Gaussian beams and optical resonators

Fortgeschrittenenpraktikum I

The aim of the experiment is to study Gaussian beams and optical resonators. In the first part of the experiment, students will investigate the fundamental properties of a laser beam, as well as the influence of optical elements, such as lenses, on the beam profile. In the second part of the experiment, the optical modes in a confocal resonator and their spatial and spectral properties will be studied.

The script contains preparation questions at the end of each section to prepare students for the preliminary discussion about the experiment, in other words, students should be able to answer them. Moreover, the experimental tasks are summarized together with the required analysis at the end of the script. For a successful laboratory experience, please, make sure to understand the main experimental tasks in section 7.

1. Laser safety

The laser used in this experiment belongs to the laser safety class 2 (or 3b depending on the setup!) and can therefore be potentially dangerous to eyesight. Please never look directly or indirectly into the laser. Remove every reflective object from clothing or the body (watches, rings, etc.) before the experiment. The laser beam should never be directed upwards and should always be limited to the area of the table. Therefore, if necessary, set up beam blocks. Always turn your back to the table when bending down and never insert reflective objects (screwdrivers, mirrors, etc.) in the middle of the beam path. Additionally, laser safety goggles are available.

2. Devices

Before the experiment, the functioning of the following devices should be known:

- HeNe Laser, $\lambda = 633\,\text{nm}$, (Polytec PL606), optical isolator (Gsänger DLI-1-650).
- Waistmeter (Thorlabs WM-100).
- Mirrors, lenses, and translation stages.
- Resonator mirror: plano-concave substrate, where the concave surface has a radius of curvature $r = 150\,\text{mm}$, and a reflectivity of $\sim 98\%$.
- Piezoelectric actuator, high-voltage amplifier, function generator.
- CCD camera, photodiode, oscilloscope (Tektronix TDS2004B).

3. Introduction

Lasers have become an indispensable part of our everyday life and research. Therefore, in many areas of classical optics, but also in quantum optics, an understanding of the propagation of laser light is of fundamental importance. In particular, the study of Gaussian beams is essential since many lasers emit light in exactly one Gaussian mode, the so-called TEM$_{00}$ mode (**T**ransverse **E**lectro-**M**agnetic). The Gaussian modes describe how the transverse beam profile and the phase fronts change along the propagation direction z of the beam (see Fig. 1).

4. Gaussian beams

Gaussian beams are described by one of the possible solutions of the paraxial Helmholtz equation. This solution can be obtained by separation of space and time variables in the wave equation for the electromagnetic field, as derived from Maxwell's equations, when considering the paraxial approximation [1, 2]. Every Gaussian beam-like solution of the paraxial Helmholtz equation represents a Gaussian mode, and the set of possible Gaussian modes forms an orthonormal basis. Consequently, any light field that satisfies the Helmholtz equation can in principle be described by a suitable superposition of Gaussian modes $U_{(m,n)}(x,y,z)$. Here, x, y, z indicate the location of the complex electric field amplitude U, and (m,n) are the transverse mode indices. The fundamental TEM$_{00}$ mode is as follows:

$$\begin{aligned} U_{(0,0)}(x,y,z) = A_0 \frac{w_0}{w(z)} \exp\left(-\frac{x^2+y^2}{w^2(z)}\right) \times \\ \times \exp\left(-ikz - ik\frac{x^2+y^2}{2R(z)} + i\phi(z)\right), \end{aligned} \tag{1}$$

and the relative intensity distribution measured across the transverse plane (on a sheet of paper or a camera) is

$$I(x,y,z) \sim |U(x,y,z)|^2 = I_0 \left(\frac{w_0}{w(z)}\right)^2 \exp\left(-2\frac{x^2+y^2}{w(z)^2}\right),$$

thus showing a Gaussian shape in the $x-y$ plane, where the variance of the intensity is related to $w^2(z)$. Therefore, $w(z)$ is associated with the width of the beam intensity distribution in the transverse plane, hence the name beam radius or beam width. In particular, the beam width corresponds to the radius $r^2 = x^2 + y^2$ centred in the optical axis where the intensity is reduced by a factor e^{-2}. The beam radius varies along the optical axis according to

$$w(z) = w_0 \sqrt{1 + \left(\frac{z}{z_\mathrm{R}}\right)^2} \quad \text{with} \qquad z_\mathrm{R} = \frac{\pi w_0^2}{\lambda}, \tag{2}$$

where the quantity w_0, known as beam waist, is the minimum beam radius of the Gaussian beam during its propagation, i.e, the focal point of the beam. The distance z_R, referred to as Rayleigh length, is the point where the beam radius is a factor $\sqrt{2}$ larger than the waist value and it is related to the field regime and the wavefront curvature of the Gaussian beam. The Rayleigh length determines the boundary between the near-field and far-field regimes: for $|z| \ll z_\mathrm{R}$ one speaks of near-field regime, whereas the far-field regime is associated with $|z| \gg z_\mathrm{R}$. In the near-field regime, wavefronts are approximately flat (wavefront curvature $R(z) \approx \infty$), while in the far-field regime they approach the curvature of a spherical wave ($R(z) \approx z$). In general, the curvature of the wave fronts is given by:

$$R(z) = z\left[1 + \left(\frac{z_\mathrm{R}}{z}\right)^2\right].$$

In the far-field regime, $w(z)$ approaches a linear trend

$$w(z) \simeq w_0 \frac{z}{z_\mathrm{R}} = \theta z,$$

where θ indicates the divergence angle of the beam:

$$\theta \simeq \frac{\lambda}{\pi w_0}.$$

In addition to the trivial phase evolution ikz, Eq. (1) contains a further phase term $\phi(z)$, which is also known as Gouy phase:

$$\phi(z) = \arctan\left(\frac{z}{z_\mathrm{R}}\right).$$

4.1. Preparation

- Where do Gaussian beams occur?
- What is an optical mode?
- What approximation was made in deriving the Helmholtz equation and what does it mean? Which optical effects are "lost" as a result?
- How does $w(z)$ depend on w_0? How does the divergence angle depend on w_0? What do the divergence angle and w_0 have to do with diffraction?
- How many parameters are needed to describe Gaussian beams? Give some examples.
- What is the relationship between the maximum intensity I_0 and the power in a Gaussian beam?

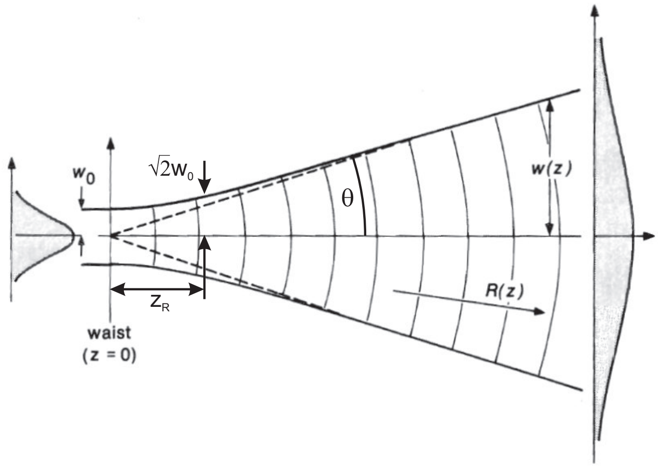

Figure 1: Propagation of a Gaussian beam. Image taken from [1].

- Discuss the significance of the terms in the phase evolution of a Gaussian beam.
- What is meant by a "collimated" beam?

5. Lenses and Gaussian beams

The shaping of a beam with the help of lenses is one of the most important tasks in optics and therefore it will be investigated in this experiment. The transformation of the transverse beam profile of a Gaussian beam upon propagation through lenses, or, in general, any optical system, can be calculated very elegantly with the help of the $ABCD$ matrix formalism [1]. For this purpose, the characteristic parameters $w(z)$ and $R(z)$ of a Gaussian beam are combined into a complex-valued quantity $q(z)$:

$$\frac{1}{q(z)} = \frac{1}{R(z)} - i\frac{\lambda}{\pi w(z)^2}.$$

One can show that a beam defined by q_1, passing through an optical system, evolves according to

$$\frac{1}{q_2} = \frac{C + D/q_1}{A + B/q_1}$$

into a beam with the complex beam parameter q_2, where A, B, C, and D denote the entries of a 2×2 matrix describing the optical system. In this formalism, the beam's transformation through any optical element is fully characterized by such a 2×2 matrix (see examples in Table 1), which, in turn, allows the description of any more complex composite optical system through simple matrix multiplication of the individual optical elements. Pay attention to the multiplication order of the subsequent $ABCD$ matrices, which must correspond to the order with which the beam propagates through the various optical elements.

For the simple example of a collimated beam ($R_1 = \infty$) with beam radius w_1 hitting a lens with focal length f, we can apply the formalism and appropriate approximations

Table 1: $ABCD$ matrices for the propagation through common optical elements.

Optical element	$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ Matrix
Free propagation for distance d	$\begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}$
Lens with focal length f	$\begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}$
Mirror with radius of curvature r	$\begin{pmatrix} 1 & 0 \\ -\frac{2}{r} & 1 \end{pmatrix}$

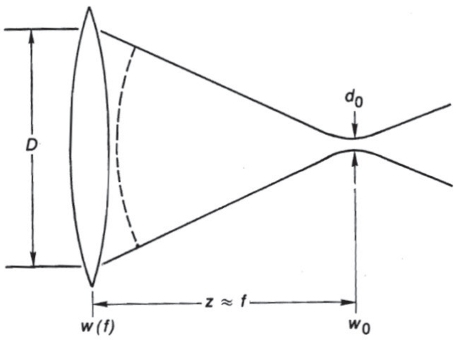

Figure 2: Focusing a Gaussian beam with a lens. Image taken from [1].

to obtain the resulting focused beam with waist w_2 (see Fig. 2 and Eq. (2)):

$$w_2 \approx \frac{\lambda f}{\pi w_1}. \tag{3}$$

5.1. Preparation

- Where does the focus shift when the beam incident on the lens is not collimated but slightly divergent?
- How can you change the size of a collimated beam with two lenses? How far apart must the lenses be? What is the relationship between the diameters of the beams and the focal lengths of the lenses?
- Use the $ABCD$ formalism and the assumption of a collimated input beam to obtain Eq. (3) after shaping by a lens with focal length f and a free-space propagation $d = f$, as shown in Fig. 2. Which approximations have been used?
- In the experiment you will measure the beam diameter with a waistmeter (Thorlabs WM-100). How does this device work and what does the signal of the photodiode in the waistmeter look like? How can the beam diameter be determined from this signal?

6. Optical resonators

Optical resonators have become indispensable in modern optics. They are essential for the realisation of lasers, pro-

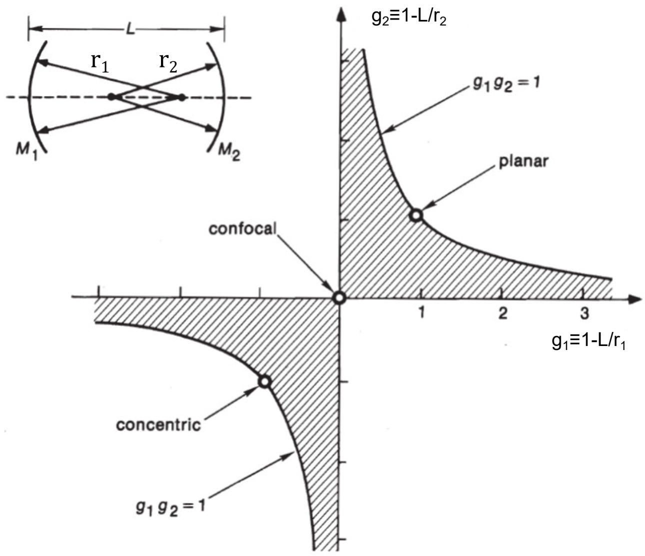

Figure 3: Stability diagram of an optical resonator. Image modified from [1].

viding optical feedback and mode selection, and they find widespread use in the field of spectroscopy as optical spectrum analysers, filters, and stable references.

6.1. Stability and spectral properties

Optical resonators, also known as optical cavities, consist of at least two mirrors, usually at least one of which has a finite curvature. Two main categories of optical cavities exist, depending on how the electromagnetic field is confined by the mirrors: standing-wave resonators and ring resonators. This experiment will focus on standing-wave resonators, where the light field coupled into the cavity is reflected back and forth between the mirrors and only specific field modes, so-called resonator modes, interfere constructively by forming standing waves. These modes are the only ones sustained by the cavity and can again be described with the help of Gaussian modes. Not all combinations of mirror curvatures $r_{1,2}$ and distances L make it possible to obtain stable resonator modes inside the cavity, hence the stability of the resonator is limited to a range of possible values. With the help of the $ABCD$ matrices, the stability condition for a stationary mode can be derived [1, 2]:

$$0 \leq g_1 g_2 \leq 1 \quad \text{where} \quad g_{1,2} = \left(1 - \frac{L}{r_{1,2}}\right) \tag{4}$$

The entire stability range is shown in Fig. 3. The transmission spectrum of the TEM_{00} modes through the optical cavity as a function of the frequency of the irradiated light or, in other terms, as a function of the mirror distance L has the distinctive aspect depicted in Fig. 4. In particular, the different TEM_{00} modes correspond to subsequent Lorentzian peaks spectrally separated by a free spectral range $\Delta\nu_{\mathrm{FSR}} = c/(2L)$ and with full width at half maxi-

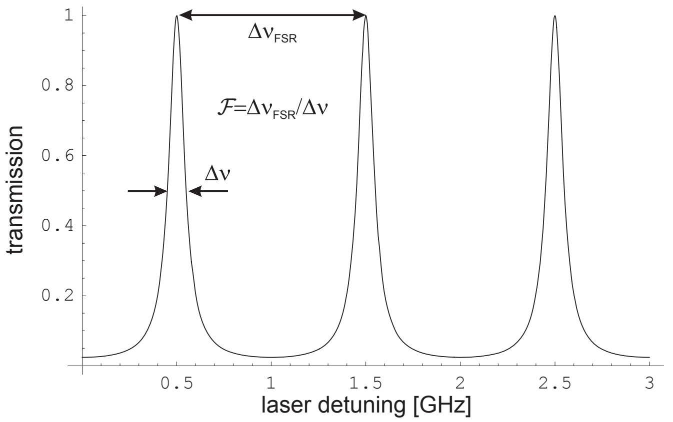

Figure 4: Transmission of an optical resonator as a function of the incident laser frequency. $\mathcal{F}$ denotes the finesse of the cavity.

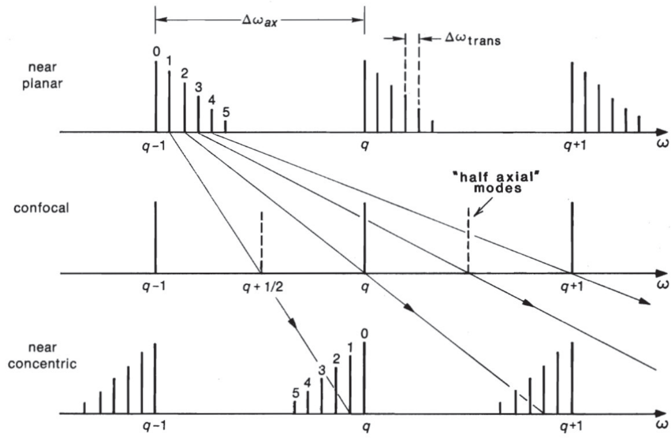

Figure 5: Resonance frequencies of higher order modes in optical resonators for different configurations. Image taken from [1].

mum (FWHM) $\Delta\nu$, the so-called bandwidth. The ratio

$$\mathcal{F} = \frac{\Delta\nu_{\mathrm{FSR}}}{\Delta\nu}$$

is the finesse of the optical cavity and represents a measure of the sharpness of mode resonances. This cavity parameter is fully determined by the cavity losses, and a high finesse value is fundamental in spectral analysis, where broad resonances can inhibit the ability to distinguish between neighbouring peaks.

6.2. Mode matching

The mode of the laser beam incident on a resonator is spectrally and spatially projected (decomposed) onto the resonator's internal modes. This is an essential property of optical resonators for spectral analysis. For an optimal spatial overlap of the laser mode with the resonator mode, a mode adjustment of the laser light has to be carried out. For this purpose, the beam waist of the laser beam needs to overlap with the beam waist of the resonator mode in size and position. This fine adjustment of the beam position and width is usually done with the help of lenses. The beam waist of the resonator mode of a symmetrical resonator ($r_1 = r_2 = r$) can be determined from

$$2z_{\mathrm{R}} = \sqrt{L(2r - L)}. \tag{5}$$

6.3. Higher order modes

In the case of mode mismatch, higher order modes (TEM_{mn} with $m, n \geq 1$) are excited in the resonator in addition to the TEM_{00} mode. The field distribution of these modes results from the multiplication of the fundamental Gaussian mode $U_{(0,0)}(x, y, z)$ with Hermite polynomials:

$$U_{(m,n)} \sim U_{(0,0)} H_m \left(\frac{\sqrt{2}}{w(z)} x \right) H_n \left(\frac{\sqrt{2}}{w(z)} y \right),$$

thus giving rise to the so-called Hermite-Gaussian beams. These modes experience an additional phase shift with each revolution in the resonator. This changes the optical resonance condition from the simplest case $\nu_q = q\frac{c}{2L}$, with $q = 1, 2, 3, \ldots$ now denoting the mode number, to the following expression:

$$\frac{\nu_{q,m,n}}{\Delta\nu_{\mathrm{FSR}}} = (q+1) + \frac{1}{\pi}(m+n+1)\arccos\left(1 - \frac{L}{r}\right). \tag{6}$$

The resulting resonant frequencies are shown in Fig. 5 for different resonator configurations.

6.4. Preparation

- What is the optical resonance condition for the TEM_{00} mode in a resonator?
- How can it be that almost 100% of the incident light can be transmitted through a resonator with negligible losses when the light has to pass through two mirrors with reflectivity $\sim 98\%$?
- What happens to the light field reflected back and forth in the case of an unstable resonator?
- What is the distinctive feature of the spectrum of a confocal resonator?
- What do the numbers (m, n) mean graphically for higher order modes?
- What determines the orientation of the intensity minima for higher-order modes?

7. Experimental tasks

7.1. Beam profiling

1. Completely determine the beam profile of the Gaussian beam emerging from the laser. For this purpose, measure the beam diameter at a series of distances

from the laser with the aid of the waistmeter and extract the beam profile via a fitting of the experimental data.
Note: this step is fundamental to later mode match the laser beam to the cavity mode, therefore, you need to prepare in advance a working fitting routine that allows you to immediately obtain the beam parameters (see task 3).
The measuring head of the waistmeter consists of a photodiode which is periodically covered by a rotating blade to perform a knife-edge measurement of the beam radius. It is important to ensure that the laser beam impinges orthogonally to the photodiode plane (why?). An electronic unit then calculates the beam diameter $2w$ from the photodiode signal.

2. Place a lens with a short focal length ($f = 30 \dots 100\,\mathrm{mm}$) at any point in the optical path of the beam and evaluate the beam profile after the lens. Which step size is useful in which range?

3. **Do in advance:** write a small script in your chosen data analysis software ahead of time to speed up the process on the day in the lab. The expected outcome is a small program in which you only need to feed in your recorded beam widths to immediately obtain a plot and a fit from which the beam waist and divergence can be extracted. Appendix A contains an example that you can use to check your fitting routine.

7.2. Mode matching

1. **Do in advance**: calculate the beam waist in the resonator in the confocal configuration for the mirrors used in the experiment with radii of curvature $r = 150\,\mathrm{mm}$ and reflectivities $\sim 98\%$.

2. Adjust the mode of the HeNe laser such that it matches the resonator mode using a selection of lenses ($f = 75\,\mathrm{mm}$, $100\,\mathrm{mm}$ and $200\,\mathrm{mm}$). Use your prepared value of the resonator waist and the calculated laser beam waist to determine which, and how many, lenses are necessary to match them. This can be easily achieved in the online toolbox *lightmachinery*.

   Place the lenses at the calculated positions, check the result with the waistmeter and verify the position of the focus. These measurements only serve as a verification of the earlier calculation and don't need to be recorded. Note: the position of the focus shifts with the lens positions and must therefore be found anew each time!

7.3. Study of cavity modes

1. Set up the two mirrors around the measured focus to finalise the experimental setup as shown in Fig. 6. The beam should hit the centre of the mirrors. Connect the function generator to the high-voltage amplifier and the amplifier to the piezo in order to periodically change the length of the resonator to allow resonance with the incident light. Check in with the supervisor before turning on the function generator and amplifier. Align the resonator by overlapping the reflections of the mirrors. Start by overlapping the back reflex of the second mirror with the incident beam on the first mirror. Then the reflex of the first mirror must be made to overlap with the incident beam on the second mirror. At this point, a mode pattern should already be observable on the mirrors or on a piece of paper after the resonator. The pattern should be made as small as possible by adjusting the mirrors.

   Now, focus the light after the resonator onto a photodiode and observe its signal together with the triangular signal of the function generator on an oscilloscope. Maximise the transmission of the TEM_{00} mode (photodiode signal) through adjustments of the resonator or the two mirrors before the resonator (only fine adjustment!). How big is the signal of the second-largest mode?

2. By slightly misaligning the coupled beam, higher-order modes can be observed. Record different modes with the CCD camera.

3. Now set up the photodiode after the resonator again. Record complete mode images (at least one full free spectral range on one flank of the function generator) for a different mirror spacing selected by yourself or specified by the supervisor. The mode matching is deliberately not readjusted and therefore a mode mismatch has to be accepted.

4. (Optional) increase again the spacing of the mirrors by steps, each time optimising the transmission of the TEM_{00} mode (highest mode in transmission). Note the increasing loss of stability when approaching the stability limit $d = 2r$.

8. Analysis

8.1. Beam profiling

Re: **1.** Determine the beam parameters with and without lens by fitting Eq. (2) to the experimental data.

Re: **2.** Compare the experimentally determined beam waist after the lens with the theoretical prediction in Eq. (3) (optionally also with the $ABCD$ matrix calculation). Is Eq. (3) well approximating the experimental results? Why?

8.2. Mode matching

Re: **2.** Detail the mode matching process conducted with *lightmachinery* and compare the experimental lens positions with the simulated ones.

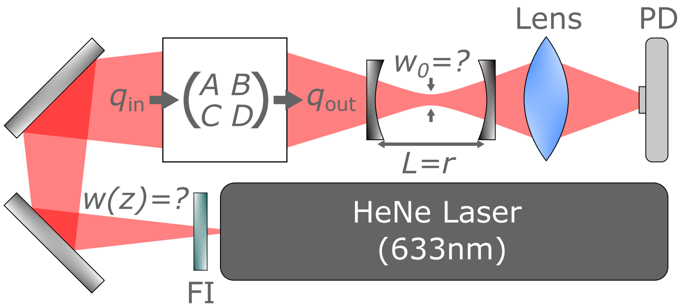

Figure 6: Schematic of the experimental setup consisting of a HeNe laser operating at $\lambda \approx 633\,\mathrm{nm}$ and a confocal cavity with mirror spacing and mirror radii $L = R$. An undetermined optical system, denoted by an $ABCD$ matrix, is employed to match the laser beam $w(z)$ to the resonator mode with a cavity beam waist of w_0. A Faraday isolator (FI) prevents backreflections from the cavity into the laser. The output signal of the cavity is focused onto a photodiode (PD) and recorded with an oscilloscope.

8.3. Study of cavity modes

Re: **2.** Compare the modes recorded with the CCD camera with the theoretically expected modes in a direct comparison in a diagram. Try to extract projections along two axes through the recorded images to further determine the intensity profile. These profiles can potentially be fitted by Hermite-Gaussian functions or otherwise analysed qualitatively.

Re: **3.** Fit the recorded transmission peaks. Determine the frequency separation of the TEM_{00} mode (strongest mode) to the TEM_{01} or the TEM_{10} mode (second strongest mode) using the recorded mode spectra via calibration of the known spectral range $\nu_{\mathrm{FSR}} = \frac{c}{2d}$ and compare the result with Eq. 6 in a graph.

References

[1] Anthony E. Siegman. *Lasers*. University Science Books, 1986.

[2] Bahaa E A Saleh and Malvin Carl Teich. *Fundamentals of photonics; 2nd ed.* Wiley, New York, NY, 2007.

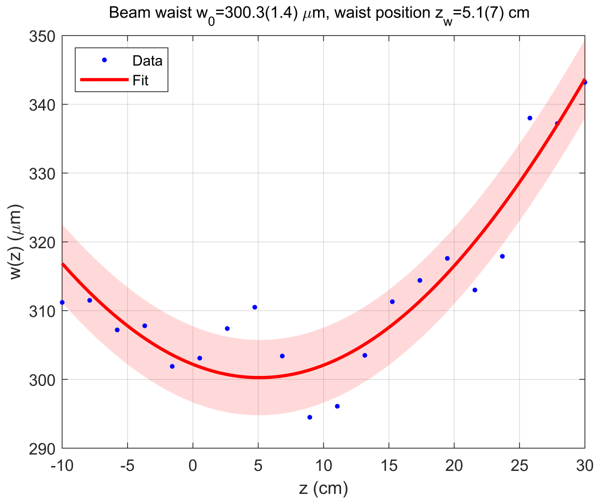

Figure 7: Fit example of a Gaussian beam profile. The top annotation reports the beam waist and its position along the optical axis, together with their uncertainties, obtained by using the data in Table 2. The red-shaded area represents the confidence interval associated with the fit uncertainty.

A. Fitting example

We report in Fig. 7 an example of data fitting that students can replicate to test their scripts to fit the experimental data measured during the lab experience. To this end, students can find in Table 2 the same data used in Fig. 7.

Table 2: Example beam width values.

z (cm)	**w(z)** (µm)
-10.00	311.2
-7.89	311.5
-5.78	307.2
-3.68	307.8
-1.57	301.9
0.53	303.1
2.63	307.4
4.74	310.5
6.84	303.4
8.95	294.5
11.05	296.1
13.16	303.5
15.26	311.3
17.37	314.4
19.47	317.6
21.58	313.0
23.68	317.9
25.79	338.0
27.89	337.2
30.00	343.2