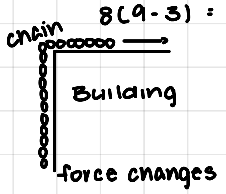
6.4 : work . force : applications: physics

work is defined by "force multiplied by distance"

example: force of 8 Newtons acts on an object and moves it from $x=3$ to $x=9$. Find the work done.

$8(9-3) = 8(6) = 48$ JOULES



our class: variable force, $F(x)$, moves an object from a to b, work done, $W = \int_a^b F(x)\,dx$

example: given $F(x) = \sin \pi x$, moves an object from $x=0$ to $x = \frac{1}{2}$.

$$\int_0^{1/2} \sin \pi x \, dx = \frac{-\cos \pi x}{\pi}\Big|_0^{1/2} = -\frac{1}{\pi}\left[\cos \pi\left(\tfrac{1}{2}\right) - \cos \pi(0)\right] = -\frac{1}{\pi}(0-1) = \boxed{\frac{1}{\pi}}$$

Joules

foot-pounds

Spring - Mass Motion

Hooke's Law

L more force, more extention

if a force stretches a string by x units

Hooke's Law: kw   within elastic limits

$\hookrightarrow$ spring constant

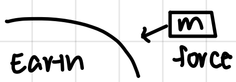
example: force of 20 pounds is used to stretch a spring by 6 inches, beyond it's natural length. find the work done in stretching the spring by 1 foot.

Hooke's Law: $F = kx \rightarrow 20 = k\left(\frac{6}{12}\right) = (20)^2 = \left(\frac{k}{2}\right)^2 \Rightarrow 40 = k$

$F = 40x$

$$\int_a^b F(x)\,dx = \int_0^1 40x\,dx = \frac{40x^2}{2} = 20x^2\Big|_0^1 = \boxed{20 \text{ foot pounds}}$$

Earth's Gravity



force of attraction $F = G\frac{mM}{x^2}$ ← mass of earth

x = distance from the center

weight of object $= mg =$ Force

$9.8\,m/s^2$ $32\,ft/s^2$

How Newton weighed the earth $G\frac{\cancel{m}M}{x^2} = \cancel{m}g \longrightarrow M = \frac{gx^2}{G}$ where x = radius of our earth ≈ 4000 miles

Pumping Water out of a tank

Force excerted by a mass

m is mg "weight"

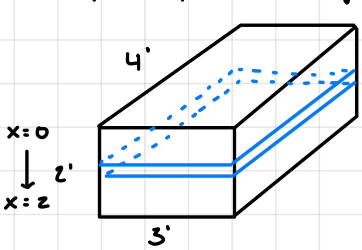
Density $= \frac{\text{mass}}{\text{volume}} = \frac{m}{V} \longrightarrow m = \rho V$ weight is a force, mass is the amount taken up

$\hookrightarrow$ symbol ρ "Rho"

water $= (\rho g) = 62.5$ pounds (weight

SI units $= \rho = 1000$ kg / meter^3

example: aquarium length = 4', width = 3', height = 2' full of water. find the work done in pumping water out.



consider a slice of water

thickness dx at a depth x

area of slab $= (3)(4) = 12$

volume of slab $= 12\,dx$

mass $= \rho V = \rho\, 12 dx$

force exerted by this slab = weight $= mg\ (\rho\, 12\, dx)g$

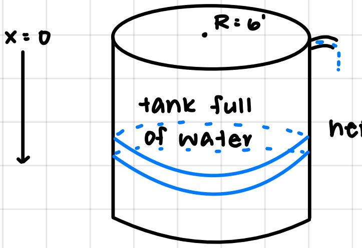
work done in pumping this slab = force · distance

$= (\rho\, 12\, dx)\, g \quad = 12 \rho g\, dx$

total work done $= \int_0^2 12 \rho g x\, dx = 12\rho g \int_0^2 x\,dx = 12\rho g\left(\frac{x^2}{2}\right)\Big|_0^2 = \left(12\rho g\left(\frac{2^2}{2}\right)\right) = \boxed{24\rho g}$

Feb 10

find the work done in pumping water out of a full cylindrical tank.



height = 8'

consider a slab of water, thickness dx, at a depth x from the top

Area of slab = $\pi R^2 = \pi 6^2 = 36\pi$

volume = area · thickness = $36\pi dx$

mass = ρ · volume = $36\pi\rho dx$ where ρ = Rho = density

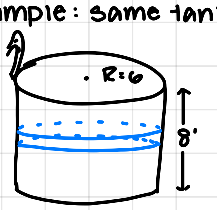
force because this slab = weight "mg" = $(36\pi \rho dx) g$

work done in pumping this slab out = force · distance = $(36\pi \rho g\, dx)\cdot(x)$

work done for the whole tank = $\int_0^8 36\pi\rho g x\, dx = 36\pi\rho g \frac{x^2}{2}\Big|_0^8 = 36\pi\rho g \frac{8^2}{2} = 36\pi\rho g\cdot 32 = \boxed{W = (32)(36)\pi\rho g}$

example: same tank; pump water out of a spout; spout is 2 feet

book: $\rho g = \delta = 62.5$



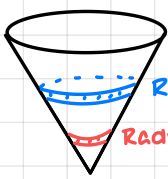
the typical slab has to be pumped x feet and another 2 feet "x + 2"

$W = \int_0^8 36\pi\rho g(x+2)\,dx = \pi\rho g\int_0^8 36(x+2)\,dx = \pi\rho g\left[\frac{36x^2}{2} + 72x\right]\Big|_0^8$

$= \pi\rho g\left[18(8)^2 + 72(8)\right] = \boxed{1728\pi\rho g \text{ or } 550 \text{ foot pounds per second}}$

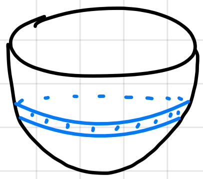
Irregular tanks

Radius of slab changes, Radius = R(x)

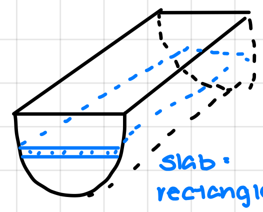


Radius = R

Radius = r

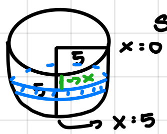


Radius = R(x)

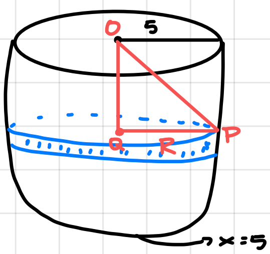


slab = rectangle

example: hemispherical tank, radius = 5. full of water. how much work is done in pumping water from the top? no spout



Solve: take a slab of water, thickness dx at a depth x from the top



|OP| = 5

In ΔOPQ, $OP^2 = OQ^2 + PQ^2$

$5^2 = x^2 + R^2 \Rightarrow R^2 = 5^2 - x^2$

$\boxed{R = \sqrt{25 - x^2}}$

Area of slab = $\pi R^2 = \pi(25 - x^2)$

volume of slab = $\pi(25 - x^2)\,dx$

mass = volume · ρ = $\rho\pi(25 - x^2)\,dx$

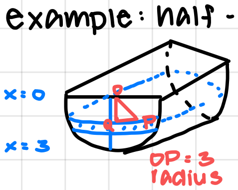
force = weight of slab = $\rho\pi(25 - x^2)\,dx$

distance to be pumped = x

work = $\rho\pi g x(25 - x^2)\,dx$

total work for the tank: $\int_0^5 \rho\pi g x(25 - x^2)\,dx = \rho\pi g\int_0^5 (25x - x^3)\,dx = \rho\pi g\left[\frac{25x^2}{2} - \frac{x^4}{4}\right]\Big|_0^5 = \frac{25(5)^2}{2} - \frac{5^4}{4} = \boxed{\frac{625}{4}\rho\pi g}$

example: half-cylinder tank; length 4; radius 3. find the work done in pumping water @ top.



Solve: typical slab of water, thickness dx, depth x from the top, looks rectangular

length = 4

width =

OQ = depth = x ; ΔOPQ $OP^2 = OQ^2 + QP^2$

$3^2 = x^2 + (QP)^2$

$\sqrt{3^2 - x^2} = \sqrt{QP^2} \Rightarrow QP = \sqrt{3^2 - x^2}$ = half width of slab

$2\sqrt{3^2 - x^2}$ = double = full width of slab

area of slab = length · width = $4\cdot 2\sqrt{9 - x^2} = 8\sqrt{9 - x^2}$

volume of slab = area · thickness = $8\sqrt{9 - x^2}\,dx$

mass of slab = $\rho V = \rho 8\sqrt{9 - x^2}\,dx$

weight of slab = force = "mg" = $\rho g 8\sqrt{9 - x^2}\,dx$

this slab has to be pumped "carried" x units

work done on this slab = force · distance = $\rho g 8x\sqrt{9 - x^2}\,dx$

entire tank: $\int_0^3 8\rho g x\sqrt{9 - x^2}\,dx = 8\rho g\int_0^3 x\sqrt{9 - x^2}\,dx$

x = 0, u = 9

$u = 9 - x^2$

$\frac{du}{-2} = \frac{-2x\,dx}{-2}$

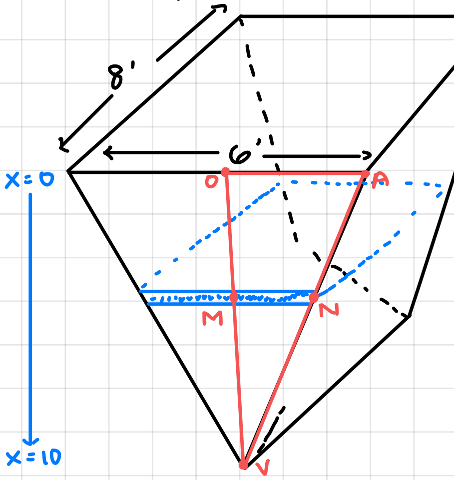
$-\frac{du}{2} = x\,dx$

$= 8\rho g\int_0^3 \sqrt{u}\left(-\tfrac{1}{2}du\right) = \frac{8\rho g}{-2}\int_0^9 u^{1/2}\,du$

$= -4\rho g\int_0^9 u^{1/2}\,du = 4\rho g\int_9^0 u^{1/2}\,du$

$4\rho g \int_9^0 u^{1/2}\,du = 4\rho g \left[\frac{u^{3/2}}{3/2}\right]\Big|_9^0 = \frac{2}{3}\cdot 4\rho g\left(9^{3/2} - 0^{3/2}\right) = \frac{8}{3}\rho g(27) = \boxed{72\rho g \text{ foot pounds}}$

last example: find the work done in pumping water through the spout



front face is a triangle

depth = 10 ft full tank

spout is another 2 feet

typical slab, thickness dx, depth x from top

slab is rectangular. length = 8' ; width = ?

length OA = 3, OV = 10

length OM = x

MV = $10-x$

similar Δ's, $\frac{MN}{OA} = \frac{MV}{OV} = \frac{MN}{3}$

MN = half width of slice → total width = "double" = $2\left(\frac{3}{10}(10-x)\right) = \frac{10-x}{10}$ => MN = $3\left(\frac{10-x}{10}\right)$

Area of slab = length · width = $8\cdot 2\cdot\frac{3}{10}(10-x) = \frac{24}{5}(10-x)$

Volume of slab = $\frac{24}{5}(10-x)\,dx$

mass of slab = ρ · volume = $\rho\frac{24}{5}(10-x)\,dx$

weight = force = $g\rho\frac{24}{5}(10-x)\,dx$

work done in pumping this slice = force · distance = $\rho g\frac{24}{5}(10-x)\,dx\,(x+2)$

spout is 2 feet

total work = $\int_0^{10}\rho g\frac{24}{5}(10-x)(x+2)\,dx = \rho g\int_0^{10}\frac{24}{5}(10x+20-x^2-2x)\,dx = \rho g\int_0^{10}\frac{24}{5}(-x^2-8x+20)\,dx = \frac{24}{5}\rho g\int_0^{10}(-x^2-8x+20$

$\frac{8x^2}{2} + 20x - \frac{x^3}{3}\Big|_0^{10} = 4(10)^2 + 20(10) - \frac{(10)^3}{3} = \frac{800}{3}\cdot\frac{24}{5} = \boxed{1280\rho g \text{ ft lbs}}$