COMP 250 Assignment #2 – Winter 2017
Alex Hale
ID: 260672475

1)

a) An array sorted in descending order will be the worst case because the swap code (i.e. inside the if condition) must run every loop.

b) Line 1: T_{assign} + T_{comp} (repeated n times) + (T_{arith} +T_{assign}) (repeated n-1 times)

⇨ $1 + 1 * n + 2 * (n - 1) = 3n - 1$

Line 2: T_{assign} + T_{comp} (repeated n-i+1 times) + (T_{arith} + T_{assign}) (repeated n-i times)

⇨ $1 + 1 * (n - 1 + 1) + 2 * (n - i) = 2 + 3n - i$

⇨ $\sum_{i=1}^{n-1} 2 + 3n - i = 2 + 3n - \sum_{i=1}^{n-1} i = 2 + \frac{7}{2}n - \frac{1}{2}n^2$

Line 3: T_{index} + T_{comp} + T_{arith} + T_{index} + T_{cond} = 5
Line 4: T_{index} + T_{assign} = 2
Line 5: T_{arith} + T_{index} + T_{assign} + T_{index} = 4
Line 6: T_{assign} + T_{arith} + T_{index} = 3

Line 3 to 6: 14 (repeated n-i times)

⇨ $14(n - i) = 14n - 14\sum_{i=1}^{n-1} i = 14n - 7(n^2 - n) = 21n - 7n^2$

$$T(n) = 3n - 1 + 2 + \frac{7}{2}n - \frac{1}{2}n^2 + 21n - 7n^2 = \boldsymbol{-\frac{15}{2}n^2 + \frac{55}{2}n + 1}$$

c) $T(n) \le -\frac{15}{2}n^2 + \frac{55}{2}n^2 + n^2 = 21n^2 = cn^2$

⇨ $\boldsymbol{T(n)}$ **is** $\boldsymbol{O(n^2)}$

2) a) given

b) $T(n)$ is $O(n)$

c) $T(n)$ is $O(n^2)$

d) $T(n)$ is $O(\log_2 n)$

e) $T(n)$ is $O(1)$

260672475
Alex Hal

Feb. 11/17

(cont.) 3) Prove $4^n < n!$ for $n \geq 9$

base case: $n = 9$ $\quad 4^n = 4^9 = 262\,144 < n! = 9! = 362\,880$

induction step: assume $4^k < k!$ is true, prove that $4^{k+1} < (k+1)!$ is tru

$$(k+1)! = \underbrace{(k+1)k! > (k+1)4^k}_{\text{since } k! > 4^k} \Rightarrow \cancel{(k+1)}k! > \cancel{(k+1)}4^k$$

↳ $k! > 4^k$

↳ we've proven our goal

$\therefore \boxed{4^n < n! \text{ for all } n \geq 9}$

4) $T(n) = \begin{cases} 1, & n = 1 \\ 3T(n-1) + 2, & n > 1 \end{cases}$ → use substitution approach

① $T(n) = 3T(n-1) + 2$
$T(n-1) = 3T(n-2) + 2$

② $T(n) = 3(3T(n-2) + 2) + 2 = 3^2 T(n-2) + 8$
$T(n-2) = 3T(n-3) + 2$

③ $T(n) = 3^2(3T(n-3) + 2) + 8 = 3^3 T(n-3) + 26$ → $2 + 3(2) + 3^2(2) = 2(3^0 + 3^1 + 3^2)$
$T(n-3) = 3T(n-4) + 2$

$$= 2\sum_{i=1}^{k} 3^{i-1}$$
$$= 3^k - 1$$

Ⓚ $T(n) = 3^k T(n-k) + 3^k - 1$ ← guess at expression

↳ base case: $n - k = 1 \Rightarrow k = n-1 \Rightarrow T(n-k) = T(1)$

↳ $T(n) = 3^{n-1} T(1) + 3^{n-1} - 1 = 3^{n-1} + 3^{n-1} - 1 = \boxed{2 \cdot 3^{n-1} - 1 = T(n)}$

↳ check: $T(1) = 1$, $T(2) = 5$, $T(3) = 17$ ✓

5) $25n + 5$ is $O(n)$ iff there exist an integer n_0 and a real number c such that for all $n \geq n_0$, $25n + 5 \leq c \cdot n$

↳ i.e. $\exists n_0 \in \mathbb{N}, \exists c \in \mathbb{R}: \forall n \geq n_0, 25n + 5 \leq c \cdot n$

$$25n + \underbrace{5 \leq 25n + 5n}_{\times n} = 30n = cn \Rightarrow c = 30$$

↳ let $n_0 = 1 \Rightarrow 25(1) + 5 = 30 = \overset{c}{30}(1)$

$\therefore 25n + 5 \leq cn$ for all $n \geq 1$ where $c = 30$

$$\boxed{\therefore 25n + 5 \text{ is } O(n)}$$

6) $f(n) = (n+10)^{2.5} + n^2 + 1$ is $O(n^{2.5})$ iff there exist an integer n_0 and a real number c such that for all $n \geq n_0$, $f(n) \leq c \cdot n^{2.5}$

$$(n+10)^{2.5} + n^2 + 1 \leq n^{2.5} + \underset{\times n^{0.5}}{n^{2.5}} + \underset{\times 2.5}{n^{2.5}} = 3n^{2.5} = cn^{2.5} \Rightarrow c = 3$$

additional $n^{2.5}$ overpowers $(n+10)$ for large values of n

↳ let $n_0 = 20 \Rightarrow (20+10)^{2.5} + (20)^2 + 1 \doteq 5330 < 5367 \doteq 3(20)^{2.5}$

$\therefore (n+10)^{2.5} + n^2 + 1 \leq cn^{2.5}$ for all $n \geq 20$ where $c = 3$

$$\therefore \boxed{(n+10)^{2.5} + n^2 + 1 \text{ is } O(n^{2.5})}$$

↳Assignment # 2

260672475

Feb. 12/17 Alex Ha

7) $(n+1)^2$ is not $O(n)$ iff for any n_0 and c, there exists an $n \geq n_0$ such that $(n+1)^2 > c \cdot n$

$$(n+1)^2 = n^2 + 2n + 1 > c \cdot n \iff \frac{n^2+2n+1}{n} > c$$

$$\frac{n^2+2n+1}{n} > \frac{n^2+2n}{n} = n+2 > c$$

$\therefore$ if we choose $n = c+2 \implies (n+1)^2 = c^2 + 4c + 4 > c^2 + 2c = c \cdot n$

$\boxed{\therefore (n+1)^2 \text{ is not } O(n)}$

8) Announcement: "Some people have a blue face" $\iff$ at least one person has a blue fa

↳if there were only two people w/ blue faces: Person A sees that Person B has a blue face. For all A knows, B is the only one with a blue face. A also knows that, since they didn't see anybody else with a blue face, there are two possible cases:

1. There is one person w/ a blue face: person B
2. There are two people w/ blue faces: Person A and Person B

↳B has the same thought process, with reversed roles. Therefore, nobody dies on the first night. The second day A sees that B is alive

↳see typed response on next page

Hilroy

8) Announcement = "Some people have a blue face" ⇔ *At least* one person has a blue face

- if there had only been one person on the island with a blue face:
  - day 1: Person A doesn't see anybody on the island with a blue face. There must be one person with a blue face, so they must be the one and only person with a blue face.
  - night 1: person A dies
- if there had been two people on the island with a blue face:
  - Day 1: Person A sees that Person B has a blue face, and nobody else. Since there is *at least* one person with a blue face, Person A knows that there are two cases: a) B is the one person with a blue face, or b) both A and B have blue faces. B undergoes the same logical process in reverse
  - Night 1: A and B do not know if they have a blue face. Nobody dies
  - Day 2: A sees that B is still alive. If B had not seen anybody else with a blue face on Day 1, then they would have died on Night 1. Therefore, A knows that case b) is true, and they have a blue face. B undergoes the same logical process
  - Night 2: Person A and Person B die
- if there had been k people on the island with a blue face:
  - by inductive reasoning:
    - one person with blue face => they would have died first night
    - two people with blue faces => they both would have died 2^{nd} night
    - …
    - k people with blue faces => they will all die the k^{th} night

- There are 10 people on the island with a blue face. They will each become certain of whether they have a blue face on the 10^{th} day, because the 9 people they can see with blue faces will all be alive or all be dead. They all realize that they have blue faces on the 10^{th} day, so they all die on the 10^{th} night.