1) $f: \mathbb{R}^2 \to \mathbb{R}^2 \ \begin{pmatrix} x \\ y \end{pmatrix} \mapsto (x^3+y^2)\, e^{-x^2-y^2}$

$\frac{\partial f}{\partial x} = (2x^4+2y^2x-3x^2) - e^{-x^2-y^2}$

$\frac{\partial f}{\partial y} = (2y^3+2xy^3-2y) - e^{-x^2-y^2}$

$D_{(v,w)} f\begin{pmatrix} x \\ y \end{pmatrix} = \frac{\partial f\begin{pmatrix} x \\ y \end{pmatrix}}{\partial x} v + \frac{\partial f\begin{pmatrix} x \\ y \end{pmatrix}}{\partial y} w =$

$= (3x^2-2x^4-2xy^2)\, e^{-x^2-y^2} v + (2y-2xy^3-2y^3)\, e^{-x^2-y^2} w =$

$= e^{-x^2-y^2}\left[(3x^2-2x^4-2xy^2)v + (2y-2xy^3-2y^3)w\right]$

3) $f: \mathbb{R}^2 \to \mathbb{R} \ \begin{pmatrix} x \\ y \end{pmatrix} \mapsto x^4+4x^2y+2y^2 \qquad g: \mathbb{R}^2 \to \mathbb{R}^2 : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} xy^2 \\ x^2+y \end{pmatrix}$

$f \circ g = x^4y^8 + 4x^2y^4(x^2+y) + 2(x^4+2x^2y+y^2) = x^4y^8 + 4x^4y^4 + 4x^2y^3 + 2x^4 + 4x^2y + 2y^2$

$(f\circ g)' = \begin{pmatrix} 3x^3y^8 + 16x^3y^4 + 8xy^3 + 8x^3 + 8xy^7 \\ 8x^4y^7 + 16x^4y^3 + 12x^2y^2 + 4x^2 + 4y \end{pmatrix}$

$\frac{\partial f}{\partial x} = 3x^4 + 8xy$

$f(g(x)) = f\begin{pmatrix} xy^2 \\ x^2+y \end{pmatrix}$

4) $f: \cancel{\mathbb{R}}D \to \mathbb{R}^2 \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 1 - \frac{1}{2}y \\ 1 + \frac{1}{3}x \end{pmatrix}$

$D \in [0;2]$

$\forall x,y \in [0;2]: f_1\begin{pmatrix} x \\ y \end{pmatrix} \in [0;1] \subset D \wedge f_2\begin{pmatrix} x \\ y \end{pmatrix} \in [1; 1\frac{2}{3}] \subset D$ $\square$

L.stetig $\forall x,y \in D\ \exists L > 0: \left\| f\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} - f\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right\| \leq L \left\| \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} - \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right\|$

$\|\cdot\| := \|\cdot\|_\infty$

$\Rightarrow \max\{|1 - \frac{1}{2}y_1 - 1 + \frac{1}{2}y_2|, |1 + \frac{1}{3}x_1 - 1 - \frac{1}{3}x_2|\} \leq L \max\{|x_1 - x_2|, |y_1 - y_2|\}$

$= \max\{\frac{1}{2}|y_1 - y_2|, \frac{1}{3}|x_1 - x_2|\} \leq L \max\{|x_1 - x_2|, |y_1 - y_2|\}$

set $L = \max\{\frac{1}{2}, \frac{1}{3}\} = \underline{\frac{1}{2}}$

Fixpunkt: $\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 - \frac{1}{2}b \\ 1 + \frac{1}{3}a \end{pmatrix} \Rightarrow a = 1 - \frac{1}{2}b, \Rightarrow b = 1 + \frac{1}{3}a \Rightarrow a = 1 - \frac{1}{2} - \frac{1}{6}a \Rightarrow \underline{a = \cancel{\tfrac{6}{14}}\ \tfrac{3}{7}}$

$b = 1 + \frac{1}{3} \cdot \frac{\cancel{6}\,3}{\cancel{14}\,7} \Rightarrow \underline{\underline{b = 1 + \frac{1}{7}}}$

2) $f: \mathbb{R}^2 \to \mathbb{R}: \begin{pmatrix} x \\ y \end{pmatrix} \mapsto x^2y - xy^3$

$$D_{v,w} f\begin{pmatrix} a \\ b \end{pmatrix} = \lim_{h \to 0}\left(\frac{f(a+hv, b+hw) - f(a,b)}{h}\right) = \lim_{h \to 0}\left(\frac{(a+hv)^2(b+hw) - (a+hv)(b+hw)^3 - a^2b + ab^3}{h}\right) =$$

$$= \lim_{h \to 0}\left(\frac{a^2hw + 2abhv + 2ah^2vw + h^2v^2b + h^3wv^2 - ah^3w^3 - 3ab^2hw - 3abh^2w^2 - hvb^3 - 3hvb^2hw \cdots}{h}\right)$$

$$= \lim_{h \to 0}\left(a^2w + 2abv + 2ahvw + hv^2b + hwv^2 - ah^2w^3 - 3ab^2w - 3abhw^2 - b^3v - \cdots\right) =$$

$$= 2abv - b^3v + a^2w - 3ab^2w = \underline{(2ab - b^3)v + (a^2 - 3ab^2)w}$$

$$D_{vw} f\begin{pmatrix} a \\ b \end{pmatrix} = \frac{\partial}{\partial x} f\begin{pmatrix} a \\ b \end{pmatrix} v + \frac{\partial}{\partial y} f\begin{pmatrix} a \\ b \end{pmatrix} w = \underline{(2ab - b^3)v + (a^2 - 3ab^2)w}$$

$\gamma : (0 : 2\pi) \to \mathbb{R}, \; t \mapsto \begin{pmatrix} \sin(t) \\ \cos(t) \end{pmatrix}$

$\cancel{f \circ g}\; f \circ \gamma = \sin^2(t)\cos(t) - \sin(t)\cos^3(t)$

$(f \circ \gamma)' = 2\cos^2(t)\sin(t) - \cos^4(t) + 3\cos^2(t)\sin^2(t) - \sin^3(t)$

Für $\sigma, \omega = 1$