COMP 251 Assignment 4
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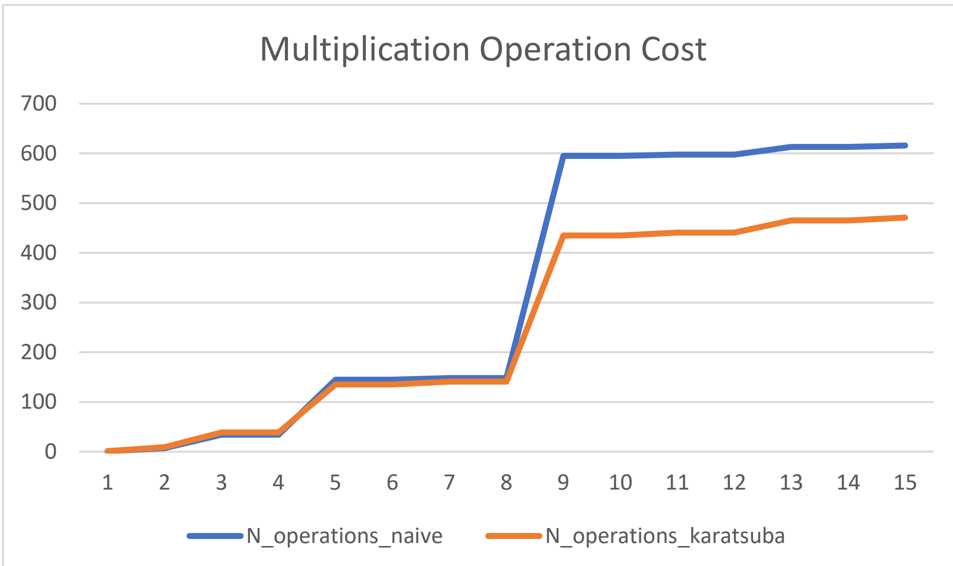
1) The Java implementation of the naïve and Karatsuba divide-and-conquer multiplication algorithms can be found in the attached *Multiply.java* file. The raw data of the number of operations required for each algorithm can be found in the attached *karatsuba.csv* file. The plot of the data in *karatsuba.csv* can be found in the table below:



Note that the Karatsuba algorithm is increasingly faster than the naïve algorithm for increasingly large integer bit lengths. Notice also that the algorithms experience "plateaus", where the running time does not increase when an extra bit is added. This occurs because the number of recursive calls required does not increase when an extra bit is added.

2)

a. $k = \log_5(25) = 2$
Case 1: $f(n) = n = O\left(n^{k-\varepsilon}\right) = O(n^{2-\varepsilon})$ is true for $0 < \varepsilon$ (e.g. $\varepsilon = 1$).
Therefore, $\boldsymbol{T(n) = \Theta(n^2)}$

b. $k = \log_3(2)$
Case 3, part 1: $f(n) = n\log(n) = \Omega\left(n^{\log_3(2)+\varepsilon}\right)$ is true for $0 < \varepsilon$ (e.g. $\varepsilon = 0.1$).
Case 3, part 2: $2f\left(\frac{n}{3}\right) \le cf(n) \to 2\left(\frac{n}{3}\right)\log\left(\frac{n}{3}\right) \le cn\log(n) \to \frac{2}{3}\log(n) - \frac{2}{3}\log(3) \le \log(n)$ is true for $c > \frac{2}{3}$ (for large n)
Therefore, $\boldsymbol{T(n) = \Theta(n\log(n))}$

c. $k = \log_{\frac{3}{4}}(1) = 0$
Case 2: $f(n) = 1 = \Theta(n^0 \log^0 n) = \Theta(1)$ is true.

Therefore, $\boldsymbol{T(n) = \Theta\left(n^0\ log^1(n)\right) = \Theta(log(n))}$

d. $k = \log_3(7)$
Case 3, part 1: $f(n) = n^3 = \Omega\left(n^{\log_3(7)+\varepsilon}\right)$ is true for $0 < \varepsilon$ (e.g. $\varepsilon > 3 - \log_3(7)$).
Case 3, part 2: $7f\left(\frac{n}{3}\right) \leq cf(n) \rightarrow 7\left(\frac{n}{3}\right)^3 \leq cn^3 \rightarrow \frac{7}{27}n^3 \leq \mathrm{c}n^3$ is true for $c < 1$
Therefore, $\boldsymbol{T(n) = \Theta(n^3)}$

e. $k = \log_2(1) = 0$
Case 3, part 1: $f(n) = n(2 - \cos(n)) = \Omega(n^{0+\varepsilon})$ is true for $0 < \varepsilon$.
Case 3, part 2: Let $n = 2\pi m$, m odd. $1f\left(\frac{n}{2}\right) \leq cf(n) \rightarrow \frac{n}{2}\left(2 - \cos\left(\frac{n}{2}\right)\right) \leq$
$c(n(2 - \cos(n)) \rightarrow \frac{2\pi m}{2}\left(2 - \cos\left(\frac{2\pi m}{2}\right)\right) \leq c(2\pi m(2 - \cos(2\pi m)) \rightarrow 3\pi m \leq c(2\pi m)$
is only true for $c > \frac{3}{2} > 1$
Therefore, the Master Theorem doesn't apply.

3) Apply the Master theorem to T_A:

$k = \log_2(7)$
Case 1: $f(n) = n^2 = O\left(n^{\log_2(7)-\varepsilon}\right)$ is true for $0 < \varepsilon < (2 - \log_2(7)$.
Therefore, $\boldsymbol{T_A(n) = \Theta(n^{log_2(7)})}$

Apply the start of the Master theorem to T_B. Case 1 also applies.

$k = \log_4(\alpha)$
Case 1: $f(n) = n^2 = O\left(n^{\log_4(\alpha)-\varepsilon}\right)$

Choose α to meet the following:
A: $\log_4(\alpha) > 2$ (to satisfy the Master theorem)
⇨ $\alpha > 4^2 = 16$

B: $\log_4(\alpha) < \log_2(7)$ (to be faster than T_A)
⇨ $\alpha < 4^{\log_2(7)} = 49$

Therefore, $16 < \alpha < 49$.

Choose the highest possible integer value of α, **which is 48**.

Therefore, the largest integer value of α for which T_B is asymptotically faster than T_A is **48**.

Please see the following pages of this PDF for rough work.

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$O(x)$ = upper bound
$\Theta(x)$ = upper & lower bound (exact)
$\Omega(x)$ = lower bound

2) a) $k = \log_5(25) = 2$

$f(n) = n = O(n^{k-2}) = O(n) \Rightarrow \boxed{T(n) = \Theta(n^2) \quad (\text{case } 1)}$

b) $k = \log_3(2) = \log 2 / \log 3$

~~② $f(n) = n\log(n) = \Theta(n^{\log_3(2)} \log^2(n))$ nope X~~

~~② $f(n) = n\log(n) = \Theta(n^{\log_2 3} \log(n)) \Rightarrow \frac{\log(n)}{(\log(n))^p} = \frac{n^{\log_2 3}}{n}$ nope X~~

~~③ $f(n) = n\log(n) = \Omega(n^{\log_2(3)+\varepsilon})$ X~~

~~$(\log(n))^{1-p} = n^{\log_2(3)-1}$~~

~~① $f(n) = n\log(n) = O(n^{\log_2(3)-\varepsilon})$ ✓~~

~~$\rightarrow T(n) = \Theta(n^{\log}$~~

~~can't be applied~~

c) $k = \log_{3/4}(1) = \log(1)/\log(3/4) = 0$

③ $f(n) = 1 = \Omega(n^{0+1}) = \Omega(n^1)$ X
& $(1) f(n/b) \le c f(n) = 1 \le c(1)$ X not satisfied

~~can't be applied~~

② ~~$f(n) = 1 = \Theta(n^0 \log(n))$~~ X $f(n) = 1 = \Theta(n^0 \log^0(n))$ ✓ $\Rightarrow T(n) = \Theta(n^0 \log^1(n)) = \Theta(\log(n))$

① $f(n) = 1 = O(n^{0-\varepsilon}) = O(n^{-1}) \rightarrow$ false for any $n > 1$

d) $k = \log_3(7)$

① $f(n) = n^3 = O(n^{\log_3(7)-\varepsilon})$ X nope

③ $f(n) = n^3 = \Omega(n^{\log_3(7)+\varepsilon})$ ✓ & $7f(\frac{n}{3}) \le c f(n)$

$7(\frac{n}{3})^3 \le cn^3$

$\frac{7}{27} n^3 \le cn^3$ ✓

$\Rightarrow \boxed{T(n) = \Theta(n^3) \quad (\text{case } 3)}$

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e) $k = \log_2(1) = 0$

$f(n) = n(2-\cos(n))$

Use case ③, let $n = 2\pi m$, m odd.

$2n - n\cos(n) = \Omega(n^{0+\varepsilon})$ ✓

(1) $f(\frac{n}{2}) \le c f(n)$

$\frac{n}{2}(2-\cos(\frac{n}{2})) \le c(n(2-\cos(n)))$

$\frac{2\pi m}{2}(2-\cos(\frac{2\pi m}{2})) \le c(2\pi m(2-\cos(2\pi m)))$

↳ m odd

↳ $\pi m(2+1) \le c(2\pi m(2-1))$

$3\pi m \le c(2\pi m)$

↳ for this to be true, $c \ge \frac{3}{2} > 1$ ✗

Master theorem doesn't apply

(1) $f(\frac{n}{2}) \le c f(n)$

$\frac{n}{2}(2-\cos(\frac{n}{2})) \le c(n(2-\cos(n)))$

$\frac{2\pi m}{2}(2-\cos(\frac{2\pi m}{2})) \le c(n(2-\cos(n)))$

$2\pi m - \pi m \underbrace{\cos(\pi m)}_{m \text{ odd}} \le 2cn - cn\cos(n)$

$2\pi m + \pi m \le 2c(2\pi m) - cn\underbrace{\cos(2\pi m)}_{m \text{ odd}}$

$3\pi m \le 4c\pi m + c(2\pi m)$

$3\pi m \le c(6\pi m)$

b) $k = \log_3(2)$ ① $f(n) = \log(n)n = O(n^{\log_3(2)-\varepsilon})$ ✗ nope

② $f(n) = n\log(n) = \Theta(n^{\log_3(2)}\log^p(n))$ ✗ nope, $n^{\log_3(2)}$ throws it off

③ $f(n) = n\log(n) = \Omega(n^{\log_3(2)+\varepsilon})$ ✓

$2f(\frac{n}{3}) \le cf(n) \to 2(\frac{n}{3})\log(\frac{n}{3}) \le cn(\log(n)) \to \frac{2}{3}(\log(n) - \log(3)) \le \log(n)$

$\frac{2}{3}\log(n) \le c\log(n) + \frac{2}{3}\log(3)$

↳ for large n, true for $c > \frac{2}{3}$ ✓

↳ $\boxed{T(n) = \Theta(n\log(n))}$

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3) (A) $T_A(n) = 7T_A\left(\frac{n}{2}\right) + n^2$ $\qquad$ $T_B(n) = \alpha T_B\left(\frac{n}{4}\right) + n^2$

→ apply Master theorem to both cases

$k = \log_2(7)$ $\quad f(n) = n^2$

① $n^2 = O(n^{\log_2(7)-\varepsilon})$ ✓

② $n^2 = \Theta(n^{\log_2(7)} \log^p(n))$ X

③ $n^2 = \Omega(n^{\log_2(7)+\varepsilon})$ ✓

$7f\left(\frac{n}{2}\right) \leq cf(n)$

$7\left(\frac{n}{2}\right)^2 \leq cn^2$

$\frac{7}{4}n^2 \leq cn^2$

↳ $c \geq \frac{7}{4} > 1$ X

↳ doesn't apply

$T_A(n) = \Theta(n^{\log_2(7)})$

$k = \log_4(\alpha)$ $\quad f(n) = n^2$

① $O(n^{\log_4(\alpha)-\varepsilon})$ → choose ~~the~~ α such that the following are satisfied:

Ⓐ $\log_4(\alpha) > 2$

Ⓑ $\log_4(\alpha) < \log_2(7)$

Ⓐ $\log_4(\alpha) > 2 \Rightarrow \alpha > 4^2 = 16$

~~$\Rightarrow \alpha > 2^4 \Rightarrow \alpha > 16$~~

Ⓑ $\log_4(\alpha) < \log_2(7)$

~~$\Rightarrow \alpha < (\log_2(7))$~~ $\Rightarrow \alpha < 4^{\log_2(7)}$

~~$\Rightarrow \alpha < 49$~~ $\quad \Rightarrow \alpha < 49$

$\therefore 16 < \alpha < 49$

choose largest available $\Rightarrow \boxed{\alpha = 48}$

⇓

$T_B(n) = \Theta(n^{\log_4(48)}$

$a = b^c$

$\log_b(a) = c$

2.807

2.807354922