In[1]:=
```
(*Below are definitions needed to define phi_{T,u_} in Lemma 3.2. Note
 that these are the same expressions found in definitions.sage file*)
alp2[a_, b_, d_] = (16) * (3 * b^2 * d + a^2);
alp3[c_, d_, e_, b_] = (-1) * (- (c^3 * d^2 * e) + 24 * b) * (c^3 * d^2 * e) ^3;
alp4[c_, d_, b_] = (c^2 * d) ^2 * ((c^2 * d) ^2 + 16 * (c^2 * d) * b + 16 * b^2);
alp5[a_, b_] = a^4 + 12 * a^3 * b + 14 * a^2 * b^2 - 12 * a * b^3 + b^4;
alp6[a_, b_] = (a + 3 * b) * (a^3 + 9 * a^2 * b + 3 * a * b^2 + 3 * b^3);
alp7[a_, b_] = (a^2 - a * b + b^2) *
    (a^6 + 5 * a^5 * b - 10 * a^4 * b^2 - 15 * a^3 * b^3 + 30 * a^2 * b^4 - 11 * a * b^5 + b^6);
alp8[a_, b_] = a^8 - 16 * a^7 * b + 96 * a^6 * b^2 - 288 * a^5 * b^3 +
   480 * a^4 * b^4 - 448 * a^3 * b^5 + 224 * a^2 * b^6 - 64 * a * b^7 + 16 * b^8;
alp9[a_, b_] = (a^3 - 3 * a * b^2 + b^3) * (a^9 - 9 * a^7 * b^2 + 27 * a^6 * b^3 -
      45 * a^5 * b^4 + 54 * a^4 * b^5 - 48 * a^3 * b^6 + 27 * a^2 * b^7 - 9 * a * b^8 + b^9);
alp10[a_, b_] = a^12 - 8 * a^11 * b + 16 * a^10 * b^2 + 40 * a^9 * b^3 - 240 * a^8 * b^4 +
   432 * a^7 * b^5 - 256 * a^6 * b^6 - 288 * a^5 * b^7 + 720 * a^4 * b^8 -
   720 * a^3 * b^9 + 416 * a^2 * b^10 - 128 * a * b^11 + 16 * b^12;
alp12[a_, b_] = (a^4 - 6 * a^3 * b + 12 * a^2 * b^2 - 12 * a * b^3 + 6 * b^4) *
    (a^12 - 18 * a^11 * b + 144 * a^10 * b^2 - 684 * a^9 * b^3 + 2154 * a^8 * b^4 -
      4728 * a^7 * b^5 + 7368 * a^6 * b^6 - 8112 * a^5 * b^7 + 6132 * a^4 * b^8 -
      3000 * a^3 * b^9 + 864 * a^2 * b^10 - 144 * a * b^11 + 24 * b^12);
alp22[a_, b_, d_] = (16) * d^2 * (a^2 - a * b + b^2);
alp24[a_, b_] = a^4 + 16 * a^3 * b + 80 * a^2 * b^2 + 128 * a * b^3 + 256 * b^4;
alp26[a_, b_] = (21 * a^2 - 6 * a * b + b^2) * (6861 * a^6 - 2178 * a^5 * b -
      825 * a^4 * b^2 + 180 * a^3 * b^3 + 75 * a^2 * b^4 - 18 * a * b^5 + b^6);
alp28[a_, b_] = a^16 + 32 * a^15 * b + 448 * a^14 * b^2 + 3584 * a^13 * b^3 + 17 664 * a^12 * b^4 +
   51 200 * a^11 * b^5 + 51 200 * a^10 * b^6 - 237 568 * a^9 * b^7 - 1 183 744 * a^8 * b^8 -
   1 900 544 * a^7 * b^9 + 3 276 800 * a^6 * b^10 + 26 214 400 * a^5 * b^11 + 72 351 744 * a^4 * b^12 +
   117 440 512 * a^3 * b^13 + 117 440 512 * a^2 * b^14 + 67 108 864 * a * b^15 + 16 777 216 * b^16;
bet2[a_, b_, d_] = (-64) * a * (9 * b^2 * d - a^2);
bet3[c_, d_, e_, b_] =
  (-1) * (c^3 * d^2 * e) ^4 * ((c^3 * d^2 * e) ^2 - 36 * (c^3 * d^2 * e) * b + 216 * b^2);
bet4[c_, d_, b_] = ((c^2 * d) + 8 * b) * (c^2 * d) ^3 *
    (- (c^2 * d) ^2 - 16 * (c^2 * d) * b + 8 * b^2);
bet5[a_, b_] = (-1) * (a^2 + b^2) * (a^4 + 18 * a^3 * b + 74 * a^2 * b^2 - 18 * a * b^3 + b^4);
bet6[a_, b_] =
  (-1) * (a^2 + 6 * a * b - 3 * b^2) * (a^4 + 12 * a^3 * b + 30 * a^2 * b^2 + 36 * a * b^3 + 9 * b^4);
bet7[a_, b_] = (-1) * (a^12 + 6 * a^11 * b - 15 * a^10 * b^2 - 46 * a^9 * b^3 +
      174 * a^8 * b^4 - 222 * a^7 * b^5 + 273 * a^6 * b^6 - 486 * a^5 * b^7 +
      570 * a^4 * b^8 - 354 * a^3 * b^9 + 117 * a^2 * b^10 - 18 * a * b^11 + b^12);
bet8[a_, b_] = (-1) * (a^4 - 8 * a^3 * b + 16 * a^2 * b^2 - 16 * a * b^3 + 8 * b^4) *
    (a^8 - 16 * a^7 * b + 96 * a^6 * b^2 - 288 * a^5 * b^3 + 456 * a^4 * b^4 -
      352 * a^3 * b^5 + 80 * a^2 * b^6 + 32 * a * b^7 - 8 * b^8);
bet9[a_, b_] = (-1) * (a^18 - 18 * a^16 * b^2 + 42 * a^15 * b^3 + 27 * a^14 * b^4 -
      306 * a^13 * b^5 + 735 * a^12 * b^6 - 1080 * a^11 * b^7 + 1359 * a^10 * b^8 - 2032 * a^9 * b^9 +
      3240 * a^8 * b^10 - 4230 * a^7 * b^11 + 4128 * a^6 * b^12 - 2970 * a^5 * b^13 +
      1557 * a^4 * b^14 - 570 * a^3 * b^15 + 135 * a^2 * b^16 - 18 * a * b^17 + b^18);
```

```
bet10[a_, b_] = (-1) * (a^2 - 2 * a * b + 2 * b^2) * (a^4 - 2 * a^3 * b + 2 * b^4) *
    (a^4 - 2 * a^3 * b - 6 * a^2 * b^2 + 12 * a * b^3 - 4 * b^4) * (a^8 - 6 * a^7 * b + 4 * a^6 * b^2 +
      48 * a^5 * b^3 - 146 * a^4 * b^4 + 176 * a^3 * b^5 - 104 * a^2 * b^6 + 32 * a * b^7 - 4 * b^8);
bet12[a_, b_] = (-1) * (a^8 - 12 * a^7 * b + 60 * a^6 * b^2 - 168 * a^5 * b^3 +
      288 * a^4 * b^4 - 312 * a^3 * b^5 + 216 * a^2 * b^6 - 96 * a * b^7 + 24 * b^8) *
    (a^16 - 24 * a^15 * b + 264 * a^14 * b^2 - 1776 * a^13 * b^3 + 8208 * a^12 * b^4 -
      27 696 * a^11 * b^5 + 70 632 * a^10 * b^6 - 138 720 * a^9 * b^7 + 211 296 * a^8 * b^8 -
      248 688 * a^7 * b^9 + 222 552 * a^6 * b^10 - 146 304 * a^5 * b^11 + 65 880 * a^4 * b^12 -
      17 136 * a^3 * b^13 + 1008 * a^2 * b^14 + 576 * a * b^15 - 72 * b^16);
bet22[a_, b_, d_] = (-32) * (a - 2 * b) * (a + b) * (2 * a - b) * d^3;
bet24[a_, b_] =
  (-1) * (a^2 + 8 * a * b - 16 * b^2) * (a^2 + 8 * a * b + 8 * b^2) * (a^2 + 8 * a * b + 32 * b^2);
bet26[a_, b_] = (-1) * (183 * a^4 - 36 * a^3 * b - 30 * a^2 * b^2 + 12 * a * b^3 - b^4) *
    (393 * a^4 - 156 * a^3 * b + 30 * a^2 * b^2 - 12 * a * b^3 + b^4) *
    (759 * a^4 - 228 * a^3 * b - 30 * a^2 * b^2 + 12 * a * b^3 - b^4);
bet28[a_, b_] = (-1) * (a^8 + 16 * a^7 * b + 96 * a^6 * b^2 + 256 * a^5 * b^3 -
      256 * a^4 * b^4 - 4096 * a^3 * b^5 - 12 288 * a^2 * b^6 - 16 384 * a * b^7 - 8192 * b^8) *
    (a^8 + 16 * a^7 * b + 96 * a^6 * b^2 + 256 * a^5 * b^3 + 128 * a^4 * b^4 -
      1024 * a^3 * b^5 - 3072 * a^2 * b^6 - 4096 * a * b^7 - 2048 * b^8) *
    (a^8 + 16 * a^7 * b + 96 * a^6 * b^2 + 256 * a^5 * b^3 + 512 * a^4 * b^4 +
      2048 * a^3 * b^5 + 6144 * a^2 * b^6 + 8192 * a * b^7 + 4096 * b^8);
delta21[a_, b_, d_] = 2^8 * b^2 * d * (b^2 * d - a^2);
delta22[a_, b_, d_] = 4 * b^2 * d * (b^2 * d - a^2);
delta24[a_, b_, d_] = 1 / 64 * b^2 * d * (b^2 * d - a^2);
delta3c2d[c_, d_, e_, b_] = 3 * b * d^2 * e^4 * (c^3 * d^2 * e - 27 * b);
delta4c[c_, d_, b_] = 2 * b * c * d^3 * (16 * b + c^2 * d);
delta42c[c_, d_, b_] = 1 / 16 * b * c * d^3 * (16 * b + c^2 * d);
delta51[a_, b_] = b * a * (a^2 + 11 * a * b - b^2);
delta61[a_, b_] = (a + 9 * b) * a * (a + b) * b;
delta62[a_, b_] = 1 / 8 * (a + 9 * b) * a * (a + b) * b;
delta71[a_, b_] = b * a * (a - b) * (a^3 + 5 * a^2 * b - 8 * a * b^2 + b^3);
delta81[a_, b_] = a * (a - 2 * b) * b * (a - b) * (a^2 - 8 * a * b + 8 * b^2);
delta82[a_, b_] = 1 / 8 * a * (a - 2 * b) * b * (a - b) * (a^2 - 8 * a * b + 8 * b^2);
delta91[a_, b_] = b * a * (a - b) * (a^2 - a * b + b^2) * (a^3 + 3 * a^2 * b - 6 * a * b^2 + b^3);
delta101[a_, b_] = a * (a - 2 * b) * b * (a - b) * (a^2 + 2 * a * b - 4 * b^2) * (a^2 - 3 * a * b + b^2);
delta102[a_, b_] =
  1 / 4 * a * (a - 2 * b) * b * (a - b) * (a^2 + 2 * a * b - 4 * b^2) * (a^2 - 3 * a * b + b^2);
delta121[a_, b_] = a * (a - 2 * b) * b * (a - b) * (a^2 - 6 * a * b + 6 * b^2) *
    (a^2 - 2 * a * b + 2 * b^2) * (a^2 - 3 * a * b + 3 * b^2);
delta122[a_, b_] = 1 / 8 * a * (a - 2 * b) * b * (a - b) * (a^2 - 6 * a * b + 6 * b^2) *
    (a^2 - 2 * a * b + 2 * b^2) * (a^2 - 3 * a * b + 3 * b^2);
delta221[a_, b_, d_] = 2^6 * b * a * (a - b) * d^3;
delta222[a_, b_, d_] = b * a * (a - b) * d^3;
delta241[a_, b_] = 8 * a * (a + 8 * b) * b * (a + 4 * b);
delta242[a_, b_] = 1 / 2 * a * (a + 8 * b) * b * (a + 4 * b);
delta244[a_, b_] = 1 / 32 * a * (a + 8 * b) * b * (a + 4 * b);
delta261[a_, b_] = (9 * a - b) * (3 * a - b) * (3 * a + b) * (5 * a - b) * (a - b) * a;
```

```
delta264[a_, b_] = 1 / 8 * (9 * a - b) * (3 * a - b) * (3 * a + b) * (5 * a - b) * (a - b) * a;
delta2616[a_, b_] = 1 / 512 * (9 * a - b) * (3 * a - b) * (3 * a + b) * (5 * a - b) * (a - b) * a;
delta281[a_, b_] = 2 * b * a * (a + 2 * b) * (a + 4 * b) *
    (a^2 - 8 * b^2) * (a^2 + 8 * a * b + 8 * b^2) * (a^2 + 4 * a * b + 8 * b^2);
delta2816[a_, b_] = 1 / 128 * b * a * (a + 2 * b) * (a + 4 * b) * (a^2 - 8 * b^2) *
    (a^2 + 8 * a * b + 8 * b^2) * (a^2 + 4 * a * b + 8 * b^2);
delta2864[a_, b_] = 1 / 4096 * b * a * (a + 2 * b) * (a + 4 * b) * (a^2 - 8 * b^2) *
    (a^2 + 8 * a * b + 8 * b^2) * (a^2 + 4 * a * b + 8 * b^2);
l2 = 1.5;
l3 = 2;
l4 = 2.4;
l5 = 3;
l6 = 3;
l7 = 4;
l8 = 4;
l9 = 4.5;
l10 = 4.5;
l12 = 4.8;
l22 = 2;
l24 = 3;
l26 = 4;
l28 = 4.8;
```

In[99]:=

```
(*Below we define phi_{T,u_T}. We note that when T=C_2 x C_8 or T=C_N,
where N=2,4,9,10,12, our expression for phi_{T,u_T} differs slightly
  from that given in the statement of Lemma 3.2. This is due to the Reduce
  function in Mathematica being unable to process non-integer exponents. Thus,
l_T is is replaced with l_T*j_T where j_T is the least positive
 integer such that l_T*j_T is an integer. Upon exponentiating
 the expression corresponding to u_T^(-12)max() by j_T,
we observe that our function phi_{T,u_T} is nonnegative if and
 only if the function phi in Lemma 3.2 is nonnegative *)
phi21[x_] = (Max[Abs[alp2[1, 1, x]]^3, bet2[1, 1, x]^2])^2 - (Abs[delta21[1, 1, x]])^3;
phi22[x_] =
  (2^(-12) * Max[Abs[alp2[1, 1, x]]^3, bet2[1, 1, x]^2])^2 - (Abs[delta22[1, 1, x]])^3;
phi24[x_] = (2^(-12) * Max[Abs[alp2[1, 1, x]]^3, bet2[1, 1, x]^2])^2 -
    (Abs[delta24[1, 1, x]])^3;
phi3c2d[x_] = (Max[Abs[alp3[1, 1, 1, x]]^3, bet3[1, 1, 1, x]^2]) -
    (Abs[delta3c2d[1, 1, 1, x]])^l3;
phi4c[x_] = (Max[Abs[alp4[1, 1, x]]^3, bet4[1, 1, x]^2])^5 - (Abs[delta4c[1, 1, x]])^12;
phi42c[x_] =
  (2^(-12) * Max[Abs[alp4[1, 1, x]]^3, bet4[1, 1, x]^2])^5 - (Abs[delta42c[1, 1, x]])^12;
phi51[x_] = (Max[Abs[alp5[1, x]]^3, bet5[1, x]^2]) - (Abs[delta51[1, x]])^l5;
phi61[x_] = (Max[Abs[alp6[1, x]]^3, bet6[1, x]^2]) - (Abs[delta61[1, x]])^l6;
phi62[x_] = (2^(-12) * Max[Abs[alp6[1, x]]^3, bet6[1, x]^2]) - (Abs[delta62[1, x]])^l6;
phi71[x_] = (Max[Abs[alp7[1, x]]^3, bet7[1, x]^2]) - (Abs[delta71[1, x]])^l7;
```

```
phi81[x_] = (Max[Abs[alp8[1, x]]^3, bet8[1, x]^2]) - (Abs[delta81[1, x]])^l8;
phi82[x_] = (2^(-12) * Max[Abs[alp8[1, x]]^3, bet8[1, x]^2]) - (Abs[delta82[1, x]])^l8;
phi91[x_] = (Max[Abs[alp9[1, x]]^3, bet9[1, x]^2])^2 - (Abs[delta91[1, x]])^9;
phi101[x_] = (Max[Abs[alp10[1, x]]^3, bet10[1, x]^2])^2 - (Abs[delta101[1, x]])^9;
phi102[x_] =
  (2^(-12) * Max[Abs[alp10[1, x]]^3, bet10[1, x]^2])^2 - (Abs[delta102[1, x]])^9;
phi121[x_] = (Max[Abs[alp12[1, x]]^3, bet12[1, x]^2])^5 - (Abs[delta121[1, x]])^24;
phi122[x_] =
  (2^(-12) * Max[Abs[alp12[1, x]]^3, bet12[1, x]^2])^5 - (Abs[delta122[1, x]])^24;
phi221[x_] = (Max[Abs[alp22[1, x, 1]]^3, bet22[1, x, 1]^2]) -
   (Abs[delta221[1, x, 1]])^l22;
phi222[x_] = (2^(-12) * Max[Abs[alp22[1, x, 1]]^3, bet22[1, x, 1]^2]) -
   (Abs[delta222[1, x, 1]])^l22;
phi241[x_] = (Max[Abs[alp24[1, x]]^3, bet24[1, x]^2]) - (Abs[delta241[1, x]])^l24;
phi242[x_] =
  (2^(-12) * Max[Abs[alp24[1, x]]^3, bet24[1, x]^2]) - (Abs[delta242[1, x]])^l24;
phi244[x_] = (4^(-12) * Max[Abs[alp24[1, x]]^3, bet24[1, x]^2]) -
   (Abs[delta244[1, x]])^l24;
phi261[x_] = (Max[Abs[alp26[1, x]]^3, bet26[1, x]^2]) - (Abs[delta261[1, x]])^l26;
phi264[x_] =
  (4^(-12) * Max[Abs[alp26[1, x]]^3, bet26[1, x]^2]) - (Abs[delta264[1, x]])^l26;
phi2616[x_] = (16^(-12) * Max[Abs[alp26[1, x]]^3, bet26[1, x]^2]) -
   (Abs[delta2616[1, x]])^l26;
phi281[x_] = (Max[Abs[alp28[1, x]]^3, bet28[1, x]^2])^5 - (Abs[delta281[1, x]])^24;
phi2816[x_] =
  (16^(-12) * Max[Abs[alp28[1, x]]^3, bet28[1, x]^2])^5 - (Abs[delta2816[1, x]])^24;
phi2864[x_] = (64^(-12) * Max[Abs[alp28[1, x]]^3, bet28[1, x]^2])^5 -
   (Abs[delta2864[1, x]])^24;
```

In[127]:=
```
(*The code below shows that phi_{T,u_T} is nonnegative for all rational numbers x*)
Reduce[phi21[x] > 0, x, Reals]
Reduce[phi22[x] > 0, x, Reals]
Reduce[phi24[x] > 0, x, Reals]
Reduce[phi3c2d[x] > 0, x, Reals]
Reduce[phi4c[x] > 0, x, Reals]
Reduce[phi42c[x] > 0, x, Reals]
Reduce[phi51[x] > 0, x, Reals]
Reduce[phi61[x] > 0, x, Reals]
Reduce[phi61[x] > 0, x, Reals]
Reduce[phi71[x] > 0, x, Reals]
Reduce[phi81[x] > 0, x, Reals]
Reduce[phi81[x] > 0, x, Reals]
Reduce[phi91[x] > 0, x, Reals]
Reduce[phi101[x] > 0, x, Reals]
Reduce[phi101[x] > 0, x, Reals]
Reduce[phi121[x] > 0, x, Reals]
Reduce[phi121[x] > 0, x, Reals]
Reduce[phi221[x] > 0, x, Reals]
Reduce[phi222[x] > 0, x, Reals]
Reduce[phi241[x] > 0, x, Reals]
Reduce[phi242[x] > 0, x, Reals]
Reduce[phi244[x] > 0, x, Reals]
Reduce[phi261[x] > 0, x, Reals]
Reduce[phi264[x] > 0, x, Reals]
Reduce[phi2616[x] > 0, x, Reals]
Reduce[phi281[x] > 0, x, Reals]
Reduce[phi2816[x] > 0, x, Reals]
Reduce[phi2864[x] > 0, x, Reals]
```

Out[127]= True

Out[128]= True

Out[129]= True

Out[130]= True

Out[131]= True

Out[132]= True

Out[133]= True

Out[134]= True

Out[135]= True

Out[136]= True

Out[137]= True

Out[138]= True

Out[139]= True

Out[140]= True

Out[141]= True

Out[142]= True

Out[143]= True

Out[144]= True

Out[145]= True

Out[146]= $x < \frac{1}{8} \times \left(-3 - \sqrt{5}\right) \,||\, \frac{1}{8} \times \left(-3 - \sqrt{5}\right) < x < \frac{1}{8} \times \left(1 - \sqrt{5}\right) \,||\, \frac{1}{8} \times \left(1 - \sqrt{5}\right) < x < \frac{1}{8} \times \left(-3 + \sqrt{5}\right) \,||\, \frac{1}{8} \times \left(-3 + \sqrt{5}\right) < x < \frac{1}{8} \times \left(1 + \sqrt{5}\right) \,||\, x > \frac{1}{8} \times \left(1 + \sqrt{5}\right)$

Out[147]= $x < \frac{1}{8} \times \left(-3 - \sqrt{5}\right) \,||\, \frac{1}{8} \times \left(-3 - \sqrt{5}\right) < x < \frac{1}{8} \times \left(1 - \sqrt{5}\right) \,||\, \frac{1}{8} \times \left(1 - \sqrt{5}\right) < x < \frac{1}{8} \times \left(-3 + \sqrt{5}\right) \,||\, \frac{1}{8} \times \left(-3 + \sqrt{5}\right) < x < \frac{1}{8} \times \left(1 + \sqrt{5}\right) \,||\, x > \frac{1}{8} \times \left(1 + \sqrt{5}\right)$

Out[148]= $x < \frac{1}{8} \times \left(-3 - \sqrt{5}\right) \,||\, \frac{1}{8} \times \left(-3 - \sqrt{5}\right) < x < \frac{1}{8} \times \left(1 - \sqrt{5}\right) \,||\, \frac{1}{8} \times \left(1 - \sqrt{5}\right) < x < \frac{1}{8} \times \left(-3 + \sqrt{5}\right) \,||\, \frac{1}{8} \times \left(-3 + \sqrt{5}\right) < x < \frac{1}{8} \times \left(1 + \sqrt{5}\right) \,||\, x > \frac{1}{8} \times \left(1 + \sqrt{5}\right)$

Out[149]= True

Out[150]= True

Out[151]= True

Out[152]= True

Out[153]= True

Out[154]= True