

# JUSTIFICATION IN THE DERIVATION OF THE BLACK-SCHOLES MODEL

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When working with stochastic processes, an interesting result is that  $(dX)^2$  can be replaced by  $dt$ . In Article No.1, I used this fact to show Itô's Lemma, but here I want to justify the result properly. The purpose of this article is simple: to show why the quadratic variation of a Brownian motion leads to  $dt$ . I would classify this as a foundational idea that underpins the framework of the Black-Scholes model and much of modern quantitative finance.

## WHY $(dX)^2$ BECOMES $dt$ : A JUSTIFICATION USING STOCHASTIC CALCULUS

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In Article No.1, I quoted the fact that we can replace  $(dX)^2$  by  $dt$ , and this holds true for all such equations whenever we come across them. I will give an informal demonstration of the derivation. Consider the typical time interval of 0 to  $t$  and break this interval into  $n$  equal time steps of  $dt$ , where  $n \in \mathbb{N}$  (i.e. a natural number). Consider the time at each interval to be  $t_j$  where:

$$t_j = \frac{jt}{n}$$

Where  $j \in \mathbb{N}$  varying from 1 to  $n$ . Choose  $dt = \frac{t}{n}$ , so that each time interval is of length  $dt$  as required.

Let  $dX(t_j)$  correspond to the value of the random variable at time  $t_j$  and let it be a random variable selected from a normal distribution with mean 0 and variance equal to  $dt$  (i.e. standard deviation equal to  $\sqrt{dt}$ ). Therefore, we define the expectation of the random function  $g(dX)$  as:

$$\mathbb{E}[g(dX)] = \int_{-\infty}^{\infty} g(x)f(x)dx,$$

Where

$$f(x) = \frac{1}{\sqrt{2\pi dt}} \exp\left(-\frac{x^2}{2dt}\right)$$

Note that  $f(x)$  is just the probability density function (pdf) of the normal distribution with mean 0 and variance equal to  $dt$ . The expectation is the mean of the resulting function, which we can use this definition to calculate:

$$\begin{aligned} \mathbb{E}\left[\left(\sum_{j=1}^n (dX(t_j))^2 - t\right)^2\right] &= \mathbb{E}\left[\left(\sum_{j=1}^n (dX(t_j))^2\right)^2 - 2t \sum_{j=1}^n (dX(t_j))^2 + t^2\right] \\ &= n\mathbb{E}[(dX)^4] + n(n-1)\left(\mathbb{E}[(dX)^2]\right)^2 - 2tn\mathbb{E}[(dX)^2] + t^2\mathbb{E}[1] \end{aligned} \quad (1)$$

By initially multiplying out the brackets, then by using the Linearity of Expectation and the fact that each  $dX(t_j)$  comes from the same normal distribution for every value of  $j$ , i.e.  $\mathbb{E}[(dX(t_j))^2] = \mathbb{E}[(dX(t_i))^2] \forall i, j \in \mathbb{N}$ ; also note that this fact is also true to the power of four  $\forall i, j \in \mathbb{N}$ . The first two terms in equation (1) come from the term:

$$\left(\sum_{j=1}^n (dX(t_j))^2\right)^2$$

When squaring this and expanding out the brackets, we obtain  $n$  terms of the form  $(dX(t_j))^4$ , thus we obtain the  $n\mathbb{E}[(dX)^4]$  term, and we obtain  $n(n-1)$  terms of the form  $(dX(t_j))^2(dX(t_i))^2$  where  $i \neq j$ , thus we obtain the  $n(n-1)\left(\mathbb{E}[(dX)^2]\right)^2$  term. Thus, we have written that  $\mathbb{E}[(dX)^2] = \mathbb{E}[(dX(t_j))^2]$  and  $\mathbb{E}[(dX)^4] = \mathbb{E}[(dX(t_j))^4] \forall j$ . We also have used the result:

$$\begin{aligned}\mathbb{E} \left[ (dX(t_j))^2 (dX(t_i))^2 \right] &= \mathbb{E} \left[ (dX(t_j))^2 \right] \mathbb{E} \left[ (dX(t_i))^2 \right] \\ &= \left( \mathbb{E} \left[ (dX)^2 \right] \right)^2\end{aligned}$$

Which can be proved using the fact that random numbers  $dX(t_i)$  and  $(t_j)$  are selected at different times (i.e. at  $t_i$  and  $t_j$ ) and hence, are independent events. Perhaps, to give an example, will help the intuition behind why this works, consider a case where  $n = 3$ . Thus:

$$\begin{aligned}&\mathbb{E} \left[ \left( \sum_{j=1}^n (dX(t_j))^2 \right)^2 - 2t \sum_{j=1}^n (dX(t_j))^2 + t^2 \right] \\ &= \mathbb{E} \left[ \left( (dX(t_1))^2 + (dX(t_2))^2 + (dX(t_3))^2 \right)^2 - 2t \left( (dX(t_1))^2 + (dX(t_2))^2 + (dX(t_3))^2 \right) + t^2 \right] \\ &= \mathbb{E} \left[ (dX(t_1))^4 + (dX(t_2))^4 + (dX(t_3))^4 + 2(dX(t_1))^2 (dX(t_2))^2 + 2(dX(t_1))^2 (dX(t_3))^2 + 2(dX(t_1))^2 (dX(t_3))^2 \right. \\ &\quad \left. - 2t \left( (dX(t_1))^2 + (dX(t_2))^2 + (dX(t_3))^2 \right) + t^2 \right] \\ &= 3\mathbb{E} \left[ (dX)^4 \right] + 6 \left( \mathbb{E} \left[ (dX)^2 \right] \right)^2 - 6t\mathbb{E} \left[ (dX)^2 \right] + t^2\mathbb{E}[1]\end{aligned}$$

And so, we can see in the case  $n = 3$ , we have the correct form.

Let us now evaluate equation (1). First note that:

$$\begin{aligned}\mathbb{E}[1] &= \int_{-\infty}^{\infty} f(x)dx = 1 \\ \mathbb{E} \left[ (dX)^2 \right] &= \int_{-\infty}^{\infty} x^2 f(x)dx = dt \\ \mathbb{E} \left[ (dX)^4 \right] &= \int_{-\infty}^{\infty} x^4 f(x)dx = 3(dt)^2\end{aligned}$$

Using integration by parts. Substituting these results back into equation (1), we obtain:

$$\mathbb{E} \left[ \left( \sum_{j=1}^n (dX(t_j))^2 - t \right)^2 \right] = n \left( 3(dt)^2 \right) + n(n-1)(dt)^2 - 2tn(dt) + t^2$$

But since  $dt = \frac{t}{n}$ , we obtain:

$$\begin{aligned}\mathbb{E} \left[ \left( \sum_{j=1}^n (dX(t_j))^2 - t \right)^2 \right] &= n \left( 3 \left( \frac{t}{n} \right)^2 \right) + n(n-1) \left( \frac{t}{n} \right)^2 - 2tn \left( \frac{t}{n} \right) + t^2 \\ &= \frac{2t^2}{n}\end{aligned}$$

Let  $n \rightarrow \infty$ , so:

$$\mathbb{E} \left[ \left( \sum_{j=1}^n (dX(t_j))^2 - t \right)^2 \right] = \frac{2t^2}{n} \rightarrow 0$$

as  $n \rightarrow \infty$ . Since the left-hand side (LHS) corresponds to an integral over the square of a quantity, the unique way that this integral is 0, in the limit  $n \rightarrow \infty$ , is when:

$$\sum_{j=1}^n (dX(t_j))^2 = t$$

as  $n \rightarrow \infty$ . This is the mean square limit.

Define the stochastic integral as:

$$\int_0^t h(t) dX = \lim_{n \rightarrow \infty} \sum_{j=1}^n h(t_j) dX(t_j)$$

for a function  $h(t)$ . Intuitively speaking, we can think of the integral on the LHS as an infinite sum of infinitesimally small areas under the curve. Thus:

$$\lim_{n \rightarrow \infty} \sum_{j=1}^n (dX(t_j))^2 = \int_0^t (dX)^2$$

by choosing  $h(t) = dX(t)$ . Thus:

$$\int_0^t (dX)^2 = t$$

can be rewritten as:

$$\int_0^t (dX)^2 = \int_0^t dt$$

Therefore, we obtain:

$$\int_0^t ((dX)^2 - dt) = 0$$

And so, we can conclude that  $(dX)^2 = dt$  when written under an integral.

## CONCLUSION

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The key idea is that Brownian motion builds up variance linearly with time, which means  $(dX)^2$  behaves like  $dt$  when we look at the limit  $dt \rightarrow 0$ . With that in mind, the rule that  $(dX)^2 = dt$  isn't a trick, it follows directly from the fact that Brownian increments are of order  $(dt)^{\frac{1}{2}}$  (noting that  $dX = \phi(dt)^{\frac{1}{2}}$ ). This result is what makes Itô's Lemma work and why models like the Black-Scholes model can be derived cleanly. With this result established, we can use the rule without deriving it each time.

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