Magnetisches Moment von Kernen

a) $\boldsymbol{j} = \boldsymbol{l} + \boldsymbol{s}$

$$\boldsymbol{j}^2 = (\boldsymbol{l} + \boldsymbol{s})^2 = \boldsymbol{l}^2 + \boldsymbol{s}^2 + 2\boldsymbol{l}\cdot\boldsymbol{s} = \boldsymbol{l}^2 + \boldsymbol{s}^2 + 2\boldsymbol{l}\cdot(\boldsymbol{j} - \boldsymbol{l}) = \boldsymbol{s}^2 - \boldsymbol{l}^2 + 2\boldsymbol{j}\cdot\boldsymbol{l}$$

$$\Rightarrow \boldsymbol{j}\cdot\boldsymbol{l} = \frac{1}{2}\left(\boldsymbol{j}^2 + \boldsymbol{l}^2 - \boldsymbol{s}^2\right)$$

b)

$$\langle j m_j | g_l \boldsymbol{j}^2 | j m_j \rangle = j(j+1)$$

$$\langle j m_j | g_l \boldsymbol{j}\cdot\boldsymbol{l} | j m_j \rangle = \langle j m_j | g_l \frac{1}{2}\left(\boldsymbol{j}^2 + \boldsymbol{l}^2 - \boldsymbol{s}^2\right) | j m_j \rangle = \\ = \frac{g_l}{2}\Big[j(j+1) + l(l+1) - s(s+1) \Big]$$

$$\langle j m_j | g_s \boldsymbol{s}\cdot\boldsymbol{j} | j m_j \rangle = \langle j m_j | g_l \frac{1}{2}\left(\boldsymbol{j}^2 + \boldsymbol{s}^2 - \boldsymbol{l}^2\right) | j m_j \rangle = \\ = \frac{g_s}{2}\Big[j(j+1) + s(s+1) - l(l+1) \Big]$$

$$\langle j m_j | g_l \boldsymbol{j}\cdot\boldsymbol{l} + g_s \boldsymbol{s}\cdot\boldsymbol{j} | j m_j \rangle = \langle j m_j | g_l \boldsymbol{j}\cdot\boldsymbol{l} | j m_j \rangle + \langle j m_j | g_s \boldsymbol{s}\cdot\boldsymbol{j} | j m_j \rangle = \\ = \frac{1}{2}\left(g_l\Big[j(j+1) + l(l+1) - s(s+1) \Big] + g_s\Big[j(j+1) + s(s+1) - l(l+1) \Big] \right)$$

$$g_{\text{Kern}} = \frac{\langle j m_j | g_l \boldsymbol{j}\cdot\boldsymbol{l} + g_s \boldsymbol{s}\cdot\boldsymbol{j} | j m_j \rangle}{\langle j m_j | g_l \boldsymbol{j}^2 | j m_j \rangle} = \\ = \frac{g_l\big[j(j+1) + l(l+1) - s(s+1) \big] + g_s\big[j(j+1) + s(s+1) - l(l+1) \big]}{2j(j+1)}$$

c) $\Delta_{lsj} := j(j+1) + s(s+1) - l(l+1)$

$$g_{\text{Kern}} = g_{\text{Kern}} + g_l - g_l = g_l + \frac{g_l(-\Delta_{lsj} + 2j(j+1)) + g_s\Delta_{lsj} - g_l 2j(j+1)}{2j(j+1)}$$

$$= g_l + \frac{g_l(-\Delta_{lsj}) + g_s\Delta_{lsj}}{2j(j+1)} = g_l + (g_s - g_l)\frac{\Delta_{lsj}}{2j(j+1)}$$

d) $j = l \pm \frac{1}{2}; \quad s = \frac{1}{2}; \quad s(s+1) = \frac{3}{4}; \quad j(j+1) = l^2 + 2l + \frac{3}{4}$

$$\frac{\Delta_{lsj}}{2j(j+1)} = \frac{j(j+1) + s(s+1) - l(l+1)}{2j(j+1)} = \frac{l^2 + 2l + \frac{3}{4} + \frac{3}{4} - l^2 - l}{2l^2 + 4l + \frac{3}{2}}$$
$$= \frac{l + \frac{3}{2}}{2l^2 + 4l + \frac{3}{2}} = \frac{l + \frac{3}{2}}{(2l+1)\left(l + \frac{3}{2}\right)} = \frac{1}{2l+1}$$

$$\Rightarrow g_{\text{Kern}} = g_l \pm (g_s - g_l)\frac{\Delta_{lsj}}{2j(j+1)} = g_l \pm \frac{(g_s - g_l)}{2l+1}$$

e)

Der Isospin des Deuterons

a)

b)

c)

d)

e)

f)

g)

Quadrupolmoment der Kerne

$$x = ar\sin(\theta)\cos(\varphi)$$
$$y = ar\sin(\theta)\sin(\varphi)$$
$$z = br\cos(\theta)$$

a) $\rho_{\mathrm{el}} = \frac{Ze}{V} = \frac{3Ze}{4\pi a^2 b}$; $\quad \|\boldsymbol{r}\|^2 = r^2\left(a^2\sin^2(\theta) + b^2\cos^2(\theta)\right)$

$$\begin{aligned}
Q &= \int_V \rho_{\mathrm{el}}(\boldsymbol{r})\left[3z^2 - \|\boldsymbol{r}\|^2\right]\mathrm{d}V = \int_0^1\int_0^{2\pi}\int_0^{\pi} \rho_{\mathrm{el}} a^2 b r^2 \sin(\theta)\left[3z^2 - \|\boldsymbol{r}\|^2\right]\mathrm{d}\theta\,\mathrm{d}\varphi\,\mathrm{d}r \\
&= 2\pi\rho_{\mathrm{el}} a b^2 \int_0^1\int_0^{\pi} r^4 \sin(\theta)\left[-a^2\sin^2(\theta) + 2b^2\cos^2(\theta)\right]\mathrm{d}\theta\,\mathrm{d}r \\
&= 2\pi\rho_{\mathrm{el}} a b^2 \int_0^1 r^4\,\mathrm{d}r \int_0^{\pi} \sin(\theta)\left[-a^2\sin^2(\theta) + 2b^2\cos^2(\theta)\right]\mathrm{d}\theta \\
&= \frac{2}{5}\pi\rho_{\mathrm{el}} a b^2\left[-a^2\int_0^{\pi}\sin^3(\theta)\,\mathrm{d}\theta + 2b^2\int_0^{\pi}\cos^2(\theta)\sin(\theta)\,\mathrm{d}\theta\right] \\
&= \frac{2}{5}\pi\rho_{\mathrm{el}} a b^2\left[-a^2\frac{4}{3} + b^2\frac{4}{3}\right] \\
&= \frac{2}{5}(b^2 - a^2)\frac{4\rho_{\mathrm{el}}\pi a b^2}{3} \\
&= \frac{2}{5}Ze(b^2 - a^2)
\end{aligned}$$

a und b hier vertauscht (im Vergleich zur Angabe) wegen der Parametrisierung. Können aber durch neue Parametrisierung einfach getauscht werden. Der Übersichtlichkeit halber wird in der nächsten Aufgabe die Notation aus der Angabe übernommen.

b) $Q(\mathrm{Ta}) = 6\cdot 10^{-24} e\ \mathrm{cm}^2$; $\quad Q(\mathrm{Sb}) = -1.2\cdot 10^{-24} e\ \mathrm{cm}^2$

$$a = R(1+\epsilon), \quad b = \frac{R}{\sqrt{1+\epsilon}} \quad \Longrightarrow \quad a^2 - b^2 = (1+\epsilon)^2 R^2 - \frac{R^2}{1+\epsilon} \approx 3R^2\epsilon$$

$$Q(\mathrm{Ta}) = \tfrac{2}{5} Z_{\mathrm{Ta}} e(a^2 - b^2) \approx \tfrac{2}{5} 73 e 3 R^2 \epsilon_{\mathrm{Ta}} = \tfrac{438}{5} e R^2 \epsilon_{\mathrm{Ta}}$$

$$\Rightarrow \epsilon_{\mathrm{Ta}} = \tfrac{5}{438} \tfrac{Q(\mathrm{Ta})}{eR^2} = 0.146$$

$$Q(\mathrm{Sb}) = \tfrac{2}{5} Z_{\mathrm{Sb}} e(a^2 - b^2) \approx \tfrac{2}{5} 51 e 3 R^2 \epsilon_{\mathrm{Sb}} = \tfrac{306}{5} e R^2 \epsilon_{\mathrm{Sb}}$$

$$\Rightarrow \epsilon_{\mathrm{Sb}} = \tfrac{5}{306} \tfrac{Q(\mathrm{Sb})}{eR^2} = -0.037$$

$$\left.\frac{a}{b}\right|_{\epsilon_{\mathrm{Ta}}} = \underline{\underline{1.227}}$$

$$\left.\frac{a}{b}\right|_{\epsilon_{\mathrm{Sb}}} = \underline{\underline{0.945}}$$