Hydrostatic equilibirum

$$\rho(r) = \rho_c \left(1 - \frac{r}{R} \right)$$

a)

$$\frac{\mathrm{d}m}{\mathrm{d}r} = 4\pi r^2 \rho(r) = 4\pi r^2 \rho_c \left(1 - \frac{r}{R}\right)$$

$$\int_{0}^{m(r)} 1 \, \mathrm{d}\tilde{m} = \int_{0}^{r} 4\pi \tilde{r}^{2} \rho_{c} \left(1 - \frac{\tilde{r}}{R}\right) \, \mathrm{d}\tilde{r}$$

$$m(r) = 4\pi \rho_c \left(\frac{r^3}{3} - \frac{r^4}{4R}\right)$$

b)

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -G\frac{m(r)\rho(r)}{r^2} = -G\pi\rho_c^2 \left(-\frac{4r}{3} + \frac{7r^2}{3R} - \frac{r^3}{R^2} \right)$$

$$\int\limits_{Pc}^{P} 1\,\mathrm{d}\tilde{P} = \int\limits_{0}^{r} -G\,\tfrac{m(\tilde{r})\rho(\tilde{r})}{\tilde{r}^2}\,\mathrm{d}\tilde{r}$$

$$P(r) = -G\pi\rho_c^2 \left(-\frac{2r^2}{3} + \frac{7r^3}{9R} - \frac{r^4}{4R^2} \right) - P_c$$

$$P(R) = 0 \quad \Rightarrow \quad P_c = -G\pi\rho_c^2 \left(-\frac{2R^2}{3} + \frac{7R^2}{9} - \frac{R^2}{4} \right) = G\pi\rho_c^2 \frac{5R^2}{36}$$

$$\rho_c^2 = \frac{36P_c}{5G\pi R^2}$$

$$\Rightarrow P(r) = \underbrace{P_c \left(1 - \frac{24r^2}{5R^2} + \frac{28r^3}{5R^3} - \frac{9r^4}{5R^4}\right)}_{=}$$

c)

$$\frac{P(x = \frac{r}{R})}{P_c} = 1 - \frac{24}{5}x^2 + \frac{28}{5}x^3 - \frac{9}{5}x^4$$

