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BACHELOR THESIS

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Pole Shifting Theorem in Control Theory

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Dedication.

Title: Pole Shifting Theorem in Control Theory

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Contents

Introduction

The pole shifting theorem is one of the basic results of the theory of linear dynamic systems with linear feedback. It claims that in case of controllable systems one can achieve an arbitrary asymptotic behavior by a suitably chosen feedback. To understand this crucial theorem, we must first describe a few basic concepts.

1. Dynamic Systems

1.1 Systems of First Order Differential Equations

Remark. *Let $f(t)$ be a function of time $t \in \mathbb{R}^+$. We will denote its derivative with respect to t by*

$$\dot{f}(t) = \frac{d}{dt} f(t) \ .$$

Definition. *A* **system of linear differential equations of order one with constant coefficients** *is the system*

$$\begin{aligned} \dot{x}_1(t) &= a_{1,1} x_1(t) + \ldots + a_{1,n} x_n(t) \\ &\vdots \\ \dot{x}_n(t) &= a_{n,1} x_1(t) + \ldots + a_{n,n} x_n(t) \ . \end{aligned}$$

This system can be written in the matrix form

$$\dot{x}(t) = Ax(t) \ ,$$

where $x(t) = (x_1(t), \ldots, x_n(t))^T \in \mathbb{C}^n, x_i \colon \mathbb{R}^+ \to \mathbb{C}$, is a **state vector** *(***state** *for short) of the system and the matrix $A \in \mathbb{C}^{n \times n}$, $A = (a_{i,j})$ is a* **fundamental matrix** *of the system. The* **initial condition** *of the system is the state $x(0)$.*

This system is also called a **linear autonomous system***.*

We will use the matrix form, as it is a very compact way of describing the system.

To express the solution of a linear autonomous system in a similarly compact way, we will establish the notion of the matrix exponential.

Definition. *Let X be a real or complex square matrix. The exponential of X, denoted by e^X, is the square matrix of the same type defined by the series*

$$e^X = \sum_{k=0}^{\infty} \frac{1}{k!} X^k \ ,$$

where X^0 is defined to be the identity matrix I of the same type as X.

For this definition to make sense, we need to show that the series converges for any real or complex square matrix. Firstly, we will define what it means for a matrix series to converge. In this text, we will define the convergence using the Frobenius norm.

Definition. **Frobenius norm** *is a matrix norm, denoted as $\|\cdot\|_F$, which for an arbitrary $n \times m$ matrix A is defined as*

$$\|A\|_F = \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{m} |a_{i,j}|^2} \ .$$

Remark. *In what follows,* $\mathbb{K}$ *will denote a field of either real or complex numbers.*

Lemma 1. *The Frobenius norm satisfies the following statements for any matrices* A, B, $C \in \mathbb{K}^{n\times m}$, $D \in \mathbb{K}^{m\times o}$ *and any scalar* $\alpha \in \mathbb{K}$.

1. $\|A+B\|_F \leq \|A\|_F + \|B\|_F$,
2. $\|\alpha A\|_F = |\alpha|\|A\|_F$,
3. $\|A\|_F \geq 0$ *with equality occurring if and only if* $A = O_{n\times m}$,
4. $\|CD\|_F \leq \|C\|_F\|D\|_F$.

Proof. The first three points can be simply shown using the definition of the Frobenius form and the properties of the absolute value.

The fourth point follows from the Cauchy–Schwarz inequality

$$\|CD\|_F^2 = \sum_{i=1}^{m}\sum_{j=1}^{o}\|c_i d_j\|_2^2 \leq \sum_{i=1}^{m}\sum_{j=1}^{o}\|c_i\|_2^2\|d_j\|_2^2 = \sum_{i=1}^{m}\|c_i\|_2^2\sum_{j=1}^{o}\|d_j\|_2^2 = \|C\|_F^2\|D\|_F^2 \ ,$$

where $\|\cdot\|_2$ denotes the Euclidean norm, and c_i, d_i denote the i-th column of the matrices C and D respectively. □

Lemma 2. *The absolute value of any element of a matrix is always less than or equal to the Frobenius norm of the matrix. In particular, for a matrix* $A^k = (a_{i,j}^{(k)})_{n\times n}$, *where* $A \in \mathbb{K}^{n\times n}$, *it holds for every position* (i,j) *that* $|a_{i,j}^{(k)}| \leq \|A^k\|_F \leq \|A\|_F^k$.

Proof. For an arbitrary element of the matrix $X = (x_{i,j})_{n\times m}$ it holds

$$|x_{i,j}| \leq \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{m}|x_{i,j}|^2} = \|X\|_F \ .$$

It follows

$$|a_{i,j}^{(k)}| \leq \|A^k\|_F \leq \|A\|_F^k \ ,$$

where the second inequality follows from the fourth point of Lemma 1. □

Corollary 1. *Let us have a matrix* $A^k = (a_{i,j}^{(k)})_{n\times n}$. *Then the series* $\sum_{k=0}^{\infty}\frac{b^k}{k!}a_{i,j}^{(k)}$ *converges absolutely for any* $b \in \mathbb{K}$.

Proof. By Lemma 2, for any $N \in \mathbb{N}$, we have

$$\sum_{k=0}^{N}\left|\frac{b^k}{k!}a_{i,j}^{(k)}\right| \leq \sum_{k=0}^{N}\frac{|b|^k}{k!}\left|a_{i,j}^{(k)}\right| \leq \sum_{k=0}^{N}\frac{|b|^k}{k!}\|A\|_F^k = \sum_{k=0}^{N}\frac{\|bA\|_F^k}{k} \ .$$

Then

$$\sum_{k=0}^{\infty}\left|\frac{b^k}{k!}a_{i,j}^{(k)}\right| = \lim_{N\to\infty}\sum_{k=0}^{N}\left|\frac{b^k}{k!}a_{i,j}^{(k)}\right| \leq \lim_{N\to\infty}\sum_{k=0}^{N}\frac{\|bA\|_F^k}{k} = \sum_{k=0}^{\infty}\frac{\|bA\|_F^k}{k} = e^{\|bA\|_F} \ .$$

□

Definition. *A matrix sequence* $\{A_k\}_{k=0}^{\infty}$ *of* $n \times m$ *matrices is said to* **converge** *to* $n \times m$ *matrix* A*, denoted* $A_k \longrightarrow A$*, if*

$$\forall \varepsilon \in \mathbb{R}, \varepsilon > 0 \quad \exists n_0 \in \mathbb{N} \quad \forall n \in \mathbb{N}, n \geq n_0 : ||A_n - A||_F < \varepsilon \ .$$

Lemma 3. *A matrix sequence* $\{A_k = (a_{i,j}^{(k)})_{n\times m}\}_{k=0}^{\infty}$ *converges to a matrix* $A = (a_{i,j})_{n\times m}$ *if and only if it converges elementwise, in other words*

$$\forall i \in \{1, \dots, n\} \quad \forall j \in \{1, \dots, m\} : a_{i,j}^{(k)} \xrightarrow{k\to\infty} a_{i,j} \ .$$

Proof. Let $A_k \to A$. For any $\varepsilon \in \mathbb{R}^+$ we can find such n_0 that $\|A_n - A\|_F < \varepsilon$ for every $n \geq n_0$. By Lemma 2, we then have

$$|a_{i,j}^{(n)} - a_{i,j}| \leq \|A_n - A\|_F < \varepsilon \ .$$

It follows that $\{A_k\}_{k=0}^{\infty}$ converges to A elementwise.

Conversely, let ε be a positive real number. For every position (i, j) we find such $n_{i,j}$ that

$$\forall n \geq n_{i,j} : |a_{i,j}^{(n)} - a_{i,j}| < \frac{\varepsilon}{\sqrt{nm}} \ .$$

We put $N_0 = \min\{n_{i,j}\}$. Now $\forall n \in \mathbb{N}, n \geq N_0$ it holds

$$||A_n - A||_F = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{m} |a_{i,j}^{(n)} - a_{i,j}|^2} < \sqrt{nm\frac{\varepsilon^2}{nm}} = \varepsilon \ .$$

□

Claim 1. *The matrix exponential is well defined, that is, the matrix series* $\sum_{k=0}^{\infty} \frac{1}{k!}X^k$ *converges for any matrix* X*.*

Proof. Let $X^k = (x_{i,j}^{(k)})_{n\times n}$. By Corollary 1 every element of the matrix $\sum_{k=0}^{\infty} \frac{1}{k!}X^k = \left(\sum_{k=0}^{\infty} \frac{1}{k!}x_{i,j}^{(k)}\right)_{n\times n}$ converges absolutely. Therefore, the matrix series converges elementwise to some matrix Y (we denote this matrix by e^X). □

Lemma 4. *Let* $\{A_k\}_{k=0}^{\infty}$ *be a matrix sequence, where* $A_k \in \mathbb{K}^{n\times m}$*, and let* $B \in \mathbb{K}^{r\times n}$*,* $C \in \mathbb{K}^{m\times s}$*. If* $\sum_{k=0}^{\infty} A_k$ *converges, then also* $\sum_{k=0}^{\infty} BA_kC$ *converges, and the following equality holds*

$$B\left(\sum_{k=0}^{\infty} A_k\right)C = \sum_{k=0}^{\infty} BA_kC \ .$$

Proof. We know that for $N \in \mathbb{N}$ it is true

$$\sum_{k=0}^{N} BA_kC = B\left(\sum_{k=0}^{N} A_k\right)C \ .$$

We want to now show that the left hand side converges to $B\left(\sum_{k=0}^{\infty} A_k\right)C$. Let $\varepsilon_1 \in \mathbb{R}^+$ be fixed. Since the series $\sum_{k=0}^{\infty} A^k$ converges, we can find N_0 such that for every $N \in \mathbb{N}, N \geq N_0$ it holds

$$\left\|\sum_{k=0}^{\infty} A_k - \sum_{l=0}^{N} A_l\right\| < \varepsilon_1 \ .$$

Then

$$\left\|B\left(\sum_{k=0}^{\infty}A_k\right)C-\sum_{l=0}^{N}BA_lC\right\|_F=\left\|B\left(\sum_{k=0}^{\infty}A_k\right)C-B\left(\sum_{l=0}^{N}A_l\right)C\right\|_F=$$

$$=\left\|B\left(\sum_{k=0}^{\infty}A_k-\sum_{l=0}^{N}A_l\right)C\right\|_F\leq\|B\|_F\left\|\sum_{k=0}^{\infty}A_k-\sum_{l=0}^{N}A_l\right\|_F\|C\|_F<\|B\|_F\|C\|_F\varepsilon_1\ .$$

This concludes the proof that the series $\sum_{k=0}^{\infty}BA_kC$ converges to $B\left(\sum_{k=0}^{\infty}A_k\right)C$. □

Definition. *Let us have a matrix function $X(t)\colon\mathbb{R}\to\mathbb{K}^{n\times m}$. Then the derivative of the function is*

$$\frac{d}{dt}X(t)=\left(\frac{d}{dt}x_{i,j}(t)\right)_{n\times m}=\left(\dot{x}_{i,j}(t)\right)_{n\times m}\ .$$

Lemma 5. *Let A, B and X be real or complex $n\times n$ matrices. Then*

1. *if $AB=BA$, then $e^AB=Be^A$,*
2. *if R is invertible, then $e^{R^{-1}XR}=R^{-1}e^XR$,*
3. *$\frac{d}{dt}e^{tX}=Xe^{tX}$, for $t\in\mathbb{R}$,*
4. *if $AB=BA$, then $e^{A+B}=e^Ae^B$.*

Proof. 1. Because of the convergence of the matrix exponential, we can use Lemma 4 and get

$$e^AB=\sum_{k=0}^{\infty}\frac{1}{k!}A^kB\overset{AB=BA}{=\!=\!=}\sum_{k=0}^{\infty}\frac{1}{k!}BA^k=B\sum_{k=0}^{\infty}\frac{1}{k!}A^k=Be^A\ .$$

2. Following from Lemma 4, we have

$$e^{R^{-1}XR}=\sum_{k=0}^{\infty}\frac{1}{k!}(R^{-1}XR)^k=\sum_{k=0}^{\infty}\frac{1}{k!}R^{-1}X^kR=R^{-1}\left(\sum_{k=0}^{\infty}\frac{1}{k!}X^k\right)R=R^{-1}e^XR\ .$$

3. The elements of the matrix $e^{tX}=\sum_{k=0}^{\infty}\frac{t^k}{k!}X^k=(e_{i,j}(t))_{n\times n}$ are equal to

$$e_{i,j}(t)=\sum_{k=0}^{\infty}\frac{t^k}{k!}a_{i,j}^{(k)}\ ,$$

where $X^k=(a_{i,j}^{(k)})_{n\times n}$. By Corollary 1 the series $\sum_{k=0}^{\infty}\frac{t^k}{k!}a_{i,j}^{(k)}$ is absolutely convergent for every $t\in\mathbb{K}$. We can now differentiate the individual elements (see Pick et al., 2019, Věta 8.2.2).

$$\frac{d}{dt}e_{i,j}(t)=\frac{d}{dt}\sum_{k=0}^{\infty}\frac{t^k}{k!}a_{i,j}^{(k)}=\sum_{k=1}^{\infty}\frac{t^{k-1}}{(k-1)!}a_{i,j}^{(k)}=\sum_{k=0}^{\infty}\frac{t^k}{k!}a_{i,j}^{(k+1)}\ .$$

Using Lemma 4 we get the desired result

$$\frac{d}{dt}e^{tX}=\left(\frac{d}{dt}e_{i,j}(t)\right)_{n\times n}=\left(\sum_{k=0}^{\infty}\frac{t^k}{k!}a_{i,j}^{(k+1)}\right)_{n\times n}=\sum_{k=0}^{\infty}\frac{t^k}{k!}X^{k+1}=X\sum_{k=0}^{\infty}\frac{t^k}{k!}X^k=Xe^{tX}\ .$$

4. Let $A^k B^l = (\alpha_{i,j}^{(k,l)})_{n\times n}$. Then

$$\begin{aligned} e^{A+B} &= \sum_{k=0}^{\infty} \frac{1}{k!}(A+B)^k = \sum_{k=0}^{\infty}\sum_{l=0}^{k} \binom{k}{l} \frac{1}{k!} A^l B^{k-l} = \sum_{k=0}^{\infty}\sum_{l=0}^{k} \frac{1}{l!(k-l)!} A^l B^{k-l} \\ &= \left(\sum_{k=0}^{\infty}\sum_{l=0}^{k} \frac{1}{l!(k-l)!} \alpha^{(l,k-l)} \right)_{n\times n} = \left(\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{1}{k!l!} \alpha_{i,j}^{(k,l)} \right)_{n\times n} \\ &= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{1}{k!l!} A^k B^l = \sum_{k=0}^{\infty} \left(\frac{1}{k!} A^k \sum_{l=0}^{\infty} \frac{1}{l!} B^l \right) \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} A^k \cdot \sum_{l=0}^{\infty} \frac{1}{l!} B^l = e^A e^B \ . \end{aligned}$$

The second equality holds by the assumption that $AB = BA$, and the last three equalities hold by Lemma 4. The crucial point is the fifth equality, in which we reorder the elements of the series as depicted in Figure 1.1. The

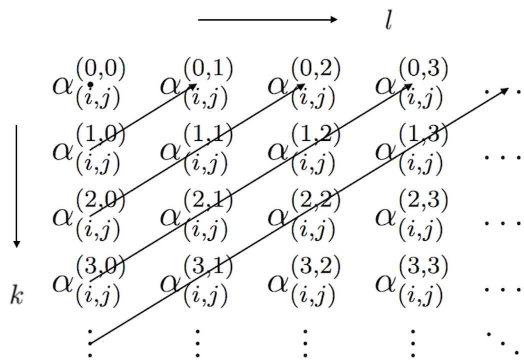

Figure 1.1

equality holds as long as the original series is absolutely convergent, which is satisfied since every element of the matrix e^{A+B} converges absolutely by Corollary 1.

$\square$

Lemma 6. *For any $a \in \mathbb{K}$ we have $e^{aI} = e^a I$.*

Proof. Follows straight from the definition of the matrix exponential.

$$e^{aI} = \sum_{k=0}^{\infty} \frac{a^k}{k!} I^k = \left(\delta_{i,j} \sum_{k=0}^{\infty} \frac{a^k}{k!} \right)_{n\times n} = (\delta_{i,j} e^a)_{n\times n} = e^a I$$

$\square$

Now, using the properties in Lemma 5, we can see that $\dot{x}(t) = Ax(t)$ is solved by $x(t) = e^{tA}x(0)$. The solution is unique which follows from the general theory of linear differential equations (see Pick et al., 2019, Věta 13.5.1).

Claim 2. *The autonomous linear system $\dot{x}(t) = Ax(t)$ with an initial condition $x(0)$ is uniquely solved by $x(t) = e^{tA}x(0)$.*

1.1.1 Stability of Linear Autonomous Systems

Typically, we require the autonomous system to stabilize itself back into its stable state after some disturbances.

Definition. *The linear autonomous system $\dot{x}(t) = Ax(t)$ is* **stable***, if for any initial state $x(0) \in \mathbb{K}^n$ the system converges to o for $t \to \infty$.*

Let A be a real or complex matrix. Then there is a regular matrix $R \in \mathbb{K}^{n\times n}$ such that the matrix

$$J = R^{-1}AR$$

is in a Jordan normal form. By substituting $x(t) = Ry(t)$, which is equivalent to changing the basis of our system, we get

$$\begin{aligned} R\dot{y}(t) &= ARy(t) \\ \dot{y}(t) &= R^{-1}ARy(t) \\ \dot{y}(t) &= Jy(t) \ . \end{aligned}$$

Therefore, by Claim 2, the unique solution is

$$y(t) = e^{tJ}y(0) \ .$$

It is sufficient to determine when $y(t)$ converges to o, because since R is an invertible matrix, $x(t)$ converges to o if and only if $y(t)$ converges to o.

We know that every Jordan block $J_{\lambda,n}$ in the matrix J is of the form $J_{\lambda,n} = \lambda I_n + N_n$, $n \in \mathbb{N}$, where N_n is the $n \times n$ nilpotent matrix satisfying $n_{i,j} = \delta_{i,j-1}$. For example, in case of $n = 4$ we have

$$N_4 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ (N_4)^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ (N_4)^3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

It is also true that $(N_n)^k_{i,j} = \delta_{i,j-k}$ and $(N_n)^n = O_{n\times n}$, since every right multiplication by the matrix N shifts the multiplied matrix's columns to the right by one column, that is, it maps matrix $(v_1, \ldots, v_n)$ onto $(o, v_1, \ldots, v_{n-1})$.

By Lemma 5, for each Jordan block $J_{\lambda,n}$, we now have

$$e^{tJ_{\lambda,n}} = e^{t(\lambda I+N)} = e^{t\lambda I}e^{tN} = e^{\lambda t}e^{tN} \ .$$

Let $\lambda = a + ib$ where $a,b \in \mathbb{R}$. Then we have

$$e^{tJ_{\lambda,n}} = e^{at}e^{ibt}e^{tN} \ .$$

We know that $|e^{ibt}| = 1$ and that

$$e^{tN} = \sum_{k=0}^{\infty} \frac{t^k}{k!}N^k = \sum_{k=0}^{n-1} \frac{t^k}{k!}N^k$$

since $(N_n)^n = O_{n\times n}$. Therefore, we can see that every element of the matrix e^{tN} is a polynomial in t of degree less than n. It follows that $e^{tJ_{\lambda,n}}$ approaches $O_{n\times n}$ for $t \to \infty$ if and only if

$$\lim_{t\to\infty} e^{at}t^{n-1} = 0 \ .$$

This holds for any $n \in \mathbb{N}$ if and only if $a < 0$.

Since any block diagonal matrix to the power of any natural number preserves its block form, we can write

$$J = \begin{pmatrix} J_{\lambda_1,n_1} & 0 & \cdots & 0 \\ 0 & J_{\lambda_2,n_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_{\lambda_r,n_r} \end{pmatrix}, \quad e^J = \begin{pmatrix} e^{J_{\lambda_1,n_1}} & 0 & \cdots & 0 \\ 0 & e^{J_{\lambda_2,n_2}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{J_{\lambda_r,n_r}} \end{pmatrix},$$

where the zeroes in the matrices represent zero matrices of appropriate sizes. Therefore, since $y(0)$ is a constant vector, we see that $y(t) = e^{tJ}y(0)$ converges to o if (and only if, because of the uniqueness of the solution) all the eigenvalues λ_i of the matrix A have negative real parts. As the last step, we calculate $x(t) = Ry(t)$ and $x(0) = Ry(0)$. We formulate this result into a theorem.

Theorem 1. *The system $\dot{x} = Ax(t)$ is stable if and only if all eigenvalues of the matrix A have negative real parts.*

1.1.2 Linear System With Control

Definition. *A* **continuous dynamic linear system with control u** *is a system of linear differential equations of first order with constant coefficients in the form*

$$\dot{x}(t) = Ax(t) + Bu(t) \ ,$$

where the function $x(t)\colon \mathbb{R}^+ \to \mathbb{K}^n$ is a **state vector** *(***state** *for short) of the system, $A \in \mathbb{K}^{n\times n}$ is a* **fundamental matrix** *of the system, $B \in \mathbb{K}^{n\times m}$ is a* **control matrix** *of the system and the continuous function $u(t)\colon \mathbb{R}^+ \to \mathbb{K}^m$ is a* **control vector** *of the system. The* **initial condition** *of the system is the state $x(0)$.*

We will call this system the (A, B) system for short.

In a general case, this is called an **open-loop control** system because the control is not dependent on the previous state of the system.

We can imagine this system as follows. The first part of the right-hand side, $Ax(t)$, of the equation $\dot{x}(t) = Ax(t) + Bu(t)$ can be thought of as the model of the machine or the event that we want to control and the second part, $Bu(t)$, as our control mechanism. The B matrix is then our "control board" and the control vector $u(t)$ is us deciding, which "levers" and "buttons" we want to push.

Of course, if we want this system to be self-regulating, we cannot input our own values into $u(t)$, and therefore $u(t)$ has to be calculated from the current state of our system.

Definition. *Let us have a linear differential system with the control $u(t)$ defined as*

$$u(t) = Fx(t) \ ,$$

where $F \in \mathbb{C}^{m\times n}$ is a **feedback matrix***. This system is then called a* **closed-loop control** *system or a* **linear feedback control** *system.*

For short, we will call this system the (A, B, F) system.

Usually, we are given an autonomous system and we need to find a feedback matrix F such that the resulting system has some desired behavior. The feedback control system can be expressed as the linear autonomous system

$$\dot{x}(t) = Ax(t) + BFx(t) = (A + BF)x(t) \ .$$

Definition. *The linear feedback system* (A, B, F) *is* **stable***, if the linear autonomous system* $\dot{x}(t) = (A + BF)x(t)$ *is stable.*

By Theorem 1, we now know that the system is stable (converges to o for $t \to \infty$) if all eigenvalues of the matrix $A + BF$ have negative real parts. Therefore, we are left to provide a suitable feedback matrix $F \in \mathbb{C}^{n\times n}$. This requirement can also be expressed through the characteristic polynomial of the matrix $A + BF$, since the roots of the characteristic polynomial of a matrix are precisely eigenvalues of the matrix.

Definition. *Let A be a $n \times n$ matrix. Then the* **characteristic polynomial** *of A, denoted by χ_A, is defined as*

$$\chi_A(s) = \det(sI_n - A) \ .$$

Through our observations we got to a conclusion, that we need to find a feedback matrix F such that the characteristic polynomial of the matrix $A+BF$ is

$$\chi_{A+BF} = (x - \lambda_1)(x - \lambda_2)\cdots(x - \lambda_n) \ ,$$

where all its roots $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{C}$ have negative real parts. This leads to an important definition.

Definition. *Let $\mathbb{K}$ be a field and let $A \in \mathbb{K}^{n\times n}$, $B \in \mathbb{K}^{n\times m}$, $n, m \in \mathbb{N}$. We say that a polynomial χ is* **assignable for the pair** (A, B) *if there exists such a matrix $F \in \mathbb{K}^{m\times n}$ that*

$$\chi_{A+BF} = \chi \ .$$

The pole shifting theorem states, that if A and B are "sensible" in a sense that we will discuss in the next section, then an arbitrary monic polynomial χ of degree n can be assigned to the pair (A, B). It also claims that it is immaterial over what field A and B are.

1.2 Controllable pairs

In this section we will establish the notion of controllability. We will first explain this concept for *discrete-time systems* and then we will show that the requirement for controllability of *continuous-time systems* is the same as the one for *discrete-time systems.*

1.2.1 Discrete-time systems

Let us have a continuous dynamic system $\dot{x}(t) = A_1x(t)$, where A_1 is a real or complex square matrix. We *discretize* the time, that is, instead of using continuous real-time values of $x(t)$ and $\dot{x}(t)$, we are interested in these values only at discrete *sampling times* $0, \delta, 2\delta, \ldots, k\delta, \ldots$ where $\delta \in \mathbb{R}^+$. We will denote the states at each sampling time as

$$x_k = x(k\delta) \ , k \in \mathbb{N}_0 \ .$$

The solution of this system is by Theorem 2 precisely $x(t) = e^{tA_1}x(0)$. For some $k \in \mathbb{N}$ we get $x_k = x(k\delta) = e^{k\delta A_1}x(0)$. Using the fourth point of Lemma 5 we obtain

$$\begin{aligned} x_{k+1} &= e^{(k+1)\delta A_1}x(0) \\ &= e^{\delta A_1 + k\delta A_1}x(0) \\ &= e^{\delta A_1}e^{k\delta A_1}x(0) \\ &= e^{\delta A_1}x_k \\ &= Ax_k \end{aligned}$$

by choosing $A = e^{\delta A_1}$. We see that the value of x at the sample time k can be calculated from its previous value. We will now define such a system. The definition holds for any field $\mathbb{K}$.

Definition. *A* **discrete dynamic linear system** *is a system of equations*

$$x_{k+1} = Ax_k, \ k \in \mathbb{N}_0 \ ,$$

where $x_k \in \mathbb{K}^n$ is a **state vector** *(***state** *for short) of the system and the matrix $A \in \mathbb{K}^{n \times n}$ is a* **fundamental matrix** *of the system. The* **initial condition** *of the system is the state $x(0)$.*

Similarly, we can define a discrete dynamic linear system with control.

Definition. *A* **discrete dynamic linear system with control u** *is a system of equations*

$$x_{k+1} = Ax_k + Bu_k, \ k \in \mathbb{N}_0 \ ,$$

where $x_k \in \mathbb{K}^n$ is a **state vector** *(***state** *for short) of the system, $A \in \mathbb{K}^{n \times n}$ is a fundamental matrix, $B \in \mathbb{K}^{n \times m}$ is a control matrix and $u_k \in \mathbb{K}^m$ is a control vector. The* **initial condition** *of the system is the state x_0.*

We will call this system the **discrete** *(A, B) system.*

Definition. *We say that a state x can be* **reached** *in a time $k \in \mathbb{N}_0$ if there exists such a sequence of control vectors $u_0, u_1, \ldots, u_{k-1}$ that for the initial condition $x_0 = o$ we get $x = x_k$.*

States that we can reach in time $k \in \mathbb{N}$ in open-loop control discrete-time systems can be derived as follows. The initial condition is $x_0 = o$ and we can choose arbitrary $u_0, u_1, \ldots, u_k$. Then for $k = 1$ we have

$$x_1 = Ax_0 + Bu_0 = Bu_0 \in \mathrm{Im}B \ .$$

For $k = 2$ we get

$$x_2 = Ax_1 + Bu_1 = ABu_0 + Bu_1 \in \mathrm{Im}(AB|B) \ .$$

It is clear, that for every $k \in \mathbb{N}$

$$x_k \in \mathrm{Im}(A^{k-1}B|\cdots|AB|B) \ .$$

We can observe that

$$\mathrm{Im}(B|AB|\cdots|A^kB) \subseteq \mathrm{Im}(B|AB|\cdots|A^{k+1}B), \ \forall k \in \mathbb{N} \ .$$

By the Cayley–Hamilton theorem we know that $\chi_A(A) = O_{n\times n}$. That means, that A^n can be expressed as a linear combination of the matrices $\{I, A, \ldots, A^{n-1}\}$ which implies that A^nB can be expressed as a linear combination of the matrices $\{B, AB, \ldots, A^{n-1}B\}$. We now see that

$$\mathrm{Im}(B|AB|\cdots|A^nB) \subseteq \mathrm{Im}(B|AB|\cdots|A^{n-1}B) \ .$$

It follows

$$\mathrm{Im}(B|AB|\cdots|A^{n-1}B) = \mathrm{Im}(B|AB|\cdots|A^{n-1}B|A^nB) \ . \tag{1.1}$$

For an arbitrary $k \in \mathbb{N}, k > n$ we have

$$A^kB = A^{k-n}A^nB = A^{k-n}\sum_{i=0}^{n-1}\alpha_iA^iB = \sum_{i=0}^{n-1}\alpha_iA^{k-n+i}B \in \mathrm{Im}(B|AB|\ldots|A^{k-1}B) \ ,$$

for some $\alpha_0, \ldots, \alpha_{n-1} \in \mathbb{K}$. Therefore, all the states we could reach in any time $k \in \mathbb{N}$ are already in the space

$$\mathrm{Im}(B|AB|\cdots|A^{n-1}B) \ .$$

Definition. *Let $\mathbb{K}$ be a field and let $A \in \mathbb{K}^{n\times n}$, $B \in \mathbb{K}^{n\times m}$, $n, m \in \mathbb{N}$. The matrix*

$$\mathbf{R}(A, B) = (B|AB|\cdots|A^{n-1}B)$$

is called the **rechability matrix** *of (A, B). We define the* **reachable space** *$\mathcal{R}(A, B)$ of the pair (A, B) as Im$(B|AB|\cdots|A^{n-1}B)$.*

We have seen that by left multiplying $\mathcal{R}(A, B)$ by A, we get a subspace which is already included in $\mathcal{R}(A, B)$. This leads to an important property of some subspaces.

Definition. *Let V be a vector space, W be its subspace and let f be a mapping from V to V. We call W an* **invariant subspace** *of f if $f(W) \subseteq W$. We also say that W is* **f-invariant**.

If $f = f_A$ for some matrix A, we also say that W is **A-invariant** *for short.*

Lemma 7. *$\mathcal{R}(A, B)$ is an A-invariant subspace.*

Proof. It follows from the discussion above. □

This leads us to an important property of the pair (A, B). We want to be able to get the system into any state in the state space by controlling it with our control u, i.e., choosing an appropriate sequence $u_0, \ldots, u_{n-1}$. Therefore, we desire that $\mathcal{R}(A, B) = \mathbb{K}^n$. An equivalent condition is dim$\mathcal{R}(A, B) = n$.

Definition. *Let $\mathbb{K}$ be a field and let $A \in \mathbb{K}^{n\times n}$, $B \in \mathbb{K}^{n\times m}$, $n, m \in \mathbb{N}$. The pair (A, B) is* **controllable** *if* dim$\mathcal{R}(A, B) = n$.

1.2.2 Continuous-time systems

Remark. *In this section we assume, that $\mathbb{K}$ is a field of either real or complex numbers and that $A \in \mathbb{K}^{n\times n}$, $B \in \mathbb{K}^{n\times m}$.*

We will now show that the condition for controllability of *discrete-time systems* also characterizes controllable *continuous-time systems.* For this we have to express the solution of such system using the matrices A^iB for $i \in \mathbb{N}_0$.

Definition. *Let us have a vector function $v(t)\colon \mathbb{R} \to \mathbb{K}^n$. Then the definite integral of the function on an interval $[a,b]$, $a,b \in \mathbb{R}$ is*

$$\int_a^b v(t)dt = \left(\int_a^b v_1(t)dt\ ,\ \ldots\ ,\ \int_a^b v_n(t)dt\right)^T .$$

Lemma 8. *For a matrix function $A(t)\colon \mathbb{R} \to \mathbb{K}^{n\times m}$ and a vector function $v(t)\colon \mathbb{R} \to \mathbb{K}^m$ it holds*

$$\frac{d}{dt}\left(A(t)v(t)\right) = \left(\frac{d}{dt}A(t)\right)v(t) + A(t)\frac{d}{dt}v(t)$$

Proof. Can be simply shown by rewriting the vector $A(t)v(t)$ elementwise. □

We utilize the matrix exponential in solving the inhomogeneous linear system $\dot{x}(t) = Ax(t) + Bu(t)$. By left multiplying it by e^{-tA} we get

$$\begin{aligned} e^{-tA}\dot{x}(t) - e^{-tA}Ax(t) &= e^{-tA}Bu(t) \\ \frac{d}{dt}(e^{-tA}x(t)) &= e^{-tA}Bu(t)\ . \end{aligned}$$

Note that we used Lemma 8 and the fact that $-AA = A(-A) \Rightarrow e^{-tA}A = Ae^{-tA}$ from Lemma 5. After integrating both sides with respect to t on interval (t_0, t_1) we obtain

$$\begin{aligned} [e^{-tA}x(t)]_{t_0}^{t_1} &= \int_{t_0}^{t_1} e^{-tA}Bu(t)dt \\ e^{-t_1A}x(t_1) - e^{-t_0A}x(t_0) &= \int_{t_0}^{t_1} e^{-tA}Bu(t)dt \\ x(t_1) &= e^{(t_1-t_0)A}x(t_0) + \int_{t_0}^{t_1} e^{(t_1-t)A}Bu(t)dt\ . \end{aligned}$$

The last integral makes sense since $u(t)$ is required to be continuous.

Now it is clear that in the system where $x(0) = o$, every state in time $t \in \mathbb{R}^+$ can be expressed as

$$x(t) = \int_0^t e^{(t-s)A}Bu(s)ds\ . \tag{1.2}$$

Definition. *We say that a state $x \in \mathbb{K}^n$ can be* **reached in time** *t, if there exists a control $u(x)\colon [0,t] \to \mathbb{K}^m$ such that*

$$x = \int_0^t e^{(t-s)A}Bu(s)ds\ .$$

The set of all states that can be reached in time t is denoted by $\mathcal{R}^t$. The set $\mathcal{R} = \cup_{t\in\mathbb{R}^+}\mathcal{R}^t$ of all states that can be reached, is called a **reachable space***.*

Definition. *An n-dimensional continuous-time linear system is* **controllable***, if* $\mathcal{R} = \mathbb{K}^n$.

Theorem 2. *The n-dimensional continuous-time linear system is controllable if and only if* $dim\mathcal{R}(A,B) = n$.

Proof. From the discussion above we have

$$x(t) = \int_0^t e^{(t-s)A} Bu(s)ds = \int_0^t \sum_{k=0}^{\infty} \frac{(t-s)^k}{k!} A^k Bu(s)ds \ .$$

The n-dimensional vector $w^{(k)}(s) = A^k Bu(s)$ has the elements

$$w_i^{(k)}(s) = \sum_{j=1}^{m} \alpha_{i,j}^{(k)} u_j(s) \ ,$$

where $\alpha_{i,j}^{(k)}$ is the element of the matrix $A^k B$ on the position (i,j). Therefore, the i-th element of $x(t)$ is equal to

$$x_i(t) = \int_0^t \sum_{k=0}^{\infty} \frac{(t-s)^k}{k!} w_i^{(k)} ds = \int_0^t \sum_{k=0}^{\infty} \sum_{j=1}^{m} \frac{(t-s)^k}{k!} \alpha_{i,j}^{(k)} u_j(s)ds \ .$$

Now, in order to be able to modify this expression, we will prove that the series $\sum_{k=0}^{\infty} \frac{(t-s)^k}{k!} \alpha_{i,j}^{(k)} u_j(s)$ is absolutely convergent for every position (i,j). Following from Lemma 2, we have

$$\sum_{k=0}^{N} \left| \frac{(t-s)^k}{k!} \alpha_{i,j}^{(k)} u_j(s) \right| \le \sum_{k=0}^{N} \frac{|t-s|^k}{k!} |\alpha_{i,j}^{(k)}||u_j(s)| \le |u_j(s)| \|B\|_F \sum_{k=0}^{N} \frac{\|(t-s)A\|_F^k}{k!} \ .$$

This gives us

$$\sum_{k=0}^{\infty} \left| \frac{(t-s)^k}{k!} \alpha_{i,j}^{(k)} u_j(s) \right| = \lim_{N\to\infty} \sum_{k=0}^{N} \left| \frac{(t-s)^k}{k!} \alpha_{i,j}^{(k)} u_j(s) \right| \le |u_j(s)| \|B\|_F e^{\|(t-s)A\|_F} \ .$$

Because of the absolute convergence, we can now swap the integral and the series:

$$\begin{aligned} x_i(t) &= \int_0^t \sum_{k=0}^{\infty} \sum_{j=1}^{m} \frac{(t-s)^k}{k!} \alpha_{i,j}^{(k)} u_j(s)ds = \sum_{k=0}^{\infty} \int_0^t \sum_{j=1}^{m} \frac{(t-s)^k}{k!} \alpha_{i,j}^{(k)} u_j(s)ds \\ &= \sum_{k=0}^{\infty} \sum_{j=1}^{m} \int_0^t \frac{(t-s)^k}{k!} \alpha_{i,j}^{(k)} u_j(s)ds = \sum_{k=0}^{\infty} \sum_{j=1}^{m} \frac{\alpha_{i,j}^{(k)}}{k!} \int_0^t (t-s)^k u_j(s)ds \\ &= \sum_{k=0}^{\infty} \sum_{j=1}^{m} \frac{\alpha_{i,j}^{(k)}}{k!} v_j^{(k)}(t) \ , \end{aligned}$$

where $v^{(k)}(t) = \int_0^t (t-s)^k u(s)ds$ is a vector of length m. Therefore, we have

$$x(t) = \sum_{k=0}^{\infty} \frac{1}{k!} A^k B v^{(k)}(t) = \sum_{k=0}^{\infty} \frac{1}{k!} A^k B \int_0^t (t-s)^k u(s)ds \ .$$

By the Cayley-Hamilton theorem it then holds

$$x(t) = \sum_{i=0}^{\infty} \frac{1}{i!} A^i B v^{(i)}(t) = \sum_{i=0}^{\infty} \sum_{j=0}^{n-1} \frac{1}{i!} \alpha_{i,j} A^j B v^{(i)}(t) = \sum_{j=0}^{n-1} \left(\sum_{i=0}^{\infty} \frac{1}{i!} \alpha_{i,j} \right) A^j B v^{(i)}(t) \ .$$

The third equality follows from the absolute convergence of the resulting series, which can be shown in a similar way as above.

Now we see that

$$x(t) \in \mathrm{Im}(B|AB|\ldots|A^{n-1}B) = \mathcal{R}(A,B) \ .$$

If the system is controllable, then $x(t)$ can be equal to any of the vectors of an arbitrary basis of $\mathbb{K}^n$. Therefore, we know that n linearly independent vectors belong into $\mathcal{R}(A,B)$, and it follows that $\dim\mathcal{R}(A,B) = n$.

For the proof of the "if" part we use Sontag (1998, Theorem 3).

Conversely, if controllability fails, then there exists a non-trivial complement $\mathcal{S}$ to the reachable space $\mathcal{R}$. For any time $t \in \mathbb{R}^+$ and any vector $\rho \in \mathcal{S}$ it holds that $\rho^* x(t) = o$. By choosing the control $u(t) = B^* e^{(t-s)A^*}\rho$ on the interval $[0,s]$, which is continuous, we get by the equation (1.2)

$$o = \rho^* x(t) = \int_0^t \rho^* e^{(t-s)A} BB^* e^{(t-s)A^*} \rho ds = \int_0^t \left\| B^* e^{(t-s)A^*} \rho \right\|^2 .$$

This implies

$$0 = \left\| B^* e^{(t-s)A^*} \rho \right\|^2 = \left\| \rho^* e^{(t-s)A} B \right\|^2$$

and hence

$$o = \rho^* e^{(t-s)A} B = \sum_{k=0}^{\infty} \frac{(t-s)^k}{k!} \rho^* A^k B \ .$$

This implies, by similar usage of Cayley-Hamilton as in the "only if" part of the proof, that the vector ρ is perpendicular to $\mathcal{R}(A,B)$ and therefore $\dim\mathcal{R}(A,B)$ cannot be equal to n. □

1.2.3 Decomposition theorem

Lemma 9. *Let W be an invariant subspace of a linear mapping $f\colon V \to V$. Then there exists a basis C of V such that*

$$[f]_C^C = \begin{pmatrix} F_1 & F_2 \\ 0 & F_3 \end{pmatrix} ,$$

where F_1 is $r \times r$, $r = dimW$.

Proof. Let $(w_1, \ldots, w_r)$ be an arbitrary basis of the subspace W. We complete this sequence into basis C of V with vectors $v_1, \ldots, v_{n-r}$ where $n = \dim V$, thus $C = (w_1, \ldots, w_r, v_1, \ldots, v_{n-r})$. We know that

$$[f]_C^C = ([f(w_1)]_C, \ldots, [f(w_r)]_C, [f(v_1)]_C, \ldots, [f(v_{n-r})]_C) \ .$$

Since W is an A-invariant subspace, it holds that $f(w_i) \in W$ and therefore, because of our choice of the basis C, the matrix $[f]_C^C$ is of the desired form. □

If (A,B) is not controllable, then there exists a part of the state space that is not affected by the input. This can be shown using the following theorem.

Theorem 3. *Let (A, B) represent a dynamic system and let $dim\mathcal{R}(A, B) = r \leq n$. Then there exists an invertible $n \times n$ matrix T over $\mathbb{K}$ such that the matrices $\widetilde{A} := T^{-1}AT$ and $\widetilde{B} := T^{-1}B$ have the block structures*

$$\widetilde{A} = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix} \qquad \widetilde{B} = \begin{pmatrix} B_1 \\ 0 \end{pmatrix} \ , \tag{1.3}$$

where $A_1 \in \mathbb{K}^{r\times r}$ and $B_1 \in \mathbb{K}^{r\times m}$.

Proof. We know that $\mathcal{R}(A, B)$ is an A-invariant subspace (Lemma 7). Using Lemma 9 on the matrix mapping f_A we get a basis C for which it holds that

$$[f_A]_C^C = [\mathrm{id}]_C^K [f_A]_K^K [\mathrm{id}]_K^C = [\mathrm{id}]_C^K A[\mathrm{id}]_K^C$$

is in a block upper triangular form. By putting $T = [\mathrm{id}]_K^C$ we get that $\widetilde{A} = [f_A]_C^C$ is in the desired form.

Now, let us consider the matrix mapping f_B. We have

$$\widetilde{B} = TB = [\mathrm{id}]_C^{K_n} [f_B]_{K_n}^{K_m} = [f_B]_C^{K_m} = ([f_B(e_1)]_C, \ldots, [f_B(e_m)]_C) \ .$$

Since $f_B(e_i)$ is the i-th column of the matrix B, and trivially by definition of a reachable space it holds that $\mathrm{Im}(B) \subseteq \mathcal{R}(A, B)$, we see that $\widetilde{B}$ is in the requested form. $\square$

We achieved the new form of matrices A and B by changing the basis of our state space. We now define the relation between (A, B) and $(\widetilde{A}, \widetilde{B})$.

Definition. *Let (A, B) and $(\widetilde{A}, \widetilde{B})$ be pairs as the ones described in Theorem 3 above. Then (A, B)* **is similar to** *$(\widetilde{A}, \widetilde{B})$, denoted $(A, B) \sim (\widetilde{A}, \widetilde{B})$, if there exists an invertible matrix T for which it holds that*

$$\widetilde{A} = T^{-1}AT \quad \textit{and} \quad \widetilde{B} = T^{-1}B \ .$$

Lemma 10. *Let A and B be similar matrices, that is, there exists an invertible matrix R such that $A = R^{-1}BR$. Then $\chi_A = \chi_B$.*

Proof. We will use properties of the matrix determinant:

$$\begin{aligned} \chi_A &= \det(sI - A) = \det(sI - R^{-1}BR) \\ &= \det(sR^{-1}IR - R^{-1}BR) = \det(R^{-1}(sI - B)R) \\ &= (\det R)^{-1}\det(sI - B)\det R = \det(sI - B) \\ &= \chi_B \ . \end{aligned}$$

$\square$

Lemma 11. *If $(A, B) \sim (\widetilde{A}, \widetilde{B})$, then the assignable polynomials for the pairs (A, B) and $(\widetilde{A}, \widetilde{B})$ are the same.*

Proof. Let T be a regular matrix over $\mathbb{K}$ such that $\widetilde{A} = T^{-1}AT$ and $\widetilde{B} = T^{-1}B$. Then for any feedback matrix F we have

$$T^{-1}(A + BF)T = T^{-1}ATT^{-1}BFT = \widetilde{A} + \widetilde{B}\widetilde{F} \ ,$$

where $\widetilde{F} = FT$. It follows from Lemma 10 that

$$\chi_{A+BF} = \chi_{\widetilde{A}+\widetilde{B}\widetilde{F}} \ .$$

$\square$

Theorem 3 has the following interesting consequence. Let (A, B) be a dynamic system with initial condition $x(0) = o$, and let T be a regular matrix over $\mathbb{K}$ as in Theorem 3. By putting $x(t) = Ty(t)$ we change the basis and get

$$T\dot{y}(t) = ATy(t) + Bu(t) \ ,$$

which can be rewriten as

$$\dot{y}(t) = T^{-1}ATy(t) + T^{-1}Bu(t) = \widetilde{A}y(t) + \widetilde{B}u(t) \ .$$

This gives us

$$\begin{aligned} \dot{y}_1(t) &= A_1y_1(t) + A_2y_2(t) + B_1u(t) \\ \dot{y}_2(t) &= \qquad\qquad A_3y_2(t) \qquad\qquad , \end{aligned}$$

where $y(t) = (y_1(t), y_2(t))^T$, $y_1(t) \in \mathbb{K}^r$ and $y_2(t) \in \mathbb{K}^{n-r}$. The component $y_2(t)$ cannot be controlled and it is, for the initial condition $y(0) = Tx(0) = o$, always equal to o, since it does not depend on the control vector $u(t)$.

It is also true that the system (A_1, B_1) from Theorem 3 is a controllable pair, which we will state as a lemma.

Lemma 12. *The pair (A_1, B_1) is controllable.*

Proof. We know that $\dim\mathcal{R}(A, B) = r$. We desire that $\dim\mathcal{R}(A_1, B_1) = r$. We will show that $\mathcal{R}(\widetilde{A}, \widetilde{B}) = \mathcal{R}(A, B)$ and that each vector in $\mathcal{R}(\widetilde{A}, \widetilde{B})$ has its last $n - r$ elements equal to 0. Therefore, $\mathcal{R}(\widetilde{A}, \widetilde{B})$ restricted to its first r coordinates is equal to $\mathcal{R}(A_1, B_1)$. First, we have

$$\begin{aligned} \mathcal{R}(\widetilde{A}, \widetilde{B}) &= \mathrm{Im}(\widetilde{A}^{n-1}\widetilde{B}|\cdots|\widetilde{A}\widetilde{B}|\widetilde{B}) \\ &= \mathrm{Im}((T^{-1}AT)^{n-1}T^{-1}B|\cdots|T^{-1}ATT^{-1}B|T^{-1}B) \\ &= \mathrm{Im}(T^{-1}A^{n-1}B|\cdots|T^{-1}AB|T^{-1}B) \\ &= \{(T^{-1}A^{n-1}B|\cdots|T^{-1}AB|T^{-1}B) \cdot v | v \in \mathbb{K}^{n \cdot m}\} \\ &= \{T^{-1}(A^{n-1}B|\cdots|AB|B) \cdot v | v \in \mathbb{K}^{n \cdot m}\} \\ &= T^{-1} \cdot \{(A^{n-1}B|\cdots|AB|B) \cdot v | v \in \mathbb{K}^{n \cdot m}\} \\ &= T^{-1} \cdot (\mathrm{Im}(A^{n-1}B|\cdots|AB|B)) \\ &= T^{-1} \cdot (\mathcal{R}(A, B)) \ , \end{aligned}$$

where the mapping $\cdot : \mathbb{K}^{n \times m} \times V \to W$, where V, W are vector spaces, is defined as $A \cdot V = \{Av | v \in V\}$. Since T is an invertible matrix, which preserves linear independence, we have

$$\dim\mathcal{R}(\widetilde{A}, \widetilde{B}) = \dim(T^{-1}\mathcal{R}(A, B)) = \dim(\mathcal{R}(A, B)) = r \ .$$

Now let us focus on the structure of $\mathcal{R}(\widetilde{A}, \widetilde{B})$. We know that the last $n - r$ rows of $\widetilde{B}$ are o. Also, because of the structure of $\widetilde{A}$, for an arbitrary matrix $X \in \mathbb{K}^{r \times m}$ we have that

$$\widetilde{A}\begin{pmatrix} X \\ 0 \end{pmatrix} = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}\begin{pmatrix} X \\ 0 \end{pmatrix} = \begin{pmatrix} A_1X \\ 0 \end{pmatrix} \ ,$$

where, again, the last $n-r$ rows are equal to o. Therefore, for any positive integer k we have

$$\widetilde{A}^k\widetilde{B} = \begin{pmatrix} A_1^k B_1 \\ 0 \end{pmatrix} \ , A_1^k B_1 \in \mathbb{K}^{r\times r} \ .$$

It follows

$$\mathcal{R}(\widetilde{A}, \widetilde{B}) = \left(\begin{pmatrix} A_1^{n-1}B_1 \\ 0 \end{pmatrix} \middle| \cdots \middle| \begin{pmatrix} A_1 B_1 \\ 0 \end{pmatrix} \middle| \begin{pmatrix} B_1 \\ 0 \end{pmatrix} \right) \ .$$

By the equation (1.1) we have that the restriction to the first r coordinates of $\mathcal{R}(\widetilde{A}, \widetilde{B})$ is equal to $\mathcal{R}(A_1, B_1)$. Finally, it follows that

$$\dim\mathcal{R}(A_1, B_1) = \dim\mathcal{R}(\widetilde{A}, \widetilde{B}) = \dim\mathcal{R}(A, B) = r \ .$$

□

Now we can see that the decomposition described in Theorem 3 decomposes the matrix A into the "controllable" and the "uncontrollable" parts A_1 and A_3 respectively.

Corollary 2. *Let (A, B) be a dynamic system, and let T be a regular matrix and $\widetilde{A} = T^{-1}AT$ as in Theorem 3. Then it holds*

$$\chi_A = \chi_{\widetilde{A}} = \chi_{A_1}\chi_{A_3} = \chi_c\chi_u \ .$$

Proof. Follows from Theorem 3 and Lemma 10. □

Definition. *The polynomials χ_c and χ_u are respectively the* **controllable** *and the* **uncontrollable parts** *of the characteristic polynomial χ_A with respect to the pair (A, B). In the case where $r = 0$ we put $\chi_c = 1$, and in the case where $r = n$ we put $\chi_u = 1$.*

For this definition to be correct, we need to show that polynomials χ_{A_1} and χ_{A_3} are not dependent on the choice of the regular matrix T from Theorem 3. Since $\chi_{A_3} = \chi_A/\chi_{A_1}$, it is sufficient only to show that χ_{A_1} is independent of the choice.

Lemma 13. *Let A be a square matrix over $\mathbb{K}$. Then the controllable part χ_c of its characteristic polynomial is independent of the choice of the basis for $\mathcal{R}(A, B)$.*

Proof. Let $C = (c_1, \ldots, c_n)$ and $D = (d_1, \ldots, d_n)$ be two bases for $\mathbb{K}^n$ as constructed in the proof of Theorem 3. Then we have

$$[f_A]_C^C = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix} \ , \quad [f_A]_D^D = \begin{pmatrix} A_1' & A_2' \\ 0 & A_3' \end{pmatrix} \ ,$$

as in (1.3). We want to show that $\chi_{A_1} = \chi_{A_1'}$.

It is true that

$$[f_A]_C^C = [id]_C^D[f_A]_D^D[id]_D^C \ ,$$

where

$$[id]_D^C = ([c_1]_D, \ldots, [c_n]_D) \ .$$

We know that the vectors $c_1, \dots, c_r$ form a basis of the subspace $\mathcal{R}(A, B)$ and that the vectors $d_1, \dots, d_r$ form another basis of the same subspace. Therefore

$$[id]_D^C = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \ , \quad [id]_C^D = ([id]_D^C)^{-1} = \begin{pmatrix} T_1^{-1} & X \\ 0 & T_3^{-1} \end{pmatrix} \ ,$$

where $T_1 \in \mathbb{K}^{r \times r}$ is a regular matrix, $T_3 \in \mathbb{K}^{n-r \times n-r}$ and $T_2, X \in \mathbb{K}^{r \times n-r}$. It follows

$$\begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix} = \begin{pmatrix} T_1^{-1} & X \\ 0 & T_3^{-1} \end{pmatrix} \begin{pmatrix} A'_1 & A'_2 \\ 0 & A'_3 \end{pmatrix} \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \ ,$$

which implies that

$$A_1 = T_1^{-1} A'_1 T_1 \ .$$

By Lemma 10 it then holds that $\chi_{A_1} = \chi_{A'_1}$. $\square$

2. The Pole Shifting Theorem

The following chapter is based on the first section of the fifth chapter of Sontag (1998).

Definition. *The* **controller form** *associated to the pair* (A, b) *is the pair*

$$A^\flat = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_n \end{pmatrix}, \quad b^\flat = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

where $s^n - \alpha_n s^{n-1} - \ldots - \alpha_2 s - \alpha_1$ *is the characteristic polynomial of* A.

Lemma 14. *The characteristic polynomial of* $A^\flat$ *is* $s^n - \alpha_n s^{n-1} - \ldots - \alpha_2 s - \alpha_1$.

Proof. It can be shown using simple properties of the matrix determinant. □

Lemma 15. *The pair* $(A^\flat, b^\flat)$ *is controllable.*

Proof. Because of the form of the vector $b^\flat$, the matrix $(A^\flat)^k b^\flat$ is equal to the last column of $(A^\flat)^k$, that is

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & 1 & \beta_{k-1} & \cdots & \beta_1 \end{pmatrix}^T$$

for some $\beta_1, \ldots, \beta_{k-1} \in \mathbb{K}$. Therefore $\mathcal{R}(A^\flat, b^\flat) = n$. □

Lemma 16. *Let* $\mathbb{K}$ *be a field and let* $A_1, A_2 \in \mathbb{K}^{n\times n}$ *and* $b_1, b_2 \in \mathbb{K}^n$*, such that the pairs* $(A_1, b_1), (A_2, b_2)$ *are controllable. If the characteristic polynomials of* A_1 *and* A_2 *are the same, then the pairs* $(A_1, b_1), (A_2, b_2)$ *are similar.*

Proof. Let us have a pair

$$A^\dagger = (A^\flat)^T = \begin{pmatrix} 0 & 0 & \cdots & 0 & \alpha_1 \\ 1 & 0 & \cdots & 0 & \alpha_2 \\ 0 & 1 & \cdots & 0 & \alpha_3 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & \alpha_n \end{pmatrix} \qquad b^\dagger = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} .$$

The characteristic polynomial of the matrix $A^\dagger$ is the same as the one of the matrix $A^\flat$ since transposing a matrix preserves its characteristic polynomial. Therefore, by Cayley-Hamilton theorem and by Lemma 14, it holds that

$$O = \chi_{A^\dagger}(A) = \chi_{A^\flat}(A) = A^n - \alpha_n A^{n-1} - \ldots - \alpha_2 A - \alpha_1 ,$$

implying

$$A^n = \alpha_n A^{n-1} + \ldots + \alpha_2 A + \alpha_1 .$$

It then follows

$$\mathbf{R}(A, b)A^\dagger = \begin{pmatrix} b & Ab & \ldots & A^{n-1}b \end{pmatrix} A^\dagger = \begin{pmatrix} Ab & A^2b & \ldots & A^nb \end{pmatrix} = A\mathbf{R}(A, b) .$$

By the controllability of the pair (A, b), the column space of the matrix $\mathbf{R}(A, b)$ is of dimension n, which means, that the matrix is invertible. Therefore, we can write

$$A = \mathbf{R}(A, b) A^\dagger \mathbf{R}(A, b)^{-1} \ .$$

We see that the matrices A and $A^\dagger$ are similar. It is also true that

$$\mathbf{R}(A, b) b^\dagger = b \ .$$

Therefore $(A, b) \sim (A^\dagger, b^\dagger)$.

Since the pair $(A^\dagger, b^\dagger)$ depends only on the characteristic polynomial of the matrix A, we conclude by transitivity of the matrix similarity, that any two controllable pairs with the same characteristic polynomials are similar to each other. □

Corollary 3. *If the* **single-input** *$(m = 1)$ pair (A, b) is controllable, then it is similar to its controller form.*

Proof. Follows from Lemmas 14, 15 and 16. □

Theorem 4. *Let $\mathbb{K}$ be a field. Let $A \in \mathbb{K}^{n \times n}$, $B \in \mathbb{K}^{n \times m}$. The assignable polynomials for the pair (A, B) are precisely of the form*

$$\chi_{AB+F} = \chi \chi_u$$

where χ is an arbitrary monic polynomial of degree $r = dim \mathcal{R}(A, B)$ and χ_u is the uncontrollable part of the assignable polynomial.

In particular, the pair (A, B) is controllable if and only if every nth degree monic polynomial can be assigned to it.

Proof. By Theorem 3 and Lemma 11 we can assume that the pair (A, B) is in the same form as $(\widetilde{A}, \widetilde{B})$ in (1.3). Let us write $F = (F_1, F_2) \in \mathbb{K}^{m \times n}$, where $F_1 \in \mathbb{K}^{m \times r}, F_2 \in \mathbb{K}^{m \times (n-r)}$. Then

$$\begin{aligned} A + BF &= \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix} + \begin{pmatrix} B_1 \\ 0 \end{pmatrix} \begin{pmatrix} F_1 & F_2 \end{pmatrix} = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix} + \begin{pmatrix} B_1 F_1 & B_1 F_2 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} A_1 + B_1 F_1 & A_2 + B_1 F_2 \\ 0 & A_3 \end{pmatrix} \end{aligned}$$

It follows

$$\chi_{A+BF} = \chi_{A_1 + B_1 F_1} \chi_{A_3} = \chi_{A_1 + B_1 F_1} \chi_u$$

We see that any assignable polynomial is a multiple of the uncontrollable part χ_u.

Conversely, we want to show that the first factor can be made arbitrary by a suitable choice of F_1. This makes sense only for $r > 0$, otherwise the assignable polynomial is equal to χ_u, which cannot be changed by modifying the matrix F. Assume that we are given a monic polynomial χ. If we find such a matrix F_1 that

$$\chi_{A_1 + B_1 F_1} = \chi$$

then by putting $F = (F_1, 0)$ we get the desired characteristic polynomial, that is, $\chi_{A+BF} = \chi \chi_u$. Since the pair (A_1, B_1) is controllable as shown in Lemma

12, it is sufficient only to prove that controllable systems can be assigned an arbitrary monic polynomial χ of respective degree. Therefore, from this point on, we assume that the pair (A, B) is controllable.

We will first prove the theorem for $m = 1$ and then we will generalize it. That will conclude the proof.

Let $m = 1$. By Lemma 11 and Corollary 3 we can consider the pair (A, b) to be in the controller form. For a vector

$$f = \begin{pmatrix} f_1 & f_2 & \dots & f_n \end{pmatrix}$$

we have

$$A + bf = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ \alpha_1 & \alpha_2 & \alpha_3 & \dots & \alpha_n \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \begin{pmatrix} f_1 & f_2 & \dots & f_n \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ \alpha_1 + f_1 & \alpha_2 + f_2 & \alpha_3 + f_3 & \dots & \alpha_n + f_n \end{pmatrix} .$$

One can see that given a monic polynomial

$$\chi = s^n - \beta_n s^{n-1} - \ldots - \beta_2 s - \beta_1 ,$$

we can choose

$$f = \begin{pmatrix} \beta_1 - \alpha_1 & \beta_2 - \alpha_2 & \dots & \beta_n - \alpha_n \end{pmatrix} ,$$

and the equality $\chi_{A+bf} = \chi$ will be satisfied. We have shown that for the case where $m = 1$, the controllable pair (A, b) can be assigned an arbitrary monic polynomial of degree n.

For the general case, where m is arbitrary, we choose any vector $v \in \mathbb{K}^m$ such that $Bv \neq o$ and let $b = Bv$. TODODODO We will use the fact that for any matrix $G \in \mathbb{K}^{m \times n}$ the pair $(A + BG, b)$ can be assigned the same polynomials as the pair (A, B), because for any $f \in \mathbb{K}^{1 \times n}$ it holds

$$A + BG + bf = A + BG + Bvf = A + B(G + vf)$$

and therefore, for $F = G + vf$ we have

$$\chi_{A+BG+bf} = \chi_{A+BF} .$$

Using the result for $m = 1$, the proof will be concluded by showing that the pair $(A + BG, b)$ is controllable.

Let us have a sequence of linearly independent vectors $\{Bv = x_1, \dots, x_k\}$, $x_i \in \mathbb{K}^n$, where

$$x_i = Ax_{i-1} + Bu_{i-1}, \; i \in \{1, \dots, k\} \tag{2.1}$$

for some $u_i \in \mathbb{K}^m$, and $x_0 = o$, and assume that k is as large as possible. We denote $\mathrm{Im}(x_1, \ldots, x_k)$ by $\mathcal{V}$. By maximality of k we have $x_{k+1} \in \mathcal{V}$, which implies that

$$Ax_k + Bu = x_{k+1} \in \mathcal{V} \tag{2.2}$$

for any $u \in \mathbb{K}^m$. Therefore, in particular for $u = o$, we get

$$Ax_k \in \mathcal{V} \ . \tag{2.3}$$

It follows by (2.2) and (2.3), that for any $u \in \mathbb{K}^m$ it holds

$$Bu = x_{k+1} - Ax_k \in \mathcal{V} \ ,$$

which implies that the column space $\mathcal{B} = \mathrm{Im}B$ is included in $\mathcal{V}$. Following from this and the equality (2.1), we have

$$Ax_{i-1} = x_i - Bu_{i-1} \in \mathcal{V}$$

for $i \in \{1, \ldots, k\}$. This result together with the equation (2.3) shows that for any $i \in \{1, \ldots, k\}$ it is true that $Ax_i \in \mathcal{V}$. This means, that $\mathcal{V}$ is an A-invariant subspace containing $\mathcal{B}$. Using these two facts and the fact that for any two matrices $A_{n\times m}, B_{m\times o}$ it holds that $A(\mathrm{Im}(B)) \overset{def.}{=\!=} \{Av | v \in \mathrm{Im(B)}\} = \mathrm{Im}(AB)$, one can see that

$$\begin{aligned} \mathcal{B} &\subseteq \mathcal{V} \\ A\mathcal{B} &\subseteq A\mathcal{V} \subseteq \mathcal{V} \\ A^2\mathcal{B} &\subseteq A(A\mathcal{V}) \subseteq \mathcal{V} \\ &\vdots \\ A^{n-1}\mathcal{B} &\subseteq \mathcal{V} \ . \end{aligned}$$

Therefore, it holds

$$\mathcal{R}(A, B) = \mathrm{Im}(B|AB|A^2B|\ldots|A^{n-1}B) \subseteq \mathcal{V} \ .$$

By controllability of $\mathcal{R}(A, B)$, we obtain

$$n = \dim\mathcal{R}(A, B) \leq \dim\mathcal{V} = k \leq \dim\mathbb{K}^n = n \ .$$

This implies that $k = n$, $\mathcal{V} = \mathbb{K}^n$.

Finally, we need to show that

$$\dim\mathcal{R}(A + BG, x_1) = n \ .$$

Let us define a linear mapping $g \colon \mathcal{V} \to \mathcal{B} \subset \mathcal{V}$ by the equation $g(x_i) = u_i$ for every $i \in \{1, \ldots, n-1\}$, where u_i is such an element that $Ax_i + Bu_i = x_{i+1}$. This definition is correct and unique since the vectors v_i form a basis of $\mathcal{V}$ (see Barto and Tůma, 2019, Tvrzení 6.4). Let G be the matrix of the linear mapping g with respect to the standard basis. Then for every $i \in \{1, \ldots, n-1\}$ we have

$$(A + BG)x_i = Ax_i + BGx_i = Ax_i + Bu_i = x_{i+1} \ .$$

It follows

$$\mathcal{R}(A + BG, x_1) = \mathrm{Im}(x_1, (A + BG)x_1, \ldots, (A + BG)^{n-1}x_1) = \mathrm{Im}(x_1, x_2, \ldots, x_n) \ .$$

Finally, by linear independence of the vectors $x_1, \ldots, x_n$, it holds that $\dim\mathcal{R}(A + BG, x_1) = n$. We have shown that the pair $\mathcal{R}(A + BG, Bv)$ is controllable, and thus the proof is concluded. $\square$

Conclusion

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