a)

b)

c)

Unendlich tiefer Potentialtopf

$$V(x) = \begin{cases} \infty & \text{für } x \le 0 \\ 0 & \text{für } 0 \le x \le a \\ \infty & \text{für } a \le x \end{cases}$$

a) For $V(x) = \infty$: $\psi_{\mathrm{I}}(x) = 0$

For $V(x) = 0$: $\psi_{\mathrm{II}}(x) = A\,\mathrm{e}^{-ikx} + B\,\mathrm{e}^{ikx}$

Boundary conditions:

$$\psi_{\mathrm{I}}(x = 0) = 0 = \psi_{\mathrm{I}}(x = a)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\psi_{\mathrm{I}}(x = 0) = 0 = \frac{\mathrm{d}}{\mathrm{d}x}\psi_{\mathrm{I}}(x = a)$$

$$\Rightarrow A + B = 0 \wedge \underbrace{-ikA}_{=0}\underbrace{\left(\mathrm{e}^{-ikx} + \mathrm{e}^{ikx}\right)}_{\neq 0} = 0 \quad \Rightarrow \quad A = B = 0$$

b) For $A = -B$

$$\psi(x) = -B\,\mathrm{e}^{-ikx} + B\,\mathrm{e}^{ikx} = 2i\,B\sin(kx) = C\sin(kx)$$
$$\psi(a) = 0 \quad \Rightarrow \quad k = \tfrac{n\pi}{a} \quad , n \in \mathbb{N}$$

$$\int_0^a |\psi(x)|^2\,\mathrm{d}x = 1 \quad \Rightarrow \quad C = \sqrt{\tfrac{2}{a}}$$

$$\Rightarrow \psi_n(x) = \underline{\underline{\sqrt{\tfrac{2}{a}}\sin\left(\tfrac{n\pi x}{a}\right)}}$$

$$E_n = \tfrac{\hbar^2 k^2}{2m} = \underline{\underline{\tfrac{\hbar^2\pi^2}{2ma^2}n^2}}$$

c)

d) $\lambda = 451$ nm; $\Delta E = \frac{hc}{\lambda}$

$$\Delta E = E_2 - E_1 = \tfrac{3\hbar^2\pi^2}{2ma^2} = \tfrac{hc}{\lambda}$$
$$a = \sqrt{\tfrac{3\lambda}{2mhc}}\,\hbar\pi = \underline{\underline{6.41 \times 10^{-10}\ \mathrm{m}}}$$

Vertauschungsrelationen

$$\left[\hat{A}, \hat{B}\right] = \hat{A}\hat{B} - \hat{B}\hat{A}$$

a) $\hat{x} = x; \quad \hat{p}_x = -i\hbar\frac{\partial}{\partial x}$

$$[\hat{x}, \hat{p}_x]\,\psi = \left(-xi\hbar\frac{\partial}{\partial x} + i\hbar\frac{\partial}{\partial x}x\right)\psi = -i\hbar\left(x\frac{\partial\psi}{\partial x} - \left(\frac{\partial\psi}{\partial x} + x\frac{\partial\psi}{\partial x}\right)\right) = \underline{\underline{i\hbar\psi}}$$

We have just (re)discovered Heisenberg's uncertainty principle!

b) $\hat{L} = \hat{\vec{r}} \times \hat{\vec{p}} = -i\hbar\left(\hat{r} \times \vec{\nabla}\right)$

$$\begin{aligned}
\left[\hat{L}_y, \hat{L}_x\right] &= \hat{L}_y\hat{L}_x - \hat{L}_x\hat{L}_y = \\
&= (\hat{z}\hat{p}_x - \hat{x}\hat{p}_z)(\hat{x}\hat{p}_y - \hat{y}\hat{p}_x) - (\hat{x}\hat{p}_y - \hat{y}\hat{p}_x)(\hat{z}\hat{p}_x - \hat{x}\hat{p}_z) \\
&= \hat{z}\,[\hat{x}, \hat{p}_x]\,\hat{p}_y - (\hat{z}\hat{y}\hat{p}_x^2 - \hat{y}\hat{z}\hat{p}_x^2) - \left(\hat{x}^2\hat{p}_z\hat{p}_y - \hat{x^2}\hat{p}_y\hat{p}_z\right) + \hat{y}\,[\hat{x}, \hat{p}_x]\,\hat{p}_z = \\
&= -i\hbar\hat{z}\hat{p}_y + i\hbar\hat{y}\hat{p}_z = \\
&= \underline{\underline{i\hbar\hat{L}_x}}
\end{aligned}$$

$$\begin{aligned}
\left[\hat{\vec{L}}^2, \hat{L}_z\right] &= \left[\hat{L}_x^2, \hat{L}_z\right] + \left[\hat{L}_y^2, \hat{L}_z\right] + \left[\hat{L}_z^2, \hat{L}_z\right] = \\
&= \hat{L}_x\left[\hat{L}_x, \hat{L}_z\right] + \left[\hat{L}_x, \hat{L}_z\right]\hat{L}_x + \hat{L}_y\left[\hat{L}_y, \hat{L}_z\right] + \left[\hat{L}_y, \hat{L}_z\right]\hat{L}_y = \\
&= -i\hbar\hat{L}_x\hat{L}_y - i\hbar\hat{L}_x\hat{L}_y + i\hbar\hat{L}_x\hat{L}_y + i\hbar\hat{L}_x\hat{L}_y = \\
&= \underline{\underline{0}}
\end{aligned}$$

We can conclude that we can not measure the x and y components of the angular momentum at the same time, whilst the z component and the absolute value can be measured simultaneously.