Untitled

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The Model

The observation equation:

$$y_t = exp\left(\frac{h_t}{2}\right) + \epsilon_t, \qquad \epsilon_t \sim N(0,1)$$

which can be rewritten to

$$\log(y_t^2) = h_t + log(\epsilon_t^2)$$

if we approximate $log(\epsilon_t^2)$ by a gaussian mixture rv υ with $\upsilon_i|s_i \sim N(d_{s_i}, \sigma_{s_i}^2)$ the model is conditional on the mixture component indicator linear and Gaussian.

System of state equations:

$$\begin{aligned}
h_0 &= \mu + \sqrt{\frac{\sigma^2}{1-\phi^2}}u_0 \\
h_1 &= X_1\beta + (1-\phi)\mu + \phi h_0 + \sigma u_1 \\
&\vdots \\
h_T &= X_T\beta + (1-\phi)\mu + \phi h_{T-1} + \sigma u_T
\end{aligned}$$

where $u_t \sim N(0,1)$.

Rewritten in matrix notation:

$$H_\phi h = X\beta + \gamma + \Sigma^{\frac{1}{2}}u$$

with

$$H_\phi = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ -\phi & 1 & 0 & \dots & 0 \\ 0 & -\phi & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & -\phi & 1 \end{bmatrix}, \quad \gamma = \begin{bmatrix} \mu \\ (1-\phi)\mu \\ \vdots \\ (1-\phi)\mu \end{bmatrix}, \quad X = \begin{bmatrix} 0 & \dots & 0 \\ x_{11} & \dots & x_{1k} \\ \vdots & \vdots & \vdots \\ x_{T1} & \dots & x_{Tk} \end{bmatrix}, \quad \beta = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_k \end{bmatrix}$$

and $\Sigma = \operatorname{diag}\left(\frac{\sigma^2}{1-\phi^2}, \sigma^2, \dots, \sigma^2\right)$.

Conditional on X and the parameters

$$\underline{\hat{h}} = E[h|\phi,\mu,\beta,X] = H_\phi^{-1}(X\beta + \gamma)$$

$$\underline{\Sigma}_h = Var[h|\phi,\sigma^2] = (H_\phi' \Sigma_u^{-1} H_\phi)^{-1} = \begin{bmatrix} \frac{1}{\sigma^2} & \frac{-\phi}{\sigma^2} & 0 & \dots & 0 \\ \frac{-\phi}{\sigma^2} & \frac{1+\phi^2}{\sigma^2} & \frac{-\phi}{\sigma^2} & \ddots & \vdots \\ 0 & \frac{-\phi}{\sigma^2} & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \frac{1+\phi^2}{\sigma^2} & \frac{-\phi}{\sigma^2} \\ 0 & \dots & 0 & \frac{-\phi}{\sigma^2} & \frac{1}{\sigma^2} \end{bmatrix}$$

and hence

$$h|\phi,\mu,\beta,X,\sigma^2 \sim N(\underline{\hat{h}}, \underline{\Sigma}_h)$$

Priors:

$$\frac{(\phi+1)}{2} \sim \mathcal{B}(a_0, b_0)$$

and hence

$$p(\phi) = \frac{1}{2B(a_0,b_0)} \left(\frac{1+\phi}{2}\right)^{a_0-1} \left(\frac{1-\phi}{2}\right)^{b_0-1}$$

$$\sigma^2 \sim B_\sigma \chi_1^2 = \mathcal{G}(\infty/\in, \infty/(\in\mathcal{B}_\sigma)$$

$$\beta,\mu \sim N\left(\begin{pmatrix}\underline{\hat{\beta}} \\ \underline{\hat{\mu}}\end{pmatrix}, \begin{pmatrix}\underline{\sigma_\beta^2} & 0 \\ 0 & \underline{\sigma_\alpha^2}\end{pmatrix}\right)$$

Gibbs Sampler

Sample h (including h_0) from

$$h|y,s,\phi,\mu,\sigma^2,\beta \sim N(\overline{\hat{h}}, \overline{\Sigma}_h)$$

where the precision matrix

$$\overline{\Sigma}_h^{-1} = (\Sigma_y^{-1} + \underline{\Sigma}_h^{-1}) = \begin{bmatrix} \frac{1}{\sigma^2} & \frac{-\phi}{\sigma^2} & 0 & \dots & 0 \\ \frac{-\phi}{\sigma^2} & \frac{1+\phi^2}{\sigma^2} + \frac{1}{\sigma_{s_1}^2} & \frac{-\phi}{\sigma^2} & \ddots & \vdots \\ 0 & \frac{-\phi}{\sigma^2} & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \frac{1+\phi^2}{\sigma^2} + \frac{1}{\sigma_{s_{T-1}}^2} & \frac{-\phi}{\sigma^2} \\ 0 & \dots & 0 & \frac{-\phi}{\sigma^2} & \frac{1}{\sigma^2} + \frac{1}{\sigma_{s_T}^2} \end{bmatrix}$$

.

$$\overline{\hat{h}} = \overline{\Sigma}_h(\Sigma_y^{-1}(y-d) + \underline{\Sigma}_h^{-1}\underline{\hat{h}})$$

and the mean vector

$$\begin{aligned} \overline{\hat{h}} &= \overline{\Sigma}_h(\Sigma_y^{-1}(y-d) + \underline{\Sigma}_h^{-1}\underline{\hat{h}}) \\ &= \overline{\Sigma}_h(\Sigma_y^{-1}(y-d) + H_\phi' \Sigma_u^{-1}(X\beta+\gamma)) \end{aligned}$$

To efficiently sample from this distribuition we exploit the special structure of the precision matrix.

$$H_\phi' \Sigma_u^{-1} = \begin{bmatrix} \frac{1-\phi^2}{\sigma^2} & \frac{-\phi}{\sigma^2} & 0 & \cdots & 0 \\ 0 & \frac{1}{\sigma^2} & \frac{-\phi}{\sigma^2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \frac{1}{\sigma^2} & \frac{-\phi}{\sigma^2} \\ 0 & \cdots & \cdots & 0 & \frac{1}{\sigma^2} \end{bmatrix}$$