COMP 250 - Homework #3 Mathieu Blanchette

Question 3. (15 points)

Consider the following recurrence:

$$T(n) = \begin{cases} 1 & \text{if } n = 1 \\ 2 * T(n-1) + n + 1 & \text{if } n > 1 \end{cases}$$

a) (10 points) Obtain an explicit formula for the following recurrence using one of the techniques seen in class. In the process of doing so, you will possibly come across the summation $\Sigma_{i=1\ldots n}$ ($i * 2^i$), which can be simplified as $2 * (1 + 2^n * (n-1))$.

Let's start by computing the first few values of T(n)

$T(1) = 1$
$T(2) = 2 * 1 + 2 + 1 = 5$
$T(3) = 2*5 + 3 + 1 = 14$
$T(4) = 2*14 + 4 + 1 = 33$
$T(5) = 2*33 + 5 + 1 = 72$

$T(n) = 2 * T(n-1) + n + 1$
$= 2 * (2 * T(n-2) + (n-1) + 1) + n + 1= 4\, T(n-2) + 2\,(n-1) + n + 3$
$= 4\,(2*T(n-3) + (n-2) + 1) + 2\,(n-1) + 2 + n + 1= 8\, T(n-3) + 4\,(n-2) + 2\,(n-1) + n + 7$
$= 8\,(2\,T(n-4) + (n-3) +1) + 4\,(n-2) + 2\,(n-1) + n + 7=16\, T(n-4) + 8\,(n-3) + 4\,(n-2) + 2\,(n-1) + n + 15$
…
(after k substitutions)
$= 2^k\, T(n-k) + (\Sigma_{i=o\ldots k-1} (n-i) * 2^i) + (2^k – 1)$
$= 2^k\, T(n-k) + n\,(\Sigma_{i=o\ldots k-1} 2^i) - (\Sigma_{i=o\ldots k-1} i * 2^i) + (2^k – 1)$
$= 2^k\, T(n-k) + n\,(2^k –1) – 2*(1+2^{k-1} (n-2)) + (2^k – 1)$

Recursion stops when n-k=1, i.e. k=n-1. We then get

$T(n) = 2^{n-1}\, T(1) + n\,(2^{n-1} –1) – 2*(1+2^{n-2} (n-3)) + (2^{n-1} – 1)$
$= 2^{n-1} + n\, 2^{n-1} –n – 2 - 2^{n-1} (n-3) + 2^{n-1} – 1$
$= 2^n + 3\, 2^{n-1} – n – 3$

Conclusion: $T_{explicit}(n) = 2^n + 3 * 2^{n-1} – n – 3$

Checking the explicit formula:
$T(1) = 2^1 + 3 *1 – 1 – 3 = 1$
$T(2) = 2^2 + 3 *2 – 2 - 3 = 5$
$T(3) = 2^3 + 3 * 4 – 3 - 3 = 14$
$T(4) = 2^4 + 3 * 8 – 4 - 3 = 33$
$T(5) = 2^5 + 3 * 16 – 5 - 3 = 72$

b) (5 points) Using induction, prove that your explicit formula always takes the same values as the recurrence, for all values of n >=1.

We need to prove that $T(n) = 2^n + 3 * 2^{n-1} – n – 3$ for all n >=1. We prove this by induction on n.

Base case: For n = 1, the recurrence defines T(1) = 1, which is equal to $2^1 + 3*2^0 – 1 – 3 = 1$

Induction step: Assume $T(k) = 2^k + 3 * 2^{k-1} – k – 3$ for some $k >=1$
We want to prove that this implies that $T(k+1) = 2^{(k+1)} + 3 * 2^{(k+1)-1} – (k+1) – 3$

$$
\begin{aligned}
T(k+1) &= 2 * T(k) + (k+1) + 1 \\
&= 2 * (2^k + 3 * 2^{k-1} – k – 3) + (k+1) +1 \quad \text{(because of IH)} \\
&= 2^{k+1} + 3 * 2^k - k – 4 \\
&= 2^{k+1} + 3 * 2^{(k+1)-1} - (k+1) – 3
\end{aligned}
$$

Conclusion: The explicit formula holds for all values of $n>=1$.

Question 4. (10 points)

Prove that $\log(n!) \in \Theta(n \log(n))$ (that big Theta, not big O)

First, we show that $\log(n!) \in O(n \log n)$. For this, we need to find c and N such that for all $n \geq n_0$, $\log(n!) \leq c\, n \log(n)$. This is the easy part, as c=1 and $n_o = 1$ will work:

$$
\begin{aligned}
\log(n!) &= \log (n * (n-1) * (n-2) ...* 2 * 1) \\
&= \log(n) + \log (n-1) + \log(n-2) + ... + \log(2) + \log(1) \\
&\leq \log(n) + \log(n) + \log(n) + ... + \log(n) + \log(n) \\
&= n \log(n) \\
&= c\, n \log(n)
\end{aligned}
$$

Thus, $\log(n!) \in O(n \log n)$.

Now, to prove that $n \log n \in O(\log(n!))$, we need to find c and n_o such that for all $n \geq n_o$, $n \log n \leq c \log(n!)$. This time, it will be easier to start with the right side of the inequality.

$$
\begin{aligned}
\log(n!) &= \underbrace{\log(n) + \log (n-1) + \log(n-2) + ... + \log(n/2 +1)} + \underbrace{\log(n/2) + ...+ \log(2) +\log(1)} \\
&\geq \qquad\qquad n/2 \log(n/2) \qquad\qquad + \qquad\qquad n/2 \log(1) \\
&= n/2 \log(n/2) \\
&= n/2 (\log(n) – 1) \\
&\geq n/2 (\log(n) – ½ \log(n)) \qquad \text{if } n \geq 4 \\
&= 1/4\, n \log(n)
\end{aligned}
$$

Thus, for $n_0=4$, c = 4, we get that $n \log(n) \leq 4 n \log(n)$ for all $n \geq n_0$