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(*The code below verifies Lemmas 4.2, 4.3,
6.1 of Good elliptic curves with a specified torsion subgroup. We start
 by defining the quantities A_T, B_T, D_T, \hat{D}_T, f_t, g_t, h_T,
and z_T. We use the following naming convention: If T=C_N (resp. C_2 \times C_{2N}),
then we write X_T has XN (resp. X2N) for X =/= \hat{D}. For \hat{D}_T,
we use the name naming convention, but write instead DhN or Dh2N. *)

In[124]:= A1[a_, b_] = (a^3 - 3 * a^2 * b - 6 * a * b^2 - b^3) *
    (a^9 - 225 * a^8 * b - 855 * a^7 * b^2 - 1866 * a^6 * b^3 - 2844 * a^5 * b^4 -
      3123 * a^4 * b^5 - 2265 * a^3 * b^6 - 981 * a^2 * b^7 - 234 * a * b^8 - b^9);
A2[a_, b_] = a^16 - 240 * a^15 * b + 2152 * a^14 * b^2 - 5040 * a^13 * b^3 +
    4572 * a^12 * b^4 + 1680 * a^11 * b^5 - 3112 * a^10 * b^6 + 6480 * a^9 * b^7 -
    6970 * a^8 * b^8 - 6480 * a^7 * b^9 - 3112 * a^6 * b^10 - 1680 * a^5 * b^11 +
    4572 * a^4 * b^12 + 5040 * a^3 * b^13 + 2152 * a^2 * b^14 + 240 * a * b^15 + b^16;
A3[a_, b_] = (a^3 - 3 * a^2 * b - 6 * a * b^2 - b^3) * (a^3 + 3 * a^2 * b - b^3) *
    (a^6 + 12 * a^5 * b + 69 * a^4 * b^2 + 88 * a^3 * b^3 + 24 * a^2 * b^4 - 6 * a * b^5 + b^6);
A4[a_, b_] = (a^4 - 2 * a^3 * b - 2 * a * b^3 + b^4) *
    (a^12 - 6 * a^11 * b + 12 * a^10 * b^2 - 14 * a^9 * b^3 + 243 * a^8 * b^4 -
      468 * a^7 * b^5 + 456 * a^6 * b^6 - 468 * a^5 * b^7 + 243 * a^4 * b^8 -
      14 * a^3 * b^9 + 12 * a^2 * b^10 - 6 * a * b^11 + b^12);
A5[a_, b_] = 1152921504606846976 * b^80 - 422212465065984 * b^60 +
    15032385536 * b^40 + 393216 * b^20 + 1;
A6[a_, b_] = (a^4 + 4 * a^3 * b - 6 * a^2 * b^2 + 4 * a * b^3 + b^4) *
    (a^12 - 12 * a^11 * b + 78 * a^10 * b^2 - 188 * a^9 * b^3 +
      111 * a^8 * b^4 + 264 * a^7 * b^5 - 444 * a^6 * b^6 + 264 * a^5 * b^7 +
      111 * a^4 * b^8 - 188 * a^3 * b^9 + 78 * a^2 * b^10 - 12 * a * b^11 + b^12);
A7[a_, b_] = (a^2 + a * b + b^2) * (a^6 + 11 * a^5 * b + 30 * a^4 * b^2 +
      15 * a^3 * b^3 - 10 * a^2 * b^4 - 5 * a * b^5 + b^6);
A8[a_, b_] = a^16 - 8 * a^14 * b^2 + 12 * a^12 * b^4 + 8 * a^10 * b^6 -
    10 * a^8 * b^8 + 8 * a^6 * b^10 + 12 * a^4 * b^12 - 8 * a^2 * b^14 + b^16;
A9[a_, b_] = (a^3 + 3 * a^2 * b - b^3) * (a^9 + 9 * a^8 * b + 27 * a^7 * b^2 +
      48 * a^6 * b^3 + 54 * a^5 * b^4 + 45 * a^4 * b^5 + 27 * a^3 * b^6 + 9 * a^2 * b^7 - b^9);
A10[a_, b_] = (1 / 16) * (a^12 + 16 * a^11 * b + 104 * a^10 * b^2 + 360 * a^9 * b^3 +
      720 * a^8 * b^4 + 816 * a^7 * b^5 + 416 * a^6 * b^6 - 96 * a^5 * b^7 -
      240 * a^4 * b^8 - 80 * a^3 * b^9 + 64 * a^2 * b^10 + 64 * a * b^11 + 16 * b^12);
A12[a_, b_] = (a^4 - 2 * a^3 * b - 2 * a * b^3 + b^4) * (a^12 - 6 * a^11 * b +
      12 * a^10 * b^2 - 14 * a^9 * b^3 + 3 * a^8 * b^4 + 12 * a^7 * b^5 - 24 * a^6 * b^6 +
      12 * a^5 * b^7 + 3 * a^4 * b^8 - 14 * a^3 * b^9 + 12 * a^2 * b^10 - 6 * a * b^11 + b^12);
A22[a_, b_] = a^16 + 232 * a^14 * b^2 + 732 * a^12 * b^4 - 1192 * a^10 * b^6 +
    710 * a^8 * b^8 - 1192 * a^6 * b^10 + 732 * a^4 * b^12 + 232 * a^2 * b^14 + b^16;
A24[a_, b_] = (a^8 - 4 * a^7 * b + 4 * a^6 * b^2 + 28 * a^5 * b^3 +
      6 * a^4 * b^4 - 28 * a^3 * b^5 + 4 * a^2 * b^6 + 4 * a * b^7 + b^8) *
    (a^8 + 4 * a^7 * b + 4 * a^6 * b^2 - 28 * a^5 * b^3 + 6 * a^4 * b^4 +
      28 * a^3 * b^5 + 4 * a^2 * b^6 - 4 * a * b^7 + b^8);
A26[a_, b_] = (a^4 - 2 * a^3 * b + 6 * a^2 * b^2 - 2 * a * b^3 + b^4) *
    (a^12 - 6 * a^11 * b + 6 * a^10 * b^2 + 10 * a^9 * b^3 +
      15 * a^8 * b^4 - 36 * a^7 * b^5 + 84 * a^6 * b^6 - 36 * a^5 * b^7 +
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      15 * a^4 * b^8 + 10 * a^3 * b^9 + 6 * a^2 * b^10 - 6 * a * b^11 + b^12);
A28[a_, b_] = a^16 - 8 * a^14 * b^2 + 12 * a^12 * b^4 + 8 * a^10 * b^6 +
    230 * a^8 * b^8 + 8 * a^6 * b^10 + 12 * a^4 * b^12 - 8 * a^2 * b^14 + b^16;
B1[a_, b_] = (-1) * (a^18 + 522 * a^17 * b - 8433 * a^16 * b^2 -
      56 382 * a^15 * b^3 - 174 843 * a^14 * b^4 - 433 494 * a^13 * b^5 -
      1 084 008 * a^12 * b^6 - 2 541 474 * a^11 * b^7 - 4 836 168 * a^10 * b^8 -
      7 036 328 * a^9 * b^9 - 7 787 457 * a^8 * b^10 - 6 599 304 * a^7 * b^11 -
      4 265 121 * a^6 * b^12 - 2 050 470 * a^5 * b^13 - 692 973 * a^4 * b^14 -
      148 722 * a^3 * b^15 - 17 154 * a^2 * b^16 - 504 * a * b^17 + b^18);
B2[a_, b_] = (-1) * (a^8 - 24 * a^7 * b + 20 * a^6 * b^2 - 24 * a^5 * b^3 -
      26 * a^4 * b^4 + 24 * a^3 * b^5 + 20 * a^2 * b^6 + 24 * a * b^7 + b^8) *
    (a^16 + 528 * a^15 * b - 3992 * a^14 * b^2 + 11 088 * a^13 * b^3 - 7716 * a^12 * b^4 -
      3696 * a^11 * b^5 + 3032 * a^10 * b^6 - 14 256 * a^9 * b^7 + 17 606 * a^8 * b^8 +
      14 256 * a^7 * b^9 + 3032 * a^6 * b^10 + 3696 * a^5 * b^11 - 7716 * a^4 * b^12 -
      11 088 * a^3 * b^13 - 3992 * a^2 * b^14 - 528 * a * b^15 + b^16);
B3[a_, b_] = (-1) * (a^6 + 12 * a^5 * b + 15 * a^4 * b^2 - 20 * a^3 * b^3 - 30 * a^2 * b^4 -
      6 * a * b^5 + b^6) * (a^12 + 6 * a^11 * b + 48 * a^10 * b^2 + 428 * a^9 * b^3 +
      1899 * a^8 * b^4 + 3636 * a^7 * b^5 + 3030 * a^6 * b^6 + 720 * a^5 * b^7 -
      288 * a^4 * b^8 - 58 * a^3 * b^9 + 48 * a^2 * b^10 + 6 * a * b^11 + b^12);
B4[a_, b_] = (-1) * (a^8 - 4 * a^7 * b + 28 * a^6 * b^2 - 52 * a^5 * b^3 +
      46 * a^4 * b^4 - 52 * a^3 * b^5 + 28 * a^2 * b^6 - 4 * a * b^7 + b^8) *
    (a^16 - 8 * a^15 * b + 104 * a^13 * b^3 - 220 * a^12 * b^4 + 216 * a^11 * b^5 -
      728 * a^10 * b^6 + 1144 * a^9 * b^7 - 1026 * a^8 * b^8 + 1144 * a^7 * b^9 - 728 * a^6 *
        b^10 + 216 * a^5 * b^11 - 220 * a^4 * b^12 + 104 * a^3 * b^13 - 8 * a * b^15 + b^16);
B5[a_, b_] = (-1) * (8 * b^4 - 4 * b^2 + 1) * (8 * b^4 + 4 * b^2 + 1) *
    (4096 * b^16 - 2048 * b^14 + 512 * b^12 - 64 * b^8 + 8 * b^4 - 4 * b^2 + 1) *
    (4096 * b^16 + 2048 * b^14 + 512 * b^12 - 64 * b^8 + 8 * b^4 + 4 * b^2 + 1) *
    (1 152 921 504 606 846 976 * b^80 - 633 318 697 598 976 * b^60 +
      79 456 894 976 * b^40 + 589 824 * b^20 + 1);
B6[a_, b_] = (-1) * (a^8 - 4 * a^7 * b + 4 * a^6 * b^2 + 20 * a^5 * b^3 -
      26 * a^4 * b^4 + 20 * a^3 * b^5 + 4 * a^2 * b^6 - 4 * a * b^7 + b^8) *
    (a^16 - 8 * a^15 * b + 24 * a^14 * b^2 - 568 * a^13 * b^3 + 2684 * a^12 * b^4 -
      4776 * a^11 * b^5 + 2344 * a^10 * b^6 + 4840 * a^9 * b^7 -
      8826 * a^8 * b^8 + 4840 * a^7 * b^9 + 2344 * a^6 * b^10 - 4776 * a^5 * b^11 +
      2684 * a^4 * b^12 - 568 * a^3 * b^13 + 24 * a^2 * b^14 - 8 * a * b^15 + b^16);
B7[a_, b_] = (-1) * (a^12 + 18 * a^11 * b + 117 * a^10 * b^2 + 354 * a^9 * b^3 +
      570 * a^8 * b^4 + 486 * a^7 * b^5 + 273 * a^6 * b^6 + 222 * a^5 * b^7 +
      174 * a^4 * b^8 + 46 * a^3 * b^9 - 15 * a^2 * b^10 - 6 * a * b^11 + b^12);
B8[a_, b_] = (-1) * (a^8 - 4 * a^6 * b^2 - 2 * a^4 * b^4 - 4 * a^2 * b^6 + b^8) *
    (a^16 - 8 * a^14 * b^2 + 12 * a^12 * b^4 + 8 * a^10 * b^6 -
      34 * a^8 * b^8 + 8 * a^6 * b^10 + 12 * a^4 * b^12 - 8 * a^2 * b^14 + b^16);
B9[a_, b_] = (-1) * (a^18 + 18 * a^17 * b + 135 * a^16 * b^2 + 570 * a^15 * b^3 +
      1557 * a^14 * b^4 + 2970 * a^13 * b^5 + 4128 * a^12 * b^6 + 4230 * a^11 * b^7 +
      3240 * a^10 * b^8 + 2032 * a^9 * b^9 + 1359 * a^8 * b^10 + 1080 * a^7 * b^11 + 735 * a^6 *
        b^12 + 306 * a^5 * b^13 + 27 * a^4 * b^14 - 42 * a^3 * b^15 - 18 * a^2 * b^16 + b^18);
B10[a_, b_] = (-1 / 64) * (a^2 + 2 * a * b + 2 * b^2) *
    (a^4 + 6 * a^3 * b + 6 * a^2 * b^2 - 4 * a * b^3 - 4 * b^4) *
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    (a^4 + 6 * a^3 * b + 12 * a^2 * b^2 + 8 * a * b^3 + 2 * b^4) * (a^8 + 10 * a^7 * b +
      32 * a^6 * b^2 + 40 * a^5 * b^3 + 14 * a^4 * b^4 + 8 * a^2 * b^6 - 4 * b^8);
B12[a_, b_] = (-1) * (a^8 - 4 * a^7 * b + 4 * a^6 * b^2 - 4 * a^5 * b^3 -
      2 * a^4 * b^4 - 4 * a^3 * b^5 + 4 * a^2 * b^6 - 4 * a * b^7 + b^8) *
    (a^16 - 8 * a^15 * b + 24 * a^14 * b^2 - 40 * a^13 * b^3 + 44 * a^12 * b^4 -
      24 * a^11 * b^5 - 32 * a^10 * b^6 + 88 * a^9 * b^7 - 114 * a^8 * b^8 +
      88 * a^7 * b^9 - 32 * a^6 * b^10 - 24 * a^5 * b^11 + 44 * a^4 * b^12 -
      40 * a^3 * b^13 + 24 * a^2 * b^14 - 8 * a * b^15 + b^16);
B22[a_, b_] = (-1) * (a^8 + 20 * a^6 * b^2 - 26 * a^4 * b^4 + 20 * a^2 * b^6 + b^8) *
    (a^8 - 24 * a^7 * b + 20 * a^6 * b^2 - 24 * a^5 * b^3 - 26 * a^4 * b^4 + 24 * a^3 * b^5 +
      20 * a^2 * b^6 + 24 * a * b^7 + b^8) * (a^8 + 24 * a^7 * b + 20 * a^6 * b^2 +
      24 * a^5 * b^3 - 26 * a^4 * b^4 - 24 * a^3 * b^5 + 20 * a^2 * b^6 - 24 * a * b^7 + b^8);
B24[a_, b_] = (-1) * (a^4 - 4 * a^3 * b - 6 * a^2 * b^2 + 4 * a * b^3 + b^4) *
    (a^4 + 4 * a^3 * b - 6 * a^2 * b^2 - 4 * a * b^3 + b^4) *
    (a^8 - 4 * a^6 * b^2 + 22 * a^4 * b^4 - 4 * a^2 * b^6 + b^8) *
    (a^8 + 20 * a^6 * b^2 - 26 * a^4 * b^4 + 20 * a^2 * b^6 + b^8);
B26[a_, b_] = (-1) * (a^8 - 4 * a^7 * b + 4 * a^6 * b^2 - 28 * a^5 * b^3 +
      22 * a^4 * b^4 - 28 * a^3 * b^5 + 4 * a^2 * b^6 - 4 * a * b^7 + b^8) *
    (a^8 - 4 * a^7 * b + 4 * a^6 * b^2 - 4 * a^5 * b^3 - 2 * a^4 * b^4 - 4 * a^3 * b^5 +
      4 * a^2 * b^6 - 4 * a * b^7 + b^8) * (a^8 - 4 * a^7 * b + 4 * a^6 * b^2 +
      20 * a^5 * b^3 - 26 * a^4 * b^4 + 20 * a^3 * b^5 + 4 * a^2 * b^6 - 4 * a * b^7 + b^8);
B28[a_, b_] = (-1) * (a^8 - 4 * a^6 * b^2 - 26 * a^4 * b^4 - 4 * a^2 * b^6 + b^8) *
    (a^8 - 4 * a^6 * b^2 - 2 * a^4 * b^4 - 4 * a^2 * b^6 + b^8) *
    (a^8 - 4 * a^6 * b^2 + 22 * a^4 * b^4 - 4 * a^2 * b^6 + b^8);
D1[a_, b_] = (-1) * b * a * (a + b) * (a^2 + a * b + b^2)^3 *
    (a^3 + 6 * a^2 * b + 3 * a * b^2 - b^3)^9;
D2[a_, b_] = (-1) * b * a * (a - b) * (a + b) * (a^2 + b^2)^2 *
    (a^2 - 2 * a * b - b^2)^4 * (a^2 + 2 * a * b - b^2)^16;
D3[a_, b_] = (-1) * b^3 * a^3 * (a + b)^3 * (a^2 + a * b + b^2)^9 *
    (a^3 + 6 * a^2 * b + 3 * a * b^2 - b^3)^3;
D4[a_, b_] = (a - b)^2 * b^4 * a^4 * (a + b)^6 * (a^2 + b^2) *
    (a^2 - 4 * a * b + b^2)^3 * (a^2 - a * b + b^2)^12;
D5[a_, b_] = (2^75) * b^100 * (64 * b^8 - 8 * b^4 - 1) *
    (4096 * b^16 - 1024 * b^12 + 256 * b^8 - 24 * b^4 + 1) *
    (4096 * b^16 + 1536 * b^12 + 256 * b^8 + 16 * b^4 + 1);
D6[a_, b_] = (a + b)^2 * b^3 * a^3 * (a - b)^6 * (a^2 - a * b + b^2) *
    (a^2 - 4 * a * b + b^2)^4 * (a^2 + b^2)^12;
D7[a_, b_] = (-1) * b^7 * a^7 * (a + b)^7 * (a^3 + 8 * a^2 * b + 5 * a * b^2 - b^3);
D8[a_, b_] = (a - b)^4 * (a + b)^4 * b^16 * a^16 *
    (a^2 - 2 * a * b - b^2) * (a^2 + 2 * a * b - b^2) * (a^2 + b^2)^2;
D9[a_, b_] = (-1) * b^9 * a^9 * (a + b)^9 * (a^2 + a * b + b^2)^3 *
    (a^3 + 6 * a^2 * b + 3 * a * b^2 - b^3);
D10[a_, b_] = (1 / 4096) * a^5 * (a + 2 * b)^5 * b^10 * (a + b)^10 *
    (a^2 + 6 * a * b + 4 * b^2) * (a^2 + a * b - b^2)^2;
D12[a_, b_] = (a + b)^2 * (a - b)^6 * b^12 * a^12 *
    (a^2 - 4 * a * b + b^2) * (a^2 + b^2)^3 * (a^2 - a * b + b^2)^4;
D22[a_, b_] = b^2 * a^2 * (a - b)^2 * (a + b)^2 * (a^2 + b^2)^4 *
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    (a^2 - 2*a*b - b^2)^8 * (a^2 + 2*a*b - b^2)^8;
D24[a_, b_] = b^4 * a^4 * (a - b)^4 * (a + b)^4 * (a^2 - 2*a*b - b^2)^4 *
    (a^2 + 2*a*b - b^2)^4 * (a^2 + b^2)^8;
D26[a_, b_] = (a + b)^4 * b^6 * a^6 * (a - b)^12 *
    (a^2 - 4*a*b + b^2)^2 * (a^2 - a*b + b^2)^2 * (a^2 + b^2)^6;
D28[a_, b_] = b^8 * a^8 * (a - b)^8 * (a + b)^8 * (a^2 - 2*a*b - b^2)^2 *
    (a^2 + 2*a*b - b^2)^2 * (a^2 + b^2)^4;
Dh1[a_, b_] = (-1) * (a + b) * (a^2 + a*b + b^2) * (a^3 + 6*a^2*b + 3*a*b^2 - b^3);
Dh2[a_, b_] =
   (-1) * (a - b) * (a + b) * (a^2 + b^2) * (a^2 - 2*a*b - b^2) * (a^2 + 2*a*b - b^2);
Dh3[a_, b_] = (-1) * (a + b) * (a^2 + a*b + b^2) * (a^3 + 6*a^2*b + 3*a*b^2 - b^3);
Dh4[a_, b_] =
   (-1) * (a + b) * (a - b) * (a^2 + b^2) * (a^2 - 4*a*b + b^2) * (a^2 - a*b + b^2);
Dh5[a_, b_] = (2/5) * (64*b^8 - 8*b^4 - 1) *
    (4096*b^16 - 1024*b^12 + 256*b^8 - 24*b^4 + 1) *
    (4096*b^16 + 1536*b^12 + 256*b^8 + 16*b^4 + 1);
Dh6[a_, b_] = (-1) * (a + b) * (a - b) * (a^2 + b^2) *
    (a^2 - 4*a*b + b^2) * (a^2 - a*b + b^2);
Dh7[a_, b_] = (-1) * (a + b) * (a^3 + 8*a^2*b + 5*a*b^2 - b^3);
Dh8[a_, b_] =
   (-1) * (a - b) * (a + b) * (a^2 + b^2) * (a^2 - 2*a*b - b^2) * (a^2 + 2*a*b - b^2);
Dh9[a_, b_] = (-1) * (a + b) * (a^2 + a*b + b^2) * (a^3 + 6*a^2*b + 3*a*b^2 - b^3);
Dh10[a_, b_] =
   (-1/8) * (a + b) * (a + 2*b) * (a^2 + 6*a*b + 4*b^2) * (a^2 + a*b - b^2);
Dh12[a_, b_] = (-1) * (a + b) * (a - b) * (a^2 + b^2) *
    (a^2 - 4*a*b + b^2) * (a^2 - a*b + b^2);
Dh22[a_, b_] = (-1) * (a - b) * (a + b) * (a^2 + b^2) *
    (a^2 - 2*a*b - b^2) * (a^2 + 2*a*b - b^2);
Dh24[a_, b_] = (-1) * (a - b) * (a + b) * (a^2 + b^2) *
    (a^2 - 2*a*b - b^2) * (a^2 + 2*a*b - b^2);
Dh26[a_, b_] = (-1) * (a + b) * (a - b) * (a^2 + b^2) *
    (a^2 - 4*a*b + b^2) * (a^2 - a*b + b^2);
Dh28[a_, b_] = (-1) * (a - b) * (a + b) * (a^2 + b^2) *
    (a^2 - 2*a*b - b^2) * (a^2 + 2*a*b - b^2);
f3[x_] = B3[1, x]^2 / (1728*D3[1, x]) - x;
f4[x_] = B4[1, x]^2 / (1728*D4[1, x]) - x;
f6[x_] = B6[1, x]^2 / (1728*D6[1, x]) - x;
f7[x_] = B7[1, x]^2 / (1728*D7[1, x]) - x;
f8[x_] = B8[1, x]^2 / (1728*D8[1, x]) - x;
f9[x_] = B9[1, x]^2 / (1728*D9[1, x]) - x;
f10[x_] = B10[1, x]^2 / (1728*D10[1, x]) - x;
f12[x_] = B12[1, x]^2 / (1728*D12[1, x]) - x;
f22[x_] = B22[1, x]^2 / (1728*D22[1, x]) - x;
f24[x_] = B24[1, x]^2 / (1728*D24[1, x]) - x;
f26[x_] = B26[1, x]^2 / (1728*D26[1, x]) - x;
f28[x_] = B28[1, x]^2 / (1728*D28[1, x]) - x;
g3[x_] = A3[1, x]^2 + B3[1, x] * Dh3[1, x];
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g4[x_] = A4[1, x]^2 + B4[1, x] * Dh4[1, x];
g6[x_] = A6[1, x]^2 + B6[1, x] * Dh6[1, x];
g7[x_] = A7[1, x]^2 + B7[1, x] * Dh7[1, x];
g8[x_] = A8[1, x]^2 + B8[1, x] * Dh8[1, x];
g9[x_] = A9[1, x]^2 + B9[1, x] * Dh9[1, x];
g10[x_] = A10[1, x]^2 + B10[1, x] * Dh10[1, x];
g12[x_] = A12[1, x]^2 + B12[1, x] * Dh12[1, x];
g22[x_] = A22[1, x]^2 + B22[1, x] * Dh22[1, x];
g24[x_] = A24[1, x]^2 + B24[1, x] * Dh24[1, x];
g26[x_] = A26[1, x]^2 + B26[1, x] * Dh26[1, x];
g28[x_] = A28[1, x]^2 + B28[1, x] * Dh28[1, x];
h1[x_] = A1[1, x]^3 - Dh1[1, x]^6;
h2[x_] = A2[1, x]^3 - Dh2[1, x]^6;
h3[x_] = A3[1, x]^3 - Dh3[1, x]^6;
h4[x_] = A4[1, x]^3 - Dh4[1, x]^6;
h5[x_] = A5[1, x]^3 - Dh5[1, x]^6;
h6[x_] = A6[1, x]^3 - Dh6[1, x]^6;
h7[x_] = A7[1, x]^3 - Dh7[1, x]^6;
h8[x_] = A8[1, x]^3 - Dh8[1, x]^6;
h9[x_] = A9[1, x]^3 - Dh9[1, x]^6;
h10[x_] = A10[1, x]^3 - Dh10[1, x]^6;
h12[x_] = A12[1, x]^3 - Dh12[1, x]^6;
h22[x_] = A22[1, x]^3 - Dh22[1, x]^6;
h24[x_] = A24[1, x]^3 - Dh24[1, x]^6;
h26[x_] = A26[1, x]^3 - Dh26[1, x]^6;
h28[x_] = A28[1, x]^3 - Dh28[1, x]^6;
z1[x_] = Max[Abs[A1[1, x]^3], B1[1, x]^2] - Dh1[1, x]^6;
z2[x_] = Max[Abs[A2[1, x]^3], B2[1, x]^2] - Dh2[1, x]^6;
z3[x_] = Max[Abs[A3[1, x]^3], B3[1, x]^2] - Dh3[1, x]^6;
z4[x_] = Max[Abs[A4[1, x]^3], B4[1, x]^2] - Dh4[1, x]^6;
z6[x_] = Max[Abs[A6[1, x]^3], B6[1, x]^2] - Dh6[1, x]^6;
z7[x_] = Max[Abs[A7[1, x]^3], B7[1, x]^2] - Dh7[1, x]^6;
z8[x_] = Max[Abs[A8[1, x]^3], B8[1, x]^2] - Dh8[1, x]^6;
z9[x_] = Max[Abs[A9[1, x]^3], B9[1, x]^2] - Dh9[1, x]^6;
z10[x_] = Max[Abs[A10[1, x]^3], B10[1, x]^2] - Dh10[1, x]^6;
z12[x_] = Max[Abs[A12[1, x]^3], B12[1, x]^2] - Dh12[1, x]^6;
z22[x_] = Max[Abs[A22[1, x]^3], B22[1, x]^2] - Dh22[1, x]^6;
z24[x_] = Max[Abs[A24[1, x]^3], B24[1, x]^2] - Dh24[1, x]^6;
z26[x_] = Max[Abs[A26[1, x]^3], B26[1, x]^2] - Dh26[1, x]^6;
z28[x_] = Max[Abs[A28[1, x]^3], B28[1, x]^2] - Dh28[1, x]^6;
```

```
In[114]:= (* We now consider each T separately. *)
```

In[354]:=
```
(*Case T=C_1. We first verify Lemma 4.2. The code below shows
   that (1) the greatest real root of D_T(1,x) is \theta_T≈4.41147,
(2) the greatest real root of A_T(1,x), \hat{D}_T, h_T(x) is at most \theta_T,
and (3) the continuous functions A_T(1,x), D_T(1,x), \hat{D}_T,
h_T(x) are positive on the interval (\theta_T,infinity) *)
theta1 = Max[x /. NSolve[D1[1, x] == 0, x, Reals]]; theta1
Max[Max[x /. NSolve[A1[1, x] == 0, x, Reals]], Max[x /. NSolve[Dh1[1, x] == 0, x, Reals]],
  Max[x /. NSolve[h1[x] == 0, x, Reals]]] ≤ theta1
{A1[1, 5000] > 0, D1[1, 5000] > 0, Dh1[1, 5000] > 0, h1[5000] > 0}
```

Out[354]= 4.41147

Out[355]= True

Out[356]= {True, True, True, True}

```
(*Case T=C_1. Lastly, we verify Lemma 6.1. The code
  below shows that (1) the greatest real root of  z_T(x) is \xi_T=0,
(2) the continuous functions z_T(x) is positive on the interval (\xi_T,infinity) *)
xi1 = Max[x /. NSolve[z1[x] == 0, x, Reals]]; xi1
z1[5000] > 0
```

Out[365]= 0

Out[366]= True

In[370]:=
```
(*Case T=C_2. We first verify Lemma 4.2. The code below shows
   that (1) the greatest real root of D_T(1,x) is \theta_T≈2.41421,
(2) the greatest real root of A_T(1,x), \hat{D}_T, h_T(x) is at most \theta_T,
and (3) the continuous functions A_T(1,x), D_T(1,x), \hat{D}_T,
h_T(x) are positive on the interval (\theta_T,infinity) *)
theta2 = Max[x /. NSolve[D2[1, x] == 0, x, Reals]]; theta2
Max[Max[x /. NSolve[A2[1, x] == 0, x, Reals]], Max[x /. NSolve[Dh2[1, x] == 0, x, Reals]],
  Max[x /. NSolve[h2[x] == 0, x, Reals]]] ≤ theta2
{A2[1, 5000] > 0, D2[1, 5000] > 0, Dh2[1, 5000] > 0, h2[5000] > 0}
```

Out[370]= 2.41421

Out[371]= True

Out[372]= {True, True, True, True}

```
(*Case T=C_2. Lastly, we verify Lemma 6.1. The code
  below shows that (1) the greatest real root of  z_T(x) is \xi_T=0,
(2) the continuous functions z_T(x) is positive on the interval (\xi_T,infinity) *)
xi2 = Max[x /. NSolve[z1[x] == 0, x, Reals]]; xi2
z2[5000] > 0
```

Out[373]= 0

Out[374]= True

In[357]:=
```
(*Case T=C_3. We first verify Lemma 4.2. The code below shows
   that (1) the greatest real root of D_T(1,x) is \theta_T≈4.41147,
(2) the greatest real root of A_T(1,x), \hat{D}_T, h_T(x) is at most \theta_T,
and (3) the continuous functions A_T(1,x), D_T(1,x), \hat{D}_T,
h_T(x) are positive on the interval (\theta_T,infinity) *)
theta3 = Max[x /. NSolve[D3[1, x] == 0, x, Reals]]; theta3
Max[Max[x /. NSolve[A3[1, x] == 0, x, Reals]], Max[x /. NSolve[Dh3[1, x] == 0, x, Reals]],
  Max[x /. NSolve[h3[x] == 0, x, Reals]]] ≤ theta3
{A3[1, 5000] > 0, D3[1, 5000] > 0, Dh3[1, 5000] > 0, h3[5000] > 0}
```

Out[357]= 4.41147

Out[358]= True

Out[359]= {True, True, True, True}

In[360]:=
```
(*Case T=C_3. Next, we verify Lemma 4.3. The code below shows
  that (1) the greatest real root of  f_T(x) is \delta_T≈43.4033,
(2) the greatest real root of g_T(x), \-B_T(1,x) is at most \delta_T,
and (3) the continuous functions f_T(x), g_T(x),
-B_T(1,x) are positive on the interval (\delta_T,infinity) *)
delta3 = Max[x /. NSolve[f3[x] == 0, x, Reals]]; delta3
Max[Max[x /. NSolve[g3[x] == 0, x, Reals]],
  Max[x /. NSolve[-B3[1, x] == 0, x, Reals]]] ≤ delta3
{f3[5000] > 0, g3[5000] > 0, -B3[1, 5000] > 0}
```

Out[360]= 43.4033

Out[361]= True

Out[362]= {True, True, True}

In[363]:=
```
(*Case T=C_3. Lastly, we verify Lemma 6.1. The code below
  shows that (1) the greatest real root of  z_T(x) is \xi_T≈0.1686,
(2) the continuous functions z_T(x) is positive on the interval (\xi_T,infinity) *)
xi3 = Max[x /. NSolve[z3[x] == 0, x, Reals]]; xi3
z3[5000] > 0
```

Out[363]= 0.168612

Out[364]= True

In[378]:=
```
(*Case T=C_4. We first verify Lemma 4.2. The code below shows
   that (1) the greatest real root of D_T(1,x) is \theta_T≈3.73205,
(2) the greatest real root of A_T(1,x), \hat{D}_T, h_T(x) is at most \theta_T,
and (3) the continuous functions A_T(1,x), D_T(1,x), \hat{D}_T,
h_T(x) are positive on the interval (\theta_T,infinity) *)
theta4 = Max[x /. NSolve[D4[1, x] == 0, x, Reals]]; theta4
Max[Max[x /. NSolve[A4[1, x] == 0, x, Reals]],
  Max[x /. NSolve[Dh4[1, x] == 0, x, Reals]], Max[x /. NSolve[h4[x] == 0, x, Reals]]] ≤ theta4
{A4[1, 5000] > 0, D4[1, 5000] > 0, Dh4[1, 5000] > 0, h4[5000] > 0}
```

Out[378]= 3.73205

Out[379]= True

Out[380]= {True, True, True, True}

In[384]:=
```
(*Case T=C_4. Next, we verify Lemma 4.3. The code below shows
  that (1) the greatest real root of  f_T(x) is \delta_T≈13.5934,
(2) the greatest real root of g_T(x), \-B_T(1,x) is at most \delta_T,
and (3) the continuous functions f_T(x), g_T(x),
-B_T(1,x) are positive on the interval (\delta_T,infinity) *)
delta4 = Max[x /. NSolve[f4[x] == 0, x, Reals]]; delta4
Max[Max[x /. NSolve[g4[x] == 0, x, Reals]],
  Max[x /. NSolve[-B4[1, x] == 0, x, Reals]]] ≤ delta4
{f4[5000] > 0, g4[5000] > 0, -B4[1, 5000] > 0}
```

Out[384]= 13.5934

Out[385]= True

Out[386]= {True, True, True}

```
(*Case T=C_4. Lastly, we verify Lemma 6.1. The code
  below shows that (1) the greatest real root of  z_T(x) is \xi_T=0,
(2) the continuous functions z_T(x) is positive on the interval (\xi_T,infinity) *)
xi4 = Max[x /. NSolve[z4[x] == 0, x, Reals]]; xi4
z4[5000] > 0
```

Out[387]= 0

Out[388]= True

In[413]:=
```
(*Case T=C_5. We verify Lemma 4.2. The code below shows
   that (1) the greatest real root of D_T(1,x) is \theta_T≈0.0.67062,
(2) the greatest real root of A_T(1,x), \hat{D}_T, h_T(x) is at most \theta_T,
and (3) the continuous functions A_T(1,x), D_T(1,x), \hat{D}_T,
h_T(x) are positive on the interval (\theta_T,infinity) *)
theta5 = Max[x /. NSolve[D5[1, x] == 0, x, Reals]]; theta5
Max[Max[x /. NSolve[A5[1, x] == 0, x, Reals]],
  Max[x /. NSolve[Dh5[1, x] == 0, x, Reals]], Max[x /. NSolve[h5[x] == 0, x, Reals]]] ≤ theta5
{A5[1, 5000] > 0, D5[1, 5000] > 0, Dh5[1, 5000] > 0, h5[5000] > 0}
```

Out[413]= 0.670617

Out[414]= True

Out[415]= {True, True, True, True}

In[416]:=
```
(*Case T=C_6. We first verify Lemma 4.2. The code below shows
   that (1) the greatest real root of D_T(1,x) is \theta_T≈3.73205,
(2) the greatest real root of A_T(1,x), \hat{D}_T, h_T(x) is at most \theta_T,
and (3) the continuous functions A_T(1,x), D_T(1,x), \hat{D}_T,
h_T(x) are positive on the interval (\theta_T,infinity) *)
theta6 = Max[x /. NSolve[D6[1, x] == 0, x, Reals]]; theta6
Max[Max[x /. NSolve[A6[1, x] == 0, x, Reals]],
  Max[x /. NSolve[Dh6[1, x] == 0, x, Reals]], Max[x /. NSolve[h6[x] == 0, x, Reals]]] ≤ theta6
{A6[1, 5000] > 0, D6[1, 5000] > 0, Dh6[1, 5000] > 0, h6[5000] > 0}
```

Out[416]= 3.73205

Out[417]= True

Out[418]= {True, True, True, True}

In[419]:=
```
(*Case T=C_6. Next, we verify Lemma 4.3. The code below shows
  that (1) the greatest real root of  f_T(x) is \delta_T≈43.3677,
(2) the greatest real root of g_T(x), \-B_T(1,x) is at most \delta_T,
and (3) the continuous functions f_T(x), g_T(x),
-B_T(1,x) are positive on the interval (\delta_T,infinity) *)
delta6 = Max[x /. NSolve[f6[x] == 0, x, Reals]]; delta6
Max[Max[x /. NSolve[g6[x] == 0, x, Reals]],
  Max[x /. NSolve[-B6[1, x] == 0, x, Reals]]] ≤ delta6
{f6[5000] > 0, g6[5000] > 0, -B6[1, 5000] > 0}
```

Out[419]= 43.3677

Out[420]= True

Out[421]= {True, True, True}

In[422]:=
```
(*Case T=C_6. Lastly, we verify Lemma 6.1. The code
  below shows that (1) the greatest real root of  z_T(x) is \xi_T=0,
(2) the continuous functions z_T(x) is positive on the interval (\xi_T,infinity) *)
xi6 = Max[x /. NSolve[z6[x] == 0, x, Reals]]; xi6
z6[5000] > 0
```

Out[422]= 0

Out[423]= True

In[424]:=
```
(*Case T=C_7. We first verify Lemma 4.2. The code below
   shows that (1) the greatest real root of D_T(1,x) is \theta_T≈6.2959,
(2) the greatest real root of A_T(1,x), \hat{D}_T, h_T(x) is at most \theta_T,
and (3) the continuous functions A_T(1,x), D_T(1,x), \hat{D}_T,
h_T(x) are positive on the interval (\theta_T,infinity) *)
theta7 = Max[x /. NSolve[D7[1, x] == 0, x, Reals]]; theta7
Max[Max[x /. NSolve[A7[1, x] == 0, x, Reals]],
  Max[x /. NSolve[Dh7[1, x] == 0, x, Reals]], Max[x /. NSolve[h7[x] == 0, x, Reals]]] ≤ theta7
{A7[1, 5000] > 0, D7[1, 5000] > 0, Dh7[1, 5000] > 0, h7[5000] > 0}
```

Out[424]= 6.2959

Out[425]= True

Out[426]= {True, True, True, True}

In[427]:=
```
(*Case T=C_7. Next, we verify Lemma 4.3. The code below shows
  that (1) the greatest real root of  f_T(x) is \delta_T≈7.07956,
(2) the greatest real root of g_T(x), \-B_T(1,x) is at most \delta_T,
and (3) the continuous functions f_T(x), g_T(x),
-B_T(1,x) are positive on the interval (\delta_T,infinity) *)
delta7 = Max[x /. NSolve[f7[x] == 0, x, Reals]]; delta7
Max[Max[x /. NSolve[g7[x] == 0, x, Reals]],
  Max[x /. NSolve[-B7[1, x] == 0, x, Reals]]] ≤ delta7
{f7[5000] > 0, g7[5000] > 0, -B7[1, 5000] > 0}
```

Out[427]= 7.07956

Out[428]= True

Out[429]= {True, True, True}

```
(*Case T=C_7. Lastly, we verify Lemma 6.1. The code below
  shows that (1) the greatest real root of  z_T(x) is \xi_T≈4.3444,
(2) the continuous functions z_T(x) is positive on the interval (\xi_T,infinity) *)
xi7 = Max[x /. NSolve[z7[x] == 0, x, Reals]]; xi7
z7[5000] > 0
```

Out[430]= 4.34442

Out[431]= True

In[432]:=
```
(*Case T=C_8. We first verify Lemma 4.2. The code below shows
   that (1) the greatest real root of D_T(1,x) is \theta_T≈2.41421,
(2) the greatest real root of A_T(1,x), \hat{D}_T, h_T(x) is at most \theta_T,
and (3) the continuous functions A_T(1,x), D_T(1,x), \hat{D}_T,
h_T(x) are positive on the interval (\theta_T,infinity) *)
theta8 = Max[x /. NSolve[D8[1, x] == 0, x, Reals]]; theta8
Max[Max[x /. NSolve[A8[1, x] == 0, x, Reals]],
  Max[x /. NSolve[Dh8[1, x] == 0, x, Reals]], Max[x /. NSolve[h8[x] == 0, x, Reals]]] ≤ theta8
{A8[1, 5000] > 0, D8[1, 5000] > 0, Dh8[1, 5000] > 0, h8[5000] > 0}
```

Out[432]= 2.41421

Out[433]= True

Out[434]= {True, True, True, True}

In[435]:=
```
(*Case T=C_8. Next, we verify Lemma 4.3. The code below shows
  that (1) the greatest real root of  f_T(x) is \delta_T≈2.48383,
(2) the greatest real root of g_T(x), \-B_T(1,x) is at most \delta_T,
and (3) the continuous functions f_T(x), g_T(x),
-B_T(1,x) are positive on the interval (\delta_T,infinity) *)
delta8 = Max[x /. NSolve[f8[x] == 0, x, Reals]]; delta8
Max[Max[x /. NSolve[g8[x] == 0, x, Reals]],
  Max[x /. NSolve[-B8[1, x] == 0, x, Reals]]] ≤ delta8
{f8[5000] > 0, g8[5000] > 0, -B8[1, 5000] > 0}
```

Out[435]= 2.48383

Out[436]= True

Out[437]= {True, True, True}

In[440]:=
```
(*Case T=C_8. Lastly, we verify Lemma 6.1. The code below
  shows that (1) the greatest real root of  z_T(x) is \xi_T≈2.0198,
(2) the continuous functions z_T(x) is positive on the interval (\xi_T,infinity) *)
xi8 = Max[x /. NSolve[z8[x] == 0, x, Reals]]; xi8
z8[5000] > 0
```

Out[440]= 2.01982

Out[441]= True

In[442]:=
```
(*Case T=C_9. We first verify Lemma 4.2. The code below shows
   that (1) the greatest real root of D_T(1,x) is \theta_T≈4.41147,
(2) the greatest real root of A_T(1,x), \hat{D}_T, h_T(x) is at most \theta_T,
and (3) the continuous functions A_T(1,x), D_T(1,x), \hat{D}_T,
h_T(x) are positive on the interval (\theta_T,infinity) *)
theta9 = Max[x /. NSolve[D9[1, x] == 0, x, Reals]]; theta9
Max[Max[x /. NSolve[A9[1, x] == 0, x, Reals]],
  Max[x /. NSolve[Dh9[1, x] == 0, x, Reals]], Max[x /. NSolve[h9[x] == 0, x, Reals]]] ≤ theta9
{A9[1, 5000] > 0, D9[1, 5000] > 0, Dh9[1, 5000] > 0, h9[5000] > 0}
```

Out[442]= 4.41147

Out[443]= True

Out[444]= {True, True, True, True}

In[445]:=
```
(*Case T=C_9. Next, we verify Lemma 4.3. The code below shows
  that (1) the greatest real root of  f_T(x) is \delta_T≈4.75552,
(2) the greatest real root of g_T(x), \-B_T(1,x) is at most \delta_T,
and (3) the continuous functions f_T(x), g_T(x),
-B_T(1,x) are positive on the interval (\delta_T,infinity) *)
delta9 = Max[x /. NSolve[f9[x] == 0, x, Reals]]; delta9
Max[Max[x /. NSolve[g9[x] == 0, x, Reals]],
  Max[x /. NSolve[-B9[1, x] == 0, x, Reals]]] ≤ delta9
{f9[5000] > 0, g9[5000] > 0, -B9[1, 5000] > 0}
```

Out[445]= 4.75552

Out[446]= True

Out[447]= {True, True, True}

In[448]:=
```
(*Case T=C_9. Lastly, we verify Lemma 6.1. The code below
  shows that (1) the greatest real root of  z_T(x) is \xi_T≈3.2938,
(2) the continuous functions z_T(x) is positive on the interval (\xi_T,infinity) *)
xi9 = Max[x /. NSolve[z9[x] == 0, x, Reals]]; xi9
z9[5000] > 0
```

Out[448]= 3.29383

Out[449]= True

In[450]:=
```
(*Case T=C_10. We first verify Lemma 4.2. The code below shows
   that (1) the greatest real root of D_T(1,x) is \theta_T≈1.61803,
(2) the greatest real root of A_T(1,x), \hat{D}_T, h_T(x) is at most \theta_T,
and (3) the continuous functions A_T(1,x), D_T(1,x), \hat{D}_T,
h_T(x) are positive on the interval (\theta_T,infinity) *)
theta10 = Max[x /. NSolve[D10[1, x] == 0, x, Reals]]; theta10
Max[Max[x /. NSolve[A10[1, x] == 0, x, Reals]], Max[x /. NSolve[Dh10[1, x] == 0, x, Reals]],
  Max[x /. NSolve[h10[x] == 0, x, Reals]]] ≤ theta10
{A10[1, 5000] > 0, D10[1, 5000] > 0, Dh10[1, 5000] > 0, h10[5000] > 0}
```

Out[450]= 1.61803

Out[451]= True

Out[452]= {True, True, True, True}

In[453]:=
```
(*Case T=C_10. Next, we verify Lemma 4.3. The code below
  shows that (1) the greatest real root of  f_T(x) is \delta_T≈3.06311,
(2) the greatest real root of g_T(x), \-B_T(1,x) is at most \delta_T,
and (3) the continuous functions f_T(x), g_T(x),
-B_T(1,x) are positive on the interval (\delta_T,infinity) *)
delta10 = Max[x /. NSolve[f10[x] == 0, x, Reals]]; delta10
Max[Max[x /. NSolve[g10[x] == 0, x, Reals]],
  Max[x /. NSolve[-B10[1, x] == 0, x, Reals]]] ≤ delta10
{f10[5000] > 0, g10[5000] > 0, -B10[1, 5000] > 0}
```

Out[453]= 3.06311

Out[454]= True

Out[455]= {True, True, True}

In[462]:=
```
(*Case T=C_10. Lastly, we verify Lemma 6.1. The code below shows
  that z_T is a continuous function with no real roots. Since z_T(5000)>0,
we conclude that z_T is positive throughout its domain *)
NSolve[z10[x] == 0, x, Reals]
z10[5000] > 0
```

Out[462]= { }

Out[463]= True

In[464]:=
```
(*Case T=C_12. We first verify Lemma 4.2. The code below shows
   that (1) the greatest real root of D_T(1,x) is \theta_T≈3.73205,
(2) the greatest real root of A_T(1,x), \hat{D}_T, h_T(x) is at most \theta_T,
and (3) the continuous functions A_T(1,x), D_T(1,x), \hat{D}_T,
h_T(x) are positive on the interval (\theta_T,infinity) *)
theta12 = Max[x /. NSolve[D12[1, x] == 0, x, Reals]]; theta12
Max[Max[x /. NSolve[A12[1, x] == 0, x, Reals]], Max[x /. NSolve[Dh12[1, x] == 0, x, Reals]],
  Max[x /. NSolve[h12[x] == 0, x, Reals]]] ≤ theta12
{A12[1, 5000] > 0, D12[1, 5000] > 0, Dh12[1, 5000] > 0, h12[5000] > 0}
```

Out[464]= 3.73205

Out[465]= True

Out[466]= {True, True, True, True}

In[470]:=
```
(*Case T=C_12. Next, we verify Lemma 4.3. The code below
  shows that (1) the greatest real root of  f_T(x) is \delta_T≈3.89418,
(2) the greatest real root of g_T(x), \-B_T(1,x) is at most \delta_T,
and (3) the continuous functions f_T(x), g_T(x),
-B_T(1,x) are positive on the interval (\delta_T,infinity) *)
delta12 = Max[x /. NSolve[f12[x] == 0, x, Reals]]; delta12
Max[Max[x /. NSolve[g12[x] == 0, x, Reals]],
  Max[x /. NSolve[-B12[1, x] == 0, x, Reals]]] ≤ delta12
{f12[5000] > 0, g12[5000] > 0, -B12[1, 5000] > 0}
```

Out[470]= 3.89418

Out[471]= True

Out[472]= {True, True, True}

In[473]:=
```
(*Case T=C_12. Lastly, we verify Lemma 6.1. The code below
  shows that (1) the greatest real root of  z_T(x) is \xi_T≈2.9354,
(2) the continuous functions z_T(x) is positive on the interval (\xi_T,infinity) *)
xi12 = Max[x /. NSolve[z12[x] == 0, x, Reals]]; xi12
z12[5000] > 0
```

Out[473]= 2.93543

Out[474]= True

In[513]:=
```
(*Case T=
 C_2 \times C_2. We first verify Lemma 4.2. The code below shows that (1) the greatest
     real root of D_T(1,x) is \theta_T≈2.41421,
(2) A_T(1,x) has no real roots and the greatest real root of \hat{D}_T,
h_T(x) is at most \theta_T,
and (3) the continuous functions A_T(1,x),
D_T(1,x), \hat{D}_T,
h_T(x) are positive on the interval (\theta_T,infinity) *)
theta22 = Max[x /. NSolve[D22[1, x] == 0, x, Reals]]; theta22
NSolve[A22[1, x] == 0, x, Reals]
Max[Max[x /. NSolve[Dh22[1, x] == 0, x, Reals]],
  Max[x /. NSolve[h22[x] == 0, x, Reals]]] ≤ theta22
{A22[1, 5000] > 0, D22[1, 5000] > 0, Dh22[1, 5000] > 0, h22[5000] > 0}
```

Out[513]= 2.41421

Out[514]= { }

Out[515]= True

Out[516]= {True, True, True, True}

In[509]:=
```
(*Case T=C_2 \times C_2. Next, we verify Lemma 4.3. The code below
  shows that (1) the greatest real root of  f_T(x) is \delta_T≈1728.57,
(2) g_T(x) has no real roots and the greatest real root of \-B_T(1,x)
 is at most \delta_T, and (3) the continuous functions f_T(x),
g_T(x), -B_T(1,x) are positive on the interval (\delta_T,infinity) *)
delta22 = Max[x /. NSolve[f22[x] == 0, x, Reals]]; delta22
NSolve[g22[x] == 0, x, Reals]
Max[x /. NSolve[-B22[1, x] == 0, x, Reals]] ≤ delta22
{f22[5000] > 0, g22[5000] > 0, -B22[1, 5000] > 0}
```

Out[509]= 1728.57

Out[510]= { }

Out[511]= True

Out[512]= {True, True, True}

In[507]:=
```
(*Case T=C_2 \times C_2. Lastly, we verify Lemma 6.1. The code
  below shows that (1) the greatest real root of  z_T(x) is \xi_T=0,
(2) the continuous functions z_T(x) is positive on the interval (\xi_T,infinity) *)
xi22 = Max[x /. NSolve[z22[x] == 0, x, Reals]]; xi22
z22[5000] > 0
```

Out[507]= 0

Out[508]= True

In[517]:=
```
(*Case T=C_2 \times C_4. We first verify Lemma 4.2. The code below
   shows that (1) the greatest real root of D_T(1,x) is \theta_T≈2.41421,
(2) A_T(1,x) has no real roots and the greatest real root of \hat{D}_T,
h_T(x) is at most \theta_T, and (3) the continuous functions A_T(1,x),
D_T(1,x), \hat{D}_T, h_T(x) are positive on the interval (\theta_T,infinity) *)
theta24 = Max[x /. NSolve[D24[1, x] == 0, x, Reals]]; theta24
NSolve[A24[1, x] == 0, x, Reals]
Max[Max[x /. NSolve[Dh24[1, x] == 0, x, Reals]],
  Max[x /. NSolve[h24[x] == 0, x, Reals]]] ≤ theta24
{A24[1, 5000] > 0, D24[1, 5000] > 0, Dh24[1, 5000] > 0, h24[5000] > 0}
```

Out[517]= 2.41421

Out[518]= { }

Out[519]= True

Out[520]= {True, True, True, True}

In[527]:=
```
(*Case T=C_2 \times C_4. Next, we verify Lemma 4.3. The code below
  shows that (1) the greatest real root of  f_T(x) is \delta_T≈12.2907,
(2) g_T(x) has no real roots and the greatest real root of \-B_T(1,x)
 is at most \delta_T, and (3) the continuous functions f_T(x),
g_T(x), -B_T(1,x) are positive on the interval (\delta_T,infinity) *)
delta24 = Max[x /. NSolve[f24[x] == 0, x, Reals]]; delta24
NSolve[g24[x] == 0, x, Reals]
Max[x /. NSolve[-B24[1, x] == 0, x, Reals]] ≤ delta24
{f24[5000] > 0, g24[5000] > 0, -B24[1, 5000] > 0}
```

Out[527]= 12.2907

Out[528]= { }

Out[529]= True

Out[530]= {True, True, True}

In[531]:=
```
(*Case T=C_2 \times C_4. Lastly, we verify Lemma 6.1. The code
  below shows that (1) the greatest real root of  z_T(x) is \xi_T=0,
(2) the continuous functions z_T(x) is positive on the interval (\xi_T,infinity) *)
xi24 = Max[x /. NSolve[z24[x] == 0, x, Reals]]; xi24
z24[5000] > 0
```

Out[531]= 0

Out[532]= True

```
(*Case T=C_2 \times C_6. We first verify Lemma 4.2. The code below
   shows that (1) the greatest real root of D_T(1,x) is \theta_T≈3.73205,
(2) A_T(1,x) has no real roots and the greatest real root of \hat{D}_T,
h_T(x) is at most \theta_T, and (3) the continuous functions A_T(1,x),
D_T(1,x), \hat{D}_T, h_T(x) are positive on the interval (\theta_T,infinity) *)
theta26 = Max[x /. NSolve[D26[1, x] == 0, x, Reals]]; theta26
NSolve[A26[1, x] == 0, x, Reals]
Max[Max[x /. NSolve[Dh26[1, x] == 0, x, Reals]],
  Max[x /. NSolve[h26[x] == 0, x, Reals]]] ≤ theta26
{A26[1, 5000] > 0, D26[1, 5000] > 0, Dh26[1, 5000] > 0, h26[5000] > 0}
```

Out[535]= 3.73205

Out[536]= { }

Out[537]= True

Out[538]= {True, True, True, True}

```
(*Case T=C_2 \times C_6. Next, we verify Lemma 4.3. The code below
  shows that (1) the greatest real root of  f_T(x) is \delta_T≈3.89418,
(2) the greatest real root of g_T(x), \-B_T(1,x) is at most \delta_T,
and (3) the continuous functions f_T(x), g_T(x),
-B_T(1,x) are positive on the interval (\delta_T,infinity) *)
delta26 = Max[x /. NSolve[f26[x] == 0, x, Reals]]; delta26
Max[Max[x /. NSolve[g26[x] == 0, x, Reals]],
  Max[x /. NSolve[-B26[1, x] == 0, x, Reals]]] ≤ delta26
{f26[5000] > 0, g26[5000] > 0, -B26[1, 5000] > 0}
```

Out[543]= 6.00485

Out[544]= True

Out[545]= {True, True, True}

In[533]:=
```
(*Case T=C_2 \times C_6. Lastly, we verify Lemma 6.1. The code
  below shows that (1) the greatest real root of  z_T(x) is \xi_T=0,
(2) the continuous functions z_T(x) is positive on the interval (\xi_T,infinity) *)
xi26 = Max[x /. NSolve[z26[x] == 0, x, Reals]]; xi26
z26[5000] > 0
```

Out[533]= 0

Out[534]= True

In[546]:=
```
(*Case T=C_2 \times C_8. We first verify Lemma 4.2. The code below
   shows that (1) the greatest real root of D_T(1,x) is \theta_T≈2.41421,
(2) A_T(1,x) has no real roots and the greatest real root of \hat{D}_T,
h_T(x) is at most \theta_T, and (3) the continuous functions A_T(1,x),
D_T(1,x), \hat{D}_T, h_T(x) are positive on the interval (\theta_T,infinity) *)
theta28 = Max[x /. NSolve[D28[1, x] == 0, x, Reals]]; theta28
NSolve[A28[1, x] == 0, x, Reals]
Max[Max[x /. NSolve[Dh28[1, x] == 0, x, Reals]],
  Max[x /. NSolve[h28[x] == 0, x, Reals]]] ≤ theta28
{A28[1, 5000] > 0, D28[1, 5000] > 0, Dh28[1, 5000] > 0, h28[5000] > 0}
```

Out[546]= 2.41421

Out[547]= { }

Out[548]= True

Out[549]= {True, True, True, True}

```
(*Case T=C_2 \times C_8. Next, we verify Lemma 4.3. The code below
  shows that (1) the greatest real root of  f_T(x) is \delta_T≈3.38169,
(2) g_T(x) has no real roots and the greatest real root of \-B_T(1,x)
 is at most \delta_T, and (3) the continuous functions f_T(x),
g_T(x), -B_T(1,x) are positive on the interval (\delta_T,infinity) *)
delta28 = Max[x /. NSolve[f28[x] == 0, x, Reals]]; delta28
NSolve[g28[x] == 0, x, Reals]
Max[x /. NSolve[-B28[1, x] == 0, x, Reals]] ≤ delta28
{f28[5000] > 0, g28[5000] > 0, -B28[1, 5000] > 0}
```

Out[565]= 3.38169

Out[566]= { }

Out[567]= True

Out[568]= {True, True, True}

In[569]:=
```
(*Case T=C_2 \times C_8. Lastly, we verify Lemma 6.1. The code
  below shows that (1) the greatest real root of  z_T(x) is \xi_T=0,
(2) the continuous functions z_T(x) is positive on the interval (\xi_T,infinity) *)
xi28 = Max[x /. NSolve[z28[x] == 0, x, Reals]]; xi28
z28[5000] > 0
```

Out[569]= 0

Out[570]= True