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In[1]:= (*This notebook verifies that the following claim from Theorem 4.3:
   for n an integer with |n|>1, max{|c_{4,T}^3|,
  c_{6,T}^2=\mathfrak{h}_T where \mathfrak{h}_T=c_{6,T}^2 with T=C_N where N=3,
  5,7,8,9 and |\mathfrak{h}_T=c_{4,T}^3| for the remaining T. To this end,
  we define c_{4,T}(n) c_{6,T}(n) below:*)

In[2]:= c41[n_] = -144 * n - 48;
c61[n_] = -216;
c42[n_] = 192 * n + 1;
c62[n_] = 576 * n - 1;
c43[n_] = -24 * n + 1;
c63[n_] = -216 * n^2 + 36 * n - 1;
c44[n_] = 5136 * n^4 - 264 * n^2 + 1;
c64[n_] = - (10 224 * n^4 - 504 * n^2 - 1) * (6 * n + 1) * (6 * n - 1);
c45[n_] = 145 * n^4 + 220 * n^3 + 110 * n^2 + 20 * n + 1;
c65[n_] = - (421 * n^4 + 526 * n^3 + 206 * n^2 + 26 * n + 1) * (5 * n^2 + 4 * n + 1);
c46[n_] = (120 * n^3 + 84 * n^2 + 18 * n + 1) * (6 * n + 1);
c66[n_] = - (792 * n^4 + 648 * n^3 + 192 * n^2 + 24 * n + 1) * (24 * n^2 + 12 * n + 1);
c47[n_] =
  (967 * n^6 + 2167 * n^5 + 1920 * n^4 + 855 * n^3 + 200 * n^2 + 23 * n + 1) * (7 * n^2 + 5 * n + 1);
c67[n_] = -577 801 * n^12 - 2 519 622 * n^11 - 4 989 285 * n^10 - 5 920 782 * n^9 - 4 680 102 * n^8 -
   2 590 434 * n^7 - 1 027 173 * n^6 - 293 286 * n^5 - 59 682 * n^4 - 8414 * n^3 - 777 * n^2 - 42 * n - 1;
c48[n_] = -752 * n^8 - 3392 * n^7 - 4128 * n^6 - 1984 * n^5 -
   160 * n^4 + 224 * n^3 + 96 * n^2 + 16 * n + 1;
c68[n_] = - (2696 * n^8 + 5984 * n^7 + 5424 * n^6 + 2272 * n^5 + 184 * n^4 -
      224 * n^3 - 96 * n^2 - 16 * n - 1) * (56 * n^4 + 16 * n^3 - 16 * n^2 - 8 * n - 1);
c49[n_] = (219 * n^9 + 1107 * n^8 + 2475 * n^7 + 3240 * n^6 + 2736 * n^5 + 1539 * n^4 +
     573 * n^3 + 135 * n^2 + 18 * n + 1) * (3 * n^3 + 9 * n^2 + 6 * n + 1);
c69[n_] = -22 329 * n^18 - 242 514 * n^17 - 1 250 883 * n^16 - 4 061 502 * n^15 -
   9 272 961 * n^14 - 15 760 494 * n^13 - 20 613 420 * n^12 - 21 173 562 * n^11 -
   17 291 556 * n^10 - 11 299 356 * n^9 - 5 916 807 * n^8 - 2 474 496 * n^7 -
   819 423 * n^6 - 211 626 * n^5 - 41 607 * n^4 - 5994 * n^3 - 594 * n^2 - 36 * n - 1;
c410[n_] = 635 920 * n^12 + 2 733 440 * n^11 + 5 299 680 * n^10 + 6 129 200 * n^9 + 4 710 480 * n^8 +
   2 534 880 * n^7 + 979 520 * n^6 + 273 840 * n^5 + 54 960 * n^4 + 7720 * n^3 + 720 * n^2 + 40 * n + 1;
c610[n_] = - (5116 * n^8 + 15 328 * n^7 + 19 736 * n^6 + 14 224 * n^5 + 6254 * n^4 +
      1712 * n^3 + 284 * n^2 + 26 * n + 1) * (130 * n^4 + 160 * n^3 + 72 * n^2 + 14 * n + 1) *
   (76 * n^4 + 124 * n^3 + 66 * n^2 + 14 * n + 1) * (10 * n^2 + 6 * n + 1);
c412[n_] = (42 787 896 * n^12 + 129 338 064 * n^11 + 173 452 752 * n^10 + 137 824 296 * n^9 +
     72 709 428 * n^8 + 26 936 592 * n^7 + 7 205 496 * n^6 + 1 405 032 * n^5 + 198 498 * n^4 +
     19 836 * n^3 + 1332 * n^2 + 54 * n + 1) * (366 * n^4 + 348 * n^3 + 120 * n^2 + 18 * n + 1);
c612[n_] = - (15 494 994 936 * n^16 + 61 887 719 232 * n^15 + 113 312 451 024 * n^14 +
      126 922 389 840 * n^13 + 97 696 909 944 * n^12 + 54 936 477 216 * n^11 +
      23 387 872 248 * n^10 + 7 700 419 728 * n^9 + 1 983 804 096 * n^8 + 401 575 488 * n^7 +
      63 706 248 * n^6 + 7 841 616 * n^5 + 734 544 * n^4 + 50 640 * n^3 + 2424 * n^2 + 72 * n + 1) *
   (126 456 * n^8 + 248 736 * n^7 + 207 144 * n^6 + 96 456 * n^5 + 27 648 * n^4 +
     5016 * n^3 + 564 * n^2 + 36 * n + 1);
c422[n_] = 208 * n^2 - 8 * n + 1;
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c622[n_] = -(28*n-1)*(20*n+1)*(4*n-1);
c424[n_] = 976*n^4+672*n^3+200*n^2+24*n+1;
c624[n_] = -(52*n^2+12*n+1)*(28*n^2+12*n+1)*(4*n^2+12*n+1);
c426[n_] =
  (439104*n^6+1005408*n^5+958080*n^4+486360*n^3+138720*n^2+21078*n+1333)*
   (84*n^2+66*n+13);
c626[n_] = -(12144*n^4+18672*n^3+10752*n^2+2748*n+263)*
   (5856*n^4+8928*n^3+5088*n^2+1284*n+121)*
   (3144*n^4+4872*n^3+2832*n^2+732*n+71);
c428[n_] = 51361*n^16+180064*n^15+301720*n^14+511840*n^13+1140780*n^12+
   2129632*n^11+2812328*n^10+2658400*n^9+1853894*n^8+973088*n^7+
   387560*n^6+116768*n^5+26220*n^4+4256*n^3+472*n^2+32*n+1;
c628[n_] = -(431*n^8+592*n^7-204*n^6-944*n^5-
      854*n^4-400*n^3-108*n^2-16*n-1)*
   (337*n^8+944*n^7+1356*n^6+1328*n^5+902*n^4+400*n^3+108*n^2+16*n+1)*
   (47*n^8-176*n^7-780*n^6-1136*n^5-878*n^4-400*n^3-108*n^2-16*n-1);

(*The code below shows that if n is an integer with |n|>1,
  then c_{6,T}^2 - |c_{4,T}|^3≥0 if T=C_N where N=3,5,7,8,
  9 or |c_{4,T}|^3-c_{6,T}^2>=0 for the remaining T. Thus max
    {|c_{4,T}^3|,c_{6,T}^2=\mathfrak{h}_T for each integer n with |n|>1*)

In[32]:= Reduce[Abs[c41[x]]^3-Abs[c61[x]]^2 ≥ 0, x, Reals]
Reduce[Abs[c42[x]]^3-Abs[c62[x]]^2 ≥ 0, x, Reals]
Reduce[Abs[c63[x]]^2-Abs[c43[x]]^3 ≥ 0, x, Reals]
Reduce[Abs[c44[x]]^3-Abs[c64[x]]^2 ≥ 0, x, Reals]
Reduce[Abs[c65[x]]^2-Abs[c45[x]]^3 ≥ 0, x, Reals]
Reduce[Abs[c46[x]]^3-Abs[c66[x]]^2 ≥ 0, x, Reals]
Reduce[Abs[c67[x]]^2-Abs[c47[x]]^3 ≥ 0, x, Reals]
Reduce[Abs[c68[x]]^2-Abs[c48[x]]^3 ≥ 0, x, Reals]
Reduce[Abs[c69[x]]^2-Abs[c49[x]]^3 ≥ 0, x, Reals]
Reduce[Abs[c410[x]]^3-Abs[c610[x]]^2 ≥ 0, x, Reals]
Reduce[Abs[c412[x]]^3-Abs[c612[x]]^2 ≥ 0, x, Reals]
Reduce[Abs[c422[x]]^3-Abs[c622[x]]^2 ≥ 0, x, Reals]
Reduce[Abs[c424[x]]^3-Abs[c624[x]]^2 ≥ 0, x, Reals]
Reduce[Abs[c426[x]]^3-Abs[c626[x]]^2 ≥ 0, x, Reals]
Reduce[Abs[c428[x]]^3-Abs[c628[x]]^2 ≥ 0, x, Reals]
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Out[32]= $x \le -\frac{7}{12} \,||\, x \ge -\frac{1}{12}$

Out[33]= $x \le$ -0.0638… $||\, x \ge 0$

Out[34]= $x \le 0 \,||\, \frac{1}{27} \le x \le$ 0.0654… $||\, x \ge$ 0.491…

Out[35]= $x \le -\frac{1}{2\sqrt{5}} \,||$ -0.195… $\le x \le$ -0.128… $||\, x == 0 \,||$ 0.128… $\le x \le$ 0.195… $||\, x \ge \frac{1}{2\sqrt{5}}$

Out[36]= $x \le -0.559\ldots \;||\; -0.530\ldots \le x \le \frac{1}{10}\times\left(-3-\sqrt{5}\right) \;||\; -\frac{1}{2} \le x \le -0.149\ldots \;||\; -0.0922\ldots \le x \le \frac{1}{10}\times\left(-3+\sqrt{5}\right) \;||\; x \ge 0$

Out[37]= $x \le -\frac{1}{4} \;||\; -0.125\ldots \le x \le -0.0937\ldots \;||\; x \ge -\frac{1}{12}$

Out[38]= $x \le -0.581\ldots \;||\; -0.551\ldots \le x \le -0.543\ldots \;||\; -\frac{1}{2} \le x \le -0.360\ldots \;||\; -0.352\ldots \le x \le -0.349\ldots \;||\; -\frac{1}{3} \le x \le -0.152\ldots \;||\; -0.119\ldots \le x \le -0.108\ldots \;||\; x \ge 0$

Out[39]= $x \le -0.365\ldots \;||\; -0.356\ldots \le x \le -\frac{1}{2\sqrt{2}} \;||\; x == -\frac{1}{3} \;||\; x == -\frac{1}{4} \;||\; x == 0 \;||\; \frac{1}{2\sqrt{2}} \le x \le 0.453\ldots \;||\; x \ge 1.18\ldots$

Out[40]= $x \le -1.53\ldots \;||\; -1.34\ldots \le x \le -1.29\ldots \;||\; -1 \le x \le -0.574\ldots \;||\; -0.556\ldots \le x \le -0.551\ldots \;||\; -\frac{1}{2} \le x \le -0.205\ldots \;||\; -0.168\ldots \le x \le -0.156\ldots \;||\; x \ge 0$

Out[41]= $x \le -\frac{1}{2} \;||\; -0.374\ldots \le x \le -0.365\ldots \;||\; \frac{1}{20}\times\left(-5-\sqrt{5}\right) \le x \le -\frac{1}{4} \;||\; -0.165\ldots \le x \le -0.145\ldots \;||\; x \ge \frac{1}{20}\times\left(-5+\sqrt{5}\right)$

Out[42]= $x \le \frac{1}{6}\times\left(-3-\sqrt{3}\right) \;||\; -0.659\ldots \le x \le -0.475\ldots \;||\; x == -\frac{1}{4} \;||\; -0.215\ldots \le x \le -0.212\ldots \;||\; x \ge \frac{1}{6}\times\left(-3+\sqrt{3}\right)$

Out[43]= True

Out[44]= True

Out[45]= True

Out[46]= True