FP1: EOM, extra notes, by Bo Huang

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Polarization of medium

We consider the polarisation density induced linearly by $\vec{E}$ field.

$\vec{P} := \varepsilon_0 \overset{=}{\chi} \vec{E}$; then $\overset{=}{\chi}$ is a 3×3 Tensor.

$P_i = \sum_j \chi_{ij} E_j \varepsilon_0$

$\vec{D} := \vec{P} + \varepsilon_0 \vec{E} = \varepsilon_0 (\mathbb{I} + \overset{=}{\chi}) \cdot \vec{E} := \varepsilon_0 \overset{=}{\epsilon} \cdot \vec{E}$ $\qquad \epsilon_{ij} = \delta_{ij} + \chi_{ij}$

$D_i = \varepsilon_0 \varepsilon_{ij} E_j$

for symmetric ϵ_{ij}, we can diagonalise it as.

$\overset{=}{\varepsilon} = \begin{pmatrix} \epsilon_{11} & 0 & 0 \\ 0 & \epsilon_{22} & 0 \\ 0 & 0 & \epsilon_{33} \end{pmatrix}$ with a proper choice of coordinates.

Then $D_i = \varepsilon_0 \varepsilon_{ij} E_j$

The corresponding Maxwell eqns, and wave equation.

① $\nabla \cdot \vec{E} = \rho/\varepsilon_0 \quad \Rightarrow \quad \nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} - \frac{\nabla \cdot \vec{P}}{\varepsilon_0} \quad \Rightarrow \quad \nabla \cdot \vec{D} = 0$

② $\nabla \cdot \vec{B} = 0$

③ $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ $\qquad$ ignore magnetization

④ $\nabla \times \vec{B} = \cancel{\mu_0 \vec{j}} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \quad \Rightarrow \quad \nabla \times \vec{B} = \mu_0 \frac{\partial \vec{D}}{\partial t}$

use Ansatz $\vec{E} \Rightarrow \vec{E} e^{-i\omega t + i\vec{k}\cdot\vec{x}}$, we get from ③ ④

$$\begin{cases} i\vec{k} \times \vec{E} = i\omega \vec{B} \\ i\vec{k} \times \vec{B} = -i\omega \mu_0 \varepsilon_0 \overset{=}{\varepsilon} \cdot \vec{E} \end{cases} ; \quad c^2 = \frac{1}{\mu_0 \varepsilon_0}$$

$\Rightarrow \vec{k} \times (\vec{k} \times \vec{E}) = -\frac{\omega^2}{c^2} \overset{=}{\varepsilon} \cdot \vec{E}$;

$\Rightarrow \vec{k}(\vec{k} \cdot \vec{E}) - \vec{E} k^2 = -\frac{\omega^2}{c^2} \overset{=}{\varepsilon} \cdot \vec{E}$; $\qquad \vec{k} = k \hat{q}$; $|\hat{q}| = 1$

$\omega^2 / k^2 = v^2$

$\Rightarrow \left(\hat{q}\hat{q} - \mathbb{I} + \frac{\overset{=}{\varepsilon}}{n^2} \right) \cdot \vec{E} = 0$ $\qquad \frac{\omega^2}{k^2} \frac{1}{c^2} := n^{-2}$

given $\hat{q}$, the direction of $\vec{k}$, we need $\vec{E} \neq 0$, so:

$$\boxed{\det \left| \hat{q}\hat{q} - \mathbb{I} + \frac{\bar{\bar{\varepsilon}}}{n^2} \right| = 0 \quad \Leftarrow \text{ equation for } n(\hat{q})}$$

In general, $\nabla \cdot \vec{D} = \rho_{free}$; if $\rho_{free} = 0$, we get $i\vec{k} \cdot \vec{D} = 0 \quad \therefore \vec{k} \perp \vec{D}$

Define $\eta = \varepsilon^{-1}$ where : $\begin{cases} \vec{D} = \varepsilon_0 \bar{\bar{\varepsilon}} \cdot \vec{E} \\ \vec{E} = \varepsilon_0^{-1} \bar{\bar{\eta}} \cdot \vec{D} \end{cases}$

choose the coordinates which diagonalize $\bar{\bar{\varepsilon}}$; then

$$\bar{\bar{\varepsilon}} = \begin{pmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{pmatrix} \quad \text{and} \quad \bar{\bar{\eta}} = \begin{pmatrix} \varepsilon_1^{-1} & 0 & 0 \\ 0 & \varepsilon_2^{-1} & 0 \\ 0 & 0 & \varepsilon_3^{-1} \end{pmatrix} = \begin{pmatrix} \eta_1 & 0 & 0 \\ 0 & \eta_2 & 0 \\ 0 & 0 & \eta_3 \end{pmatrix}$$

$$\hat{q}\hat{q} \cdot \vec{E} - \vec{E} + \frac{\vec{D}/\varepsilon_0}{n^2} = 0 \quad \underset{*\varepsilon_0}{\overset{\text{left dot } \vec{D}}{\Longrightarrow}} \quad -\vec{D} \cdot \vec{E}\varepsilon_0 + \frac{|\vec{D}|^2}{n^2} = 0$$

set $|\vec{D}|^2 = 1$, $\vec{D} = (D_1, D_2, D_3)$

$$\therefore \vec{D} \cdot \eta \cdot \vec{D} = \frac{1}{n^2} \Rightarrow \sum_{i=1}^{3} D_i^2 \eta_i = \frac{1}{n^2} \quad \text{or} \quad \sum_{i=1}^{3} D_i^2 / \varepsilon_i = \frac{1}{n^2}$$

many people write $|\vec{D}| = 1$ and $\vec{D} = (x, y, z)$

and:

$$\frac{x^2}{\varepsilon_{11}} + \frac{y^2}{\varepsilon_{22}} + \frac{z^2}{\varepsilon_{33}} = \frac{1}{n^2} \quad \text{or} \quad \boxed{\frac{x^2}{n_x^2} + \frac{y^2}{n_y^2} + \frac{z^2}{n_z^2} = \frac{1}{n^2}}$$

where $n_i^2 = \varepsilon_{ii}$. This is the "Index Ellipsoid".

With this Ellipsoid, we can calculate n from the direction of $\vec{D} = (x, y, z)$ and (n_x, n_y, n_z) of the material.
note: $\vec{D} \perp \vec{k}$

For Example, $n_x = n_y = n_o$; $n_z = n_e$.

In this case, $\hat{z}$ is the symmetry axis. If we apply a static (much slower than optical frequency) field $\vec{E}$ along $\hat{z}$,

The leading terms of new $\tilde{n}_e$ and $\tilde{n}_o$ are:

$$\begin{cases} \tilde{n}_e = n_e + \frac{1}{2} \gamma_{ee} n_e^3 E_z \\ \tilde{n}_o = n_o + \underbrace{\frac{1}{2} \gamma_{oe} n_o^3 E_z}_{\text{1st Taylor term}} \end{cases}$$

Here we assume no "mixing" induced by E_z, i.e. no off-diagonal terms in $\bar{\bar{\varepsilon}}$.

Further Examples.

Direction-dependened Refractive Index:

$$M := \begin{pmatrix} q_1^2 & q_1 q_2 & q_1 q_3 \\ q_2 q_1 & q_2^2 & q_2 q_3 \\ q_3 q_1 & q_3 q_2 & q_3^2 \end{pmatrix} - \mathbb{I} + \frac{1}{n^2} \begin{pmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{22} & 0 \\ 0 & 0 & \varepsilon_{33} \end{pmatrix};$$

note: $\varepsilon_{11} = n_1^2$
$\varepsilon_{22} = n_2^2$
$\varepsilon_{33} = n_3^2$

we want: $\overset{=}{M} \cdot \vec{E} = 0$, but $\vec{E} \neq 0$

Consider $\varepsilon_{11} \neq \varepsilon_{22} = \varepsilon_{33}$

$$\vec{k} \parallel \vec{q} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}$$

$$M = \begin{pmatrix} -\sin^2\theta & \sin\theta \cdot \cos\theta & 0 \\ \sin\theta \cdot \cos\theta & -\cos^2\theta & 0 \\ 0 & 0 & -1 \end{pmatrix} + \frac{1}{n^2} \begin{pmatrix} \varepsilon_{11} & & \\ & \varepsilon_{22} & \\ & & \varepsilon_{22} \end{pmatrix}$$

Case I: $\boxed{\vec{E} = \begin{pmatrix} 0 \\ 0 \\ E \end{pmatrix}}$ $\quad \varepsilon_{11} = n_1^2$, $\varepsilon_{22} = n_2^2$

$$M \cdot \vec{E} = 0 \Rightarrow \frac{\varepsilon_{22}}{n^2} = 1 \Rightarrow \boxed{n = n_2^2}$$

Case II: $\vec{E} = \begin{pmatrix} E_1 \\ E_2 \\ 0 \end{pmatrix}$

$$M \cdot \vec{E} = 0 \Rightarrow \begin{vmatrix} -\sin^2\theta + \frac{n_1^2}{n^2} & \sin\theta\cos\theta \\ \sin\theta \cdot \cos\theta & -\cos^2\theta + \frac{n_2^2}{n^2} \end{vmatrix} = 0$$

$$\Rightarrow \boxed{\frac{\sin^2\theta}{n_1^2} + \frac{\cos^2\theta}{n_2^2} = \frac{1}{n_{(\theta)}^2}}$$

if $\theta = 0$ $\quad n^2 = n_2^2$

$\theta = \frac{\pi}{2}$ $\quad n^2 = n_1^2$