2. (a) Let

$$T(n) = 25 \cdot T\left(\frac{n}{5}\right) + n$$

Then $b = 5$, $a = 25$, and $f(n) = n$. So $\log_b a = \log_5 25 = 2$. By letting $\epsilon = 1 > 0$, we have that

$$f(n) = n = n^{2-1} = n^{\log_5 25 - 1} = n^{\log_b a - \epsilon}$$

holds for $n \geq 1$ and so $f(n) = O(n^{\log_b a - \epsilon})$. By case 1 of the Master Theorem, we have that

$$T = \Theta(n^{\log_b a}) = \Theta(n^2)$$

(b) Let

$$T(n) = 2 \cdot T\left(\frac{n}{3}\right) + n \cdot \log n$$

Then $b = 3$, $a = 2$, and $f(n) = n \log n$. So $\log_b a = \log_3 2 \approx 0.631$. Let $\epsilon = 1 - \log_3 2$. Note that $\epsilon = 1 - \log_3 2 > 1 - \log_3 3 = 1 - 1 = 0$. We have that

$$n^{\log_b a + \epsilon} = n^{\log_3 2 + (1 - \log_3 2)} = n$$

and since $\log n \geq 1$ for n sufficiently large, it follows that

$$f(n) = n \log n \geq n = n^{\log_b a + \epsilon}$$

holds for n sufficiently large. Therefore, $f(n) \in \Omega(n^{\log_b a + \epsilon})$. Also, by letting $c = \frac{2}{3} < 1$ and $n_0 = 1$, we have that for all $n > n_0$

$$af\left(\frac{n}{b}\right) = 2f\left(\frac{n}{3}\right) = \frac{2n}{3} \log \frac{n}{3} \leq \frac{2n}{3} \log n = cn \log n = cf(n)$$

[Note: the inequality above uses the fact that $\log x$ is monotone increasing on $(0, \infty)$]. Therefore, by case 3 of the Master Theorem, we have that

$$T = \Theta(f(n)) = \Theta(n \log n)$$

(c) Let

$$T(n) = T\left(\frac{3n}{4}\right) + 1$$

Then $b = \frac{4}{3}$, $a = 1$, and $f(n) = 1$. So $\log_b a = \log_{\frac{4}{3}} 1 = 0$. Letting $p = 0$, it follows that for $n \geq 1$

$$f(n) = 1 = n^0 \log^0 n = n^{\log_b a} \log^p n$$

and so $f(n) = \Theta(n^{\log_b a} \log^p n)$. By case 2 of the Master Theorem, we have that

$$T(n) = \Theta(n^{\log_b a} \log^{p+1} n) = \Theta(\log n)$$

(d) Let

$$T(n) = 7 \cdot T\left(\frac{n}{3}\right) + n^3$$

Then $b = 3$, $a = 7$, and $f(n) = n^3$. So $\log_b a = \log_3 7 \approx 1.77$. Let $\epsilon = 3 - \log_3 7$. Note that $\epsilon = 3 - \log_3 7 > 3 - \log_3 27 = 3 - 3 = 0$. We have that for $n \geq 1$

$$f(n) = n^3 = n^{\log_3 7 + (3 - \log_3 7)} = n^{\log_b a + \epsilon}$$

and so $f(n) \in \Omega(n^{\log_b a+\epsilon})$. Also, by letting $c = \frac{7}{27} < 1$ and $n_0 = 1$, it follows that for all $n > n_0$

$$af\left(\frac{n}{b}\right) = 7f\left(\frac{n}{3}\right) = 7\left(\frac{n}{3}\right)^3 = \frac{7}{27}n^3 = cn^3 = cf(n)$$

Therefore, $af\left(\frac{n}{b}\right) \leq cf(n)$ for all $n > n_0$. By case 3 of the Master Theorem, we have that

$$T = \Theta(f(n)) = \Theta(n^3)$$

(e) Let

$$T(n) = T\left(\frac{n}{2}\right) + n\,(2 - \cos n)$$

Then $b = 2$, $a = 1$, and $f(n) = n\,(2 - \cos n)$. So $\log_b a = \log_2 1 = 0$. We show that case 1, 2, and 3 of the Master Theorem are not applicable for $T(n)$.

For case 1, we begin by noting that for any $\epsilon > 0$

$$n^{\log_b a - \epsilon} = n^{-\epsilon} = \frac{1}{n^\epsilon}$$

We show that $f(n) \neq O(\underbrace{n^{\log_b a-\epsilon}}_{\frac{1}{n^\epsilon}})$ by contradiction. Suppose that $f(n) = n\,(2 - \cos n) = O(\frac{1}{n^\epsilon})$. Then there exists positive numbers c and n_0 such that, for all $n \geq n_0$, $f(n) = n\,(2 - \cos n) \leq \frac{c}{n^\epsilon}$. By dividing both sides by n gives an equivalent inequality

$$2 - \cos n \leq \frac{c}{n^{1+\epsilon}} \tag{1}$$

Note that $2 - \cos n \geq 2 - 1 = 1$ and so $2 - \cos n > 0$. Also, $n^{1+\epsilon} > 0$. So inequality (1) implies

$$n^{1+\epsilon} \leq \frac{c}{2 - \cos n} \leq \frac{c}{2-1} = c$$

holds for all $n \geq n_0$. However, $n^{1+\epsilon} \leq c$ does not hold for any $n > c$, contradicting the assumption that it holds for all $n \geq n_0$. Therefore, $f(n) \neq O(\underbrace{n^{\log_b a-\epsilon}}_{\frac{1}{n^\epsilon}})$

For case 2, by using the bound $-1 \leq \cos n \leq 1$ for all $n \geq 1$, we have that

$$f(n) = 2n - n\cos n \leq 2n + n = 3n$$
$$f(n) = 2n - n\cos n \geq 2n - n = n$$

hold for all $n \geq 1$. It then follows that $f(n) = \Theta(n)$, and it was discussed during office hours with Professor Waldispuhl that we can immediately conclude from this that $f(n) \neq \Theta(\log^p n)$ for any $p \geq 0$. That is, $f(n) \neq \Theta(\underbrace{n^{\log_b a}}_{=1} \log^p n)$ for any $p \geq 0$.

Lastly, we show that the conditions for case 3 of the Master Theorem are not satisfied. Let $k = 2k\pi$ where k is an odd integer. Then

$$f(n) = f(2k\pi) = 2k\pi\,(2 - \cos(2k\pi)) = 2k\pi\,(2-1) = 2k\pi$$

and

$$af\left(\frac{n}{b}\right) = f\left(\frac{2k\pi}{2}\right) = f(k\pi) = k\pi\,(2 - \cos(k\pi)) = k\pi\,(2-(-1)) = 3k\pi$$

Suppose that there exists a $c < 1$ such that $af\left(\frac{n}{b}\right) \leq cf(n)$ for all $n > n_0$ for some $n_0 > 0$. For any $n_0 > 0$, let k be an odd integer larger than n_0, and so

$$3k\pi = af\left(\frac{n}{b}\right) \leq cf(n) = 2k\pi \implies c \geq \frac{3}{2}$$

which is a contradiction to $c < 1$. Therefore, there does not exist a $c < 1$ such that $f\left(\frac{n}{b}\right) \leq cf(n)\ \forall n > n_0$ for some $n_0 > 0$. So case 3 of the Master Theorem is not applicable.

In summary, cases 1, 2, are 3 of the Master Theorem are not applicable for $T(n)$.

3. We first find the asymptotic running time of algorithm A. We are given that

$$T_A = 7 \cdot T_A\left(\frac{n}{2}\right) + n^2$$

Then $\log_b a = \log_2 7 \approx 2.807$. Let $\epsilon = \log_2 7 - 2$ and note that $\epsilon = \log_2 7 - 2 > \log_2 4 - 2 = 2 - 2 = 0$. Then

$$f(n) = n^2 = n^{\log_2 7 - (\log_2 7 - 2)} = n^{\log_a b - \epsilon}$$

and so $f(n) = O(n^{\log_a b - \epsilon})$. By case 1 of the Master Theorem, we have that

$$T_A = \Theta(n^{\log_b a}) = \Theta(n^{\log_2 7})$$

Next, we find the asymptotic running time of algorithm B. Let $\alpha > 16$. It then follows that $\log_4 \alpha > \log_4 16 = 2$. Let $\epsilon = \log_4 \alpha - 2$ and note that $\epsilon = \log_4 \alpha - 2 > 2 - 2 = 0$. We then have that

$$f(n) = n^2 = n^{\log_4 \alpha - (\log_4 \alpha - 2)} = n^{\log_4 \alpha - \epsilon}$$

and so $f(n) = O(n^{\log_4 \alpha - \epsilon})$. By case 1 of the Master Theorem, we have for $\alpha > 16$

$$T_B = \Theta(n^{\log_4 \alpha})$$

By using the property that $x^{\log_b y} = y^{\log_b x}$, it follows that

$$4^{\log_2 7} = 7^{\log_2 4} = 7^2 = 49$$

We then have that for $\alpha > 16$

$$\begin{aligned}
\log_4 \alpha < \log_2 7 &\iff \alpha = 4^{\log_4 \alpha} < 4^{\log_2 7} = 49 \\
\log_4 \alpha = \log_2 7 &\iff \alpha = 4^{\log_4 \alpha} = 4^{\log_2 7} = 49 \\
\log_4 \alpha > \log_2 7 &\iff \alpha = 4^{\log_4 \alpha} > 4^{\log_2 7} = 49
\end{aligned}$$

It then follows that for $n \geq 1$

$$\begin{aligned}
n^{\log_4 \alpha} &< n^{\log_2 7} \quad \text{for } 16 < \alpha < 49 \\
n^{\log_4 \alpha} &= n^{\log_2 7} \quad \text{for } \alpha = 49 \\
n^{\log_4 \alpha} &> n^{\log_2 7} \quad \text{for } \alpha > 49
\end{aligned}$$

Recall that $T_A = \Theta(n^{\log_2 7})$ and, for $\alpha > 16$, $T_B = \Theta(n^{\log_4 \alpha})$. Therefore, algorithm B is asymptotically faster than algorithm A when $16 < \alpha < 49$ and is asymptotically slower than algorithm A when $\alpha > 49$, and both algorithms have the same asymptotic running time when $\alpha = 49$. Therefore, the largest value of α such that $T_B(n)$ is asymptotically faster than $T_A(n)$ is 48. [Note that algorithms A and B have the same asymptotic running time when $\alpha = 49$].

As a side note, we did not need to examine the asymptotic running time of algorithm B for $a \leq 16$. This was because we are only looking for the largest α such that algorithm B is asymptotically faster than algorithm A, and we found that for α such that $16 < \alpha < 49$ algorithm B is asymptotically faster than algorithm A. Therefore, values of α less than or equal to 16 are not possible solutions.