Introduction and Derivation of the Black-Scholes Model

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A Brief History of Options

A financial option is a financial derivative which represents a contract between the buyer (holder) and the seller (writer). The buyer has the right, but not the obligation, to buy or sell an underlying asset (such as stocks, currencies, or commodities) at a predetermined price (strike price) within a specific time-frame. The holder pays a fee, known as a premium, to the seller for this right.

Interestingly, options trading can be traced all the way back to Ancient Greece (c. 600 B.C.), where a philosopher, Thales of Miletus, secured the right, but not obligation, to rent olive presses, forming the first known call option, and amassing a fortune.

A famous account of options trading came in the 17th century, known as the Tulip Mania of the Dutch Golden Age. Tulips were introduced to Europe in 1554 and became a status symbol for the Dutch elite. In 1636, prices increased dramatically, which was fuelled by a futures market where people traded contracts for bulbs still in the ground. In 1637, buyers refused to honour contracts, leading to a collapse in prices and confidence, which is often recognised as the first speculative bubble.

Options trading continued into the early 18th century and due to lessons learnt, opposition to trading these contracts resulted in a ban from 1734-1860. In 1973, the Chicago Board Options Exchange (CBOE) was established, launching the first standardised, listed stock options. That same year, Fischer Black, Myron Scholes, and Robert Merton developed the Black-Scholes pricing model. This paper (*The Pricing of Options and Corporate Liabilities* published in the *Journal of Political Economy*) fundamentally changed finance.

The Black-Scholes Model

The Black–Scholes model is one of the most influential equations in quantitative finance. The underlying framework that the Black-Scholes model provides is rich in foundational ideas for quantitative finance. I will only go through the basics of the Black-Scholes model in this article. It is a framework for the valuation of a fair price for European options by modelling the evolution of asset prices under uncertainty and showing how to construct a risk-free hedging strategy.

2.1 Assumptions

The Black-Scholes model has multiple assumptions, which do yield some limitations.

1. **Modelling Asset Prices with Stochastic Processes:** The underlying asset S follows a geometric Brownian motion (lognormal random walk of the form):

$$\frac{dS}{S} = \mu dt + \sigma dX \tag{1}$$

   Where μ is the drift (growth) rate of S, σ is the volatility and X is a R.V. given by $dX = \phi(dt)^{\frac{1}{2}}$ where $\phi \sim N(0,1)$ (i.e. follows a standard normal distribution).

2. **No-Arbitrage and Replication:** The Black-Scholes equation is derived under the assumption of Delta Hedging by considering a portfolio of the form $\pi = V - \Delta S$, where the holder of the portfolio continuously buys or sells shares in S so that Δ is instantaneously equal to $\frac{\partial V}{\partial S}$. This portfolio is then risk-free with $\frac{d\pi}{dt} = r\pi$.

3. **Parameter Constants:** The parameters (μ, σ, r) are all assumed to be constant. Note that r is taken to be the risk-free rate.

2.2 The Black-Scholes Formula

A European option can only be exercised at the expiry date T. Thus, for a European call option with strike price E (I do note that many sources use K instead), we clearly see that the payoff function is given by:

$$\text{Payoff} = \max(S - E, 0) \tag{2}$$

Where S is the value of the underlying asset at expiry. This yields a piecewise linear ramp function. In the Black–Scholes model, the fair price of a European-style call option is given by the following:

$$C = S\Phi\left(d_1\right) - Ee^{-r(T-t)}\Phi\left(d_2\right) \tag{3}$$

Where

$$d_1 = \frac{\ln\left(\frac{S}{E}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}, \qquad d_2 = d_1 - \sigma\sqrt{T-t} \tag{4}$$

Where $\Phi(x)$ denotes the cumulative distribution function (CDF) of the standard normal distribution, please note that some use $n(x)$ and $N(x)$ to represent the probability density function (PDF) and CDF of the standard normal distribution respectively. At a very high level, this formula is the weighted difference between the stock and strike prices.

2.3 Deriving the Black-Scholes Equation

Suppose that $V = V(S,t)$ is a function of the underlying asset S and time t. Here, V is a financial derivative that derives its value from the underlying. The underlying asset S follows a geometric Brownian motion. Using Taylor's theorem, we write:

$$\begin{aligned} dV &= V(S+dS, t+dt) - V(S,t) \\ &= \left(\frac{\partial V}{\partial S}dS + \frac{\partial V}{\partial t}dt\right) + \frac{1}{2}\left(\frac{\partial^2 V}{\partial S^2}dS^2 + 2\frac{\partial^2 V}{\partial S \partial t}dSdt + \frac{\partial^2 V}{\partial t^2}dt^2\right) + \ldots \end{aligned}$$

Where "+..." refers to the terms which are negligible as dt, $dS \to 0$. All derivatives are calculated at the point (S,t). Using $dS = S(\mu dt + \sigma dX)$ and replacing $(dX)^2 = dt$ (this justification can take time so I will omit for this article, if interested refer to Article No.3). Thus, we now obtain:

$$\begin{aligned} dV = \frac{\partial V}{\partial S}\left(S\left(\mu dt + \sigma dX\right)\right) + \frac{\partial V}{\partial t}dt + \\ \frac{1}{2}\left[\frac{\partial^2 V}{\partial S^2}\left(S^2\left(\mu^2\left(dt\right)^2 + 2\mu\sigma dtdX + \sigma^2\left(dX\right)^2\right)\right) + 2S\left(\mu dt + \sigma dX\right)dt\frac{\partial^2 V}{\partial S \partial t} + \frac{\partial^2 V}{\partial t^2}\left(dt\right)^2\right] + \ldots \end{aligned}$$

In the limit as $dt \to 0$, we have $(dt)^2 \ll dt$ and we can assume that $dXdt \ll dt$.

Thus, to leading order, we obtain:

$$\begin{aligned} dV &= \sigma S\frac{\partial V}{\partial S}dX + \left(\mu S\frac{\partial V}{\partial S} + \frac{1}{2}\mu^2 S^2\frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t}\right)dt \\ &= \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} \end{aligned}$$

This is known as Itô's Lemma. Now consider a Delta-Hedged portfolio with $\pi = V - \Delta S$, where Δ is a constant over a time interval $t \to t + dt$.

$$\begin{aligned} d\pi &= dV - \Delta dS \\ &= \sigma S\frac{\partial V}{\partial S}dX + \left(\mu S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 s^2\frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t}\right)dt - \Delta\left(S\mu dt + S\sigma dX\right) \end{aligned} \tag{5}$$

Where dt is deterministic and dX is random. We can rearrange and choose $\Delta = \frac{\partial V}{\partial S}$ such that the coefficient of the random term dX vanishes.

$\Rightarrow$ Since $d\pi$ is no longer random, it is deterministic and hence risk-free. By the principle of arbitrage we have that $d\pi = r\pi dt$. Thus, we obtain:

$$\begin{aligned} d\pi &= r\left(V - \Delta S\right)dt \\ &= r\left(V - \frac{\partial V}{\partial S}S\right)dt \end{aligned} \tag{6}$$

Combining equations (5) and (6) and dividing by dt we obtain:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \tag{7}$$

The Black-Scholes Equation (7).

Conclusion

Although the Black-Scholes model relies on simplifying assumptions such as constant volatility, lognormal returns, and frictionless markets, its development fundamentally shaped modern options trading and is rightly recognised by its Nobel Prize in Economics. Its limitations are well-known, yet they do not diminish its importance. The model introduces several foundational principles that underpin quantitative finance: risk-neutral valuation, replication, no-arbitrage pricing, and the use of stochastic calculus to model uncertainty. More advanced frameworks, from local volatility models to stochastic volatility and jump-diffusion approaches, build directly on these ideas. Understanding the Black–Scholes model is therefore not only historically significant but also essential for developing the conceptual intuition required in modern quantitative finance.

The Original Paper of the Black-Scholes Model

Black, Fischer and Scholes, Myron (1973). "The Pricing of Options and Corporate Liabilities". In: *Journal of Political Economy* 81 (3), pp. 637–654. DOI: 10.1086/260062.

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