Bachelor thesis

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0.1 Basics

Pole shifting theorem deals with generalized linear differential systems and claims, that it is possible to achieve arbitrary asymptotic behavior. To understand this basic theorem of control theory, we must first describe few basic concepts.

The state of linear system can be represented by system of $n \in \mathbb{N}$ differential equations

$$\dot{x}(t) = Ax(t) + Bu(t),$$

where $x(t)$ is the n-dimensional state vector $(\varphi(t), \dot{\varphi}(t), \ldots, \varphi^{(n-1)}(t))^T$, $u(t)$ is the m-dimensional *input* or *control* column vector, $A \in \mathbb{C}^{n\times n}$ is matrix of coefficients and $B \in \mathbb{C}^{m\times n}$ is *input* or *control* matrix. The control vector $u(t)$ is acquired from $x(t)$ by multiplying a control matrix $F \in \mathbb{C}^{m\times n}$ by $x(t)$. All the possible states of $x(t)$ create **state space**, which usually is equal to $\mathbb{C}^n$.

We can imagine this system as follows. The first part of the equation $\dot{x}(t) = Ax(t)$ can be thought of as the model of machine or event that we want to control and $Bu(t)$ as our control mechanism. The B matrix is our "control board" and $u(t)$ is us deciding, which levers and buttons we want to push. Of course, if we want this system to be self-regulating, we cannot input our own values into $u(t)$ and therefore it has to be calculated from the current state of our system. Therefore we have $u(t) = Fx(t)$. The whole system can then be rewritten as

$$\dot{x}(t) = Ax(t) + BFx(t) = (A + BF)x(t).$$

If $A + BF$ is diagonalizable matrix, then we can write

$$\Lambda = R^{-1}(A + BF)R,$$

where $R \in \mathbb{C}^{n\times n}$ is an invertible matrix and $\Lambda \in \mathbb{C}^{n\times n}$ is a diagonal matrix. We can write that

$$\dot{x}(t) = RR^{-1}(A + BF)RR^{-1}x(t) = R\Lambda R^{-1}x(t),$$

it follows, that

$$R^{-1}\dot{x}(t) = (\dot{R^{-1}x}) = \Lambda R^{-1}x(t).$$

By substituting $y(t) = R^{-1}x(t)$ we get

$$\dot{y}(t) = \Lambda y(t).$$

This equation represents system of simple linear differential equations. If we denote elements on diagonal of Λ by $\lambda_1, \lambda_2, \ldots, \lambda_n$ the resulting equations are

$$\begin{aligned} \dot{y}_1(t) &= \lambda_1 y_1(t) \\ \dot{y}_2(t) &= \lambda_2 y_2(t) \\ &\vdots \\ \dot{y}_n(t) &= \lambda_n y_n(t) \end{aligned}$$

Solution to each of these equations is in the form

$$y_k(t) = y_k(0)e^{\lambda_k t}, k \in \{1, 2, \ldots, n\}.$$

Let $\lambda_k = a_k + b_k i$ where $a_k, b_k \in \mathbb{R}$, then

$$y_k(0)e^{\lambda_k t} = y_k(0)e^{at}e^{bit}.$$

We know, that $|e^{bit}| = 1$ and that $y_k(0)$ is a constant, so the crucial part is e^{at}. This converges to 0 if and only if a is negative. Therefore we can stabilize our "machine" if we find such matrix $F \in \mathbb{C}^{n \times n}$ that $A + BF$ is diagonalizable with eigenvalues with negative real part. This can be expressed through characteristic polynomial of matrix $A + BF$. We will denote characteristic polynomial of a matrix A by χ_A. Through our observations we got to a conclusion that we need to satisfy

$$\chi_{A+BF} = (x - \lambda_1)(x - \lambda_2) \cdots (x - \lambda_n),$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{C}$ and their real part is negative. This leads to important definition.

Definition. *We say that polynomial χ is* **assignable** *for the pair (A, B) if there exists such matrix F that*

$$\chi_{A+BF} = \chi$$

The pole shifting theorem states, that if A and B are "sensible" in a sense that we will discuss in the next section, then arbitrary polynomial χ of dimension that depends on how "sensible" A and B are, can be assigned to pair (A, B).

0.2 Controllable pairs

States that we can reach in set number of iterations can be derived as follows. From state x_k and control vector u_k is the next state x_{k+1} computed by equation

$$x_{k+1} = Ax_k + Bu_k.$$

The starting condition is $x_0 = \mathbf{o}$ and we can choose arbitrary u_k. Then, for $k = 0$ we have

$$x_1 = Ax_0 + Bu_0 = Bu_0 \in \mathrm{Im}B.$$

For $k = 2$ we get

$$x_2 = Ax_1 + Bu_1 = ABu_0 + Bu_1 \in \mathrm{Im}(AB|B).$$

It is clear, that

$$x_k \in \mathrm{Im}(A^{k-1}B| \ldots |AB|B).$$

We can observe that $\mathrm{Im}(B|AB| \ldots |A^k B) \subseteq \mathrm{Im}(B|AB| \ldots |A^{k+1}B)$. Then, from Cayley-Hamilton theorem we know, that

$$\mathrm{Im}(B|AB| \ldots |A^{n-1}B) = \mathrm{Im}(B|AB| \ldots |A^{n-1}B|A^n B).$$

Therefore all the states we could ever reach are already in space

$$\mathrm{Im}(B|AB| \ldots |A^{n-1}B).$$

Definition. *Let $\mathbb{K}$ be a field and let $A \in \mathbb{K}^{n \times n}$, $B \in \mathbb{K}^{n \times m}$, $n, m \in \mathbb{N}$. We define* **reachable space**$\mathcal{R}(A, B)$ *as $Im(B|AB| \ldots |A^{n-1}B)$.*

From observations above it is clear that for arbitrary $v \in \mathcal{R}(A, B)$ we have $Av \in \mathcal{R}(A, B)$. This property is called A-*invariance.*

The maximum dimension of $\mathcal{R}(A, B)$ is, of course, n. This leads us to important property of pair (A, B), where we want to able to get the "machine" into any state by controlling it with our control matrix B. Therefore we desire that the dimension of reachable space is equal to n.

Definition. *Let $\mathbb{K}$ be a field and let $A \in \mathbb{K}^{n\times n}$, $B \in \mathbb{K}^{n\times m}$, $n, m \in \mathbb{N}$. The pair (A, B) is* **controllable** *or* **reachable** *if* $\dim \mathcal{R}(A, B) = n$.

If (A, B) are not controllable then there exists subspace of our state space that is not affected by our input. This can be shown using following theorem.

Theorem 1. *Assume that (A, B) is not controllable. Let $\dim\mathcal{R}(A, B) = r < n$. Then there exists invertible $n \times n$ matrix T over $\mathbb{K}$ such that the matrices $\widetilde{A} := T^{-1}AT$ and $\widetilde{B} := T^{-1}B$ have the block structure*

$$\widetilde{A} = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix} \qquad \widetilde{B} = \begin{pmatrix} B_1 \\ 0 \end{pmatrix}$$

where A_1 is $r \times r$ and B_1 is $r \times m$.

Proof. Let $\mathcal{S}$ be any subspace that

$$\mathcal{R}(A, B) \oplus \mathcal{S} = \mathbb{K}^n.$$

Let $\{v_1, \ldots, v_r\}$ be the basis of $\mathcal{R}(A, B)$ and $\{w_1, \ldots, w_{n-r}\}$ be the basis of $\mathcal{S}$, then we put $K = (v_1, \ldots, v_r, w_1, \ldots, w_{n-r})$ as the basis of $\mathbb{K}^n$ and we put

$$T := (v_1|\ldots|v_r|w_1|\ldots|w_{n-r}) = [\mathrm{id}]^K_C$$

where C is the canonical basis and $[\mathrm{id}]^K_C$ is the transition matrix from basis K to basis C. We have $\mathrm{Im}T = \mathbb{K}^n$ therefore T is an invertible matrix. Now we have

$$\widetilde{B} = T^{-1}B = ([\mathrm{id}]^K_C)^{-1}B = [\mathrm{id}]^C_K B$$

We know that $\mathrm{Im}B \subseteq \mathcal{R}(A, B)$ therefore every column of matrix B can be uniquely expressed as linear combination of vectors in basis B. From our choice of T we can clearly see that $\widetilde{B}$ will be of the desired form.

As for $\widetilde{A}$ we have

$$\widetilde{A} = T^{-1}AT = [\mathrm{id}]^C_K A [\mathrm{id}]^K_C$$

From the fact that $\mathcal{R}(A, B)$ is A-invariant it follows that

$$AT = (u_1|\ldots|u_r|z_1|\ldots|z_{n-r})$$

where $u_i \in \mathcal{R}(A, B)$ and $z_i \in \mathcal{S}$. Therefore, when we express these vectors in the basis K (by left multiplying AT by $T^{-1} = [\mathrm{id}]^C_K$) we get the required structure of matrix $\widetilde{A}$. ☐

We can interpret the above decomposition as follows. Consider our system $\dot{x}(t) = Ax(t) + Bu(t)$. By changing the basis by putting $x(t) = Ty(t)$ we get

$$T\dot{y}(t) = ATy(t) + Bu(t)$$

which we can write as

$$\dot{y}(t) = T^{-1}ATy(t) + T^{-1}Bu(t) = \widetilde{A}y(t) + \widetilde{B}u(T)$$

Which gives us

$$\begin{aligned}\dot{y}_1(t) &= A_1 y_1(t) + A_2 y_2(t) + B_1 u_1(t)\\ \dot{y}_2(t) &= \qquad\qquad\; A_3 y_2(t)\end{aligned}$$

where $y_1(t)$ is the first r elements of $y(t)$, $y_2(t)$ is the other elements of $y(t)$ and $u_1(t)$ is the first r elements of $u(t)$. It is clear that we cannot control $\dot{y}_2$ by any means.

It is also true that (A_1, B_1) is a controllable pair which we will state in a lemma.

Lemma 1. *The pair (A_1, B_1) is controllable.*

Proof. We know that $\dim\mathcal{R}(A, B) = r$. We desire $\dim\mathcal{R}(A_1, B_1) = r$. We will show that $\mathcal{R}(\widetilde{A}, \widetilde{B}) = \mathcal{R}(A, B)$ and that each vector in $\mathcal{R}(\widetilde{A}, \widetilde{B})$ has its last $n - r$ elements equal to 0 and that $\mathcal{R}(\widetilde{A}, \widetilde{B})$ restricted on its first r coordinates is equal to $\mathcal{R}(A_1, B_1)$.

$$\begin{aligned}\mathcal{R}(\widetilde{A}, \widetilde{B}) &= \mathrm{Im}(\widetilde{A}^{n-1}\widetilde{B}| \dots |\widetilde{A}\widetilde{B}|\widetilde{B})\\ &= \mathrm{Im}((T^{-1}AT)^{n-1}T^{-1}B| \dots |T^{-1}ATT^{-1}B|T^{-1}B)\\ &= \mathrm{Im}(T^{-1}A^{n-1}B| \dots |T^{-1}AB|T^{-1}B)\\ &= T^{-1} \cdot \mathrm{Im}(A^{n-1}B| \dots |AB|B)\\ &= T^{-1} \cdot \mathcal{R}(A, B)\end{aligned}$$

Since T is an invertible matrix, which preserves linear independence, we have

$$\dim\mathcal{R}(\widetilde{A}, \widetilde{B}) = \dim(T^{-1}\mathcal{R}(A, B)) = \dim(\mathcal{R}(A, B)) = r$$

Now, for the structure of $\mathcal{R}(\widetilde{A}, \widetilde{B})$: We know that last $n - r$ rows of $\widetilde{B}$ are $\mathbf{o}$. Also because of structure of $\widetilde{A}$ we have for arbitrary matrix $X \in \mathbb{K}^{r \times m}$ that

$$\widetilde{A}\begin{pmatrix} X \\ 0 \end{pmatrix} = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}\begin{pmatrix} X \\ 0 \end{pmatrix} = \begin{pmatrix} A_1 X \\ 0 \end{pmatrix}$$

which again has the last $n - r$ rows equal to $\mathbf{o}$. Therefore we see that for any positive integer k we have

$$\widetilde{A}^k\widetilde{B} = \begin{pmatrix} A_1^k B_1 \\ 0 \end{pmatrix}$$

From Cayle-Hemilton theorem we therefore again have that the first r coordinates (those which are not 0) of $\mathcal{R}(\widetilde{A}, \widetilde{B})$ are equal to $\mathcal{R}(A_1, B_1)$. It follows that

$$\dim\mathcal{R}(A_1, B_1) = \dim\mathcal{R}(\widetilde{A}, \widetilde{B}) = \dim\mathcal{R}(A, B) = r$$

□

Now we can see that the decomposition from Theorem 1 decomposes the matrix A into "controllable" and "uncontrollable" parts A_1 and A_3 respectively.

We also see that

$$\begin{aligned}\chi_{\widetilde{A}} &= \det(sI - \widetilde{A}) = \det(sI - T^{-1}AT) \\ &= \det(sT^{-1}IT - T^{-1}AT) = \det(T^{-1}(sI - A)T) \\ &= (\det T)^{-1}\det(sI - A)\det T = \det(sI - A) \\ &= \chi_A\end{aligned}$$

therefore it holds

$$\chi_A = \chi_{A_1}\chi_{A_3}$$

Definition. *The characteristic polynomial of matrix A splits into* **controllable** *and* **uncontrollable parts** *with respect to pair (A, B) which we denote by χ_c and χ_u respectively. We define these polynomials as*

$$\chi_c = \chi_{A_1} \qquad \chi_u = \chi_{A_3}$$

In case $r = 0$ we put $\chi_c = 1$ and in case $r = n$ we put $\chi_u = 1$.

Definition. *Let (A, B) and $(\widetilde{A}, \widetilde{B})$ be pairs as above. Then (A, B)* **is similar to** *$(\widetilde{A}, \widetilde{B})$, denoted*

$$(A, B) \sim (\widetilde{A}, \widetilde{B})$$

if there exists invertible matrix T for which it holds that

$$\widetilde{A} = T^{-1}AT \quad \textit{and} \quad \widetilde{B} = T^{-1}B$$