Magnetisches Moment von Kernen

a) $\boldsymbol{j} = \boldsymbol{l} + \boldsymbol{s}$

$$\boldsymbol{j}^2 = (\boldsymbol{l} + \boldsymbol{s})^2 = \boldsymbol{l}^2 + \boldsymbol{s}^2 + 2\boldsymbol{l} \cdot \boldsymbol{s} = \boldsymbol{l}^2 + \boldsymbol{s}^2 + 2\boldsymbol{l} \cdot (\boldsymbol{j} - \boldsymbol{l}) = \boldsymbol{s}^2 - \boldsymbol{l}^2 + 2\boldsymbol{j} \cdot \boldsymbol{l}$$

$$\Rightarrow \boldsymbol{j} \cdot \boldsymbol{l} = \frac{1}{2}\left(\boldsymbol{j}^2 + \boldsymbol{l}^2 - \boldsymbol{s}^2\right)$$

b)

$$\langle j m_j | g_l \boldsymbol{j}^2 | j m_j \rangle = j(j+1)$$

$$\begin{aligned}\langle j m_j | g_l \boldsymbol{j} \cdot \boldsymbol{l} | j m_j \rangle &= \langle j m_j | g_l \frac{1}{2}\left(\boldsymbol{j}^2 + \boldsymbol{l}^2 - \boldsymbol{s}^2\right) | j m_j \rangle = \\ &= \frac{g_l}{2}\Big[j(j+1) + l(l+1) - s(s+1) \Big]\end{aligned}$$

$$\begin{aligned}\langle j m_j | g_s \boldsymbol{s} \cdot \boldsymbol{j} | j m_j \rangle &= \langle j m_j | g_l \frac{1}{2}\left(\boldsymbol{j}^2 + \boldsymbol{s}^2 - \boldsymbol{l}^2\right) | j m_j \rangle = \\ &= \frac{g_s}{2}\Big[j(j+1) + s(s+1) - l(l+1) \Big]\end{aligned}$$

$$\langle j m_j | g_l \boldsymbol{j} \cdot \boldsymbol{l} + g_s \boldsymbol{s} \cdot \boldsymbol{j} | j m_j \rangle = \langle j m_j | g_l \boldsymbol{j} \cdot \boldsymbol{l} | j m_j \rangle + \langle j m_j | g_s \boldsymbol{s} \cdot \boldsymbol{j} | j m_j \rangle =$$
$$= \frac{1}{2}\left(g_l \Big[j(j+1) + l(l+1) - s(s+1) \Big] + g_s \Big[j(j+1) + s(s+1) - l(l+1) \Big] \right)$$

$$\begin{aligned} g_{\text{Kern}} &= \frac{\langle j m_j | g_l \boldsymbol{j} \cdot \boldsymbol{l} + g_s \boldsymbol{s} \cdot \boldsymbol{j} | j m_j \rangle}{\langle j m_j | g_l \boldsymbol{j}^2 | j m_j \rangle} = \\ &= \frac{g_l\big[j(j+1) + l(l+1) - s(s+1)\big] + g_s\big[j(j+1) + s(s+1) - l(l+1)\big]}{2j(j+1)}\end{aligned}$$

c) $\Delta_{lsj} := j(j+1) + s(s+1) - l(l+1)$

$$\begin{aligned} g_{\text{Kern}} &= g_{\text{Kern}} + g_l - g_l = g_l + \frac{g_l(-\Delta_{lsj} + 2j(j+1)) + g_s \Delta_{lsj} - g_l 2j(j+1)}{2j(j+1)} \\ &= g_l + \frac{g_l(-\Delta_{lsj}) + g_s \Delta_{lsj}}{2j(j+1)} = g_l + (g_s - g_l)\frac{\Delta_{lsj}}{2j(j+1)}\end{aligned}$$

d) $j = l \pm \frac{1}{2}; \quad s = \frac{1}{2}; \quad s(s+1) = \frac{3}{4}; \quad j(j+1) = l^2 + 2l + \frac{3}{4}$

$$\frac{\Delta_{lsj}}{2j(j+1)} = \frac{j(j+1) + s(s+1) - l(l+1)}{2j(j+1)} = \frac{l^2 + 2l + \frac{3}{4} + \frac{3}{4} - l^2 - l}{2l^2 + 4l + \frac{3}{2}}$$
$$= \frac{l + \frac{3}{2}}{2l^2 + 4l + \frac{3}{2}} = \frac{l + \frac{3}{2}}{(2l+1)\left(l + \frac{3}{2}\right)} = \frac{1}{2l+1}$$

$$\Rightarrow g_{\text{Kern}} = g_l \pm (g_s - g_l)\frac{\Delta_{lsj}}{2j(j+1)} = g_l \pm \frac{(g_s - g_l)}{2l+1}$$

e) $\mu_N = \frac{e\hbar}{2m_{\mathrm{p}}}$

Der Isospin des Deuterons

a)

$$\begin{aligned} T_z = 1 &: \ \psi_{pp} = |\uparrow\uparrow\rangle \\ T_z = 0 &: \ \psi_{pn} = |\uparrow\downarrow\rangle \\ T_z = -1 &: \ \psi_{nn} = |\downarrow\downarrow\rangle \end{aligned}$$

b)

$$\begin{aligned} \psi(T_z = 1) &= |\uparrow\uparrow\rangle \\ \psi_1(T_z = 0) &= \tfrac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right) \\ \psi_2(T_z = 0) &= \tfrac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right) \\ \psi(T_z = -1) &= |\downarrow\downarrow\rangle \end{aligned}$$

c)

Symmetrie	ψ	T_z	T
Antisymmetrisch	$\psi = \frac{1}{\sqrt{2}}(\lvert\uparrow\downarrow\rangle - \lvert\downarrow\uparrow\rangle)$	0	0
Symmetrisch	$\psi = \lvert\uparrow\uparrow\rangle$	1	1
	$\psi = \frac{1}{\sqrt{2}}(\lvert\uparrow\downarrow\rangle + \lvert\downarrow\uparrow\rangle)$	0	1
	$\psi = \lvert\downarrow\downarrow\rangle$	-1	1

d) $\lvert\Psi_{\text{Ort}}\rangle$ is symmetrisch, da $L = 0$

e) $\lvert\Psi_{\text{Spin}}\rangle\lvert\Psi_{\text{Isospin}}\rangle$ muss antisymmetrisch sein, damit $\lvert\Psi\rangle$ antisymmetrisch ist.

f) Da $S = 1$ ist die Spin-Wellenfunktion symmetrisch, weshalb die Isospin-Wellenfunktion antisymmetrisch sein muss.

g)

Quadrupolmoment der Kerne

$$x = ar\sin(\theta)\cos(\varphi)$$
$$y = ar\sin(\theta)\sin(\varphi)$$
$$z = br\cos(\theta)$$

a) $\rho_{\text{el}} = \frac{Ze}{V} = \frac{3Ze}{4\pi a^2 b}$; $\quad \|\boldsymbol{r}\|^2 = r^2\left(a^2\sin^2(\theta) + b^2\cos^2(\theta)\right)$

$$
\begin{aligned}
Q &= \int_V \rho_{\text{el}}(\boldsymbol{r})\left[3z^2 - \|\boldsymbol{r}\|^2\right]\mathrm{d}V = \int_0^1\int_0^{2\pi}\int_0^{\pi} \rho_{\text{el}} a^2 b r^2 \sin(\theta)\left[3z^2 - \|\boldsymbol{r}\|^2\right]\mathrm{d}\theta\,\mathrm{d}\varphi\,\mathrm{d}r \\
&= 2\pi\rho_{\text{el}} a b^2 \int_0^1\int_0^{\pi} r^4 \sin(\theta)\left[-a^2\sin^2(\theta) + 2b^2\cos^2(\theta)\right]\mathrm{d}\theta\,\mathrm{d}r \\
&= 2\pi\rho_{\text{el}} a b^2 \int_0^1 r^4\,\mathrm{d}r \int_0^{\pi} \sin(\theta)\left[-a^2\sin^2(\theta) + 2b^2\cos^2(\theta)\right]\mathrm{d}\theta \\
&= \frac{2}{5}\pi\rho_{\text{el}} a b^2 \left[-a^2\int_0^{\pi}\sin^3(\theta)\,\mathrm{d}\theta + 2b^2\int_0^{\pi}\cos^2(\theta)\sin(\theta)\,\mathrm{d}\theta\right] \\
&= \frac{2}{5}\pi\rho_{\text{el}} a b^2 \left[-a^2\frac{4}{3} + b^2\frac{4}{3}\right] \\
&= \frac{2}{5}(b^2 - a^2)\frac{4\rho_{\text{el}}\pi a b^2}{3} \\
&= \frac{2}{5}Ze(b^2 - a^2)
\end{aligned}
$$

a und b hier vertauscht (im Vergleich zur Angabe) wegen der Parametrisierung. Können aber durch neue Parametrisierung einfach getauscht werden. Der Übersichtlichkeit halber wird in der nächsten Aufgabe die Notation aus der Angabe übernommen.

b) $Q(\text{Ta}) = 6\cdot 10^{-24} e\ \text{cm}^2$; $\quad Q(\text{Sb}) = -1.2\cdot 10^{-24} e\ \text{cm}^2$; $\quad R = 1.3\cdot A^{\frac{1}{3}}\ \text{fm}$

$$
\begin{aligned}
a &= R(1+\epsilon) \\
b &= \frac{R}{\sqrt{1+\epsilon}}
\end{aligned}
\quad \Longrightarrow \quad
\begin{aligned}
a^2 - b^2 &= (1+\epsilon)^2 R^2 - \frac{R^2}{1+\epsilon} \approx 3R^2\epsilon \\
\frac{a}{b} &= (1+\epsilon)^{3/2}
\end{aligned}
$$

$$Q(\mathrm{Ta}) = \tfrac{2}{5} Z_{\mathrm{Ta}} e(a^2 - b^2) \approx \tfrac{2}{5} 73 e 3R^2 \epsilon_{\mathrm{Ta}} = \tfrac{438}{5} eR^2 \epsilon_{\mathrm{Ta}}$$

$$\Rightarrow \epsilon_{\mathrm{Ta}} = \tfrac{5}{438} \tfrac{Q(\mathrm{Ta})}{eR^2} = 0.146$$

$$Q(\mathrm{Sb}) = \tfrac{2}{5} Z_{\mathrm{Sb}} e(a^2 - b^2) \approx \tfrac{2}{5} 51 e 3R^2 \epsilon_{\mathrm{Sb}} = \tfrac{306}{5} eR^2 \epsilon_{\mathrm{Sb}}$$

$$\Rightarrow \epsilon_{\mathrm{Sb}} = \tfrac{5}{306} \tfrac{Q(\mathrm{Sb})}{eR^2} = -0.053$$

$$\left.\frac{a}{b}\right|_{\epsilon_{\mathrm{Ta}}} = \underline{\underline{1.227}}$$

$$\left.\frac{a}{b}\right|_{\epsilon_{\mathrm{Sb}}} = \underline{\underline{0.921}}$$