently degenerate 'staightened' form

(15) $$\dot{w}_k = \frac{\partial \mathscr{H}}{\partial J_k} = \text{const.} = \nu_k$$

(16) $$\dot{J}_k = -\frac{\partial \mathscr{H}}{\partial w_k} = 0,$$

where the angles $w_k(t) = \nu_k t + \beta_k$ depend linearly on time, the initial angles β_k are arbitrary, the Hamiltonian $\mathscr{H}(\mathbf{J})$ depends only on the actions, and $k = 1, \ldots, N$. So the Hamiltonian framework tells us that the quantity J_m canonically conjugate to a cyclical variable w_m (satisfying $\partial\mathscr{H}/\partial w_m = 0$) is 'conserved' in the sense expressed by (16). Today we make ample, highly algebraic use of Poisson brackets, and use the involutivity condition

(17) $$\{J_m, \mathscr{H}\} = 0$$

to express conservation. But since Hamilton's equations also tell us that

$$\dot{J}_m = \{J_m, \mathscr{H}\} + \frac{\partial J_m}{\partial t},$$

conditions $\dot{J}_m = 0$ and (17) are in fact equivalent (provided J_m has no explicit dependence on time); to summarize sloppily, *the quantity conjugate to a cyclical variable is conserved*. And the conservation (15) of angular velocity $\dot{w}_m$ follows from the conservation (17) of the momentum $J_m = \partial\mathscr{L}/\partial\dot{w}_m$ dual to it; or without indices, if the 'covariant celerity' (momentum) $J = \dot{w}^\flat$ is conserved, the 'contravariant celerity' (velocity) $\dot{w} = J^\sharp$ dual to it will be too.[74]

2.5 Tori and perturbations

It could be argued that Einstein's 'toroidal' revolution had already been begun by Charlier (1902, p. 94), with the introduction of action-angle variables: not only do N angles

[73] Schwarzschild (1916) pp. 549ff

[74] In simple mechanical cases ($\mathscr{L} = \mathbf{m}(\dot{\mathbf{q}}, \dot{\mathbf{q}})/2 - U(q)$) the mass matrix $\mathbf{m}^\flat$ and its inverse $\mathbf{m}^\sharp$ are used to lower and raise indices, to convert between momentum $\mathbf{p} = \dot{\mathbf{q}}^\flat = \mathbf{m}^\flat(\dot{\mathbf{q}})$ and velocity $\dot{\mathbf{q}} = \mathbf{p}^\sharp = \mathbf{m}^\sharp(\mathbf{p})$. Where $\mathbf{m}$ is constant, it cannot interfere with conservation (but more general Lagrangians can).

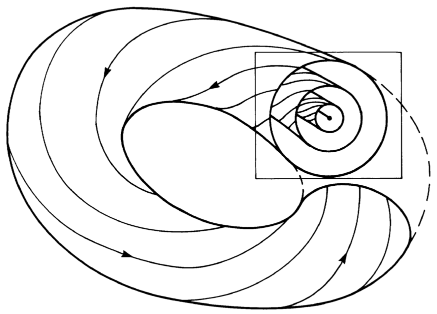

Figure 2: phase space foliated into invariant tori (Gutzwiller (1990))

yield an N-torus $\mathbb{T}_N$, it will even be conserved by the N conditions $\{\mathbf{J}, \mathscr{H}\} = 0$. But on their own, such angles express no more topological awareness than an elastic band (or two). Einstein not only constructs a torus explicitly by *Riemannisierung*, but also—with considerable topological awareness—uses homotopy classes of noncontractible loops[75] to capture its topological peculiarities.

Returning to the SSE scheme, which can now be presented with anachronistic topological awareness, the $2N$-dimensional phase space is foliated into N-dimensional tori $\mathbb{T}_N$ (see Figure 2 above); a choice of actions $\bar{\mathbf{J}}$ fixes a specific torus $\bar{\mathbb{T}}_N$ whose N constant frequencies are given by

$$\bar{\nu}_k = \frac{\partial \mathscr{H}}{\partial J_k} | \bar{J}_k.$$

One can distinguish between a *genuine* and a *merely topological* torus (the shapeless Cartesian product of shapeless loops). Einstein's *Riemannisierung* yields a merely topological torus; but the canonicity of action-angle variables gives shape and 'symplectic rigidity' to the SSE action-angle torus.

Two angles w_m, w_n pick out a two-dimensional sub-torus $\mathbb{T}_2$. If the quotient ν_m/ν_n of the corresponding frequencies is rational one speaks of "compatibility" or "resonance" or even "degeneracy," and the motion eventually closes; otherwise the motion never closes, and fills the torus ergodically.[76] More generally, such rational dependence is an equivalence relation which would partition a larger torus.

Einstein and SSE respond differently to such degeneracies. SSE use perturbations—magnetic (Zeeman effect), electric (Stark effect), relativistic[77]—to eliminate *all* degeneracies as they lead to awkward ambiguities[78] in coordinates and hence quantization. Whereas SSE want maximal ergodicity, Einstein seeks a middle ground of intermediate, partial degeneracy—which for him can be insufficient, or even excessive. An unperturbed Kepler motion—for which Sommerfeld's simpler rule (11) is almost

[75] See footnote 66 above.

[76] Einstein (1917d) may not have understood the relevance of rational dependence: "Es liege ein Problem vor, bei dem soviel Integrale

$$L(q_\nu, p_\nu) = \mathrm{const}$$

existieren, als Freiheitsgrade. Dann können die Impulse p_ν als (mehrwertige) Funktionen der q_ν ausgedrückt werden. Andererseits erfülle die Bahnkurve einen gewissen q_ν-Raum-Teil vollständig, sodass sie jedem Punkt desselben beliebig nahe kommt." Complete integrability only provides the invariant N-torus—which would then be (maximally) filled ergodically under (maximal) irrational dependence of the N frequencies.

[77] See Epstein (1918) p. 241.

[78] See §2.2.2 above.

preferable—is too resonant to justify Einstein's *Riemannisierung* (and the resulting torus), which is pointless without self-intersections. And Einstein views ergodicity as 'infinitely self-intersecting' and hence unamenable to his necessarily finite *Riemannisierung*.

Today we distinguish between gauge fixing and gauge invariance. SSE first try to 'fix the gauge' by adopting special (action-angle) coordinates, and then use perturbations to eliminate any residual ambiguities. Far from trying to fix coordinates as unambiguously as possible, Einstein rewrites the rules to make them *doubly* invariant: with respect to coordinate transformations and loop deformations.

However one chooses to apportion praise and credit, mechanics on tori certainly constitutes a topological revolution—to which Einstein made noteworthy contributions (among others, from Charlier to Kolmogorov and beyond).

2.6 Einstein and analytical mechanics

The peculiarities of Einstein's analytical mechanics deserve attention—however hard it may be to tell his own mechanical idiosyncrasies apart from those of his context. A notable peculiarity is emphasis on N-dimensional configuration (as opposed to $2N$-dimensional phase) space, and on Hamilton-Jacobi theory in configuration space. Einstein is no symplectic geometer, and we have seen (footnote 52 above) that a few months before, in December 1916, he was asking Carathéodory about canonical transformations. He thinks of the Hamiltonian as assigning a *function* $\mathscr{H}_{\mathbf{q}}$ *of momentum* $\mathbf{p}$ to every $\mathbf{q}$ in configuration space $\mathcal{Q}$—rather than a real number $\mathscr{H}(\mathbf{z})$ to every point $\mathbf{z} = (\mathbf{q}, \mathbf{p})$ of phase space Γ. Whatever his understanding of phase space and symplectic geometry, he certainly feels more at home in configuration space—which after all resembles the relativistic space-time he was so familiar with. And Hamilton-Jacobi theory reduces a dynamical congruence in phase space to one in configuration space: a hypersurface (of codimension one) in configuration space extracts[79] a congruence from the tangle produced by bringing the dynamics down from phase space. Once Einstein has a motion on configuration space, he's interested in two topological properties: *closure* (whether the motion eventually closes) and *self-intersection*. He clearly associates[80] such closure with a complete set of integrals, as numerous as the freedom degrees; one can anachronistically speak of "complete integrability." An incomplete set of integrals indicates ergodic motion which never closes,[81] fills space and prevents the application of Einstein's rule (13). If the motion closes without intersection (one can think of an unperturbed Kepler motion), coordinates can be found that satisfy Epstein's condition (12) globally—in which case Sommerfeld's rule (11) applies (and Einstein's

[79]By propagating *orthogonally* from the arbitrary W_0-*Fläche*, Schrödinger (1926, §1) automatically obtains an *untwisted* 'Hamilton-Jacobi' (Schrödinger himself would leave out Jacobi's name) congruence p_k. By not propagating orthogonally from his *Fläche des Koordinatenraumes*, Einstein (1917c,e) obtains a possibly twisted congruence L—whose curl would have to vanish, however, for it to be 'Hamilton-Jacobi' (derived in other words from the Hamilton-Jacobi equation). *Cf.* Sommerfeld (1951) p. 110: "Wäre also S irgendwie als funktion der q_k bekannt, so könnten wir nach [$p_k = \partial S/\partial q_k$] die p_k aus S ableiten. Damit ist uns aber eigentlich nicht gedient. Denn man müßte, um S als Funktion der q_k zu bestimmen, die Bewegungsgleichungen vorher integriert haben; dann wären aber außer den sukzessiven Lagen des systems auch di zugehörigen Impulskoordinaten bekannt und somit die Gl. [$p_k = \partial S/\partial q_k$] überflüssig."

[80]The intuition is reasonably correct: complete integrability does indeed confine motion to a one-dimensional manifold; but that manifold can be either closed (topologically a loop) or infinitely long (topologically $\mathbb{R}$).

[81]Einstein (1917c) p. 92: "Existieren weniger als l Integrale [...], wie dies z. B. nach POINCARÉ bei dem Problem der drei Körper der Fall ist, so sind die p_i nicht durch die q_i ausdrückbar, und es versagt die SOMMERFELD-EPSTEINsche Quantenbedingung auch in der hier gegebenen, ewtas erweiterten Form" (13).

too of course). If the motion intersects itself before closing, global satisfaction of (12) is impossible, and Riemannisation needed to restore *Eindeutigkeit*. So Einstein confines his rule to an intermediate ground caught between the intractable ergodicity of *in*complete integrability—which would prevent the (necessarily finite) Riemannisation on which the rule depends—and the non-intersecting global separability which practically favours Sommerfeld's simpler (11) by making Riemannisation as superfluous as the sum (14) and the homotopy classes **L**.

But we now know that even *complete* integrability isn't enough to rule out ergodicity, the third case (∞) of §2.3 ...

Final remarks

We have seen how Einstein uses the cosmological constant λ to produce a cylindrical topology without the spatial infinity that participates too much in the determination of inertia to satisfy **MP**. The argument is a striking combination of abstract philosophy, statistical mechanics, implacable (but informal) logic and contradictions.

And in analytical micromechanics, the methodological, mathematical, æsthetic requirement of *Koordinateninvarianz* gives rise to the torus whose N homotopy classes correspond to N highly invariant quantum conditions.

If experience and 'abstract philosophical principles' are at opposite ends of a spectrum, here we're at the philosophical end. The difference is perhaps one of generality: an individual experiment is particular, an abstract principle is distilled from many experiences.

Appendix

Einstein (1917c, pp. 87-8) in fact draws two distinctions: first between cases 1. & 2. on p. 87, then between Typus a) & Typus b). We're in an N-dimensional (region of N-dimensional) configuration space—as opposed to $2N$-dimensional phase space.

1&2: One can think of an N-dimensional harmonic but anisotropic oscillator characterized by elasticity coefficients $\kappa_1, \ldots, \kappa_N$ giving rise to Lissajous figures in an N-dimensional cube.[82]

1. All N coefficients are irrationally dependent and the motion never closes, filling (all N dimensions of) the cube ergodically.
2. Degenerate relations (rationality or even equality) between coefficients eliminate dimensions and confine the motion to lower-dimensional cubes.

a&b: One can think of a Kepler motion perturbed into a self-intersecting hypotrochoid, or *Rosettenbahn*.[83]

a) The hypotrochoid is periodic and eventually closes.
b) The hypotrochoid never closes, and eventually fills the torus ergodically.

In §2.3 I have tried to reduce the two dichotomies to a trichotomy. Again, Einstein is manifestly interested in two binary topological distinctions: *closure* and *self-intersection*—which could give rise to at most four cases. But a motion that never closes

[82] See Epstein (1916b) p. 821, Epstein (1918) pp. 240-1, Sommerfeld (1951) pp. 111-3.
[83] See Epstein (1916b) p. 821, Epstein (1918) p. 241.

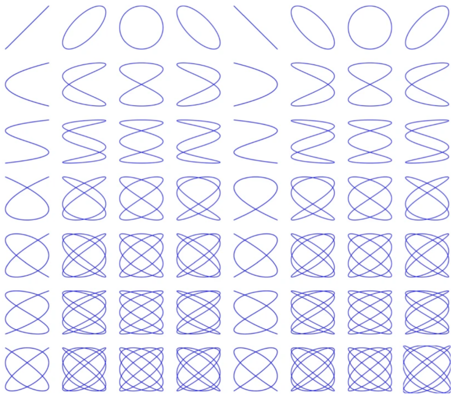

Figure 3: closed Lissajous figures characterised by rationally dependent coefficients

necessarily intersects itself (∞), which leaves only two other cases: a closed motion can intersect itself (M) or not (1).

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