The Geometric Brownian Motion of the Black-Scholes Model

8th February 2026

Geometric Brownian Motion Implies Lognormality

In the assumptions of Article No.1, I mentioned that in the Black-Scholes model, the underlying asset S follows a geometric Brownian motion. This describes how the stock price will fluctuate through time (i.e. the path).

$$\frac{dS}{S} = \mu dt + \sigma dX \tag{1}$$

This is equivalent to saying that the underlying asset S follows a lognormal random walk. When you solve the geometric Brownian motion equation, you find that the stock price can be written as an exponential of a normally distributed term (with that term being $dX = \phi(dt)^{\frac{1}{2}}$). In other words, the logarithm of the price is normally distributed, which means that the underlying asset follows a lognormal distribution. Thus, the lognormal distribution is not a separate assumption, rather it is a mathematical consequence of the underlying asset S following a geometric Brownian motion.

Standard Normal Distribution

The standard normal distribution is a probability distribution where the function is denoted by ϕ.

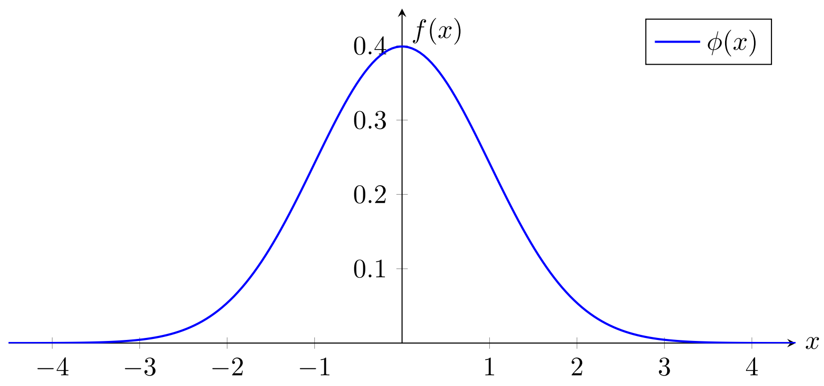

Figure 1: Standard Normal Distribution

The standard normal distribution is a bell curve with mean, $\mu = 0$, and standard deviation, $\sigma = 1$. Φ is the cumulative distribution function (CDF) of the standard normal distribution and is represented by the area underneath the probability density function, ϕ. The CDF is the probability of occurrences less than a given x value by measuring this area from $-\infty$ to x. For example, $\Phi(0) = 0.5$ because the standard normal is a symmetrical distribution and $\Phi(\infty) = 1$ because it includes all possible occurrences.

Adjusting to the Lognormal from Normal

The Black-Scholes model follows a lognormal random walk, to adjust to a lognormal distribution we just take the natural log (ln) of the values. And thus, we transform figure (1):

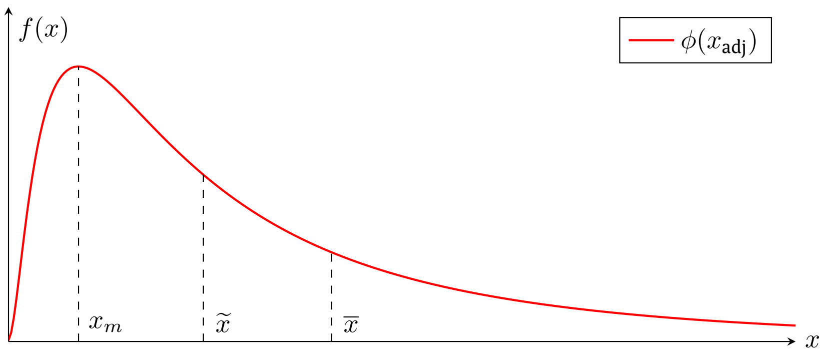

Figure 2: Lognormal Distribution

The mode, x_m, is the peak of the distribution curve and is the value at which occurrences happen most frequently. The median, $\widetilde{x}$, is the midpoint, where half the occurrences lie to the left and half to the right. The mean, $\overline{x}$ is the average of the distribution, where the total value is split equally. In a standard normal distribution, these three points coincide, but in a lognormal distribution they are separate, reflecting its skewed shape.

In a normal distribution, the relationship between the stock price and strike price is captured by taking the difference between them, $(S - E)$. However, in a lognormal distribution, we must compare the natural logarithms of both, giving $\ln(S) - \ln(E) = \ln\left(\frac{S}{E}\right)$. Thus, if $S > E$, we obtain $\ln\left(\frac{S}{E}\right) > 0$ and so, the call is in the money. If $S < E$, then we obtain $\ln\left(\frac{S}{E}\right) < 0$ and so, the call is out of the money.

We also need to adjust for other components that significantly affect option pricing: interest rates and volatility. Since options are based off the forward price of the underlying and the forward price depends on interest rates, we need to account for interest rates over the life of the option $(r(T-t))$. This ensures pricing reflects the present value of the stock's expected price at expiration. In a lognormal distribution, stock prices are right-skewed, which results in the mean is shifted slightly to the right of the mode (as seen in figure (2)). Mathematically, this equals $\frac{1}{2}\sigma^2(T-t)$, where σ is the volatility. Together, these adjustments represent the numerator for d_1 in the Black-Scholes model:

$$\begin{aligned}
&\ln\left(\frac{S}{E}\right) + r(T-t) + \frac{1}{2}\sigma^2(T-t) \\
&= \ln\left(\frac{S}{E}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)
\end{aligned}$$

To find d_1, we must convert into a number of standard deviations because standard deviations measure how far the strike price is from the mean in a lognormal distribution. Over any time interval, $T - t$, one standard deviation is equal to $\sigma\sqrt{T-t}$. The square root appears because volatility increases with time but not linearly, price dispersion widens at a slower rate as time passes. Therefore:

$$d_1 = \frac{\ln\left(\frac{S}{E}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}},$$

d_1 determines how many standard deviations the strike price is from the mean. Now, combining with the forward price of the stock gives as the average profit at expiration $\left(\text{Profit}_T = Se^{r(T-t)}\Phi(d_1)\right)$.

To figure out the likelihood that the option will be exercised, we can adjust d_1 to use the median from the mean in a lognormal distribution, this difference is given by $\sigma\sqrt{T-t}$, and so, $d_2 = d_1 - \sigma\sqrt{T-t}$. The term $\Phi(d_2)$ uses the median to estimate the probability of the option being in the money at expiration. Now, multiplying this probability by the exercise price gives the average price paid at expiration $(\text{Cost}_T = E\Phi(d_2))$. Thus, combining with the average profit at expiration, we obtain the expected value of the call option at expiration, C_T:

$$\begin{aligned}
C_T &= \text{Profit}_T - \text{Cost}_T \\
&= Se^{r(T-t)}\Phi(d_1) - E\Phi(d_2)
\end{aligned}$$

Since, we pay for the option at the present value, we must adjust by multiplying both terms by $e^{-r(T-t)}$. And so, we obtain the fair value of an option for the Black-Scholes model:

$$C = S\Phi(d_1) - Ee^{-r(T-t)}\Phi(d_2) \tag{2}$$

Conclusion

The Black–Scholes model is easier to understand once we recognise that asset prices evolve lognormally and that option values depend on how far the strike price lies within that distribution. Adjusting for interest rates, volatility, and the and how the underlying follows a geometric Brownian motion will help us understand and derive equations for d_1 and d_2. Rather than being just a formula on a page, we can see and understand how the Black–Scholes model is a way to transform the behaviour of stock prices into a fair option value.

Alexander Flann