Die Friedmann–Gleichung

$$H(t)^2 = \left(\frac{\dot{a}}{a}\right)^2 = H_0^2\left(\frac{\Omega_r}{a^4} + \frac{\Omega_m}{a^3} + \Omega_\Lambda - \frac{\Omega_0 - 1}{a^2}\right)$$

a)
- Ω_0 … total density
- Ω_r … radiation density
- Ω_m … matter density (Dark + Baryonic)
- Ω_Λ … cosmological constant (vacuum density)

b)

$$H(t)^2 = \left(\frac{\dot{a}}{a}\right)^2 = \left(\frac{\frac{\mathrm{d}a}{\mathrm{d}t}}{a}\right)^2 \quad\Rightarrow\quad a^2 H^2(t) = \left(\frac{\mathrm{d}a}{\mathrm{d}t}\right)^2 \quad\Rightarrow\quad \mathrm{d}a = \frac{\mathrm{d}t}{aH(t)}$$

$$\int 1 \,\mathrm{d}t = \int \frac{1}{aH(a)}\,\mathrm{d}a$$

c)
- $\Omega_0 \propto a^{-2}$
- $\Omega_r \propto a^{-4}$, as it gets redshifted in addition to space expanding by a^3
- $\Omega_m \propto a^{-3}$, as space expands by a^3
- $\Omega_\Lambda \propto$ const., since it seems to be an intrinsic property of vacuum/spacetime

d) Radiation dominated: $H(t) = H_0\sqrt{\frac{1}{a(t)^4}}$

$$\int_0^t 1\,\mathrm{d}\tilde{t} = \int_0^{a(t)} \frac{1}{\tilde{a}H(\tilde{a})}\,\mathrm{d}\tilde{a} = \int_0^{a(t)} \frac{\tilde{a}}{H_0}\,\mathrm{d}\tilde{a} = \frac{a(t)^2}{2H_0}$$

$$a(t) = \underline{\underline{\sqrt{2H_0 t}}}$$

Matter dominated: $H(t) = H_0\sqrt{\frac{1}{a(t)^4}}$

$$\int\limits_0^t 1\,\mathrm{d}\tilde{t} = \int\limits_0^{a(t)} \frac{1}{\tilde{a}H(\tilde{a})}\,\mathrm{d}\tilde{a} = \int\limits_0^{a(t)} \frac{\sqrt{\tilde{a}}}{H_0}\,\mathrm{d}\tilde{a} = \frac{2\sqrt{a(t)^3}}{3H_0}$$

$$a(t) = \underline{\underline{\left(\frac{3H_0t}{2}\right)^{\frac{2}{3}}}}$$

Cosmological Constant: $H(t) = H_0$

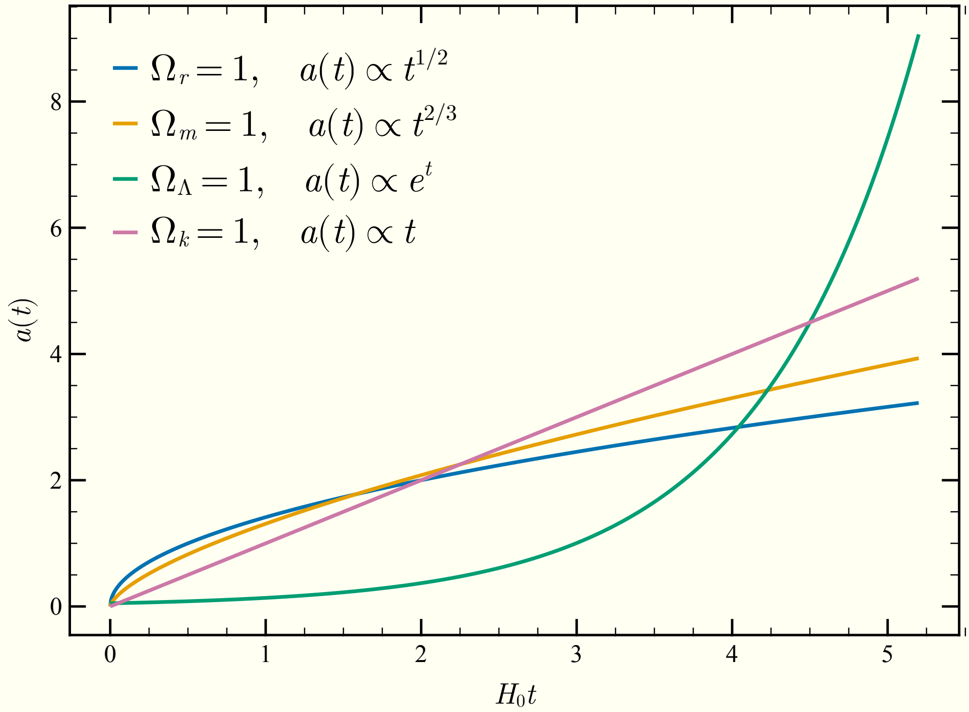
$$\int\limits_0^t 1\,\mathrm{d}\tilde{t} = \int\limits_c^{a(t)} \frac{1}{\tilde{a}H(\tilde{a})}\,\mathrm{d}\tilde{a} = \int\limits_c^{a(t)} \frac{1}{\tilde{a}H_0}\,\mathrm{d}\tilde{a} = \frac{\ln(a(t))}{H_0}$$

$$a(t) = \underline{\underline{\mathrm{e}^{H_0t+c}}}$$

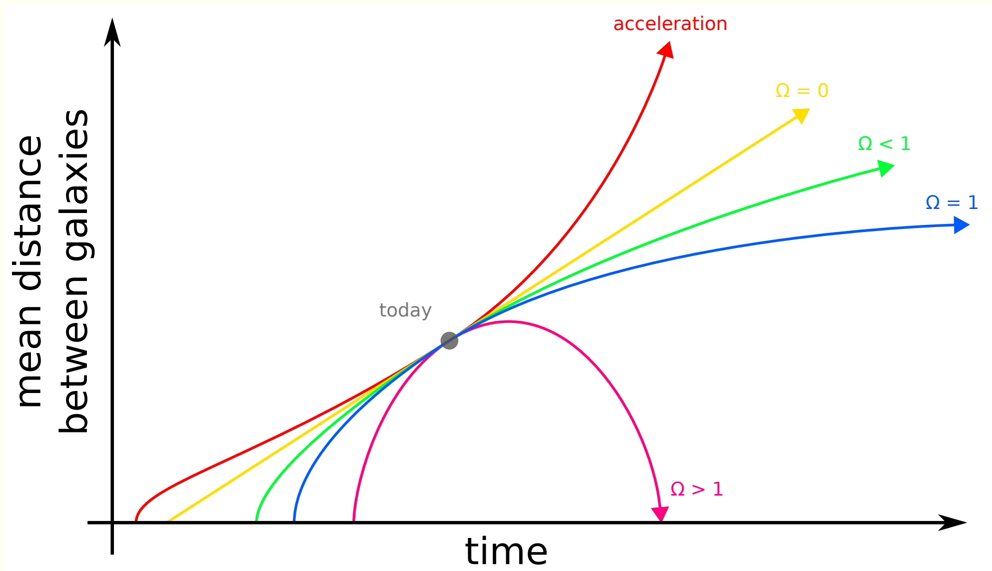
Curvature dominated: $H(t) = H_0\sqrt{\frac{1}{a(t)^2}}$

$$\int\limits_0^t 1\,\mathrm{d}\tilde{t} = \int\limits_0^{a(t)} \frac{1}{\tilde{a}H(\tilde{a})}\,\mathrm{d}\tilde{a} = \int\limits_0^{a(t)} \frac{1}{H_0}\,\mathrm{d}\tilde{a} = \frac{a(t)}{H_0}$$

$$a(t) = \underline{\underline{H_0t}}$$



e)



$\Omega_0 < 1$ leads to a negative Curvature and to an ever expanding Universe

$\Omega_0 = 1$ leads to zero Curvature and to an expansion, where the rate of expansion approaches zero asymptotically

$\Omega_0 > 1$ leads to a positive Curvature and to a collapsing Universe