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PARAMETERIZATION OF STATE FEEDBACK GAINS FOR POLE PLACEMENT

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Abstract

The pole placement problem has been a subject for research for a long time. It is well known that state feedback control is an efficient technique for the pole placement problem. For single-input systems this problem is well understood but for multi-input systems the pole placement problem is more complex. In this paper, a parameterization of state feedback gains for pole placement is characterized with respect to completeness and existence. This parameterization depends on two matrices that can be regarded as design parameters. It is shown how the degree of freedom in the pole placement problem for multi-input systems is characterized by these two matrices. It turns out that the properties of the parameterization depend on whether the characteristic polynomials of the open and the closed loop systems are coprime or not. In this paper the case when they are coprime is emphasized. It is shown that for this case every possible feedback gain can be parameterized in this way, and in this sense the parameterization is complete. The parameterization implies that a certain matrix is invertible. Necessary conditions for when this matrix is invertible are given in terms of the two design parameters.

1 Introduction

State feedback control of linear systems can be determined by specifying different control objectives. Well known design approaches like pole placement and LQ control are presented in most textbooks on the subject. While the LQ control frame work is independent of the number of inputs, there is a significant difference between the single-input and the multi-input case for the pole placement approach. The reason for this is that in the multi-input case, the feedback gain matrix is not uniquely determined by the desired closed loop pole locations. This implies that further design objectives can be stated; see for instance [8], [7] and [13] for some different approaches where other design objectives are combined with pole placement. However, opposite to the single-input case, there seems to be a lack of explicit expressions for state feedback gains for pole placement in the multi-input case, and most expressions given require that the system is transformed to some certain form.

In this paper the parameterization of state feedback gains for pole placement for multi-input systems presented in Nordström and Norlander [9] is investigated and characterized. This parameterization gives an analytical expression for the feedback gains that only involves the system matrices of the open loop system and two design parameters. The main result of the paper is that this parameterization covers every possible feedback gain when all the open loop system poles should be modified. It also contains a discussion on how the design parameters characterize the closed loop system.

The paper is confined to a treatment of the continuous-time state feedback control problem. However, since the discussion mainly concerns algebraic properties, the results are valid for discrete-time systems as well. With minor and obvious modifications they are also applicable to the observer design problem.

The paper is organized as follows. In Section 2 a problem formulation is given and necessary notations are introduced. The proposed parameterization of state feedback gains for pole placement is presented in Section 3. In Section 4 the role of the two design parameters is discussed, and it is shown that the parameterization is complete when all poles are modified. Then in Section 5 the existence of parameterized feedback gains is investigated, and some necessary conditions are given.

2 Problem formulation

Consider the continuous-time multi-input linear time-invariant system with the dynamic state equation

$$\frac{d}{dt}x(t) = Ax(t) + Bu(t), \tag{1}$$

with vectors $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$ for every fixed t, and constant matrices $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$. Without loss of generality, the matrix B is assumed to have full (column) rank.

Of fundamental importance for the behavior of the system (1) is the location of its poles, *i.e.*, the eigenvalues of the matrix A, given by the zeros of the characteristic polynomial

$$a(s) \overset{\triangle}{=} \det(sI - A) = s^n + a_1 s^{n-1} + \cdots + a_n. \tag{2}$$

For instance, the location of the poles determines entirely whether the system (1) is asymptotically stable, marginally sta-

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ble or unstable. Therefore it is desirable to be able to modify the location of the poles using state feedback control:

$$u(t) = -Lx(t), \quad L \in \mathbb{R}^{m\times n}. \tag{3}$$

Let

$$p(s) \triangleq s^n + p_1 s^{n-1} + \cdots + p_n, \tag{4}$$

where $p_i \in \mathbb{R}$ for $i = 1, \ldots, n$, be the desired characteristic polynomial of the closed loop system. The problem is now to find a feedback gain matrix L so that the characteristic polynomial of the closed loop system coincide with (4) when the control law (3) is applied to (1). That is, $\det(sI - A + BL) = p(s)$ should hold. In the single-input case ($m = 1$) the feedback gain, which is a row vector, is uniquely determined by $p(s)$, and there are several analytic expressions for it, for instance Ackermann's formula and the Bass-Gura formula [4]. In the multi-input case, however, the feedback gain is not uniquely determined by $p(s)$, and none of the mentioned formulas are applicable. In what follows, an analytic parameterization of the feedback gain for the multi-input case will be given. For convenience, though, some notations will first be introduced.

The characteristic polynomials $a(s)$ and $p(s)$ will be associated to vectors whose entries are the coefficients of the polynomials respectively:

$$\alpha \triangleq \begin{bmatrix} a_n \\ \vdots \\ a_1 \end{bmatrix}, \quad \pi \triangleq \begin{bmatrix} p_n \\ \vdots \\ p_1 \end{bmatrix}, \quad \alpha, \pi \in \mathbb{R}^n.$$

Compare with equations (2) and (4) and note the reverse order of the coefficients in α and π.

To further abbreviate the notations, the standard ON-base vectors in $\mathbb{R}^n$ will be denoted e_i for $i = 1, \ldots, n$, *i.e.*, a 1 at the ith entry and zeros elsewhere. Especially $e_n = \begin{bmatrix} 0 & \ldots & 0 & 1 \end{bmatrix}^T$ will be used frequently. The square ($n \times n$) matrix, with ones in all entries of the first subdiagonal and zeros elsewhere, will be denoted J. Thus $J = \begin{bmatrix} e_2 & \ldots & e_n & 0 \end{bmatrix}$. The identity matrix will be denoted I, and an index marks the size of it (for instance, I_m is the $m \times m$ identity matrix).

To any monic polynomial, a companion matrix can be associated. The *right companion matrices* associated to $a(s)$ and $p(s)$ are

$$K_\alpha \triangleq J - \alpha e_n^T \quad \text{and} \quad K_\pi \triangleq J - \pi e_n^T.$$

An important property of companion matrices (which explains the association to polynomials) is that

$$\det(sI - K_\alpha) = a(s) \quad \text{and} \quad \det(sI - K_\pi) = p(s).$$

A fundamental result in linear system theory is that an arbitrary characteristic polynomial for the closed loop system can be achieved by state feedback control if and only if the open loop system is controllable, *i.e.*, if and only if the controllability matrix, $\mathcal{C}$, has full rank. The controllability matrix of the system (1) is defined as

$$\mathcal{C} \triangleq \begin{bmatrix} B & AB & \ldots & A^{n-1}B \end{bmatrix}. \tag{5}$$

Throughout this paper the system (1) is assumed to be controllable.

Polynomials with a matrix in the argument will be used in the sequel. For instance, for $p(\cdot)$ in (4) we may write

$$p(A) = A^n + p_1 A^{n-1} + \cdots + p_n I.$$

Using the concept of Kronecker product, see *e.g.* [3], we have

$$p(A)B = \mathcal{C}\left[(\pi - \alpha) \otimes I_m\right], \tag{6}$$

and

$$A\mathcal{C} = \mathcal{C}\left[K_\alpha \otimes I_m\right]. \tag{7}$$

3 The parameterized state feedback gains

An explicit parameterization of the feedback gain for the multi-input state feedback pole placement problem is given by the following theorem:

Theorem 3.1 ([9], [10]) *Let $p(s)$ be a given monic polynomial, $M \in \mathbb{R}^{n\times n}$ a matrix such that $\det(sI - M) = p(s)$, and let $\Gamma \in \mathbb{R}^{nm\times n}$ satisfy the equation*

$$\left[K_\pi \otimes I_m\right]\Gamma = \Gamma M. \tag{8}$$

If the product $\mathcal{C}\Gamma$ is nonsingular, then

$$\det(sI - A + BL) = p(s),$$

with A and B as in Equation (1) and

$$L = \left[e_n^T \otimes I_m\right]\Gamma(\mathcal{C}\Gamma)^{-1}p(A). \tag{9}$$

Proof See [9] or [10]. ■

The theorem states that the closed loop system has the desired poles under the mentioned assumptions. The feedback gain is thus given by (9).

To justify that (9) truly is a closed form expression for the feedback gain, the following lemma states an explicit expression for the matrix Γ.

Lemma 3.2 ([9], [10]) *All solutions to (8) are*

$$\Gamma = \begin{bmatrix} \Gamma_0 \chi_{n-1}(M) \\ \vdots \\ \Gamma_0 \chi_1(M) \\ \Gamma_0 \end{bmatrix}, \tag{10}$$

where $\chi_j(\cdot)$, $j = 1, \ldots, n-1$ are polynomials defined as

$$\chi_j(s) \triangleq s^j + \sum_{k=1}^{j} p_k s^{j-k}, \tag{11}$$

and where $\Gamma_0 \in \mathbb{R}^{m\times n}$ is an arbitrary matrix.

Remark An equivalent definition of the polynomials in (11) is the recursive formula

$$\chi_0(s) \triangleq 1, \quad \chi_j(s) \triangleq s\chi_{j-1}(s) + p_j, \quad j = 1, \ldots, n-1.$$

■

Proof See [10]. ■

Remark In the single-input case, where $m = 1$, Γ_0 will be a row vector and Γ will be a square $n \times n$ matrix, which will be nonsingular for an appropriate choice of Γ_0 and M. Furthermore, the controllability matrix will be a square $n \times n$ matrix, which is nonsingular since the open loop system is assumed to be controllable. Then (9) turns to

$$L = e_n^T \Gamma \left(\mathcal{C}\Gamma\right)^{-1} p(A) = e_n^T \mathcal{C}^{-1} p(A),$$

which is recognized as Ackermann's formula (see for instance Kailath [4] Section 3.2). ■

Applying Lemma 3.2 to (9) yields the somewhat more convenient expression

$$L = \Gamma_0 (\mathcal{C}\Gamma)^{-1} p(A). \tag{12}$$

When the results in Theorem 3.1 and Lemma 3.2 originally were presented in [9], the right companion matrix K_π was used instead of M in the right hand side of (8). A possible choice that works in most cases is of course to set $M = K_\pi$. However, by allowing any M satisfying $\det(sI - M) = p(s)$, the results in Theorem 3.1 and Lemma 3.2 become more general; see Theorem 4.1 and the following discussion.

The parameterization of state feedback gains given in Theorem 3.1 and Lemma 3.2 gives rise to a number of questions. In the subsequent sections some of these questions are discussed. For instance, how efficient is the parameterization? That is, does the parameterization cover all possible feedback gains? If not, which feedback gains can, or cannot, be parameterized in this way? These questions are discussed in Section 4.

The matrices M and Γ_0 in the feedback gain parameterization may both be regarded as design parameters. The restriction $\det(sI - M) = p(s)$ perhaps suggests that M should be chosen in a first turn, leaving Γ_0 as the free design parameter. In Section 4 it is shown that this is a reasonable procedure. The next question then is how Γ_0 should be chosen. The parameterization relies on the existence of the inverse $(\mathcal{C}\Gamma)^{-1}$, so a first matter of interest is to choose Γ_0 so that $\mathcal{C}\Gamma$ is nonsingular. This is a nontrivial issue. In Section 5 the existence of $(\mathcal{C}\Gamma)^{-1}$ is discussed, and some necessary conditions for the existence are presented.

4 Characterization of the feedback gain parameterization

The discussion here is confined to the case when $p(s)$ and $a(s)$ are coprime, *i.e.*, when all poles of the system (1) are modified. This is precisely the case when the matrix $p(A)$ is nonsingular, which is of great importance in what follows.

The parameterization in Theorem 3.1 and Lemma 3.2 seems to offer two design parameters, the matrices M and Γ_0. While the choice of M is restricted by the property $\det(sI - M) = p(s)$, Γ_0 is more of a free parameter. Indeed, when using the parameterization to generate a feedback gain, the matrix M can be chosen as a first step, leaving Γ_0 as the only design parameter in succeeding steps. It may seem that the only function of M is that its eigenvalues are the desired poles of the closed loop system. However, the role of matrix M is a little more involved than that, as is stated in the following theorem:

Theorem 4.1 *Let the polynomial $p(s)$ and the matrix M be given according to the conditions in Theorem 3.1. If $p(s)$ and $a(s) = \det(sI - A)$ are coprime, and if there exists a feedback gain L given by* (9)*, then the matrices $A - BL$ and M are similar.*

Remark This means that there exists a nonsingular matrix T such that $T^{-1}(A - BL)T = M$. It also means that $A - BL$ and M have the same Jordan form. ■

See Appendix for a proof of Theorem 4.1.

Thus, the matrix M determines the Jordan form of the closed loop system matrix $A - BL$. In fact, one possible choice is to let M be the desired real Jordan form of $A - BL$. This restricts which matrices M that are feasible for the parameterization when there are multiple poles among the desired poles of the closed loop system. While the algebraic multiplicity of a desired closed loop system pole can be arbitrarily high, the *geometric multiplicity* is restricted. Particularly, the latter cannot be higher than the number of inputs. Exactly how high geometric multiplicities that are possible depends on the controllability indices as described by Rosenbrock's control structure theorem [12] (see for instance Section 7.2.2 in Kailath [4], or Section 3.1 in Kučera [5]).

With calculations very similar to the ones in the proof of Theorem 4.1 it is possible to show that $(p(A))^{-1}\,\mathcal{C}\Gamma$ is the solution to a certain linear matrix equation.

Lemma 4.2 ([10]) *If the polynomials $a(s) = \det(sI - A)$ and $p(s) = \det(sI - M)$ are coprime, then the product $(p(A))^{-1}\,\mathcal{C}\Gamma$ is the unique solution to the equation*

$$AX - XM = B\Gamma_0, \tag{13}$$

with matrices $\mathcal{C}$, Γ and Γ_0 according to (5) *and Lemma 3.2.*

Proof See [10]. ■

Notice that $\mathcal{C}\Gamma$ need not to be nonsingular, $(p(A))^{-1}\,\mathcal{C}\Gamma$ is the unique solution of (13) anyway. Linear matrix equations of the type (13) are called Sylvester equations, and have been used quite frequently in linear control theory, like in the theory for observers (see *e.g.* Luenberger [6] or O'Reilly [11]). Sylvester equations also have been used for pole placement of multi-input systems (*e.g.* in Bhattacharyya and de Souza [1]). Lemma 4.2 will be used to show that the parameterization presented here is complete in the case when $p(s)$ and $a(s)$ are coprime.

Theorem 4.3 *Every feedback gain L, such that $a(s) = \det(sI - A)$ and $p(s) = \det(sI - A + BL)$ are coprime, can be parameterized as in Theorem 3.1 and Lemma 3.2.*

See Appendix for a proof of Theorem 4.3.

Hence, the parameterization is complete for the case when all the poles of (1) are modified — complete in the sense that every possible feedback gain can be generated by (9) and appropriate choices of the matrices M and Γ_0.

Theorems 4.1 and 4.3 indicate how the design parameters M and Γ_0 should be used. The choice of M has relevance when $p(s)$ has multiple zeros, since then there are several possible Jordan forms for $A - BL$. Once M is chosen, and thus the Jordan form of $A - BL$ is determined, Γ_0 can be used to generate all, and only those, feedback gains L that will yield that particular Jordan form of $A - BL$.

A simple example illustrates how the parameterization can be used.

Example 4.1 Consider the third order system with two inputs

$$\frac{d}{dt}x = Ax + Bu, \quad A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix},$$

with all poles in the origin. Assume that $p(s) = (s+1)^3$ is the desired characteristic polynomial, *i.e.*, the closed loop system should have a triple pole in -1. There are two possible Jordan forms for $A - BL$, depending on whether one or two eigenvectors of $A - BL$ are desired. The choices

$$M_1 = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}, \quad M_2 = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

correspond to these two cases respectively, according to Theorem 4.1. The choice $M_3 = -I_3$ would correspond to having three eigenvectors of $A - BL$, and this is not possible to obtain by state feedback according to Rosenbrock's control structure theorem [12]. Using

$$\Gamma_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

with M_1 and M_2 in (12) give

$$L_1 = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad L_2 = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

respectively. It is readily verified $\det(sI - A + BL_1) = \det(sI - A + BL_2) = (s+1)^3$, and that $\operatorname{rank}(-I - A + BL_1) = 2$ and $\operatorname{rank}(-I - A + BL_2) = 1$. ◦

The parameterization is quite successful when $a(s)$ and $p(s)$ are coprime. However, the situation is less favorable for the case when $a(s) = \det(sI - A)$ and the desired characteristic polynomial $p(s)$ are *not* coprime. This is mainly due to the fact that $p(A)$, that is a part of the parameterization, is singular when some eigenvalue of A coincides with some zero of $p(s)$. A more thorough characterization of the parameterization is presented in Norlander [10], where the case when $p(s)$ and $a(s)$ are not coprime is discussed.

5 On the existence of the parameterized feedback gains

The parameterization of feedback gain matrices for pole placement presented in Theorem 3.1 and Lemma 3.2 relies on the invertability of the matrix $\mathcal{C}\Gamma$.

For $\mathcal{C}\Gamma$ to be invertible it must have rank n. There are several ways to write the product $\mathcal{C}\Gamma$, and below we will elaborate on this in order to give necessary conditions for nonsingularity. A well known property (see for instance [2]) is that $\operatorname{rank}\mathcal{C}\Gamma \le \min(\operatorname{rank}\mathcal{C}, \operatorname{rank}\Gamma)$. Thus obvious necessary conditions for nonsingularity of $\mathcal{C}\Gamma$ is that $\operatorname{rank}\mathcal{C} = n$ and that $\operatorname{rank}\Gamma = n$. Due to the controllability assumption it holds that $\operatorname{rank}\mathcal{C} = n$. However, some basic manipulations of the product $\mathcal{C}\Gamma$ reveals stronger conditions, which offer a nice interpretation in terms of linear system theory.

Introduce the matrix

$$\Pi \triangleq \begin{bmatrix} p_{n-1} & \cdots & p_1 & 1 \\ \vdots & \iddots & 1 & 0 \\ p_1 & \iddots & \iddots & \vdots \\ 1 & 0 & \cdots & 0 \end{bmatrix},$$

i.e., a Hankel matrix in the coefficients of $p(\cdot)$. Then Γ can be factorized as

$$\Gamma = \begin{bmatrix} \Pi \otimes I_m \end{bmatrix} \begin{bmatrix} \Gamma_0 \\ \Gamma_0 M \\ \vdots \\ \Gamma_0 M^{n-1} \end{bmatrix}. \tag{14}$$

The matrix $\begin{bmatrix} \Pi \otimes I_m \end{bmatrix}$ is square and has full rank, so Γ will have full rank if and only if the second matrix in (14) has full rank. The second matrix is recognized as the observability matrix for the pair (Γ_0, M).

It is now possible to write the product $\mathcal{C}\Gamma$ as

$$\begin{aligned}\mathcal{C}\Gamma &= \mathcal{C}\left[\Pi \otimes I_m\right] \begin{bmatrix} \Gamma_0 \\ \Gamma_0 M \\ \vdots \\ \Gamma_0 M^{n-1} \end{bmatrix} \\ &= \begin{bmatrix} B\Gamma_0 & AB\Gamma_0 & \dots & A^{n-1}B\Gamma_0 \end{bmatrix} \left[\Pi \otimes I_n\right] \begin{bmatrix} I_n \\ M \\ \vdots \\ M^{n-1} \end{bmatrix}. \end{aligned} \tag{15}$$

The matrix $\begin{bmatrix} I_n & M^T & \dots & (M^{n-1})^T \end{bmatrix}^T$ has full rank. The expression (15) shows that necessary conditions for the nonsingularity of $\mathcal{C}\Gamma$ are that $(A, B\Gamma_0)$ is controllable and that (Γ_0, M) is observable. If Γ_0 has full rank $(A, B\Gamma_0)$ is controllable if and only if (A, B) is controllable. There are no restrictions on the rank of Γ_0, though. In fact, by choosing Γ_0 as a rank one matrix, *i.e.*, $\Gamma_0 = q\gamma_0^T$ for some vectors q and γ_0^T, it is *sufficient* to have (A, Bq) controllable and (γ_0^T, M) observable for $\mathcal{C}\Gamma$ to be nonsingular. This of course reduces the system to a "single-input" system and it is only applicable when the system matrix A is cyclic.

The discussion so far leads to the following lemma:

Lemma 5.1 *The matrix $\mathcal{C}\Gamma$ is nonsingular only if $(A, B\Gamma_0)$ is controllable and (Γ_0, M) is observable.*

It is stressed that these conditions are necessary, but not sufficient, which is illustrated in the following example.

Example 5.1 Reconsider the system in Example 4.1, and assume again that $p(s) = (s+1)^3$ is the desired characteristic polynomial of the closed loop system. First, notice that the choice $M_3 = -I_3$ is ruled out by Lemma 5.1, since the pair $(\Gamma_0, -I_3)$ is unobservable for every $\Gamma_0 \in \mathbb{R}^{2\times 3}$. With

$$M_2 = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad \text{and} \quad \Gamma_0 = \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \end{bmatrix},$$

the conditions in Lemma 5.1 are satisfied exactly when $g_{11}g_{23} - g_{13}g_{21} \neq 0$. However,

$$\det \mathcal{C}\Gamma = g_{11}(g_{11}g_{23} - g_{13}g_{21}),$$

so in case $g_{11} = 0$ $\mathcal{C}\Gamma$ becomes singular, even though $g_{11}g_{23} - g_{13}g_{21} \neq 0$. ∘

It should be pointed out that, for the case when $a(s) = \det(sI - A)$ and the desired characteristic polynomial $p(s)$ are coprime, the matrix $\mathcal{C}\Gamma$ is nonsingular for almost every Γ_0 and any feasible M. The reason for this is that according to Theorem 4.3 there exist matrices M and Γ_0 for which $\mathcal{C}\Gamma$ is nonsingular. Say that the (feasible) matrix M is fixed and Γ_0 is allowed to be varied. The matrix $\mathcal{C}\Gamma$ then depends linearly on Γ_0, and thus $\det \mathcal{C}\Gamma$ is a polynomial in the entries of Γ_0. Now, $\mathcal{C}\Gamma$ is singular exactly when $\det \mathcal{C}\Gamma = 0$, but since there exist matrices Γ_0 for which $\det \mathcal{C}\Gamma \neq 0$, due to the continuity of polynomials $\det \mathcal{C}\Gamma \neq 0$ will hold for almost all matrices Γ_0. In this sense the nonsingularity of $\mathcal{C}\Gamma$ is generic. Particularly, if $\mathcal{C}\Gamma$ is singular for Γ_0^*, every neighborhood of Γ_0^* contains matrices Γ_0 for which $\mathcal{C}\Gamma$ is nonsingular.

6 Conclusions

A somewhat modified version of the parameterization of state feedback gains for pole placement in [9] has been presented and characterized in terms of completeness and existence. It has been shown that the parameterization is complete in the case when all poles of the open loop system should be modified. The parameterization involves two design parameters. One of these, a square matrix, determines the Jordan form of the closed loop system matrix $A - BL$, while the other one is an almost free design parameter, which *e.g.* can be used to meet additional control objectives (see for instance Chapter 7 in [10]). The existence of parameterized feedback gains has also been discussed, and some necessary conditions on the design parameters have been formulated.

Appendix

Proof of Theorem 4.1 The proof is based on the fact that it can be shown that $T^{-1}(A - BL)T = M$ holds for $T = (p(A))^{-1}\,\mathcal{C}\Gamma$. The inverse of $p(A)$ exists since $p(s)$ and $a(s)$ are coprime. The existence of a L given by (9) implies that the matrix $\mathcal{C}\Gamma$ is nonsingular. A straightforward calculation gives

$$\begin{aligned} &(\mathcal{C}\Gamma)^{-1}\, p(A)(A - BL)\,(p(A))^{-1}\,\mathcal{C}\Gamma \\ &= (\mathcal{C}\Gamma)^{-1}\left(A\mathcal{C}\Gamma - p(A)BL\,(p(A))^{-1}\,\mathcal{C}\Gamma\right) \\ &= (\mathcal{C}\Gamma)^{-1}\left(A\mathcal{C}\Gamma - p(A)B\left[e_n^T \otimes I_m\right]\Gamma\right) \\ &= (\mathcal{C}\Gamma)^{-1}\left(\mathcal{C}\left[K_\alpha \otimes I_m\right]\Gamma - \mathcal{C}\left[(\pi - \alpha)e_n^T \otimes I_m\right]\Gamma\right) \\ &= (\mathcal{C}\Gamma)^{-1}\mathcal{C}\left[K_\pi \otimes I_m\right]\Gamma = (\mathcal{C}\Gamma)^{-1}\mathcal{C}\Gamma M = M, \end{aligned}$$

which proves the theorem. The last few steps made use of (6), (7) and (8). ■

Proof of Theorem 4.3 First notice that, according to (12) and Lemma 4.2, the parameterized feedback gain can be written as $L = \Gamma_0 X$, where X is the unique solution to the Sylvester equation (13). Now let L^* be any feedback gain for which $a(s) = \det(sI - A)$ and $p(s) = \det(sI - A + BL)$ are coprime. With the particular choice $M = A - BL^*$ and $\Gamma_0 = L^*$, the identity matrix I is obviously a solution to (13). Also, $X = I$ is the unique solution, and thus the parameterized feedback gain is $L = \Gamma_0 X^{-1} = L^*$. This proves the theorem, since this holds for every such L^*. ■

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