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In[183]:= (*The code below verifies Lemmas 4.2, 4.3,
6.1 and Corollary 4.10 of Good elliptic curves with a specified torsion subgroup. We
start by defining the quantities A_T, B_T, D_T, \hat{D}_T, f_t, g_t, h_T,
and z_T. We use the following naming convention: If T=C_N (resp. C_2 \times C_{2N}),
then we write X_T has XN (resp. X2N) for X \neq \hat{D}. For \hat{D}_T,
we use the name naming convention, but write instead DhN or Dh2N. *)

In[184]:= A1[a_, b_] = (a^3 - 3*a^2*b - 6*a*b^2 - b^3) *
(a^9 - 225*a^8*b - 855*a^7*b^2 - 1866*a^6*b^3 - 2844*a^5*b^4 -
3123*a^4*b^5 - 2265*a^3*b^6 - 981*a^2*b^7 - 234*a*b^8 - b^9);
A2[a_, b_] = a^16 - 240*a^15*b + 2152*a^14*b^2 - 5040*a^13*b^3 +
4572*a^12*b^4 + 1680*a^11*b^5 - 3112*a^10*b^6 + 6480*a^9*b^7 -
6970*a^8*b^8 - 6480*a^7*b^9 - 3112*a^6*b^10 - 1680*a^5*b^11 +
4572*a^4*b^12 + 5040*a^3*b^13 + 2152*a^2*b^14 + 240*a*b^15 + b^16;
A3[a_, b_] = (a^3 - 3*a^2*b - 6*a*b^2 - b^3) * (a^3 + 3*a^2*b - b^3) *
(a^6 + 12*a^5*b + 69*a^4*b^2 + 88*a^3*b^3 + 24*a^2*b^4 - 6*a*b^5 + b^6);
A4[a_, b_] = (a^4 - 2*a^3*b - 2*a*b^3 + b^4) *
(a^12 - 6*a^11*b + 12*a^10*b^2 - 14*a^9*b^3 + 243*a^8*b^4 -
468*a^7*b^5 + 456*a^6*b^6 - 468*a^5*b^7 + 243*a^4*b^8 -
14*a^3*b^9 + 12*a^2*b^10 - 6*a*b^11 + b^12);
A5[a_, b_] = 1152921504606846976*b^80 - 422212465065984*b^60 +
15032385536*b^40 + 393216*b^20 + 1;
A6[a_, b_] = (a^4 + 4*a^3*b - 6*a^2*b^2 + 4*a*b^3 + b^4) *
(a^12 - 12*a^11*b + 78*a^10*b^2 - 188*a^9*b^3 +
111*a^8*b^4 + 264*a^7*b^5 - 444*a^6*b^6 + 264*a^5*b^7 +
111*a^4*b^8 - 188*a^3*b^9 + 78*a^2*b^10 - 12*a*b^11 + b^12);
A7[a_, b_] = (a^2 + a*b + b^2) * (a^6 + 11*a^5*b + 30*a^4*b^2 +
15*a^3*b^3 - 10*a^2*b^4 - 5*a*b^5 + b^6);
A8[a_, b_] = a^16 - 8*a^14*b^2 + 12*a^12*b^4 + 8*a^10*b^6 -
10*a^8*b^8 + 8*a^6*b^10 + 12*a^4*b^12 - 8*a^2*b^14 + b^16;
A9[a_, b_] = (a^3 + 3*a^2*b - b^3) * (a^9 + 9*a^8*b + 27*a^7*b^2 +
48*a^6*b^3 + 54*a^5*b^4 + 45*a^4*b^5 + 27*a^3*b^6 + 9*a^2*b^7 - b^9);
A10[a_, b_] = (1/16) * (a^12 + 16*a^11*b + 104*a^10*b^2 + 360*a^9*b^3 +
720*a^8*b^4 + 816*a^7*b^5 + 416*a^6*b^6 - 96*a^5*b^7 -
240*a^4*b^8 - 80*a^3*b^9 + 64*a^2*b^10 + 64*a*b^11 + 16*b^12);
A12[a_, b_] = (a^4 - 2*a^3*b - 2*a*b^3 + b^4) * (a^12 - 6*a^11*b +
12*a^10*b^2 - 14*a^9*b^3 + 3*a^8*b^4 + 12*a^7*b^5 - 24*a^6*b^6 +
12*a^5*b^7 + 3*a^4*b^8 - 14*a^3*b^9 + 12*a^2*b^10 - 6*a*b^11 + b^12);
A22[a_, b_] = a^16 + 232*a^14*b^2 + 732*a^12*b^4 - 1192*a^10*b^6 +
710*a^8*b^8 - 1192*a^6*b^10 + 732*a^4*b^12 + 232*a^2*b^14 + b^16;
A24[a_, b_] = (a^8 - 4*a^7*b + 4*a^6*b^2 + 28*a^5*b^3 +
6*a^4*b^4 - 28*a^3*b^5 + 4*a^2*b^6 + 4*a*b^7 + b^8) *
(a^8 + 4*a^7*b + 4*a^6*b^2 - 28*a^5*b^3 + 6*a^4*b^4 +
28*a^3*b^5 + 4*a^2*b^6 - 4*a*b^7 + b^8);
A26[a_, b_] = (a^4 - 2*a^3*b + 6*a^2*b^2 - 2*a*b^3 + b^4) *
(a^12 - 6*a^11*b + 6*a^10*b^2 + 10*a^9*b^3 +
15*a^8*b^4 - 36*a^7*b^5 + 84*a^6*b^6 - 36*a^5*b^7 +

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15 * a^4 * b^8 + 10 * a^3 * b^9 + 6 * a^2 * b^10 - 6 * a * b^11 + b^12);
A28[a_, b_] = a^16 - 8 * a^14 * b^2 + 12 * a^12 * b^4 + 8 * a^10 * b^6 +
230 * a^8 * b^8 + 8 * a^6 * b^10 + 12 * a^4 * b^12 - 8 * a^2 * b^14 + b^16;
B1[a_, b_] = (-1) * (a^18 + 522 * a^17 * b - 8433 * a^16 * b^2 -
56382 * a^15 * b^3 - 174843 * a^14 * b^4 - 433494 * a^13 * b^5 -
1084008 * a^12 * b^6 - 2541474 * a^11 * b^7 - 4836168 * a^10 * b^8 -
7036328 * a^9 * b^9 - 7787457 * a^8 * b^10 - 6599304 * a^7 * b^11 -
4265121 * a^6 * b^12 - 2050470 * a^5 * b^13 - 692973 * a^4 * b^14 -
148722 * a^3 * b^15 - 17154 * a^2 * b^16 - 504 * a * b^17 + b^18);
B2[a_, b_] = (-1) * (a^8 - 24 * a^7 * b + 20 * a^6 * b^2 - 24 * a^5 * b^3 -
26 * a^4 * b^4 + 24 * a^3 * b^5 + 20 * a^2 * b^6 + 24 * a * b^7 + b^8) *
(a^16 + 528 * a^15 * b - 3992 * a^14 * b^2 + 11088 * a^13 * b^3 - 7716 * a^12 * b^4 -
3696 * a^11 * b^5 + 3032 * a^10 * b^6 - 14256 * a^9 * b^7 + 17606 * a^8 * b^8 +
14256 * a^7 * b^9 + 3032 * a^6 * b^10 + 3696 * a^5 * b^11 - 7716 * a^4 * b^12 -
11088 * a^3 * b^13 - 3992 * a^2 * b^14 - 528 * a * b^15 + b^16);
B3[a_, b_] = (-1) * (a^6 + 12 * a^5 * b + 15 * a^4 * b^2 - 20 * a^3 * b^3 - 30 * a^2 * b^4 -
6 * a * b^5 + b^6) * (a^12 + 6 * a^11 * b + 48 * a^10 * b^2 + 428 * a^9 * b^3 +
1899 * a^8 * b^4 + 3636 * a^7 * b^5 + 3030 * a^6 * b^6 + 720 * a^5 * b^7 -
288 * a^4 * b^8 - 58 * a^3 * b^9 + 48 * a^2 * b^10 + 6 * a * b^11 + b^12);
B4[a_, b_] = (-1) * (a^8 - 4 * a^7 * b + 28 * a^6 * b^2 - 52 * a^5 * b^3 +
46 * a^4 * b^4 - 52 * a^3 * b^5 + 28 * a^2 * b^6 - 4 * a * b^7 + b^8) *
(a^16 - 8 * a^15 * b + 104 * a^13 * b^3 - 220 * a^12 * b^4 + 216 * a^11 * b^5 -
728 * a^10 * b^6 + 1144 * a^9 * b^7 - 1026 * a^8 * b^8 + 1144 * a^7 * b^9 - 728 * a^6 *
b^10 + 216 * a^5 * b^11 - 220 * a^4 * b^12 + 104 * a^3 * b^13 - 8 * a * b^15 + b^16);
B5[a_, b_] = (-1) * (8 * b^4 - 4 * b^2 + 1) * (8 * b^4 + 4 * b^2 + 1) *
(4096 * b^16 - 2048 * b^14 + 512 * b^12 - 64 * b^8 + 8 * b^4 - 4 * b^2 + 1) *
(4096 * b^16 + 2048 * b^14 + 512 * b^12 - 64 * b^8 + 8 * b^4 + 4 * b^2 + 1) *
(1152921504606846976 * b^80 - 633318697598976 * b^60 +
79456894976 * b^40 + 589824 * b^20 + 1);
B6[a_, b_] = (-1) * (a^8 - 4 * a^7 * b + 4 * a^6 * b^2 + 20 * a^5 * b^3 -
26 * a^4 * b^4 + 20 * a^3 * b^5 + 4 * a^2 * b^6 - 4 * a * b^7 + b^8) *
(a^16 - 8 * a^15 * b + 24 * a^14 * b^2 - 568 * a^13 * b^3 + 2684 * a^12 * b^4 -
4776 * a^11 * b^5 + 2344 * a^10 * b^6 + 4840 * a^9 * b^7 -
8826 * a^8 * b^8 + 4840 * a^7 * b^9 + 2344 * a^6 * b^10 - 4776 * a^5 * b^11 +
2684 * a^4 * b^12 - 568 * a^3 * b^13 + 24 * a^2 * b^14 - 8 * a * b^15 + b^16);
B7[a_, b_] = (-1) * (a^12 + 18 * a^11 * b + 117 * a^10 * b^2 + 354 * a^9 * b^3 +
570 * a^8 * b^4 + 486 * a^7 * b^5 + 273 * a^6 * b^6 + 222 * a^5 * b^7 +
174 * a^4 * b^8 + 46 * a^3 * b^9 - 15 * a^2 * b^10 - 6 * a * b^11 + b^12);
B8[a_, b_] = (-1) * (a^8 - 4 * a^6 * b^2 - 2 * a^4 * b^4 - 4 * a^2 * b^6 + b^8) *
(a^16 - 8 * a^14 * b^2 + 12 * a^12 * b^4 + 8 * a^10 * b^6 -
34 * a^8 * b^8 + 8 * a^6 * b^10 + 12 * a^4 * b^12 - 8 * a^2 * b^14 + b^16);
B9[a_, b_] = (-1) * (a^18 + 18 * a^17 * b + 135 * a^16 * b^2 + 570 * a^15 * b^3 +
1557 * a^14 * b^4 + 2970 * a^13 * b^5 + 4128 * a^12 * b^6 + 4230 * a^11 * b^7 +
3240 * a^10 * b^8 + 2032 * a^9 * b^9 + 1359 * a^8 * b^10 + 1080 * a^7 * b^11 + 735 * a^6 *
b^12 + 306 * a^5 * b^13 + 27 * a^4 * b^14 - 42 * a^3 * b^15 - 18 * a^2 * b^16 + b^18);
B10[a_, b_] = (-1 / 64) * (a^2 + 2 * a * b + 2 * b^2) *
(a^4 + 6 * a^3 * b + 6 * a^2 * b^2 - 4 * a * b^3 - 4 * b^4) *

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(a^4 + 6*a^3*b + 12*a^2*b^2 + 8*a*b^3 + 2*b^4) * (a^8 + 10*a^7*b +
  32*a^6*b^2 + 40*a^5*b^3 + 14*a^4*b^4 + 8*a^2*b^6 - 4*b^8);
B12[a_, b_] = (-1) * (a^8 - 4*a^7*b + 4*a^6*b^2 - 4*a^5*b^3 -
  2*a^4*b^4 - 4*a^3*b^5 + 4*a^2*b^6 - 4*a*b^7 + b^8) *
(a^16 - 8*a^15*b + 24*a^14*b^2 - 40*a^13*b^3 + 44*a^12*b^4 -
  24*a^11*b^5 - 32*a^10*b^6 + 88*a^9*b^7 - 114*a^8*b^8 +
  88*a^7*b^9 - 32*a^6*b^10 - 24*a^5*b^11 + 44*a^4*b^12 -
  40*a^3*b^13 + 24*a^2*b^14 - 8*a*b^15 + b^16);
B22[a_, b_] = (-1) * (a^8 + 20*a^6*b^2 - 26*a^4*b^4 + 20*a^2*b^6 + b^8) *
(a^8 - 24*a^7*b + 20*a^6*b^2 - 24*a^5*b^3 - 26*a^4*b^4 + 24*a^3*b^5 +
  20*a^2*b^6 + 24*a*b^7 + b^8) * (a^8 + 24*a^7*b + 20*a^6*b^2 +
  24*a^5*b^3 - 26*a^4*b^4 - 24*a^3*b^5 + 20*a^2*b^6 - 24*a*b^7 + b^8);
B24[a_, b_] = (-1) * (a^4 - 4*a^3*b - 6*a^2*b^2 + 4*a*b^3 + b^4) *
(a^4 + 4*a^3*b - 6*a^2*b^2 - 4*a*b^3 + b^4) *
(a^8 - 4*a^6*b^2 + 22*a^4*b^4 - 4*a^2*b^6 + b^8) *
(a^8 + 20*a^6*b^2 - 26*a^4*b^4 + 20*a^2*b^6 + b^8);
B26[a_, b_] = (-1) * (a^8 - 4*a^7*b + 4*a^6*b^2 - 28*a^5*b^3 +
  22*a^4*b^4 - 28*a^3*b^5 + 4*a^2*b^6 - 4*a*b^7 + b^8) *
(a^8 - 4*a^7*b + 4*a^6*b^2 - 4*a^5*b^3 - 2*a^4*b^4 - 4*a^3*b^5 +
  4*a^2*b^6 - 4*a*b^7 + b^8) * (a^8 - 4*a^7*b + 4*a^6*b^2 +
  20*a^5*b^3 - 26*a^4*b^4 + 20*a^3*b^5 + 4*a^2*b^6 - 4*a*b^7 + b^8);
B28[a_, b_] = (-1) * (a^8 - 4*a^6*b^2 - 26*a^4*b^4 - 4*a^2*b^6 + b^8) *
(a^8 - 4*a^6*b^2 - 2*a^4*b^4 - 4*a^2*b^6 + b^8) *
(a^8 - 4*a^6*b^2 + 22*a^4*b^4 - 4*a^2*b^6 + b^8);
D1[a_, b_] = (-1) * b * a * (a + b) * (a^2 + a*b + b^2)^3 *
(a^3 + 6*a^2*b + 3*a*b^2 - b^3)^9;
D2[a_, b_] = (-1) * b * a * (a - b) * (a + b) * (a^2 + b^2)^2 *
(a^2 - 2*a*b - b^2)^4 * (a^2 + 2*a*b - b^2)^16;
D3[a_, b_] = (-1) * b^3 * a^3 * (a + b)^3 * (a^2 + a*b + b^2)^9 *
(a^3 + 6*a^2*b + 3*a*b^2 - b^3)^3;
D4[a_, b_] = (a - b)^2 * b^4 * a^4 * (a + b)^6 * (a^2 + b^2) *
(a^2 - 4*a*b + b^2)^3 * (a^2 - a*b + b^2)^12;
D5[a_, b_] = (2^75) * b^100 * (64*b^8 - 8*b^4 - 1) *
(4096*b^16 - 1024*b^12 + 256*b^8 - 24*b^4 + 1) *
(4096*b^16 + 1536*b^12 + 256*b^8 + 16*b^4 + 1);
D6[a_, b_] = (a + b)^2 * b^3 * a^3 * (a - b)^6 * (a^2 - a*b + b^2) *
(a^2 - 4*a*b + b^2)^4 * (a^2 + b^2)^12;
D7[a_, b_] = (-1) * b^7 * a^7 * (a + b)^7 * (a^3 + 8*a^2*b + 5*a*b^2 - b^3);
D8[a_, b_] = (a - b)^4 * (a + b)^4 * b^16 * a^16 *
(a^2 - 2*a*b - b^2) * (a^2 + 2*a*b - b^2) * (a^2 + b^2)^2;
D9[a_, b_] = (-1) * b^9 * a^9 * (a + b)^9 * (a^2 + a*b + b^2)^3 *
(a^3 + 6*a^2*b + 3*a*b^2 - b^3);
D10[a_, b_] = (1 / 4096) * a^5 * (a + 2*b)^5 * b^10 * (a + b)^10 *
(a^2 + 6*a*b + 4*b^2) * (a^2 + a*b - b^2)^2;
D12[a_, b_] = (a + b)^2 * (a - b)^6 * b^12 * a^12 *
(a^2 - 4*a*b + b^2) * (a^2 + b^2)^3 * (a^2 - a*b + b^2)^4;
D22[a_, b_] = b^2 * a^2 * (a - b)^2 * (a + b)^2 * (a^2 + b^2)^4

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(a^2 - 2*a*b - b^2)^8 * (a^2 + 2*a*b - b^2)^8;
D24[a_, b_] = b^4 * a^4 * (a - b)^4 * (a + b)^4 * (a^2 - 2*a*b - b^2)^4 *
(a^2 + 2*a*b - b^2)^4 * (a^2 + b^2)^8;
D26[a_, b_] = (a + b)^4 * b^6 * a^6 * (a - b)^12 *
(a^2 - 4*a*b + b^2)^2 * (a^2 - a*b + b^2)^2 * (a^2 + b^2)^6;
D28[a_, b_] = b^8 * a^8 * (a - b)^8 * (a + b)^8 * (a^2 - 2*a*b - b^2)^2 *
(a^2 + 2*a*b - b^2)^2 * (a^2 + b^2)^4;
Dh1[a_, b_] = (-1) * (a + b) * (a^2 + a*b + b^2) * (a^3 + 6*a^2*b + 3*a*b^2 - b^3);
Dh2[a_, b_] =
(-1) * (a - b) * (a + b) * (a^2 + b^2) * (a^2 - 2*a*b - b^2) * (a^2 + 2*a*b - b^2);
Dh3[a_, b_] = (-1) * (a + b) * (a^2 + a*b + b^2) * (a^3 + 6*a^2*b + 3*a*b^2 - b^3);
Dh4[a_, b_] =
(-1) * (a + b) * (a - b) * (a^2 + b^2) * (a^2 - 4*a*b + b^2) * (a^2 - a*b + b^2);
Dh5[a_, b_] = (2 / 5) * (64 * b^8 - 8 * b^4 - 1) *
(4096 * b^16 - 1024 * b^12 + 256 * b^8 - 24 * b^4 + 1) *
(4096 * b^16 + 1536 * b^12 + 256 * b^8 + 16 * b^4 + 1);
Dh6[a_, b_] = (-1) * (a + b) * (a - b) * (a^2 + b^2) *
(a^2 - 4*a*b + b^2) * (a^2 - a*b + b^2);
Dh7[a_, b_] = (-1) * (a + b) * (a^3 + 8*a^2*b + 5*a*b^2 - b^3);
Dh8[a_, b_] =
(-1) * (a - b) * (a + b) * (a^2 + b^2) * (a^2 - 2*a*b - b^2) * (a^2 + 2*a*b - b^2);
Dh9[a_, b_] = (-1) * (a + b) * (a^2 + a*b + b^2) * (a^3 + 6*a^2*b + 3*a*b^2 - b^3);
Dh10[a_, b_] =
(-1 / 8) * (a + b) * (a + 2*b) * (a^2 + 6*a*b + 4*b^2) * (a^2 + a*b - b^2);
Dh12[a_, b_] = (-1) * (a + b) * (a - b) * (a^2 + b^2) *
(a^2 - 4*a*b + b^2) * (a^2 - a*b + b^2);
Dh22[a_, b_] = (-1) * (a - b) * (a + b) * (a^2 + b^2) *
(a^2 - 2*a*b - b^2) * (a^2 + 2*a*b - b^2);
Dh24[a_, b_] = (-1) * (a - b) * (a + b) * (a^2 + b^2) *
(a^2 - 2*a*b - b^2) * (a^2 + 2*a*b - b^2);
Dh26[a_, b_] = (-1) * (a + b) * (a - b) * (a^2 + b^2) *
(a^2 - 4*a*b + b^2) * (a^2 - a*b + b^2);
Dh28[a_, b_] = (-1) * (a - b) * (a + b) * (a^2 + b^2) *
(a^2 - 2*a*b - b^2) * (a^2 + 2*a*b - b^2);
f3[x_] = B3[1, x]^2 / (1728 * D3[1, x]) - x;
f4[x_] = B4[1, x]^2 / (1728 * D4[1, x]) - x;
f6[x_] = B6[1, x]^2 / (1728 * D6[1, x]) - x;
f7[x_] = B7[1, x]^2 / (1728 * D7[1, x]) - x;
f8[x_] = B8[1, x]^2 / (1728 * D8[1, x]) - x;
f9[x_] = B9[1, x]^2 / (1728 * D9[1, x]) - x;
f10[x_] = B10[1, x]^2 / (1728 * D10[1, x]) - x;
f12[x_] = B12[1, x]^2 / (1728 * D12[1, x]) - x;
f22[x_] = B22[1, x]^2 / (1728 * D22[1, x]) - x;
f24[x_] = B24[1, x]^2 / (1728 * D24[1, x]) - x;
f26[x_] = B26[1, x]^2 / (1728 * D26[1, x]) - x;
f28[x_] = B28[1, x]^2 / (1728 * D28[1, x]) - x;
g3[x_] = A3[1, x]^2 + B3[1, x] * Dh3[1, x];

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g4[x_] = A4[1, x]^2 + B4[1, x] * Dh4[1, x];
g6[x_] = A6[1, x]^2 + B6[1, x] * Dh6[1, x];
g7[x_] = A7[1, x]^2 + B7[1, x] * Dh7[1, x];
g8[x_] = A8[1, x]^2 + B8[1, x] * Dh8[1, x];
g9[x_] = A9[1, x]^2 + B9[1, x] * Dh9[1, x];
g10[x_] = A10[1, x]^2 + B10[1, x] * Dh10[1, x];
g12[x_] = A12[1, x]^2 + B12[1, x] * Dh12[1, x];
g22[x_] = A22[1, x]^2 + B22[1, x] * Dh22[1, x];
g24[x_] = A24[1, x]^2 + B24[1, x] * Dh24[1, x];
g26[x_] = A26[1, x]^2 + B26[1, x] * Dh26[1, x];
g28[x_] = A28[1, x]^2 + B28[1, x] * Dh28[1, x];
h1[x_] = A1[1, x]^3 - Dh1[1, x]^6;
h2[x_] = A2[1, x]^3 - Dh2[1, x]^6;
h3[x_] = A3[1, x]^3 - Dh3[1, x]^6;
h4[x_] = A4[1, x]^3 - Dh4[1, x]^6;
h5[x_] = A5[1, x]^3 - Dh5[1, x]^6;
h6[x_] = A6[1, x]^3 - Dh6[1, x]^6;
h7[x_] = A7[1, x]^3 - Dh7[1, x]^6;
h8[x_] = A8[1, x]^3 - Dh8[1, x]^6;
h9[x_] = A9[1, x]^3 - Dh9[1, x]^6;
h10[x_] = A10[1, x]^3 - Dh10[1, x]^6;
h12[x_] = A12[1, x]^3 - Dh12[1, x]^6;
h22[x_] = A22[1, x]^3 - Dh22[1, x]^6;
h24[x_] = A24[1, x]^3 - Dh24[1, x]^6;
h26[x_] = A26[1, x]^3 - Dh26[1, x]^6;
h28[x_] = A28[1, x]^3 - Dh28[1, x]^6;
z1[x_] = Max[Abs[A1[1, x]^3], B1[1, x]^2] - Dh1[1, x]^6;
z2[x_] = Max[Abs[A2[1, x]^3], B2[1, x]^2] - Dh2[1, x]^6;
z3[x_] = Max[Abs[A3[1, x]^3], B3[1, x]^2] - Dh3[1, x]^6;
z4[x_] = Max[Abs[A4[1, x]^3], B4[1, x]^2] - Dh4[1, x]^6;
z6[x_] = Max[Abs[A6[1, x]^3], B6[1, x]^2] - Dh6[1, x]^6;
z7[x_] = Max[Abs[A7[1, x]^3], B7[1, x]^2] - Dh7[1, x]^6;
z8[x_] = Max[Abs[A8[1, x]^3], B8[1, x]^2] - Dh8[1, x]^6;
z9[x_] = Max[Abs[A9[1, x]^3], B9[1, x]^2] - Dh9[1, x]^6;
z10[x_] = Max[Abs[A10[1, x]^3], B10[1, x]^2] - Dh10[1, x]^6;
z12[x_] = Max[Abs[A12[1, x]^3], B12[1, x]^2] - Dh12[1, x]^6;
z22[x_] = Max[Abs[A22[1, x]^3], B22[1, x]^2] - Dh22[1, x]^6;
z24[x_] = Max[Abs[A24[1, x]^3], B24[1, x]^2] - Dh24[1, x]^6;
z26[x_] = Max[Abs[A26[1, x]^3], B26[1, x]^2] - Dh26[1, x]^6;
z28[x_] = Max[Abs[A28[1, x]^3], B28[1, x]^2] - Dh28[1, x]^6;

```

In[297]:= (\* We now consider each T separately. \*)

```

In[298]:= (*Case T=C_1. We first verify Lemma 4.2. The code below shows
          that (1) the greatest real root of  $D_T(1,x)$  is  $\theta_T \approx 4.41147$ ,
          (2) the greatest real root of  $A_T(1,x)$ ,  $\hat{D}_T$ ,  $h_T(x)$  is at most  $\theta_T$ ,
          and (3) the continuous functions  $A_T(1,x)$ ,  $D_T(1,x)$ ,  $\hat{D}_T$ ,
           $h_T(x)$  are positive on the interval  $(\theta_T, \infty)$  *)
          theta1 = Max[x /. NSolve[D1[1, x] == 0, x, Reals]]; theta1
          Max[Max[x /. NSolve[A1[1, x] == 0, x, Reals]], Max[x /. NSolve[Dh1[1, x] == 0, x, Reals]],
          Max[x /. NSolve[h1[x] == 0, x, Reals]]] <= theta1
          {A1[1, 5000] > 0, D1[1, 5000] > 0, Dh1[1, 5000] > 0, h1[5000] > 0}

```

```
Out[298]= 4.41147
```

```
Out[299]= True
```

```
Out[300]= {True, True, True, True}
```

```

In[301]:= (*Case T=C_1. Lastly, we verify Lemma 6.1. The code
          below shows that (1) the greatest real root of  $z_T(x)$  is  $\xi_T=0$ ,
          (2) the continuous functions  $z_T(x)$  is positive on the interval  $(\xi_T, \infty)$  *)
          xi1 = Max[x /. NSolve[z1[x] == 0, x, Reals]]; xi1
          z1[5000] > 0

```

```
Out[301]= 0
```

```
Out[302]= True
```

```

In[303]:= (*Case T=C_2. We first verify Lemma 4.2. The code below shows
          that (1) the greatest real root of  $D_T(1,x)$  is  $\theta_T \approx 2.41421$ ,
          (2) the greatest real root of  $A_T(1,x)$ ,  $\hat{D}_T$ ,  $h_T(x)$  is at most  $\theta_T$ ,
          and (3) the continuous functions  $A_T(1,x)$ ,  $D_T(1,x)$ ,  $\hat{D}_T$ ,
           $h_T(x)$  are positive on the interval  $(\theta_T, \infty)$  *)
          theta2 = Max[x /. NSolve[D2[1, x] == 0, x, Reals]]; theta2
          Max[Max[x /. NSolve[A2[1, x] == 0, x, Reals]], Max[x /. NSolve[Dh2[1, x] == 0, x, Reals]],
          Max[x /. NSolve[h2[x] == 0, x, Reals]]] <= theta2
          {A2[1, 5000] > 0, D2[1, 5000] > 0, Dh2[1, 5000] > 0, h2[5000] > 0}

```

```
Out[303]= 2.41421
```

```
Out[304]= True
```

```
Out[305]= {True, True, True, True}
```

```

In[306]:= (*Case T=C_2. Lastly, we verify Lemma 6.1. The code
          below shows that (1) the greatest real root of  $z_T(x)$  is  $\xi_T=0$ ,
          (2) the continuous functions  $z_T(x)$  is positive on the interval  $(\xi_T, \infty)$  *)
          xi2 = Max[x /. NSolve[z1[x] == 0, x, Reals]]; xi2
          z2[5000] > 0

```

```
Out[306]= 0
```

```
Out[307]= True
```

```

In[308]:= (*Case T=C_3. We first verify Lemma 4.2. The code below shows
          that (1) the greatest real root of  $D_T(1,x)$  is  $\theta_T \approx 4.41147$ ,
          (2) the greatest real root of  $A_T(1,x)$ ,  $\hat{D}_T$ ,  $h_T(x)$  is at most  $\theta_T$ ,
          and (3) the continuous functions  $A_T(1,x)$ ,  $D_T(1,x)$ ,  $\hat{D}_T$ ,
           $h_T(x)$  are positive on the interval  $(\theta_T, \infty)$  *)
          theta3 = Max[x /. NSolve[D3[1, x] == 0, x, Reals]]; theta3
          Max[Max[x /. NSolve[A3[1, x] == 0, x, Reals]], Max[x /. NSolve[Dh3[1, x] == 0, x, Reals]],
          Max[x /. NSolve[h3[x] == 0, x, Reals]]] <= theta3
          {A3[1, 5000] > 0, D3[1, 5000] > 0, Dh3[1, 5000] > 0, h3[5000] > 0}

```

```
Out[308]= 4.41147
```

```
Out[309]= True
```

```
Out[310]= {True, True, True, True}
```

```

In[311]:= (*Case T=C_3. Next, we verify Lemma 4.3. The code below shows
          that (1) the greatest real root of  $f_T(x)$  is  $\delta_T \approx 43.4033$ ,
          (2) the greatest real root of  $g_T(x)$ ,  $-B_T(1,x)$  is at most  $\delta_T$ ,
          and (3) the continuous functions  $f_T(x)$ ,  $g_T(x)$ ,
           $-B_T(1,x)$  are positive on the interval  $(\delta_T, \infty)$  *)
          delta3 = Max[x /. NSolve[f3[x] == 0, x, Reals]]; delta3
          Max[Max[x /. NSolve[g3[x] == 0, x, Reals]],
          Max[x /. NSolve[-B3[1, x] == 0, x, Reals]]] <= delta3
          {f3[5000] > 0, g3[5000] > 0, -B3[1, 5000] > 0}

```

```
Out[311]= 43.4033
```

```
Out[312]= True
```

```
Out[313]= {True, True, True}
```

```

In[314]:= (*Case T=C_3. Lastly, we verify Lemma 6.1. The code below
          shows that (1) the greatest real root of  $z_T(x)$  is  $\xi_T \approx 0.1686$ ,
          (2) the continuous functions  $z_T(x)$  is positive on the interval  $(\xi_T, \infty)$  *)
          xi3 = Max[x /. NSolve[z3[x] == 0, x, Reals]]; xi3
          z3[5000] > 0

```

```
Out[314]= 0.168612
```

```
Out[315]= True
```

```

In[316]:= (*Case T=C_4. We first verify Lemma 4.2. The code below shows
          that (1) the greatest real root of  $D_T(1,x)$  is  $\theta_T \approx 3.73205$ ,
          (2) the greatest real root of  $A_T(1,x)$ ,  $\hat{D}_T$ ,  $h_T(x)$  is at most  $\theta_T$ ,
          and (3) the continuous functions  $A_T(1,x)$ ,  $D_T(1,x)$ ,  $\hat{D}_T$ ,
           $h_T(x)$  are positive on the interval  $(\theta_T, \infty)$  *)
          theta4 = Max[x /. NSolve[D4[1, x] == 0, x, Reals]]; theta4
          Max[Max[x /. NSolve[A4[1, x] == 0, x, Reals]],
          Max[x /. NSolve[Dh4[1, x] == 0, x, Reals]], Max[x /. NSolve[h4[x] == 0, x, Reals]]] <= theta4
          {A4[1, 5000] > 0, D4[1, 5000] > 0, Dh4[1, 5000] > 0, h4[5000] > 0}

```

```
Out[316]= 3.73205
```

```
Out[317]= True
```

```
Out[318]= {True, True, True, True}
```

```

In[319]:= (*Case T=C_4. Next, we verify Lemma 4.3. The code below shows
          that (1) the greatest real root of  $f_T(x)$  is  $\delta_T \approx 13.5934$ ,
          (2) the greatest real root of  $g_T(x)$ ,  $-B_T(1,x)$  is at most  $\delta_T$ ,
          and (3) the continuous functions  $f_T(x)$ ,  $g_T(x)$ ,
           $-B_T(1,x)$  are positive on the interval  $(\delta_T, \infty)$  *)
          delta4 = Max[x /. NSolve[f4[x] == 0, x, Reals]]; delta4
          Max[Max[x /. NSolve[g4[x] == 0, x, Reals]],
          Max[x /. NSolve[-B4[1, x] == 0, x, Reals]]] <= delta4
          {f4[5000] > 0, g4[5000] > 0, -B4[1, 5000] > 0}

```

```
Out[319]= 13.5934
```

```
Out[320]= True
```

```
Out[321]= {True, True, True}
```

```

In[322]:= (*Case T=C_4. Lastly, we verify Lemma 6.1. The code
          below shows that (1) the greatest real root of  $z_T(x)$  is  $\xi_T=0$ ,
          (2) the continuous functions  $z_T(x)$  is positive on the interval  $(\xi_T, \infty)$  *)
          xi4 = Max[x /. NSolve[z4[x] == 0, x, Reals]]; xi4
          z4[5000] > 0

```

```
Out[322]= 0
```

```
Out[323]= True
```



```

In[324]:= (*Case T=C_5. We verify Lemma 4.2. The code below shows
          that (1) the greatest real root of D_T(1,x) is \theta_T≈0.0.67062,
          (2) the greatest real root of A_T(1,x), \hat{D}_T, h_T(x) is at most \theta_T,
          and (3) the continuous functions A_T(1,x), D_T(1,x), \hat{D}_T,
          h_T(x) are positive on the interval (\theta_T,infinity) *)
theta5 = Max[x /. NSolve[D5[1, x] == 0, x, Reals]]; theta5
Max[Max[x /. NSolve[A5[1, x] == 0, x, Reals]],
    Max[x /. NSolve[Dh5[1, x] == 0, x, Reals]], Max[x /. NSolve[h5[x] == 0, x, Reals]]] <= theta5
{A5[1, 5000] > 0, D5[1, 5000] > 0, Dh5[1, 5000] > 0, h5[5000] > 0}

```

```
Out[324]= 0.670617
```

```
Out[325]= True
```

```
Out[326]= {True, True, True, True}
```

```

In[327]:= (*Case T=C_5. We verify Corollary 4.10. The code below shows that (1) the
          greatest root of -\hat{D}_T(1,x)*B_T(1,x) is zeta_T≈0.671318,
          (2) |\hat{D}_T(1,x)*B_T(1,x)| = -\hat{D}_T(1,x)*B_T(1,x) on the interval
          (zeta_T,infinity) since the function is continuous and positive when
          evaluated at 5000>zeta_T, (3) A_T(1,x)^2 - |\hat{D}_T(1,x)*B_T(1,x)|
          has no real roots on the interval (zeta_T,infinity),
          and (4) A_T(1,x)^2 - |\hat{D}_T(1,x)*B_T(1,x)|>0 is positive on
          the interval (zeta_T,infinity) since the function is continuous
          and A_T(1,5000)^2 - |\hat{D}_T(1,5000)*B_T(1,5000)|>0*)
zeta5 = Max[x /. NSolve[-Dh5[1, x] * B5[1, x] == 0, x, Reals]]; zeta5
-Dh5[1, 5000] * B5[1, 5000] > 0
Max[x /. NSolve[A5[1, x]^2 + 3 / 5 * Dh5[1, x] * B5[1, x] == 0, x, Reals]] < zeta5
A5[1, 5000]^2 + 3 / 5 * Dh5[1, 5000] * B5[1, 5000] > 0

```

```
Out[327]= 0.671318
```

```
Out[328]= True
```

```
Out[329]= True
```

```
Out[330]= True
```

```

In[331]:= (*Case T=C_6. We first verify Lemma 4.2. The code below shows
          that (1) the greatest real root of  $D_T(1,x)$  is  $\theta_T \approx 3.73205$ ,
          (2) the greatest real root of  $A_T(1,x)$ ,  $\hat{D}_T$ ,  $h_T(x)$  is at most  $\theta_T$ ,
          and (3) the continuous functions  $A_T(1,x)$ ,  $D_T(1,x)$ ,  $\hat{D}_T$ ,
           $h_T(x)$  are positive on the interval  $(\theta_T, \infty)$  *)
          theta6 = Max[x /. NSolve[D6[1, x] == 0, x, Reals]]; theta6
          Max[Max[x /. NSolve[A6[1, x] == 0, x, Reals]],
              Max[x /. NSolve[Dh6[1, x] == 0, x, Reals]], Max[x /. NSolve[h6[x] == 0, x, Reals]]] <= theta6
          {A6[1, 5000] > 0, D6[1, 5000] > 0, Dh6[1, 5000] > 0, h6[5000] > 0}

```

```
Out[331]= 3.73205
```

```
Out[332]= True
```

```
Out[333]= {True, True, True, True}
```

```

In[334]:= (*Case T=C_6. Next, we verify Lemma 4.3. The code below shows
          that (1) the greatest real root of  $f_T(x)$  is  $\delta_T \approx 43.3677$ ,
          (2) the greatest real root of  $g_T(x)$ ,  $-B_T(1,x)$  is at most  $\delta_T$ ,
          and (3) the continuous functions  $f_T(x)$ ,  $g_T(x)$ ,
           $-B_T(1,x)$  are positive on the interval  $(\delta_T, \infty)$  *)
          delta6 = Max[x /. NSolve[f6[x] == 0, x, Reals]]; delta6
          Max[Max[x /. NSolve[g6[x] == 0, x, Reals]],
              Max[x /. NSolve[-B6[1, x] == 0, x, Reals]]] <= delta6
          {f6[5000] > 0, g6[5000] > 0, -B6[1, 5000] > 0}

```

```
Out[334]= 43.3677
```

```
Out[335]= True
```

```
Out[336]= {True, True, True}
```

```

In[337]:= (*Case T=C_6. Lastly, we verify Lemma 6.1. The code
          below shows that (1) the greatest real root of  $z_T(x)$  is  $\xi_T=0$ ,
          (2) the continuous functions  $z_T(x)$  is positive on the interval  $(\xi_T, \infty)$  *)
          xi6 = Max[x /. NSolve[z6[x] == 0, x, Reals]]; xi6
          z6[5000] > 0

```

```
Out[337]= 0
```

```
Out[338]= True
```

```
In[339]:= (*Case T=C_7. We first verify Lemma 4.2. The code below
          shows that (1) the greatest real root of  $D_T(1,x)$  is  $\theta_T \approx 6.2959$ ,
          (2) the greatest real root of  $A_T(1,x)$ ,  $\hat{D}_T$ ,  $h_T(x)$  is at most  $\theta_T$ ,
          and (3) the continuous functions  $A_T(1,x)$ ,  $D_T(1,x)$ ,  $\hat{D}_T$ ,
           $h_T(x)$  are positive on the interval  $(\theta_T, \infty)$  *)
theta7 = Max[x /. NSolve[D7[1, x] == 0, x, Reals]]; theta7
Max[Max[x /. NSolve[A7[1, x] == 0, x, Reals]],
    Max[x /. NSolve[Dh7[1, x] == 0, x, Reals]], Max[x /. NSolve[h7[x] == 0, x, Reals]]] <= theta7
{A7[1, 5000] > 0, D7[1, 5000] > 0, Dh7[1, 5000] > 0, h7[5000] > 0}
```

```
Out[339]= 6.2959
```

```
Out[340]= True
```

```
Out[341]= {True, True, True, True}
```

```
In[342]:= (*Case T=C_7. Next, we verify Lemma 4.3. The code below shows
          that (1) the greatest real root of  $f_T(x)$  is  $\delta_T \approx 7.07956$ ,
          (2) the greatest real root of  $g_T(x)$ ,  $-B_T(1,x)$  is at most  $\delta_T$ ,
          and (3) the continuous functions  $f_T(x)$ ,  $g_T(x)$ ,
           $-B_T(1,x)$  are positive on the interval  $(\delta_T, \infty)$  *)
delta7 = Max[x /. NSolve[f7[x] == 0, x, Reals]]; delta7
Max[Max[x /. NSolve[g7[x] == 0, x, Reals]],
    Max[x /. NSolve[-B7[1, x] == 0, x, Reals]]] <= delta7
{f7[5000] > 0, g7[5000] > 0, -B7[1, 5000] > 0}
```

```
Out[342]= 7.07956
```

```
Out[343]= True
```

```
Out[344]= {True, True, True}
```

```
In[345]:= (*Case T=C_7. Lastly, we verify Lemma 6.1. The code below
          shows that (1) the greatest real root of  $z_T(x)$  is  $\xi_T \approx 4.3444$ ,
          (2) the continuous functions  $z_T(x)$  is positive on the interval  $(\xi_T, \infty)$  *)
xi7 = Max[x /. NSolve[z7[x] == 0, x, Reals]]; xi7
z7[5000] > 0
```

```
Out[345]= 4.34442
```

```
Out[346]= True
```

```

In[347]:= (*Case T=C_8. We first verify Lemma 4.2. The code below shows
          that (1) the greatest real root of  $D_T(1,x)$  is  $\theta_T \approx 2.41421$ ,
          (2) the greatest real root of  $A_T(1,x)$ ,  $\hat{D}_T$ ,  $h_T(x)$  is at most  $\theta_T$ ,
          and (3) the continuous functions  $A_T(1,x)$ ,  $D_T(1,x)$ ,  $\hat{D}_T$ ,
           $h_T(x)$  are positive on the interval  $(\theta_T, \infty)$  *)
          theta8 = Max[x /. NSolve[D8[1, x] == 0, x, Reals]]; theta8
          Max[Max[x /. NSolve[A8[1, x] == 0, x, Reals]],
              Max[x /. NSolve[Dh8[1, x] == 0, x, Reals]], Max[x /. NSolve[h8[x] == 0, x, Reals]]] <= theta8
          {A8[1, 5000] > 0, D8[1, 5000] > 0, Dh8[1, 5000] > 0, h8[5000] > 0}

Out[347]= 2.41421

Out[348]= True

Out[349]= {True, True, True, True}

In[350]:= (*Case T=C_8. Next, we verify Lemma 4.3. The code below shows
          that (1) the greatest real root of  $f_T(x)$  is  $\delta_T \approx 2.48383$ ,
          (2) the greatest real root of  $g_T(x)$ ,  $-B_T(1,x)$  is at most  $\delta_T$ ,
          and (3) the continuous functions  $f_T(x)$ ,  $g_T(x)$ ,
           $-B_T(1,x)$  are positive on the interval  $(\delta_T, \infty)$  *)
          delta8 = Max[x /. NSolve[f8[x] == 0, x, Reals]]; delta8
          Max[Max[x /. NSolve[g8[x] == 0, x, Reals]],
              Max[x /. NSolve[-B8[1, x] == 0, x, Reals]]] <= delta8
          {f8[5000] > 0, g8[5000] > 0, -B8[1, 5000] > 0}

Out[350]= 2.48383

Out[351]= True

Out[352]= {True, True, True}

In[353]:= (*Case T=C_8. Lastly, we verify Lemma 6.1. The code below
          shows that (1) the greatest real root of  $z_T(x)$  is  $\xi_T \approx 2.0198$ ,
          (2) the continuous functions  $z_T(x)$  is positive on the interval  $(\xi_T, \infty)$  *)
          xi8 = Max[x /. NSolve[z8[x] == 0, x, Reals]]; xi8
          z8[5000] > 0

Out[353]= 2.01982

Out[354]= True

```

```

In[355]:= (*Case T=C_9. We first verify Lemma 4.2. The code below shows
          that (1) the greatest real root of  $D_T(1,x)$  is  $\theta_T \approx 4.41147$ ,
          (2) the greatest real root of  $A_T(1,x)$ ,  $\hat{D}_T$ ,  $h_T(x)$  is at most  $\theta_T$ ,
          and (3) the continuous functions  $A_T(1,x)$ ,  $D_T(1,x)$ ,  $\hat{D}_T$ ,
           $h_T(x)$  are positive on the interval  $(\theta_T, \infty)$  *)
          theta9 = Max[x /. NSolve[D9[1, x] == 0, x, Reals]]; theta9
          Max[Max[x /. NSolve[A9[1, x] == 0, x, Reals]],
              Max[x /. NSolve[Dh9[1, x] == 0, x, Reals]], Max[x /. NSolve[h9[x] == 0, x, Reals]]] <= theta9
          {A9[1, 5000] > 0, D9[1, 5000] > 0, Dh9[1, 5000] > 0, h9[5000] > 0}

```

Out[355]= 4.41147

Out[356]= True

Out[357]= {True, True, True, True}

```

In[358]:= (*Case T=C_9. Next, we verify Lemma 4.3. The code below shows
          that (1) the greatest real root of  $f_T(x)$  is  $\delta_T \approx 4.75552$ ,
          (2) the greatest real root of  $g_T(x)$ ,  $-B_T(1,x)$  is at most  $\delta_T$ ,
          and (3) the continuous functions  $f_T(x)$ ,  $g_T(x)$ ,
           $-B_T(1,x)$  are positive on the interval  $(\delta_T, \infty)$  *)
          delta9 = Max[x /. NSolve[f9[x] == 0, x, Reals]]; delta9
          Max[Max[x /. NSolve[g9[x] == 0, x, Reals]],
              Max[x /. NSolve[-B9[1, x] == 0, x, Reals]]] <= delta9
          {f9[5000] > 0, g9[5000] > 0, -B9[1, 5000] > 0}

```

Out[358]= 4.75552

Out[359]= True

Out[360]= {True, True, True}

```

In[361]:= (*Case T=C_9. Lastly, we verify Lemma 6.1. The code below
          shows that (1) the greatest real root of  $z_T(x)$  is  $\xi_T \approx 3.2938$ ,
          (2) the continuous functions  $z_T(x)$  is positive on the interval  $(\xi_T, \infty)$  *)
          xi9 = Max[x /. NSolve[z9[x] == 0, x, Reals]]; xi9
          z9[5000] > 0

```

Out[361]= 3.29383

Out[362]= True

```
In[363]:= (*Case T=C_10. We first verify Lemma 4.2. The code below shows
that (1) the greatest real root of D_T(1,x) is \theta_T≈1.61803,
(2) the greatest real root of A_T(1,x), \hat{D}_T, h_T(x) is at most \theta_T,
and (3) the continuous functions A_T(1,x), D_T(1,x), \hat{D}_T,
h_T(x) are positive on the interval (\theta_T,infinity) *)
theta10 = Max[x /. NSolve[D10[1, x] == 0, x, Reals]]; theta10
Max[Max[x /. NSolve[A10[1, x] == 0, x, Reals]], Max[x /. NSolve[Dh10[1, x] == 0, x, Reals]],
Max[x /. NSolve[h10[x] == 0, x, Reals]]] ≤ theta10
{A10[1, 5000] > 0, D10[1, 5000] > 0, Dh10[1, 5000] > 0, h10[5000] > 0}
```

```
Out[363]= 1.61803
```

```
Out[364]= True
```

```
Out[365]= {True, True, True, True}
```

```
In[366]:= (*Case T=C_10. Next, we verify Lemma 4.3. The code below
shows that (1) the greatest real root of f_T(x) is \delta_T≈3.06311,
(2) the greatest real root of g_T(x), -B_T(1,x) is at most \delta_T,
and (3) the continuous functions f_T(x), g_T(x),
-B_T(1,x) are positive on the interval (\delta_T,infinity) *)
delta10 = Max[x /. NSolve[f10[x] == 0, x, Reals]]; delta10
Max[Max[x /. NSolve[g10[x] == 0, x, Reals]],
Max[x /. NSolve[-B10[1, x] == 0, x, Reals]]] ≤ delta10
{f10[5000] > 0, g10[5000] > 0, -B10[1, 5000] > 0}
```

```
Out[366]= 3.06311
```

```
Out[367]= True
```

```
Out[368]= {True, True, True}
```

```
In[369]:= (*Case T=C_10. Lastly, we verify Lemma 6.1. The code below shows
that z_T is a continuous function with no real roots. Since z_T(5000)>0,
we conclude that z_T is positive throughout its domain *)
NSolve[z10[x] == 0, x, Reals]
z10[5000] > 0
```

```
Out[369]= {}
```

```
Out[370]= True
```

```

In[371]:= (*Case T=C_12. We first verify Lemma 4.2. The code below shows
           that (1) the greatest real root of  $D_T(1,x)$  is  $\theta_T \approx 3.73205$ ,
           (2) the greatest real root of  $A_T(1,x)$ ,  $\hat{D}_T$ ,  $h_T(x)$  is at most  $\theta_T$ ,
           and (3) the continuous functions  $A_T(1,x)$ ,  $D_T(1,x)$ ,  $\hat{D}_T$ ,
            $h_T(x)$  are positive on the interval  $(\theta_T, \infty)$  *)
           theta12 = Max[x /. NSolve[D12[1, x] == 0, x, Reals]]; theta12
           Max[Max[x /. NSolve[A12[1, x] == 0, x, Reals]], Max[x /. NSolve[Dh12[1, x] == 0, x, Reals]],
           Max[x /. NSolve[h12[x] == 0, x, Reals]]] <= theta12
           {A12[1, 5000] > 0, D12[1, 5000] > 0, Dh12[1, 5000] > 0, h12[5000] > 0}

```

```
Out[371]= 3.73205
```

```
Out[372]= True
```

```
Out[373]= {True, True, True, True}
```

```

In[374]:= (*Case T=C_12. Next, we verify Lemma 4.3. The code below
           shows that (1) the greatest real root of  $f_T(x)$  is  $\delta_T \approx 3.89418$ ,
           (2) the greatest real root of  $g_T(x)$ ,  $-B_T(1,x)$  is at most  $\delta_T$ ,
           and (3) the continuous functions  $f_T(x)$ ,  $g_T(x)$ ,
            $-B_T(1,x)$  are positive on the interval  $(\delta_T, \infty)$  *)
           delta12 = Max[x /. NSolve[f12[x] == 0, x, Reals]]; delta12
           Max[Max[x /. NSolve[g12[x] == 0, x, Reals]],
           Max[x /. NSolve[-B12[1, x] == 0, x, Reals]]] <= delta12
           {f12[5000] > 0, g12[5000] > 0, -B12[1, 5000] > 0}

```

```
Out[374]= 3.89418
```

```
Out[375]= True
```

```
Out[376]= {True, True, True}
```

```

In[377]:= (*Case T=C_12. Lastly, we verify Lemma 6.1. The code below
           shows that (1) the greatest real root of  $z_T(x)$  is  $\xi_T \approx 2.9354$ ,
           (2) the continuous functions  $z_T(x)$  is positive on the interval  $(\xi_T, \infty)$  *)
           xi12 = Max[x /. NSolve[z12[x] == 0, x, Reals]]; xi12
           z12[5000] > 0

```

```
Out[377]= 2.93543
```

```
Out[378]= True
```

```

In[379]:= (*Case T=
C_2 \times C_2. We first verify Lemma 4.2. The code below shows that (1) the greatest
real root of D_T(1,x) is \theta_T\approx 2.41421,
(2) A_T(1,x) has no real roots and the greatest real root of \hat{D}_T,
h_T(x) is at most \theta_T,
and (3) the continuous functions A_T(1,x),
D_T(1,x), \hat{D}_T,
h_T(x) are positive on the interval (\theta_T,infinity) *)
theta22 = Max[x /. NSolve[D22[1, x] == 0, x, Reals]]; theta22
NSolve[A22[1, x] == 0, x, Reals]
Max[Max[x /. NSolve[Dh22[1, x] == 0, x, Reals]],
Max[x /. NSolve[h22[x] == 0, x, Reals]]] <= theta22
{A22[1, 5000] > 0, D22[1, 5000] > 0, Dh22[1, 5000] > 0, h22[5000] > 0}

```

```
Out[379]= 2.41421
```

```
Out[380]= {}
```

```
Out[381]= True
```

```
Out[382]= {True, True, True, True}
```

```

In[383]:= (*Case T=C_2 \times C_2. Next, we verify Lemma 4.3. The code below
shows that (1) the greatest real root of f_T(x) is \delta_T\approx 1728.57,
(2) g_T(x) has no real roots and the greatest real root of -B_T(1,x)
is at most \delta_T, and (3) the continuous functions f_T(x),
g_T(x), -B_T(1,x) are positive on the interval (\delta_T,infinity) *)
delta22 = Max[x /. NSolve[f22[x] == 0, x, Reals]]; delta22
NSolve[g22[x] == 0, x, Reals]
Max[x /. NSolve[-B22[1, x] == 0, x, Reals]] <= delta22
{f22[5000] > 0, g22[5000] > 0, -B22[1, 5000] > 0}

```

```
Out[383]= 1728.57
```

```
Out[384]= {}
```

```
Out[385]= True
```

```
Out[386]= {True, True, True}
```

```

In[387]:= (*Case T=C_2 \times C_2. Lastly, we verify Lemma 6.1. The code
below shows that (1) the greatest real root of z_T(x) is \xi_T=0,
(2) the continuous functions z_T(x) is positive on the interval (\xi_T,infinity) *)
xi22 = Max[x /. NSolve[z22[x] == 0, x, Reals]]; xi22
z22[5000] > 0

```

```
Out[387]= 0
```

```
Out[388]= True
```



```

In[389]:= (*Case T=C_2 \times C_4. We first verify Lemma 4.2. The code below
          shows that (1) the greatest real root of D_T(1,x) is \theta_T\approx 2.41421,
          (2) A_T(1,x) has no real roots and the greatest real root of \hat{D}_T,
          h_T(x) is at most \theta_T, and (3) the continuous functions A_T(1,x),
          D_T(1,x), \hat{D}_T, h_T(x) are positive on the interval (\theta_T,infinity) *)
          theta24 = Max[x /. NSolve[D24[1, x] == 0, x, Reals]]; theta24
          NSolve[A24[1, x] == 0, x, Reals]
          Max[Max[x /. NSolve[Dh24[1, x] == 0, x, Reals]],
              Max[x /. NSolve[h24[x] == 0, x, Reals]]] <= theta24
          {A24[1, 5000] > 0, D24[1, 5000] > 0, Dh24[1, 5000] > 0, h24[5000] > 0}

Out[389]= 2.41421

Out[390]= {}

Out[391]= True

Out[392]= {True, True, True, True}

In[393]:= (*Case T=C_2 \times C_4. Next, we verify Lemma 4.3. The code below
          shows that (1) the greatest real root of f_T(x) is \delta_T\approx 12.2907,
          (2) g_T(x) has no real roots and the greatest real root of -B_T(1,x)
          is at most \delta_T, and (3) the continuous functions f_T(x),
          g_T(x), -B_T(1,x) are positive on the interval (\delta_T,infinity) *)
          delta24 = Max[x /. NSolve[f24[x] == 0, x, Reals]]; delta24
          NSolve[g24[x] == 0, x, Reals]
          Max[x /. NSolve[-B24[1, x] == 0, x, Reals]] <= delta24
          {f24[5000] > 0, g24[5000] > 0, -B24[1, 5000] > 0}

Out[393]= 12.2907

Out[394]= {}

Out[395]= True

Out[396]= {True, True, True}

In[397]:= (*Case T=C_2 \times C_4. Lastly, we verify Lemma 6.1. The code
          below shows that (1) the greatest real root of z_T(x) is \xi_T=0,
          (2) the continuous functions z_T(x) is positive on the interval (\xi_T,infinity) *)
          xi24 = Max[x /. NSolve[z24[x] == 0, x, Reals]]; xi24
          z24[5000] > 0

Out[397]= 0

Out[398]= True

```

```

In[399]:= (*Case T=C_2 \times C_6. We first verify Lemma 4.2. The code below
           shows that (1) the greatest real root of D_T(1,x) is \theta_T\approx 3.73205,
           (2) A_T(1,x) has no real roots and the greatest real root of \hat{D}_T,
           h_T(x) is at most \theta_T, and (3) the continuous functions A_T(1,x),
           D_T(1,x), \hat{D}_T, h_T(x) are positive on the interval (\theta_T,infinity) *)
           theta26 = Max[x /. NSolve[D26[1, x] == 0, x, Reals]]; theta26
           NSolve[A26[1, x] == 0, x, Reals]
           Max[Max[x /. NSolve[Dh26[1, x] == 0, x, Reals]],
           Max[x /. NSolve[h26[x] == 0, x, Reals]]] <= theta26
           {A26[1, 5000] > 0, D26[1, 5000] > 0, Dh26[1, 5000] > 0, h26[5000] > 0}

Out[399]= 3.73205

Out[400]= {}

Out[401]= True

Out[402]= {True, True, True, True}

In[403]:= (*Case T=C_2 \times C_6. Next, we verify Lemma 4.3. The code below
           shows that (1) the greatest real root of f_T(x) is \delta_T\approx 3.89418,
           (2) the greatest real root of g_T(x), \neg B_T(1,x) is at most \delta_T,
           and (3) the continuous functions f_T(x), g_T(x),
           \neg B_T(1,x) are positive on the interval (\delta_T,infinity) *)
           delta26 = Max[x /. NSolve[f26[x] == 0, x, Reals]]; delta26
           Max[Max[x /. NSolve[g26[x] == 0, x, Reals]],
           Max[x /. NSolve[-B26[1, x] == 0, x, Reals]]] <= delta26
           {f26[5000] > 0, g26[5000] > 0, -B26[1, 5000] > 0}

Out[403]= 6.00485

Out[404]= True

Out[405]= {True, True, True}

In[406]:= (*Case T=C_2 \times C_6. Lastly, we verify Lemma 6.1. The code
           below shows that (1) the greatest real root of z_T(x) is \xi_T=0,
           (2) the continuous functions z_T(x) is positive on the interval (\xi_T,infinity) *)
           xi26 = Max[x /. NSolve[z26[x] == 0, x, Reals]]; xi26
           z26[5000] > 0

Out[406]= 0

Out[407]= True

```

```

In[408]:= (*Case T=C_2 \times C_8. We first verify Lemma 4.2. The code below
          shows that (1) the greatest real root of D_T(1,x) is \theta_T\approx 2.41421,
          (2) A_T(1,x) has no real roots and the greatest real root of \hat{D}_T,
          h_T(x) is at most \theta_T, and (3) the continuous functions A_T(1,x),
          D_T(1,x), \hat{D}_T, h_T(x) are positive on the interval (\theta_T,infinity) *)
          theta28 = Max[x /. NSolve[D28[1, x] == 0, x, Reals]]; theta28
          NSolve[A28[1, x] == 0, x, Reals]
          Max[Max[x /. NSolve[Dh28[1, x] == 0, x, Reals]],
              Max[x /. NSolve[h28[x] == 0, x, Reals]]] <= theta28
          {A28[1, 5000] > 0, D28[1, 5000] > 0, Dh28[1, 5000] > 0, h28[5000] > 0}

Out[408]= 2.41421

Out[409]= {}

Out[410]= True

Out[411]= {True, True, True, True}

In[412]:= (*Case T=C_2 \times C_8. Next, we verify Lemma 4.3. The code below
          shows that (1) the greatest real root of f_T(x) is \delta_T\approx 3.38169,
          (2) g_T(x) has no real roots and the greatest real root of -B_T(1,x)
          is at most \delta_T, and (3) the continuous functions f_T(x),
          g_T(x), -B_T(1,x) are positive on the interval (\delta_T,infinity) *)
          delta28 = Max[x /. NSolve[f28[x] == 0, x, Reals]]; delta28
          NSolve[g28[x] == 0, x, Reals]
          Max[x /. NSolve[-B28[1, x] == 0, x, Reals]] <= delta28
          {f28[5000] > 0, g28[5000] > 0, -B28[1, 5000] > 0}

Out[412]= 3.38169

Out[413]= {}

Out[414]= True

Out[415]= {True, True, True}

In[416]:= (*Case T=C_2 \times C_8. Lastly, we verify Lemma 6.1. The code
          below shows that (1) the greatest real root of z_T(x) is \xi_T=0,
          (2) the continuous functions z_T(x) is positive on the interval (\xi_T,infinity) *)
          xi28 = Max[x /. NSolve[z28[x] == 0, x, Reals]]; xi28
          z28[5000] > 0

Out[416]= 0

Out[417]= True

```