Riemann zeta function

Alexander Helbok, Matthias Lotze, Patryk Morawski, Helen Zwölfer

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Overview

1. Introduction to the zeta function
2. Riemann conjecture
3. Zeta function and primes
4. Formal definitions

Introduction to the zeta function

Definition

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots \qquad \text{, for } \operatorname{Re}(s) > 1$$

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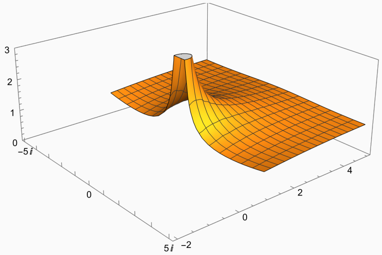

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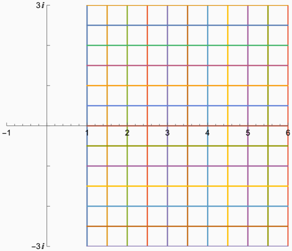

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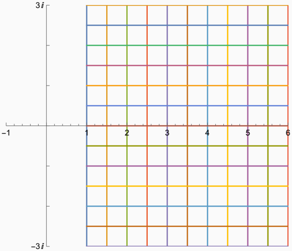

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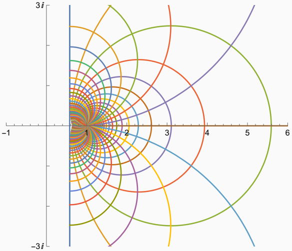

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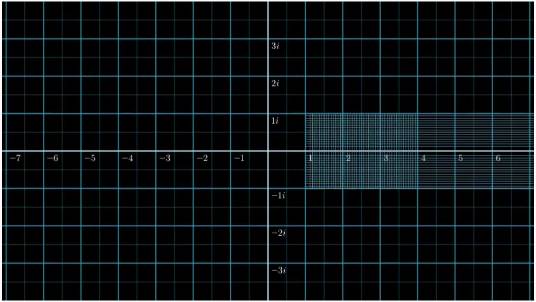
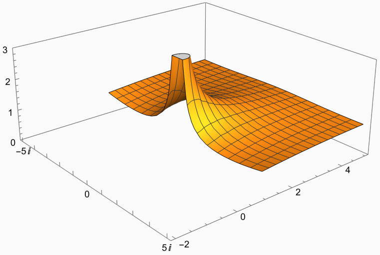


Properties

– meromorphic (holomorphic with pole at s = 1)



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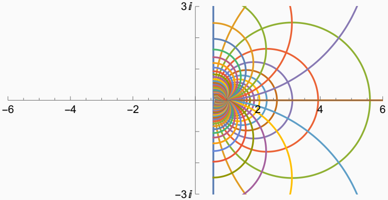

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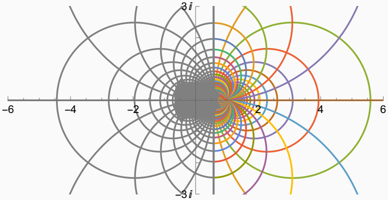

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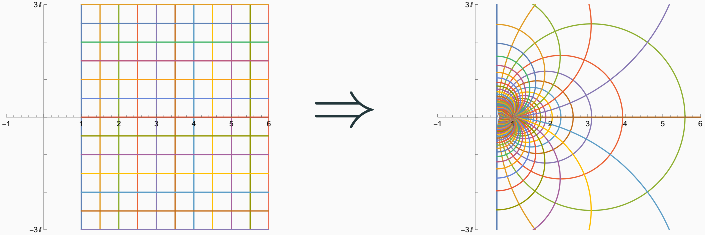

Properties

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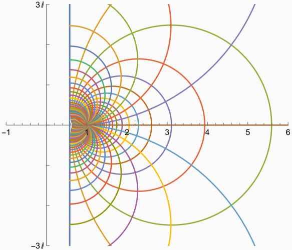
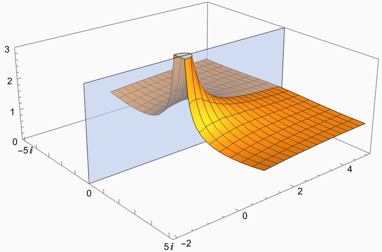
- complex differentiable in $\mathbb{C} \Rightarrow$ analytic + infinitely differentiable
- unique analytic continuation
- conformal (preserves Angles)
- symmetric along real axis





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$$\zeta(-1) = \sum_{n=1}^{\infty} \frac{1}{n^{-1}} = 1 + 2 + 3 + 4 + \cdots = -\frac{1}{12} \qquad \text{(???)}$$

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$$\zeta(-1) = \sum_{n=1}^{\infty} \frac{1}{n^{-1}} = 1 + 2 + 3 + 4 + \cdots = \text{undefined!}$$

Riemann conjecture

General Introduction to the Riemann Hypothesis

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad \text{for } s = \sigma + it \text{ with } \sigma > 1$$

„All nontrivial zeros of the Riemann zeta function have real part $\frac{1}{2}$"

- Proposed by Riemann in 1859
- One of the Clay Mathematic Institute's Millennium Prize Problems, which offers a million dollars to anyone who solves any of them

To even consider the Riemann zeta function for $s \in \mathbb{C}$ with $\mathrm{Re}(s) = \sigma < 1$, we first need to find a definition
$\Rightarrow$ Analytic Continuation

Alternative formula

$$\Gamma(s) = \int_0^\infty x^{s-1} \exp[-x] \, \mathrm{d}x$$

The Gamma function is a generalization of the factorial:
$\Gamma(1) = 1$ and $n\Gamma(n) = \Gamma(n+1)$(integration by parts)
$\Rightarrow \Gamma(n+1) = n!$ for $n \in \mathbb{N}$

Now we substitute $nu = x$, thus $n\mathrm{d}u = \mathrm{d}x$ in the Gamma function:
$\Gamma(s) = \int_0^\infty x^{s-1} \exp[-x] \, \mathrm{d}x \overset{\text{u-sub}}{=} \int_0^\infty n^s u^{s-1} \exp[-nu] \, \mathrm{d}u$

Analytic Continuation and Functional Equation of $\zeta(s)$

Taking the sum of $\frac{\Gamma(s)}{n^s}$ we get:

$$\sum_{n=1}^{\infty} \frac{\Gamma(s)}{n^s} = \sum_{n=1}^{\infty} \left[\int_0^{\infty} x^{s-1} \exp[-nx] \, \mathrm{d}x \right] \overset{(*)}{=} \int_0^{\infty} \left[\sum_{n=1}^{\infty} \exp[-nx] \right] x^{s-1} \, \mathrm{d}x$$

$(*)$ We can change the order of summation and integration because of absolute convergence

$\sum_{n=1}^{\infty} \exp[-nx]$ is a geometric series $\sum_{n=1}^{\infty} p^n = \frac{p}{1-p}$ with $p = \exp[-nx]$. Therefore:

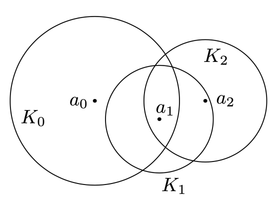
$$\int_0^{\infty} \left[\sum_{n=1}^{\infty} \exp[-nx] \right] x^{s-1} \, \mathrm{d}x = \int_0^{\infty} \frac{\exp[-x]}{1-\exp[-x]} x^{s-1} \, \mathrm{d}x = \int_0^{\infty} \frac{x^{s-1}}{\exp[-x]-1} \, \mathrm{d}x$$

Now we have: $\Gamma(s)\zeta(s) = \int_0^{\infty} \frac{x^{s-1}}{\exp[-x]-1} \, \mathrm{d}x$ for $\operatorname{Re}(s) = \sigma < 1$

Analytic Continuation

Let the function $f_0 : K_0 \to \mathbb{C}$ be holomorphic

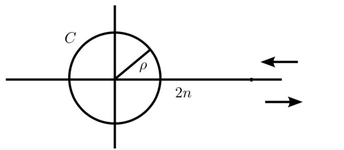
- Choose a point $a_1 \in U$ and develop the power series $\Rightarrow$ new function f_1 with convergence disk K_1, holomorphic on K_1
- Choose $a_2 \in K1 \setminus K_0$ and get f_2 with convergence disk K_2, holomorphic on K_2
- ...



Since $f_i|_{K_i \cap K_j} = f_j|_{K_i \cap K_j}$ for all i, j we get a well-defined analytical continuation of f_0 on $K_0, K_1, \ldots$

Analytic Continuation

We now continue $\zeta(s)$ analytically using contour integrals.
Let C be a contour consisting of the real axis from $+\infty$ to ρ, the circle with $|x| = \rho$, $0 < \rho < 2\pi$, and the real axis from ρ to $+\infty$.



We consider the Integral $I(s) = \int_C \frac{x^{s-1}}{\exp[x]-1}\,\mathrm{d}x$.
Reminder: We just proved $\Gamma(s)\zeta(s) = \sum_{n=1}^{\infty} \frac{\Gamma(s)}{n^s} = \int_0^{\infty} \frac{x^{s-1}}{\exp[x]-1}\,\mathrm{d}x$

By the Residue Theorem, $I(s)$ cannot depend on ρ
$\implies$ We can consider $\rho \to 0$ and deduce that the integral along the circle goes to zero. Thus only the paths along the real axis contribute.

Analytic Continuation

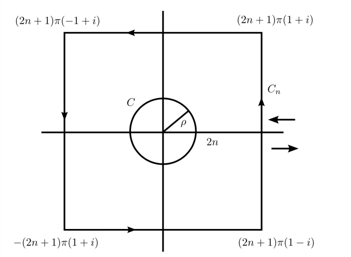
Simplifying, we get $I(s) = \frac{\zeta(s)}{\Gamma(1-s)} 2\pi i \exp[\pi i s]$.
$\implies \zeta(s) = \frac{\exp[-i\pi s]\Gamma(1-s)}{2\pi i} \int_C \frac{x^{s-1}}{\exp[x]-1} \, \mathrm{d}x$ for $\sigma > 1$

However, the integral converges uniformly for all s, so the only poles left are given by $\Gamma(1-s)$, thus for $s = 1, 2, 3, ...$

$I(s)$ vanishes for $s = 2, 3, ...$ and therefore the only pole is at $s = 1$.
$\implies$ The function is holomorphic for $s \neq 1$.

Functional Equation

We now use the same technique of contour integrals to deduce a functional equation. Consider $I_n = \int_{C_n} \frac{x^{s-1}}{\exp[x]-1}\,\mathrm{d}x$ along C_n.



Residue Theorem:

$I(s) + 2\pi i \sum$ residues $= I_n(s) \iff I(s) = I_n(s) - 2\pi i \sum$ residues

Between C and C_n we have poles at $\pm 2i\pi, \ldots, \pm 2ni\pi$. Calculating the residues, we get: $I(s) = I_n(s) + 4\pi i \exp[i\pi s] \sin\left[\frac{\pi s}{2}\right] \sum_{m=1}^{n} (2m\pi)^{s-1}$

Functional Equation

We now consider $\sigma < 0$ and $n \to \infty$. Then $I_n(s) \to 0$.

We now have:

$$\begin{aligned} I(s) &= 4\pi i \exp[i\pi s] \sin\left[\frac{\pi s}{2}\right] \sum_{m=1}^{n} (2m\pi)^{s-1} \\ &= 4\pi i \exp[i\pi s] \sin\left[\frac{\pi s}{2}\right] (2\pi)^{s-1} \zeta(1-s) \end{aligned}$$

Using $I(s) = \frac{\zeta(s)}{\Gamma(1-s)} 2\pi i \exp[\pi i s]$ which we got by analytic continuation:

$$\begin{aligned} &\frac{\zeta(s)}{\Gamma(1-s)} 2\pi i \exp[\pi i s] = I(s) = 4\pi i \exp[i\pi s] \sin\left[\frac{\pi s}{2}\right] (2\pi)^{s-1} \zeta(1-s) \\ &\implies \zeta(s) = \Gamma(1-s)\zeta(1-s) 2^s \pi^{s-1} \sin\left[\frac{\pi s}{2}\right] \end{aligned}$$

This functional equation holds for $\sigma < 0$.
So we found the “trivial“ zeros at $s = -2m$, $m \in \mathbb{N}$!

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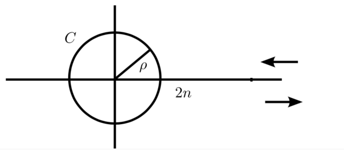
- Analytic continuation via contour integral:
  $\zeta(s) = \frac{\exp[-i\pi s]\Gamma(1-s)}{2\pi i} \int_C \frac{x^{s-1}}{\exp[x]-1}\,\mathrm{d}x$



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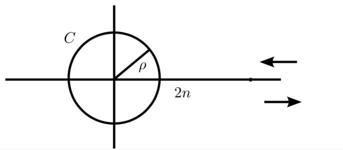
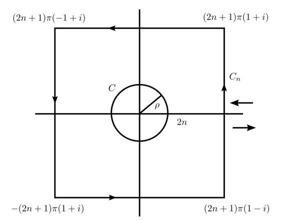
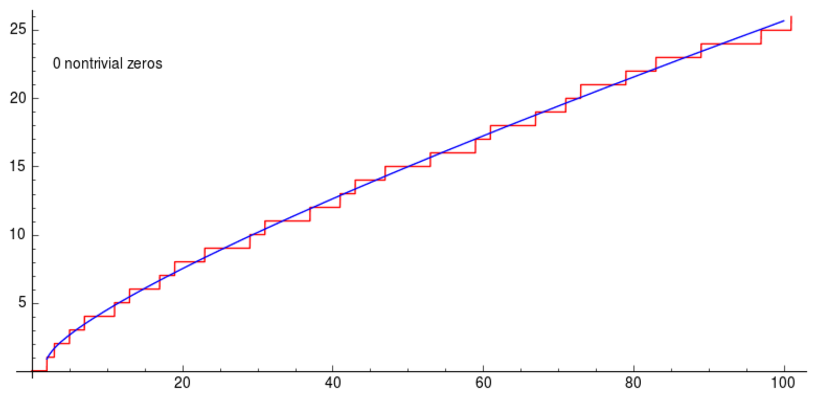
- Functional equation by considering residues:
  $\zeta(s) = \Gamma(1-s)\zeta(1-s)2^s\pi^{s-1}\sin\left[\frac{\pi s}{2}\right]$ for $\sigma < 0$





Zeta function and primes

Prime step-function



Formal definitions

(Algebraic) field $(K, +, \cdot)$

- $+, \cdot : K \times K \to K$
- $(K, +)$ abelian group
- $(K \setminus \{0\}, \cdot)$ abelian group

K-Vector space $(V, \oplus, \odot)$

- $\oplus : V \times V \to V \quad \Rightarrow (V, \oplus)$ abelian group
- $\odot : K \times V \to V \quad \Rightarrow$ scalar multiplication of the field K on $(V, \oplus)$
- $\mathrm{Hom}(V, W) := \{f \mid f\colon V \to W\}$:

$$
\begin{aligned}
&(f, g) \mapsto f \oplus g \\
&V \ni v \mapsto (f \oplus g)(v) := f(v) + g(v) \in W \\
&(\lambda, f) \mapsto \lambda \odot f \\
&V \ni v \mapsto (\lambda \odot f)(v) := \lambda f(v) \in W
\end{aligned}
$$

Algebra $(A, +, \cdot, \bullet)$

- $(A, +, \cdot)$ K-Vector space
- $\bullet : A \times A \to A$ product on $A \Rightarrow K$-bilinear map
- $(C^\infty(M), +, \cdot, \bullet)$: associative, unital, commutative algebra over $\mathbb{R}/\mathbb{C}$

$$\begin{aligned} \bullet : C^\infty(M) \times C^\infty(M) &\to C^\infty(M) \\ (f, g) &\mapsto f \bullet g \\ M \ni v &\mapsto (f \bullet g)(v) := f(v) \cdot g(v) \end{aligned}$$

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