Hydrostatic equilibirum

$$\rho(r) = \rho_c \left(1 - \tfrac{r}{R}\right)$$

a)

$$\frac{\mathrm{d}m}{\mathrm{d}r} = 4\pi r^2 \rho(r) = 4\pi r^2 \rho_c \left(1 - \tfrac{r}{R}\right)$$

$$\int_0^{m(r)} 1 \,\mathrm{d}\tilde{m} = \int_0^r 4\pi \tilde{r}^2 \rho_c \left(1 - \tfrac{\tilde{r}}{R}\right) \mathrm{d}\tilde{r}$$

$$m(r) = \underline{\underline{4\pi\rho_c \left(\tfrac{r^3}{3} - \tfrac{r^4}{4R}\right)}}$$

b)

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -G\,\tfrac{m(r)\rho(r)}{r^2} = -G\pi\rho_c^2 \left(-\tfrac{4r}{3} + \tfrac{7r^2}{3R} - \tfrac{r^3}{R^2}\right)$$

$$\int_{Pc}^{P} 1 \,\mathrm{d}\tilde{P} = \int_0^r -G\,\tfrac{m(\tilde{r})\rho(\tilde{r})}{\tilde{r}^2}\,\mathrm{d}\tilde{r}$$

$$P(r) = -G\pi\rho_c^2 \left(-\tfrac{2r^2}{3} + \tfrac{7r^3}{9R} - \tfrac{r^4}{4R^2}\right) - P_c$$

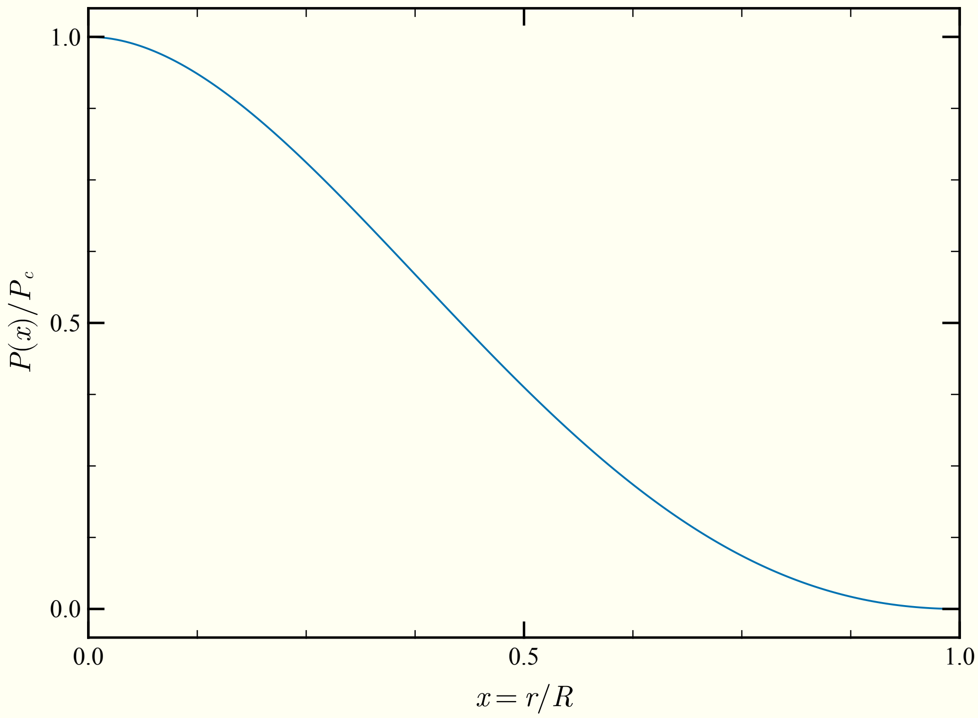
$$P(R) = 0 \quad \Rightarrow \quad P_c = -G\pi\rho_c^2 \left(-\tfrac{2R^2}{3} + \tfrac{7R^2}{9} - \tfrac{R^2}{4}\right) = G\pi\rho_c^2\,\tfrac{5R^2}{36}$$

$$\rho_c^2 = \tfrac{36P_c}{5G\pi R^2}$$

$$\Rightarrow P(r) = \underline{\underline{P_c \left(1 - \tfrac{24r^2}{5R^2} + \tfrac{28r^3}{5R^3} - \tfrac{9r^4}{5R^4}\right)}}$$

c)

$$\frac{P(x = \tfrac{r}{R})}{P_c} = 1 - \frac{24}{5}x^2 + \frac{28}{5}x^3 - \frac{9}{5}x^4$$


1.0
0.5
0.0
$P(x)/P_c$
0.0
0.5
1.0
$x = r/R$