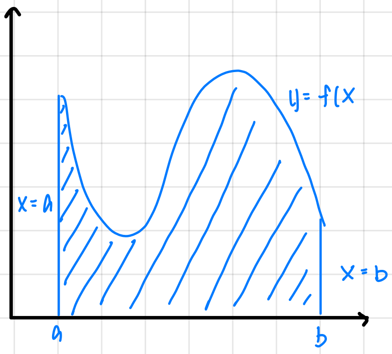
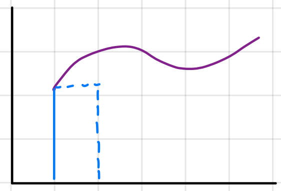
5.1 The Area and Distance Problems

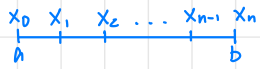


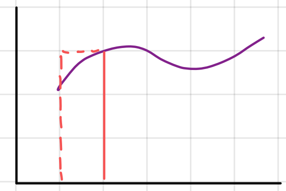
the more rectangles, the more accurate



left endpoint

$L_n = \Delta x[f(x_0)+f(x_1)+\cdots+f(x_{n-1})]$

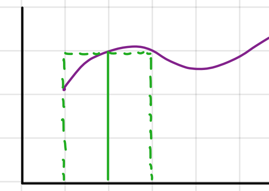




right endpoint

$R_n = \Delta x[f(x_1)+f(x_2)+\cdots+f(x_n)]$

$\Delta x = \frac{b-a}{n}$



midpoint

$S = \{(x, y) \mid a \le x \le b,\ 0 \le y \le f(x)\}$

- the area A of the region S that lies under the graph of a continuous function f is the limit of the sum of areas of approximating rectangles:

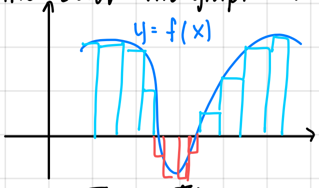
$$A = \lim_{n\to\infty} [f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x]$$

- the limit of distance problems

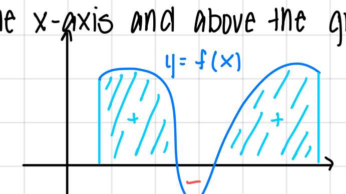
$$d = \lim_{n\to\infty} \sum_{i=1}^{n} f(t_{i-1})\Delta t = \lim_{n\to\infty}\sum_{i=1}^{n} f(t_i)\Delta t$$

5.2 The Definite Integral

- Definite Integral: if f is defined for $a \le x \le b$, we divide the integral $[a, b]$ into n subintervals of equal width $\Delta x = (b-a)/n$. let $x_0 (=a), x_1, x_2, \ldots x_n (=b)$ be the endpoints of these subintervals and we let $x_1^*, x_2^*, \ldots, x_n^*$ be any sample points in these subintervals, so x_i^* lies in the i-th subinterval $[x_{i-1}, x_i]$. then the definite integral of f from a to b is $\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x$ provided that this limit exists and gives the same value for all possible choices of sample points meaning x is integrable on $[a, b]$
- precise definition of the limit that defines the integer N such that $\left| \int_a^b f(x)\,dx - \sum_{i=1}^{n} f(x_i^*)\Delta x \right| < \epsilon$ for every integer $n > N$ and for every choice of x_i^* in $[x_{i-1}, x_i]$
- anatomy of an integral: $\int_a^b f(x)\,dx$ — integral $\to \int$; $b \leftarrow$ upper limit / bound; $a \leftarrow$ lower limit / bound
- Riemann Sum: $\sum_{i=1}^{n} f(x_i^*)\Delta x$
- a definite integral can be interpreted as a net area which is a difference of areas: $\int_a^b f(x)\,dx = A_1 - A_2$, where A_1 is the area of the region above the x-axis and below the graph of f, and A_2 is the area of the region below the x-axis and above the graph of f



$\Sigma f(x_i^*)$ is an approximation to the net area



$\int_a^b f(x)\,dx$ is the net area

- if f is integrable on $[a, b]$, then
$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$$
where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$

- if f is continuous on $[a, b]$, or if f has only a finite number of jump discontinuities, then f is integrable on $[a, b]$; that is, the definite integral $\int_a^b f(x)\,dx$ exists
- when evaluating definite integrals, make sure to work with and understand these sigma notation properties

- Sums of Powers
  1. $\sum_{i=1}^{n} 1 = n$
  2. $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
  3. $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
  4. $\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$

- Properties of Sums
  1. $\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$
  2. $\sum_{i=1}^{n} (a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$

- The Midpoint Rule: $\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(\bar{x}_i)\Delta x = \Delta x[f(\bar{x}_1) + \cdots + f(\bar{x}_n)]$ where $\Delta x = \frac{b-a}{n}$ and $\bar{x}_i = \frac{1}{2}(x_{i-1} + x_i) =$ midpoint of $[x_{i-1}, x_i]$
- if we interchange the bounds, then Δx changes from $\frac{(b-a)}{n}$ to $\frac{(a-b)}{n}$ therefore: $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$
- if $a = b$ and $\Delta x = 0$, $\int_b^a f(x)\,dx = 0$

- Properties of the Integral: assume f and g are continuous functions
  1. $\int_a^b c\,dx = c(b-a)$, where c is any constant
  2. $\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
  3. $\int_a^b c f(x)\,dx = c\int_a^b f(x)\,dx$, where c is any constant
  4. $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$, for integrals of the same function over adjacent intervals

Comparison Properties of the Integral:
1. if $f(x) \ge 0$ for $a \le x \le b$, then $\int_a^b f(x)\,dx \ge 0$
2. if $f(x) \ge g(x)$ for $a \le x \le b$, then $\int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$
3. if $m \le f(x) \le M$ for $a \le x \le b$, then $m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$

5.3 The Fundamental Theorem of Calculus

- Part 1: if f is continuous on $[a, b]$, then the function g defined by $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$

$$g(x) = \int_a^x f(t)\,dt \qquad a \le x \le b$$

is continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) = f(x)$.

- $\frac{g(x+h)-g(x)}{h} = \frac{1}{h}\int_x^{x+h} f(t)\,dt \;\to\; f(u) \le \frac{g(x+h)-g(x)}{h} \le f(v) \;\to\; g'(x) = \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} = f(x) \Rightarrow \frac{d}{dx}\int_a^x f(t)\,dt = f(x)$

- Part 2: if f is continuous on $[a, b]$, then $\int_a^x F'(t)\,dt = F(x) - F(a)$

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any antiderivative of f, that is, a function F such that $F' = f$
- $F(x) = g(x) + C$

- The Fundamental Theorem of Calculus: suppose f is continuous on $[a, b]$.
1. if $g(x) = \int_a^x f(t)\,dt$, then $g'(x) = f(x)$
2. $\int_a^b f(x)\,dx = F(b) - F(a)$, where F is any antiderivative of f. that is, $F' = f$

5.4 Indefinite Integrals and the Net Change Theorem

- indefinite integrals

$\int f(x)\,dx = F(x)$ means $F'(x) = f(x)$

a definite integral $\int_a^b f(x)\,dx$ is a number, whereas an indefinite integral $\int f(x)\,dx$ is a function

$\int_a^b f(x)\,dx = \int f(x)\,dx \Big|_a^b$

- **Table of Indefinite Integrals:**
- $\int cf(x)\,dx = c\int f(x)\,dx$
- $\int k\,dx = kx + C$
- $\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$
- $\int \frac{1}{x}\,dx = \ln|x| + C$
- $\int e^x\,dx = e^x + C$
- $\int \sin x\,dx = -\cos x + C$
- $\int \sec^2 x\,dx =$
- $\int \sec x \tan x\,dx = \sec x + C$
- $\int \sinh x\,dx = \cosh x + C$
- $\int \cosh x\,dx = \sinh x + C$
- $\int \cos x\,dx = \sin x + C$
- $\int \csc^2 x\,dx = -\cot x + C$
- $\int \csc x \cot x\,dx = -\csc x + C$

$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$

$\int b^x\,dx = \frac{b^x}{\ln b} + C$

$\int \frac{1}{x^2+1}\,dx = \tan^{-1}x + C$

$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + C$

- **Net Change Theorem:** the integral of a rate of change is the net change: $\int_a^b F'(x)\,dx = F(b) - F(a)$
(applies to all rates in 3.7)
- if $V(t)$ is the volume of water in a resivor at time t, then its derivative $V'(t)$ is the rate at which water flows into the resivor at time t.
So, $\int_{t_1}^{t_2} V'(t)\,dt = V(t_2) - V(t_1)$ is the change in the amount of water in the reservior between time t_1 and time t_2.
- if $[C](t)$ is the concentration of the product of a chemical reaction at time t, then the rate of reaction is the derivative $\frac{d[C]}{dt}$
So, $\int_{t_1}^{t_2} \frac{d[C]}{dt}\,dt = [C](t_2) - [C](t_1)$ is the change in concentration of C from time t_1 to time t_2.
- if the mass of a rod measured from the left end to a point x is $m(x)$, then the linear density is $\rho(x) = m'(x)$.
So, $\int_a^b \rho(x)\,dx = m(b) - m(a)$ is the mass of the segment of the rod that lies between $x = a$ and $x = b$
- if the rate of growth of a population is $\frac{dn}{dt}$, then $\int_{t_1}^{t_2} \frac{dn}{dt}\,dt = n(t_2) - n(t_1)$ is the net change in population during the time period from t_1 to t_2.
(The population increases when births happen and decreases when deaths occur. The net change takes into account both births and deaths.)
- if $C(x)$ is the cost of producing x units of a commodity, then the marginal cost is the derivative $C'(x)$.
So, $\int_{x_1}^{x_2} C'(x)\,dx \approx C(x_2) - C(x_1)$ is the increase in cost when production is increased from x_1 units to x_2 units.
- if an object moves along a straight line with position function $s(t)$, then its velocity is $v(t) = s'(t)$.
So, $\int_{t_1}^{t_2} v(t)\,dt = s(t_2) - s(t_1)$ is the net change of position or displacement, of the object during the time period from t_1 to t_2.
- $\int_{t_1}^{t_2} |v(t)|\,dt$ = total distance traveled
- the acceleration of the object is $a(t) = v'(t)$, so $\int_{t_1}^{t_2} a(t)\,dt = v(t_2) - v(t_1)$ is the change in velocity from time t_1 to time t_2

5.5 The Substitution Rule

- if $F' = f$, then $\int F'(g(x))\,g'(x)\,dx = F(g(x)) + C$
- **The Substitution Rule:** if $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then $\int f(g(x))\,g'(x)\,dx = \int f(u)\,du$
- $\int \tan x\,dx = \ln|\sec x| + C$
- **The Substitution Rule for Definite Integrals:** if g' continuous on the range of $u = g(x)$, then $\int_a^b f(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$
- **Integrals of Symmetric Functions:** suppose f is continuous on $[-a, a]$
- if f is even $[f(-x) = f(x)]$, then $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$
- if f is odd $[f(-x) = f(x)]$, then $\int_{-a}^{a} f(x)\,dx = 0$
- $\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx = -\int_0^{-a} f(x)\,dx + \int_0^a f(x)\,dx$
- $\int_{-a}^{a} f(x)\,dx = \int_0^a f(-u)\,du + \int_0^a f(x)\,du$