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Decision Support

An iterative approach for achieving consensus when ranking a finite set of alternatives by a group of experts

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ABSTRACT

This paper proposes a novel iterative approach for achieving consensus when a group of experts is given the task to rank a finite set of alternatives. Unlike traditional approaches which use various metrics to express expert disagreements, the proposed approach is based on a *premetric* concept to express such disagreements. This premetric approach can capture more effectively the nature of agreements or disagreements that naturally occur when experts rank alternatives. The proposed approach is very flexible in that it considers a wide spectrum of ways to approach the problem of reaching consensus. These ways are based on an assignment formulation where one may consider various alternative consensus improving strategies. Which strategy to consider depends on the nature of the group decision making (GDM) problem under consideration and it can change as the GDM process evolves. In particular, this paper examines and provides novel solutions for the following fundamental problems: (1) How to evaluate the level of consensus? (2) How to identify the most appropriate disagreements to consider next when the consensus is not at a desired level? and (3) How to derive a reasonably 'close' solution when experts are not in perfect consensus while they are not able or not willing to further improve the consensus? Furthermore, this paper provides a theoretical foundation of the proposed premetric-based approach and then it uses this theoretical foundation to compare the new approach with some traditional ones.

1. Introduction

The problem of how to aggregate individual ordinal rankings of a finite set of alternatives into a collective ranking that best reflects the ranking preferences of a group of experts has been a research topic for over two centuries, as it can be traced back to the work by the Marquis de Condorcet in the 18th century. A relatively more recent fundamental contribution is known as Arrow's Impossibility Theorem (Arrow, 1951) which has promoted research on social choice. As result, numerous social choice procedures have been developed since then (Alpern & Chen, 2017; Bottero, Ferretti, & Figueira, 2018; Čaklović & Kurdija, 2017; Cook, Kress, & Seiford, 1996; Fishburn, 1977), which can be divided into two major categories. Namely, work on ad hoc methods and work on distance-based methods (Cook, 2006). Distance-based approaches were initially advanced by Kemeny and Snell (1962), and were later adopted by some other researchers (Armstrong, Cook, & Seiford, 1982; Blin, 1976; Cook & Seiford, 1978).

Distance-based approaches aim at finding an optimal collective ranking that somehow is 'closest' to the 'individuals' preferences. The notion of 'closest' is quantified by using a distance metric which always satisfies certain mathematical properties (described as a set of axioms).

This type of approaches have some intuitive appeal and thus have attracted everlasting attention from researchers on social choice (SC) and/or group decision making (GDM) (Cook, 2006). However, as it will be explained in Section 6 of this paper, distance-based approaches cannot reflect consensus properly in SC/GDM. This is true for two fundamental reasons:

(a) The value of the distance measure is equal to 0 if and only if two preference vectors are identical. However, as it is illustrated next, two experts may reach a type of consensus even if their preference vectors are not identical.
(b) The distance measure is required to satisfy the triangle inequality. However, the relaxation of the triangle inequality requirement is compatible with the consensus intransitivity property in social choice. For instance, suppose that experts

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A and B have some preferences in common (thus they have reached some type of consensus), experts B and C have some preferences in common too, but yet experts A and C may still have no preference in common.

Ordinal rankings are frequently used by groups for social choice (see, e.g., Cook, 2006; Cook et al., 1996; Fishburn, 1977 and the references therein). Traditionally, distance-based techniques are popular methods for deriving collective rankings from experts' ordinal preferences (Blin, 1976; Cook & Seiford, 1978; Kemeny & Snell, 1962). Cook and Seiford have used an assignment model to obtain collective rankings of alternatives when the experts provide ties-permitted ordinal rankings (Cook & Seiford, 1978). In their 1978's work, their assignment model uses a distance metric and can only produce complete ranking(s) as final solution(s). Being aware of this limitation, Cook, Seiford along with Armstrong extended their model to a more general case allowing it to give ties-permitted solutions in a subsequent study (Armstrong et al., 1982). Their scheme is realized by generalizing the assignment model to a constrained transportation model within which they still use the distance metric.

Recently, a different scheme was proposed for obtaining ties-permitted solutions Hou (2015a). This scheme also uses an assignment model but minimizes a *premetric* rather than a distance metric. A premetric is not a metric but an 'almost' metric because it does not necessarily satisfy the triangle inequality which is known to be a basic requirement for any metric. Disagreement, as opposed to consensus, has also been considered in group decisions (Ray & Triantaphyllou, 1998; 1999). To reach a collective solution with the maximal consensus, frequently means to attain a collective solution with the minimal disagreement. Therefore, approaching a collective solution for a group decision making problem is frequently realized by improving the consensus or by reducing the disagreements.

Generally speaking, in order to achieve consensus in a GDM process one must consider the following aspects: (1) How can we quantify the level of consensus? An indicator for quantifying the level of consensus should be introduced; (2) If consensus is not at a desired level, disagreements should be identified in order to improve consensus; and (3) How can we derive a compromised collective solution when experts have not reached perfect consensus and they are not able or willing to further improve consensus? A model should be constructed to find a reasonably 'close' solution to this collective consensus level when it is not perfect.

This paper is organized as follows. Section 2 describes some fundamental concepts from the relevant literature. Of key importance is the notion of a preference map (or PM) (Hou, 2015a). How the preferences of the experts are analyzed is given in Sections 3 and 4. By using the consensus gap indicator (a premetric) introduced by Hou (2015a), this paper proposes several additional indicators for evaluating the level of a group's consensus, by identifying any pairs of experts who have disagreements (especially the maximum disagreement). This paper also proposes methods for identifying the alternatives for which the experts have ranking disputes (especially the alternative with the most dispute). This part of information will be included in a consensus evaluation sequence which provides key guidance for improving consensus. Section 5 describes an approach for iteratively improving consensus until an acceptable consensus level has been reached or someone gives up the improving effort. If the consensus improving process is terminated because somebody gave up, a compromised solution is proposed based on an optimization model, where the total disagreement among the experts is minimized. An example of how this approach can be applied is also illustrated. To highlight the advantages of the proposed approach, the consensus gap, which is used as a fundamental concept in the proposed approach, is considered within an axiomatic framework. This is done in Section 6. In the same section a comparison between the consensus gap and the Cook–Seiford distance is also made from an axiomatic/theoretical point of view. The paper ends with Section 7 which summarizes some key observations regarding this important problem.

2. Basic notation and definitions

We introduce some basic notation and definitions for our discussion that can be found in the literature (Hou, 2015a; 2015b; 2016).

Denote by $\mathscr{A} = \{A_1, A_2, \ldots, A_n\}$ the set with the alternatives to be ranked and by $\{E_1, E_2, \ldots, E_m\}$ the expert group, where $1 < n < +\infty$ and $1 < m < +\infty$ are positive integers. The following assumptions are needed:

1. The experts' preferences are assumed to be ties-permitted ordinal rankings;
2. The alternatives in a tie are assumed to be ranked at identical positions, and these positions are consecutive positive integers.

A ties-permitted ordinal ranking is a weak ordering based on a weak order relation $\preceq$ on the alternative set, some of whose members may be tied with each other. According to the above assumptions, a ties-permitted ordinal ranking can be represented by a preference map (PM), whose components are sets representing the possible ranking positions of the alternatives. Before the concept of a preference map (PM) is introduced formally, two other supporting concepts, first introduced in (Hou, 2015b), are presented next.

Let $\mathscr{A}$ be an alternative set where $\mathscr{A} = \{A_1, A_2, \ldots, A_n\}$ which is ordered according to a weak order relation $\preceq$.

Definition1-a (Hou, 2015b) A sequence $(\xi_i)_{n\times 1}$ is called the *predominance sequence* of the alternative set $\mathscr{A}$ with respect to the order relation $\preceq$, if and only if the following is true: $\xi_i = \{A_k \mid A_k \in \mathscr{A}, A_i \prec A_k\}$.

For instance, let $\mathscr{A} = \{A_1, A_2, A_3, A_4\}$ and the weak order relation on the set $\mathscr{A}$ be: $A_2 \succ A_3 \sim A_4 \succ A_1$. Then the corresponding predominance sequence is as follows:

$$\xi = (\{A_2, A_3, A_4\}, \emptyset, \{A_2\}, \{A_2\})^{\mathrm{T}}.$$

That is, the first entry is the set $\{A_2, A_3, A_4\}$ because the alternatives A_2, A_3 and A_4 are the ones which are "better" (i.e., are related with the relationship "$\succ$") than the first alternative (i.e., A_1). The second entry is the empty set (i.e., $\emptyset$) because no alternative from the set $\mathscr{A}$ is better than the second alternative (i.e., A_2) and so on.

Definition 1-b (Hou, 2015b) A sequence $(\eta_i)_{n\times 1}$ is called the *indifference sequence* of the alternative set $\mathscr{A}$ with respect to the order relation $\preceq$, if and only if the following is true: $\eta_i = \{A_k \mid A_k \in \mathscr{A}, A_i \sim A_k\}$.

When the data of the previous example are used here, then the corresponding indifference sequence is as follows:

$$\eta = (\{A_1\}, \{A_2\}, \{A_3, A_4\}, \{A_3, A_4\})^{\mathrm{T}}.$$

That is, the first entry is the set $\{A_1\}$ because only the alternative A_1 is of equal significance (i.e., relation "$\sim$" holds) when the first alternative (i.e., A_1) is considered, while alternatives A_3 and A_4 are of equal significance to the third alternative (i.e., A_3) and so on.

Now we are ready to formally introduce the notion of the preference map (PM) of the alternative set $\mathscr{A}$ as follows (please note that in Hou (2015b) the term PSV, for preference sequence vector, was used instead of PM):

Definition 1. (Hou, 2015b) A sequence $(\zeta_i)_{n\times 1}$ is called the *preference map* (PM) of the alternative set $\mathscr{A}$ with respect to the order relation $\preceq$, if and only if the following is true: $\zeta_i = \{|\xi_i| + 1, |\xi_i| + 2, \ldots, |\xi_i| + |\eta_i|\}$.

As before, when the data of the previous example are used, then the corresponding PM is as follows:

$\zeta = (\{4\}, \{1\}, \{2, 3\}, \{2, 3\})^{\mathrm{T}}.$

For instance, the first entry of this PM is equal to {4} because $|\xi_1| = |\{A_2, A_3, A_4\}| = 3$ and thus $|\xi_1| + 1 = 3 + 1 = 4$. The second entry is equal to {1}, because $|\xi_2| = |\emptyset| = 0$ and thus $|\xi_2| + 1 = 0 + 1 = 1$, and so on with the rest of the entries of this PM. In summary, given the previous alternative set $\mathscr{A}$ and the way the weak ordering relationship $\preceq$ is defined on this set, the corresponding PM indicates that alternative A_1 is of order 4, alternative A_2 is of order 1, while alternatives A_3 and A_4 cannot be distinguished among themselves and are of order 2 or 3 each.

A PM has the following properties:

Property 1. *(Hou (2015a)) Let $(T_i)_{n\times 1}$ be a PM of a ties-permitted ordinal ranking and $I = \{1, 2, \ldots, n\}$, then*

1. $\forall i\ (T_i \subseteq I \text{ and } T_i \neq \emptyset)$.
2. $\bigcup_{i=1}^{n} T_i = I$.
3. $\forall i, j \in I$, *either* $T_i = T_j$ *or* $T_i \cap T_j = \emptyset$.
4. *The elements of T_i represent the alternative A_i's consecutive positions when $|T_i| > 1$.*
5. *A component T_i appears $|T_i|$ times in the components of $(T_i)_{n\times 1}$.*

We explain the meaning of Item (5) of Property 1 by means of an example. Assume that

$$\begin{bmatrix} \{1\} \\ \{2,3\} \\ \{2,3\} \\ \{4\} \end{bmatrix}$$

is a PM representing a ties-permitted ordinal ranking of $A_1 \succ A_2 \sim A_3 \succ A_4$. Then the component {1} appears $|\{1\}| = 1$ time in this PM, while the component {2, 3} appears $|\{2, 3\}| = 2$ times in the same PM. We emphasize that, the components of a PM are sets of consecutive natural number(s). For instance, the sequence ({1}, {2, 3}, {2, 3}, {4}) is allowable while the sequence ({1}, {2, 4}, {3}, {2, 4}) is not allowable (because the numbers 2 and 4 in {2, 4} are not consecutive).

Because a PM is a sequence whose components are consecutive number sets, the inclusion relation (i.e., the '$\subseteq$' relation) of two PMs and such operations as the intersection (i.e., the '$\cap$' operation) and the union (i.e., the '$\cup$' operation) of two PMs can be defined or manipulated in an component-by-component manner. Moreover, a PM matrix can be deduced whose columns are PMs corresponding to the ties-permitted ordinal rankings provided by the experts.

The consensus gap between two PMs is quantified by an indicator as follows:

Definition 2. (Hou, 2015a) The consensus gap between two PMs, say $T^{(1)} = \left(T_i^{(1)}\right)_{n\times 1}$ and $T^{(2)} = \left(T_i^{(2)}\right)_{n\times 1}$, is defined as follows:

$$\begin{aligned}\Delta(T^{(1)}, T^{(2)}) &= \sum_{i=1}^{n} \delta(T_i^{(1)}, T_i^{(2)}) \\ &= \sum_{i=1}^{n} \max\{0, \min T_i^{(1)} - \max T_i^{(2)}, \min T_i^{(2)} \\ &\quad - \max T_i^{(1)}\}. \end{aligned} \tag{1}$$

It is easy to verify that the consensus gap indicator of Eq. (1) is a premetric rather than a metric (a premetric satisfies only the properties of nonnegativity and symmetry). In this paper, we also call the consensus gap defined by Eq. (1) the disagreement between two PMs.

Moreover, a dispute matrix (also called the consensus gap indicator matrix in Hou (2015a)) which is related to the experts' disagreements on alternatives can be defined as follows:

Definition 3. (Hou, 2015a) Assume that $T^{(j)} = (T_i^{(j)})_{n\times 1}$, $j = 1, 2, \ldots, m$, are the PMs of the experts. The dispute matrix is defined by $DispM = (S_{ik})_{n\times n}$, where

$$S_{ik} = \sum_{j=1}^{m} \delta(T_i^{(j)}, \{k\}) \tag{2}$$

represents the total gap of the experts if alternative A_i is to be ranked at position k.

Consensus and the consensus ranking can also be defined by using PMs.

Definition 4. (Hou (2015b)) Assume that the experts' PMs are $T^{(1)} = (T_i^{(1)})_{n\times 1}, T^{(2)} = (T_i^{(2)})_{n\times 1}, \ldots, T^{(m)} = (T_i^{(m)})_{n\times 1}$. The experts are in consensus if, and only if $\forall i (T_i^{(1)} \cap T_i^{(2)} \cap \ldots \cap T_i^{(m)} \neq \emptyset)$; or, equivalently, $\forall i, j, k (T_i^{(j)} \cap T_i^{(k)} \neq \emptyset)$. Moreover, the consensus ranking is (W_i), where

$$\begin{bmatrix} W_1 \\ W_2 \\ \vdots \\ W_n \end{bmatrix} = \begin{bmatrix} T_1^{(1)} \\ T_2^{(1)} \\ \vdots \\ T_n^{(1)} \end{bmatrix} \bigcap \begin{bmatrix} T_1^{(2)} \\ T_2^{(2)} \\ \vdots \\ T_n^{(2)} \end{bmatrix} \bigcap \cdots \bigcap \begin{bmatrix} T_1^{(m)} \\ T_2^{(m)} \\ \vdots \\ T_n^{(m)} \end{bmatrix} = \begin{bmatrix} \cap_{k=1}^{m} T_1^{(k)} \\ \cap_{k=1}^{m} T_2^{(k)} \\ \vdots \\ \cap_{k=1}^{m} T_n^{(k)} \end{bmatrix}. \tag{3}$$

The following result can be derived from Definition 2:

Property 2. *(Hou, 2015a) Let $T^{(1)}$, $T^{(2)}$ and $T^{(3)}$ be three PMs. If $T^{(2)} \subseteq T^{(3)}$, then,*

$$\Delta(T^{(1)}, T^{(2)}) \geq \Delta(T^{(1)}, T^{(3)}). \tag{4}$$

In Hou (2015a), a procedure for deriving a collective ranking or rankings was proposed by using an assignment model based on the dispute matrix as defined in Definition 3. In order to seek a possibly better solution when the assignment model produces multiple solutions, the procedure takes advantage of the inequality (4) in its last step. This is true because given three PMs, if the union map of two of them is still a PM, then the gap between the union map and the third PM can be less (if not as large as) than that between either of the original two PMs and the third PM. The concepts of consensus and consensus ranking in the consensus case were defined and discussed in Hou (2015b). This procedure has been applied to a fuzzy case (Hou, 2016) and a linguistic case (Hou, 2015b). However, the important topic of how to assist experts when working towards consensus was not touched on in Hou (2015a,b) or Hou (2016). This paper provides a complete approach by introducing some additional measures so as to evaluate the consensus level and to identify the disagreements. Moreover, this paper compares the premetric approach with the Cook–Seiford metric as a main example of the distance-based approach from an axiomatic/theoretical point of view. We also remark that the preference map (PM) was called preference sequence vector (PSV) in the previous literature. We do this change because a PM is not a vector in the mathematical sense since its components are not elements of a linear space.

3. How to evaluate the level of consensus?

We next describe a property for the consensus gap when two experts are in consensus.

Property 3. *Two experts are in consensus if, and only if the consensus gap between their PMs is 0.*

Proof. The proof is described in Section 6.3 when explaining Axiom 1 for the consensus gap. □

Now we construct a disagreement matrix to discern possible disagreements between experts. The disagreements are represented by consensus gaps between each pair of the PMs.

Definition 5. Assume that the experts' PMs are $T^{(1)} = (T_i^{(1)})_{n\times1}, T^{(2)} = (T_i^{(2)})_{n\times1}, \ldots, T^{(m)} = (T_i^{(m)})_{n\times1}$. The disagreement matrix is defined as

$$D = \left(\Delta_{jk}\right)_{m\times m}, \text{ where } \Delta_{jk} = \Delta(T^{(j)}, T^{(k)}). \tag{5}$$

The disagreement matrix is symmetric because we always have $\Delta_{jk} = \Delta_{kj}$ and hence $\forall i(\Delta_{ii} = 0)$. Therefore, sometimes we just need to consider those Δ_{jk} with $j < k$.

The elements of a disagreement matrix defined by Eq. (5) indicate whether a pair of experts is in consensus or not (by using Property 3). We can use this information to introduce an indicator called the group consensus index (GCI) which is able to measure the consensus level of the group.

Definition 6. The group consensus index (GCI) is defined as

$$GCI = \frac{2\sum_{i=1}^{m-1}\sum_{j=i+1}^{m}\rho_{ij}}{m(m-1)}, \tag{6}$$

where

$$\rho_{ij} = \begin{cases} 1, & \text{if } \Delta_{ij} = 0, \\ 0, & \text{otherwise.} \end{cases}$$

The GCI represents the proportion of the number of expert pairs that are in consensus among all possible expert pairs. We have the following results:

- $0 \le GCI \le 1$.
- If $GCI = 0$, then the experts have no consensus with each other.
- If $GCI = 1$, then the experts are in complete consensus.
- The larger the value of GCI is, the more consensus the experts will have among themselves.

The following example will be carried throughout this paper to clarify some key concepts.

Example 1. Assume that a GDM setting involves 6 experts, $\{E_1, E_2, E_3, E_4, E_5, E_6\}$, and 5 alternatives, $\{A_1, A_2, A_3, A_4, A_5\}$. The experts' preferences are assumed to be as

$$\begin{aligned}
E_1 &: \quad A_1 \sim A_2 \succ A_3 \sim A_5 \succ A_4, \\
E_2 &: \quad A_1 \succ A_2 \sim A_3 \succ A_4 \sim A_5, \\
E_3 &: \quad A_1 \sim A_2 \succ A_3 \succ A_4 \sim A_5, \\
E_4 &: \quad A_2 \succ A_1 \succ A_3 \succ A_5 \succ A_4, \\
E_5 &: \quad A_1 \succ A_2 \succ A_3 \succ A_4 \sim A_5, \\
E_6 &: \quad A_3 \succ A_2 \succ A_1 \succ A_4 \succ A_5.
\end{aligned}$$

According to Definition 1, the experts' PMs (column under $T^{(i)}$ corresponds to Expert E_i) are obtained as follows:

$$\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{array}
\begin{array}{c} T^{(1)} \\ \begin{bmatrix} \{1,2\} \\ \{1,2\} \\ \{3,4\} \\ \{5\} \\ \{3,4\} \end{bmatrix} \end{array},
\begin{array}{c} T^{(2)} \\ \begin{bmatrix} \{1\} \\ \{2,3\} \\ \{2,3\} \\ \{4,5\} \\ \{4,5\} \end{bmatrix} \end{array},
\begin{array}{c} T^{(3)} \\ \begin{bmatrix} \{1,2\} \\ \{1,2\} \\ \{3\} \\ \{4,5\} \\ \{4,5\} \end{bmatrix} \end{array},
\begin{array}{c} T^{(4)} \\ \begin{bmatrix} \{2\} \\ \{1\} \\ \{3\} \\ \{5\} \\ \{4\} \end{bmatrix} \end{array},
\begin{array}{c} T^{(5)} \\ \begin{bmatrix} \{1\} \\ \{2\} \\ \{3\} \\ \{4,5\} \\ \{4,5\} \end{bmatrix} \end{array},
\begin{array}{c} T^{(6)} \\ \begin{bmatrix} \{3\} \\ \{2\} \\ \{1\} \\ \{4\} \\ \{5\} \end{bmatrix} \end{array}.$$

By using formulae (1) and (5), we obtain the disagreement matrix as follows:

$$D = (\Delta_{ij})_{6\times6} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 2 & 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 2 & 0 & 4 \\ 5 & 3 & 3 & 6 & 4 & 0 \end{pmatrix}. \tag{*}$$

For illustrative purposes the case of $\Delta_{6,1}$ is described in detail as follows:

$$\begin{aligned}
\Delta_{6,1} &= \Delta(T^{(6)}, T^{(1)}) = \sum_{k=1}^{5}\delta(T_k^{(6)}, T_k^{(1)}) \\
&= \delta(\{3\}, \{1,2\}) + \delta(\{2\}, \{1,2\}) + \delta(\{1\}, \{3,4\}) \\
&\quad + \delta(\{4\}, \{5\}) + \delta(\{5\}, \{3,4\}) \\
&= 1 + 0 + 2 + 1 + 1 = 5.
\end{aligned}$$

The GCI which corresponds to the above disagreement matrix can be computed following formula (6) as $GCI = \frac{8}{15}$.

We remark that the GCI somehow resembles Kendall's tau coefficient which was introduced by Kendall (1938). However, the GCI is different than Kendall's tau as follows: the GCI represents the percentage of the number of expert pairs that are in consensus among all possible expert pairs; while Kendall's tau coefficient is restricted to the case of two experts and represents the percentage of pairwise comparisons of alternatives that both experts agree on. We also remark that the GCI is defined by using a premetric. Moreover, as will be explained in Section 6, it cannot be defined by using a traditional distance metric. One reason, among others, lies in that the traditional distance between two PMs is 0 if and only if their two preference maps are identical. This property renders the traditional distance ineffective for describing the consensus among experts.

4. How to identify the most suitable disagreements to consider next when the consensus is not at a desired level?

The GCI index defined by formula (6) does not reflect the actual disagreement values. In order to improve consensus when the experts are not in complete consensus, it is necessary to introduce indicators pointing at key disagreements.

4.1. Maximum experts' pairwise disagreement

When the experts are not in complete consensus, the question of how to examine the disagreement between experts needs to be studied for improving consensus.

By using the disagreement matrix defined by Definition 5, we introduce the pairwise disagreement index (PDisaI) as follows.

Definition 7. The pairwise disagreement index (PDisaI) is defined as follows:

$$PDisaI = \max_j\{\max_k\{\Delta_{jk} | j < k\}\}. \tag{7}$$

The above measure represents the maximum disagreement among the experts. The condition $j < k$ is a result of the fact that the disagreement matrix is always a symmetric one with the first diagonal comprised of zero values.

In order to point at expert pairs that have the maximum disagreement, we define a special set in the following.

Definition 7' The subscript-pair set of maximum disagreement pairs (MDP) is defined as follows:

$$MDP = \{(j,k) | \Delta_{jk} = PDisaI, \, j < k, \, \Delta_{jk} > 0\}. \tag{7'}$$

To illustrate, we consider the disagreement matrix marked by (*) in Section 3. By formula (7), we know that $PDisaI = 6$. By formula (7'), we know that $MDP = \{(4,6)\}$, which indicates that experts 4 and 6 have the maximum disagreement.

According to Definition 2, Property 3, Eqs. (5), (7), and (7′), we have the following result:

Proposition 1. *We have $PDisaI = 0$ if, and only if there is complete consensus among the experts with regard to every alternative; If $PDisaI \neq 0$, the disagreement index indicates the value of the largest disagreement, and MDP indicates the corresponding pair(s) of experts that have it. That is, if $PDisaI > 0$ and $(i^*, j^*) \in MDP$, then the largest disagreement is between experts E_{i^*} and E_{j^*}.*

4.2. Maximum dispute on alternatives

To recognize on which alternative or on which sub-group of alternatives the experts have the most dispute, we introduce the maximum dispute index (MDispI) on alternatives.

One objective of the GDM problem considered in this paper is how to rank the alternatives. At the beginning, we do not know the collective ranking. However, we can calculate the total gap for each alternative when ranking it at position i, for $i = 1, 2, \ldots, n$. Thus, as defined by Definition 3, we can obtain the dispute matrix $DispM = (S_{ik})_{n\times n}$ where S_{ik} represents the total gap of the experts if alternative A_i is to be ranked at position k.

Clearly, we have $S_{ik} \geq 0$, and, if $S_{ik} = 0$ then ranking A_i at position k is acceptable by all the experts; otherwise, some (at least one) experts disagree with A_i's ranking at position k. Therefore, we can check whether $\min_k\{S_{ik}\}$ is 0 to know whether the experts would like to rank alternative A_i at a common position. Generally speaking, the maximum of the alternatives' dispute is meaningful, because this index provides intuitive knowledge on whether or not there exist some dispute when the experts rank the alternatives, and at what level when the dispute exists. Next, we introduce the following definition.

Definition 8. The maximum dispute index (MDispI) is defined as follows:

$$MDispI = \max_i \min_k \{S_{ik}\}. \tag{8}$$

In order to point at alternatives on which the experts have the maximum dispute, we define a set in the following:

Definition 8' The subscript set of maximum dispute alternatives (MDA) is defined as follows:

$$MDA = \{i | S_{ik} = MDispI, S_{ik} > 0\}. \tag{8′}$$

To illustrate, we consider Example 1 of which the experts' PMs are presented in Section 3. Using formulae (1) and (2), we obtain the dispute matrix as

	{1}	{2}	{3}	{4}	{5}
A_1	3	3	7	13	19
A_2	3	1	6	12	18
A_3	9	5	2	7	13
A_4	20	14	8	2	1
A_5	18	12	6	1	2

From this matrix one can see that the maximum dispute index (MDispI) is as follows: $MDispI = 3$, and $MDA = \{1\}$. Therefore the maximum dispute happens for alternative A_1.

4.3. The consensus evaluation sequence: a sequence carrying indicators related to consensus and disagreements

In order to achieve consensus, we need to examine the group consensus level, and then (if the consensus level is not at an acceptable value) identify which pair of the experts have disagreements (especially the most disagreement), as well as identify on which alternative the experts have dispute (especially the alternative with the most dispute) so as to improve consensus. As previously described, we have proposed and discussed several novel indicators related to consensus and disagreement, which can be summarized in the following:

- The Group Consensus Index (GCI): a measure for examining the group consensus level.
- The Maximum Disagreement Pairs (MDP) and the Pairwise Disagreement Index (PDisaI): indicators pointing at which pair of experts have the maximum disagreement and at what level.
- The Maximum Dispute Alternatives (MDA) and the Maximum Dispute Index (MDispI): indicators focusing on the maximum disagreement from the alternative's point of view. Namely, on which alternative the experts have maximum dispute regarding its ranking and at what level.

The above indicators are not totally independent of each other. For instance, if the group is in complete consensus (i.e., $GCI = 1$), then the maximum disagreement between the experts will be 0 (i.e., $PDisaI = 0$), and, the maximum dispute on alternatives will also be 0 (i.e., $MDispI = 0$). However, these indicators do reflect various aspects associated with consensus and disagreements. A desirable indicator should carry as much relevant information as possible. However, it is almost impossible to include all the information in a single indicator. Therefore, we propose to use a sequence called the consensus evaluation sequence (CES) with more than one indicator as its elements. This concept is defined next.

Definition 9. The consensus evaluation sequence (CES) is defined as follows:

$$CES = [GCI; MDP, PDisaI; MDA, MDispI]. \tag{9}$$

To illustrate, we consider Example 1. The indicators associated with the CES have been obtained previously as follows:

$$\begin{aligned} &GCI = \frac{8}{15}, && \text{(in Section 3)};\\ &MDP = \{(4,6)\}, PDisaI = 6, && \text{(in subsection 4.1)};\\ &MDA = \{1\}, MDispI = 3, && \text{(in subsection 4.2)}. \end{aligned}$$

Therefore, the CES is:

$$CES = \Big[GCI = \frac{8}{15}; MDP = \{(4,6)\}, \\ PDisaI = 6; MDA = \{1\}, MDispI = 3\Big].$$

From this CES one can deduce the following: first, currently the group is not in complete consensus (because the group consensus level is less than 1); second, the pair of experts 4 and 6 have the maximum disagreement equal to 6; and third, the experts have the most dispute on alternative A_1 and the dispute value is 3.

5. How to derive a reasonably 'close' solution?

In this section we discuss how to elicit collective ranking(s) from the experts' PMs. When $GCI = 1$, or equivalently $PDisaI = 0$, the experts have reached complete consensus. In this case, a collective ranking can be easily elicited by formula (3). In the case of not being in consensus, we next propose an iterative process based on the indices constructed in Sections 3 and 4.

5.1. An iterative process

In a GDM or MCDM problem (for more on Multi-Criteria Decision Making (MCDM), please refer to Triantaphyllou (2000)), if we just need the final ranking, then the problem is an assignment problem to determine which alternative should be assigned to the first place, which alternative to the second, and so on. Thus the assignment model has an intrinsic value to be applied to a

GDM or an MCDM problem. The assignment model should optimize a measurement in some sense to get an optimal solution. In the approaches described in Hou (2015a, 2016)), the dispute matrix (defined by Definition 3) is used as the assignment matrix and the total consensus gap of a potential solution from the experts' PMs is minimized.

When not being in complete consensus, we propose in this paper a novel iterative process within which the indices constructed in Sections 3 and 4 are used for evaluating consensus and for identifying disagreements, and an assignment model similar to the one in Hou (2015a) is used for deducing a compromised collective ranking.

To improve consensus, one may focus on experts' pairwise disagreements only or on alternatives' disputes only or on both. Also, people may choose the maximum, the minimum or just a random value from the existing disagreements to decide which ranking to re-evaluate (disagreement to consider). Which strategy to consider depends on the coordinator/facilitator's recognition of the nature of the group decision making (GDM) problem under consideration and it can change as the GDM process evolves. For instance, it might be a good idea when there are too many arguments and one may want to start with the least contentious case, just to build some trust between the experts and thus initiate a consensus oriented process. Therefore, there can be a wide spectrum of different ways to approach the problem of reaching consensus, and this novel aspect provides the needed flexibility that the GDM process needs to be truly effective. One such procedure is elaborated next.

Step 1: Transform the experts' preferences into PMs. Ask the group to provide a threshold value of GCI for their acceptable consensus level. If the group does not provide any threshold value of GCI, we just take the value 1 (the possible maximum level of group consensus) as the threshold.

Step 2: Construct the disagreement matrix and compute the disagreement index (*PDisaI*).

Step 3: If $PDisaI = 0$ or if the *GCI* of the current experts preferences is not less than the threshold value, goto Step 4; else, select a Δ_{ij} with $i < j$ and $\Delta_{ij} > 0$. We ask the experts related to the selected Δ_{ij} to modify their preferences by means of negotiation. If the two experts do not agree to modify their rankings, then move on to another pair of experts (if any) with $\Delta_{ij} > 0$ and $i < j$. If none of these experts is willing to change his/her ranking, then goto Step 4; otherwise, after getting updated preferences, goto Step 1.

Step 4: If $PDisaI = 0$, use formula (3) to obtain the collective solution. Else, establish an assignment model (AM1) (similar to the one in Hou, 2015a) and solve it:

$$(AM1)\quad \begin{aligned} \min \quad & f = \sum_i \sum_k x_{ik} S_{ik} \\ s.t. \quad & \sum_{i=1}^{n} x_{ik} = 1, \quad \text{for all } k, \\ & \sum_{k=1}^{n} x_{ik} = 1, \quad \text{for all } i, \\ & x_{ik} = 0, 1, \quad \text{for all } i, k, \end{aligned}$$

where S_{ik} is calculated by formula (2) representing the disagreement of the experts if alternative A_i is to be ranked at position k. The objective function of the model AM1 minimizes the summation of all gaps (disagreements).

Two cases are possible:

A. If the model produces a unique solution, then the considered GDM problem has a unique solution and the solution's corresponding ranking is taken as the collective ranking.
B. If the model produces multiple solutions, then examine all possible union maps of the multiple solutions (the union of group of solutions, union of subgroup of solutions, and so on) and find out the best as the collective solution. The number of all such subgroups is equal to 2^n, where n is the number of alternative solutions. However, the value of n is rather a small number in such problems and thus this exhaustive search is feasible. Here, 'best' refers to a PM that has the minimal total gap to the experts' preferences. We emphasize that a union map should first be a PM. We also remark that the rationality of Item (B) comes from formula (4).

5.2. An illustrative example

For illustrative purposes, we apply the previous procedure to Example 1 which was presented in Section 3. It is assumed that no GCI threshold is provided. Next we carry out the proposed iterative procedure to elicit the best possible collective ranking. In order to make the process relatively simple, in this illustrative example we just focus on the pairwise disagreements and select only a possible maximum value to consider next.

Iteration #1

Transform the experts' preferences into PMs as presented in Section 3.

By using Eq. (5) we obtain the disagreement matrix as shown in Section 3 which is marked by (*). From this matrix we get the pairwise disagreement index (*PDisaI*, indicated by Eq. (7)) $PDisaI = 6$, and we recognize that the experts' preferences are not in consensus. Currently the CES is:

$$\left[GCI = \frac{8}{15}; MDP = \{(4, 6)\}, PDisaI = 6; MDA = \{1\}, MDispI = 3\right].$$

Because experts 4 and 6 have the maximum disagreement we ask experts 4 and 6 to modify their preferences by means of negotiation. It should be stated here that the coordinator/facilitator of the GDM process may choose a different pair of experts to seek a re-negotiated ranking among experts. Instead of focusing on the pair of experts with the highest disagreement, a pair of experts with the lowest disagreement may be chosen instead. This may be a good idea if there are issues of high personal antagonism and one does not wish to start with the hardest step first, but instead start with a rather simple case to build trust among the experts and create a positive momentum towards finding a good consensus. Another option is to select a pair of experts randomly as long as they have some disagreement in their rankings. As a different strategy one may start by identifying the alternative with the highest total disagreements and so on. The various options discussed in Section 4 are relevant for this point of the GDM process. This novel aspect provides the needed flexibility that the GDM process needs to be truly effective. After their discussion, assume that we get their updated preferences as follows

$E_4: A_1 \succ A_2 \succ A_3 \succ A_5 \succ A_4,$
$E_6: A_1 \succ A_2 \succ A_3 \succ A_4 \succ A_5.$

Iteration #2

Now the experts' PMs are as follows:

	E_1	E_2	E_3	E_4	E_5	E_6
A_1	$\{1, 2\}$	$\{1\}$	$\{1, 2\}$	$\{1\}$	$\{1\}$	$\{1\}$
A_2	$\{1, 2\}$	$\{2, 3\}$	$\{1, 2\}$	$\{2\}$	$\{2\}$	$\{2\}$
A_3	$\{3, 4\}$	$\{2, 3\}$	$\{3\}$	$\{3\}$	$\{3\}$	$\{3\}$
A_4	$\{5\}$	$\{4, 5\}$	$\{4, 5\}$	$\{5\}$	$\{4, 5\}$	$\{4\}$
A_5	$\{3, 4\}$	$\{4, 5\}$	$\{4, 5\}$	$\{4\}$	$\{4, 5\}$	$\{5\}$

,

from which the new disagreement matrix is calculated to be as follows:

$$D = (\Delta_{ij})_{6\times 6} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 2 & 0 & 0 \end{pmatrix}.$$

From this matrix it can be concluded that the current expert preferences are still not in consensus. But now the GCI has been improved to $GCI = 13/15$. Currently the CES is:

$$\Big[GCI = \frac{13}{15}; MDP = \{(1,6), (4,6)\},$$
$$PDisaI = 2; MDA = \{4,5\}, MDispI = 1\Big].$$

As before, because experts 1 and 6 have the maximum disagreement we ask experts 1 and 6 to re-evaluate their views and we assume that their updated preferences are:

$E_1: \ A_1 \sim A_2 \succ A_3 \succ A_4 \sim A_5,$
$E_6: \ A_1 \succ A_2 \succ A_3 \succ A_4 \succ A_5.$

- *Iteration #3*

The experts' PMs are now updated and are as follows:

	E_1	E_2	E_3	E_4	E_5	E_6
A_1	$\{1,2\}$	$\{1\}$	$\{1,2\}$	$\{1\}$	$\{1\}$	$\{1\}$
A_2	$\{1,2\}$	$\{2,3\}$	$\{1,2\}$	$\{2\}$	$\{2\}$	$\{2\}$
A_3	$\{3\}$	$\{2,3\}$	$\{3\}$	$\{3\}$	$\{3\}$	$\{3\}$
A_4	$\{4,5\}$	$\{4,5\}$	$\{4,5\}$	$\{5\}$	$\{4,5\}$	$\{4\}$
A_5	$\{4,5\}$	$\{4,5\}$	$\{4,5\}$	$\{4\}$	$\{4,5\}$	$\{5\}$

Now the disagreement matrix is

$$D = (\Delta_{ij})_{6\times 6} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \end{pmatrix}, \qquad (**)$$

which indicates that experts 4 and 6 still have disagreement. Currently, the GCI is $GCI = 14/15$, and the CES is:

$$\Big[GCI = \frac{14}{15}; MDP = \{(4,6)\}, PDisaI = 2; MDA = \{4,5\}, MDispI = 1\Big].$$

It is assumed that none of them is willing to further modify his/her preference. A compromise solution can thus be deduced as the collective ranking order in the next step.

- *Finding a compromise solution:*

Based on the up-to-the-minute experts' PMs (i.e., the PMs in the disagreement matrix marked by (**)), the assignment matrix is computed by using Eq. (2):

$$(S_{ik})_{5\times 5} = \begin{pmatrix} 0 & 4 & 10 & 16 & 22 \\ 4 & 0 & 5 & 11 & 17 \\ 11 & 5 & 0 & 6 & 12 \\ 19 & 13 & 7 & 1 & 1 \\ 19 & 13 & 7 & 1 & 1 \end{pmatrix}. \qquad (***)$$

For illustrative purposes the case of $S_{2,3}$ is described in detail as follows:

$$S_{2,3} = \sum_{j=1}^{6} \delta(T_2^{(j)}, \{3\})$$
$$= \delta(\{1,2\},\{3\}) + \delta(\{2,3\},\{3\}) + \delta(\{1,2\},\{3\}) + \delta(\{2\},\{3\})$$
$$+ \delta(\{2\},\{3\}) + \delta(\{2\},\{3\}).$$

By Eq. (1), we obtain $S_{2,3} = 1+0+1+1+1+1 = 5$.

Establish the optimization model (AM1) based upon the above matrix $(S_{ik})_{5\times 5}$. It is shown next as follows:

$$(AM1) \quad \begin{array}{ll} \min & f = \sum_{i=1}^{5}\sum_{k=1}^{5} x_{ik}S_{ik} \\ s.t. & \sum_{i=1}^{5} x_{ik} = 1, \quad \text{for } k = 1\ldots 5, \\ & \sum_{k=1}^{5} x_{ik} = 1, \quad \text{for } i = 1\ldots 5, \\ & x_{ik} = 0, 1, \quad \text{for } i = 1\ldots 5, k = 1\ldots 5. \end{array}$$

Solving this model (for instance, by using the Hungarian Method), we obtain the following two solutions:

- $\mathbf{\Lambda}^{(1)^T} = (\{1\}, \{2\}, \{3\}, \{4\}, \{5\})^T$,
- $\mathbf{\Lambda}^{(2)^T} = (\{1\}, \{2\}, \{3\}, \{5\}, \{4\})^T$.

The total disagreement between each of the above two solutions and the experts' PMs is 2. Their union map is $(\{1\}, \{2\}, \{3\}, \{4, 5\}, \{4, 5\})^T$ and it can be checked by Property 1 that it is also a PM. From formula (4), we know that the union PM may be better than either of the multiple solutions. Indeed, the union map is better as a result of the fact that its maximal disagreement between the experts preferences is 0. Thus, we take the union map as the collective PM, which indicates a compromise collective ranking as follows:

$A_1 \succ A_2 \succ A_3 \succ A_4 \sim A_5.$

By the way, if the Cook–Seiford approach is applied to the above example, it will terminate at our first iteration as a result of the fact that it is a single-iteration approach. The Cook–Seiford approach will produce a collective ranking as $A_1 \sim A_2 \succ A_3 \succ A_4 \sim A_5$ with the Cook–Seiford distance being equal to 12. If a single-iteration approach (such as the Cook–Seiford approach) is applied, most likely it would produce inferior results because it does not provide an opportunity for the experts to revise some of their rankings in an effort to reach a better consensus level. The proposed iterative approach has a mechanism to strategically identify ways to revise rankings of alternatives towards, hopefully, a better collective consensus. This was demonstrated in this illustrative example.

6. Theoretical foundation of the proposed premetric-based approach and comparison

For deriving collective rankings from the experts' ordinal preferences, a type of method is the distance-based approach aiming to find an optimal ranking that is 'closest' to the experts' preferences in a minimum distance sense. This kind of approach was initially advanced by Kemeny and Snell (1962), and was later adopted by some other researchers (Armstrong et al., 1982; Blin, 1976; Cook & Seiford, 1978). Cook and Seiford (1978) and Cook (2006) have presented a set of axioms for approaches of this kind.

In the proposed procedure of this paper, a consensus gap is minimized to derive a collective ranking. The consensus gap indicator is a premetric rather than a distance metric (Hou, 2015a). In this section, we make a comparison between the consensus gap and the Cook–Seiford distance from an axiomatic point of view. For convenience of the comparison, here we call our approach *the premetric-based approach*. We shall first put the consensus gap indicator into an axiomatic system, and we then make a comparison and present a discussion.

We remark that *in this section* ***we reproduce some words directly from the literature*** Cook (2006) *just for the purpose of comparison.* We also remark that the comparison we make here is different from the one in Hou (2015a). Hou (2015a) just made a comparison by means of a simple example between the computational result produced by his method and the result produced by the Cook-Seiford method. The objective of the comparison in Hou (2015a) is to show the similarities. However, our comparison here is made from an axiomatic point of view, and we focus on the difference of the preference representations and the difference of the two axiom systems. The objective of our comparison is to show the differences and highlight some advantages of the proposed premetric-based approach.

6.1. Preference representations: preference map (PM) and the Cook–Seiford vector (CSV)

In the Cook–Seiford distance-based approach (Cook & Seiford, 1978), an ordinal ranking is represented by a Cook–Seiford vector (CSV). In the proposed premetric-based approach, an ordinal ranking is represented by a preference map (PM, see Definition 1). A CSV assigns a common single number to all the alternatives in a tie, and this number is the middle position of the tie in the ranking; while a PM assigns all possible positions to each alternative in a tie, and these possible positions are represented by positive and consecutive integer numbers.

To illustrate, assume that $A_1 \succ A_2 \sim A_3 \succ A_4 \sim A_5 \sim A_6 \succ A_7$ is a ranking order on 7 alternatives. Its corresponding CSV and PM are respectively given by the following two column vectors:

$$\begin{array}{c} \\ A_1\\ A_2\\ A_3\\ A_4\\ A_5\\ A_6\\ A_7 \end{array} \begin{array}{c} CSV \\ \begin{bmatrix} 1\\ 2.5\\ 2.5\\ 5\\ 5\\ 5\\ 7 \end{bmatrix} \end{array}, \text{ and } \begin{array}{c} PM \\ \begin{bmatrix} \{1\}\\ \{2,3\}\\ \{2,3\}\\ \{4,5,6\}\\ \{4,5,6\}\\ \{4,5,6\}\\ \{7\} \end{bmatrix} \end{array}.$$

Evidently, if $(v_i)_{n\times1}$ is a CSV, then $v_i \in \{1, 1.5, 2, 2.5, 3, 3.5, \ldots, \frac{2n-1}{2}, n\}$. If $(T_i)_{n\times1}$ is a PM, then T_i is a set containing number(s) from $\{1, 2, \ldots, n\}$ and if $|T_i| > 1$ then T_i is a set containing consecutive numbers from $\{1, 2, \ldots, n\}$.

6.2. The Cook–Seiford distance between CSVs and the consensus gap between PMs

Let $CS^{(1)} = (v_i^{(1)})_{n\times1}$ and $CS^{(2)} = (v_i^{(2)})_{n\times1}$ be two CSVs. The Cook–Seiford distance is defined as follows (Cook & Seiford, 1978):

$$d(CS^{(1)}, CS^{(2)}) = \sum_{i=1}^{n} |v_i^{(1)} - v_i^{(2)}|. \tag{10}$$

Let $T^{(1)} = (T_i^{(1)})_{n\times1}$ and $T^{(2)} = (T_i^{(2)})_{n\times1}$ be two PMs. The consensus gap is calculated as follows (Definition 2):

$$\Delta(T^{(1)}, T^{(2)}) = \sum_{i=1}^{n} \max\{0, \min T_i^{(1)} - \max T_i^{(2)}, \min T_i^{(2)} - \max T_i^{(1)}\}. \tag{11}$$

Clearly, for *complete rankings* (i.e., rankings without ties as defined by Cook & Seiford, 1978), the consensus gap is identical to the Cook–Seiford distance. However, this assertion is not valid for ties-included ordinal rankings.

6.3. Axioms for the consensus gap and axioms for the Cook–Seiford distance

Let $CS^{(1)}$, $CS^{(2)}$ and $CS^{(3)}$ be three Cook-Seiford vectors (CSVs). Cook and Seiford (1978) and Cook (2006) present a set of axioms for the distance-based approach as follows:

- *Axiom 1:* (Nonnegativity) $d(CS^{(1)}, CS^{(2)}) \geq 0$ with equality iff $\forall i (v_i^{(1)} - v_i^{(2)} = 0)$, where $CS^{(1)} = (v_i^{(1)})_{n\times1}$ and $CS^{(2)} = (v_i^{(2)})_{n\times1}$.
- *Axiom 2:* (Symmetry) $d(CS^{(1)}, CS^{(2)}) = d(CS^{(2)}, CS^{(1)})$.
- *Axiom 3:* (Triangle inequality) $d(CS^{(1)}, CS^{(3)}) \leq d(CS^{(1)}, CS^{(2)}) + d(CS^{(2)}, CS^{(3)})$ for any three CSVs.
- *Axiom 4:* (Invariance) $d(CS^{(1)}, CS^{(2)}) = d(\widetilde{CS}^{(1)}, \widetilde{CS}^{(2)})$, where $\widetilde{CS}^{(1)}$ and $\widetilde{CS}^{(2)}$ result from $CS^{(1)}$ and $CS^{(2)}$ respectively by the same permutation of the alternatives in each case.
- *Axiom 5:* (Lifting from n to $(n+1)$-dimensional space) If $\underline{CS}^{(1)}$ and $\underline{CS}^{(2)}$ result from $CS^{(1)}$ and $CS^{(2)}$ by listing the same $(n+1)$st alternative in the last place, then $d(CS^{(1)}, CS^{(2)}) = d(\underline{CS}^{(1)}, \underline{CS}^{(2)})$.
- *Axiom 6:* (Scaling) The minimum positive distance is 1.

Let $T^{(1)}$, $T^{(2)}$ and $T^{(3)}$ be three preference maps (PMs). The consensus gap also satisfies a set of axioms similar to the above (but definitely have some differences which will be discussed in Section 6.4):

- *Axiom 1:* (Nonnegativity) $\Delta(T^{(1)}, T^{(2)}) \geq 0$ with equality iff $\forall i (T_i^{(1)} \cap T_i^{(2)} \neq \emptyset)$, where $T^{(1)} = (T_i^{(1)})_{n\times1}$ and $T^{(2)} = (T_i^{(2)})_{n\times1}$.
- *Axiom 2:* (Symmetry) $\Delta(T^{(1)}, T^{(2)}) = \Delta(T^{(2)}, T^{(1)})$.
- *Axiom 3:* (Constrained triangle inequality)
  - (a) $\Delta(T^{(1)}, T^{(3)}) \leq \Delta(T^{(1)}, T^{(2)}) + \Delta(T^{(2)}, T^{(3)})$, where $T^{(1)}$, $T^{(2)}$ and $T^{(3)}$ are three PMs of three complete rankings.
  - (b) $\Delta(T^{(1)}, T^{(3)}) \leq \Delta(T^{(1)}, T^{(2)}) + \Delta(T^{(2)}, T^{(3)})$, where $T^{(1)}$, $T^{(2)}$ and $T^{(3)}$ are three PMs and $T^{(2)} \subseteq T^{(3)}$. In addition, because we have $\Delta(T^{(2)}, T^{(3)}) = 0$ under the condition of $T^{(2)} \subseteq T^{(3)}$, thus, this constrained inequality can be shortened as: $\Delta(T^{(1)}, T^{(3)}) \leq \Delta(T^{(1)}, T^{(2)})$ where $T^{(2)} \subseteq T^{(3)}$.
- *Axiom 4:* (Invariance) $\Delta(T^{(1)}, T^{(2)}) = \Delta(\widetilde{T}^{(1)}, \widetilde{T}^{(2)})$, where $\widetilde{T}^{(1)}$ and $\widetilde{T}^{(2)}$ result from $T^{(1)}$ and $T^{(2)}$ respectively by the same permutation of the alternatives in each case.
- *Axiom 5:* (Lifting from n to $(n+1)$-dimensional space) If $\underline{T}^{(1)}$ and $\underline{T}^{(2)}$ result from $T^{(1)}$ and $T^{(2)}$ by listing the same $(n+1)$st alternative in the last place, then $\Delta(T^{(1)}, T^{(2)}) = \Delta(\underline{T}^{(1)}, \underline{T}^{(2)})$.
- *Axiom 6:* (Scaling) The minimum positive gap between two PMs is equal to 2.

Next we give some explanations for the Axioms which are consistent with the proposed notion of consensus gap.

1. Regarding Axiom 1, $\Delta(T^{(1)}, T^{(2)}) \geq 0$ comes directly from the gap's definition which takes a form of $\max\{0, \ldots\}$. The sufficient and necessary condition is deduced from the gap's definition and the fact that the components of a PM are sets containing consecutive and positive integer numbers: If $T_i^{(1)} \cap T_i^{(2)} \neq \emptyset$, we have $\min T_i^{(1)} - \max T_i^{(2)} \leq 0$ and $\min T_i^{(2)} - \max T_i^{(1)} \leq 0$, and thus we have $\max\{0, \min T_i^{(1)} - \max T_i^{(2)}, \min T_i^{(2)} - \max T_i^{(1)}\} = 0$; Conversely, if $\max\{0, \min T_i^{(1)} - \max T_i^{(2)}, \min T_i^{(2)} - \max T_i^{(1)}\} = 0$, then we have $\min T_i^{(1)} - \max T_i^{(2)} \leq 0$ and $\min T_i^{(2)} - \max T_i^{(1)} \leq 0$ and we thus know that $T_i^{(1)} \cap T_i^{(2)} \neq \emptyset$ as a result of the fact that $T_i^{(1)}$ and $T_i^{(2)}$ are sets containing consecutive and positive integer numbers from $\{1, 2, \ldots, n\}$.
2. Regarding Axiom 2, it is a reasonable one because we have
$$\Delta(T^{(2)}, T^{(1)}) = \sum_{i=1}^{n} \max\{0, \min T_i^{(2)} - \max T_i^{(1)}, \min T_i^{(1)} - \max T_i^{(2)}\} = \Delta(T^{(1)}, T^{(2)}).$$
3. Regarding Axiom 3, we next provide more detail.
   - (a) If $T^{(1)}$, $T^{(2)}$ and $T^{(3)}$ are three PMs of three complete rankings, then as shown in Section 6.2, the consensus gap

calculated by formula (11) is identical to the Cook–Seiford distance calculated by formula (10). Therefore, the first part of Axiom 3 is satisfied by the consensus gap.

(b) If $T^{(2)} \subseteq T^{(3)}$, i.e., $\forall i(T_i^{(2)} \subseteq T_i^{(3)})$, then we know $\min T_i^{(2)} \geq \min T_i^{(3)}$ and $\max T_i^{(2)} \leq \max T_i^{(3)}$, since $T_i^{(2)}$ and $T_i^{(3)}$ are sets containing consecutive and positive integer numbers. Thus we have:

$$\min T_i^{(1)} - \max T_i^{(3)} \leq \min T_i^{(1)} - \max T_i^{(2)}, \text{ and,}$$

$$\min T_i^{(3)} - \max T_i^{(1)} \leq \min T_i^{(2)} - \max T_i^{(1)}.$$

Therefore we have:

$$\max\{0, \min T_i^{(1)} - \max T_i^{(3)}, \min T_i^{(3)} - \max T_i^{(1)}\}$$
$$\leq \max\{0, \min T_i^{(1)} - \max T_i^{(2)}, \min T_i^{(2)} - \max T_i^{(1)}\}.$$

Hence we have $\Delta(T^{(1)}, T^{(3)}) \leq \Delta(T^{(1)}, T^{(2)})$ under the condition of $T^{(2)} \subseteq T^{(3)}$. Further, under the condition of $T^{(2)} \subseteq T^{(3)}$, we know that $\Delta(T^{(1)}, T^{(3)}) \leq \Delta(T^{(1)}, T^{(2)})$ can be written as $\Delta(T^{(1)}, T^{(3)}) \leq \Delta(T^{(1)}, T^{(2)}) + \Delta(T^{(2)}, T^{(3)})$, because we have, from Axiom 1, $\Delta(T^{(2)}, T^{(3)}) = 0$ under the condition of $T^{(2)} \subseteq T^{(3)}$ hence $\forall i(T_i^{(2)} \cap T_i^{(3)} \neq \emptyset)$. Hence the second part of Axiom 3 is satisfied by the consensus gap.

4. Regarding Axiom 4, it is a reasonable one as a result of the fact that the gap defined by formula (1) or (11) between two PMs depends upon only the alternatives' rankings rather than the alternatives' labels.
5. Regarding Axiom 5, from formula (11) we have:

$$\Delta(\underline{T}^{(1)}, \underline{T}^{(2)}) = \Delta(T^{(1)}, T^{(2)}) + \max\{0, \min T_{n+1}^{(1)} - \max T_{n+1}^{(2)}, \min T_{n+1}^{(2)} - \max T_{n+1}^{(1)}\}.$$

Since A_{n+1} is listed both at the $(n+1)^{\text{st}}$ places of $\underline{T}^{(1)}$ and $\underline{T}^{(2)}$, thus we have $\max\{0, \min T_{n+1}^{(1)} - \max T_{n+1}^{(2)}, \min T_{n+1}^{(2)} - \max T_{n+1}^{(1)}\} = 0$. Hence we know $\Delta(\underline{T}^{(1)}, \underline{T}^{(2)}) = \Delta(T^{(1)}, T^{(2)})$.

6. Regarding Axiom 6, it is a reasonable one because the components of a PM contain only integer numbers. A minimum gap value can happen when there are only two alternatives for which at most three different PMs can be constructed. They are:

$$PM^{(1)} = \begin{bmatrix} \{1\} \\ \{2\} \end{bmatrix}, PM^{(2)} = \begin{bmatrix} \{1,2\} \\ \{1,2\} \end{bmatrix}, \text{ and } PM^{(3)} = \begin{bmatrix} \{2\} \\ \{1\} \end{bmatrix}.$$

By using Eq. (1), the consensus gaps between the above three PMs are $\Delta(PM^{(1)}, PM^{(2)}) = 0$, $\Delta(PM^{(2)}, PM^{(3)}) = 0$, and $\Delta(PM^{(1)}, PM^{(3)}) = 2$. Hence Axiom 6 is validated.

6.4. Comparison and discussion

The main difference between the Cook–Seiford distance and the consensus gap lies in Axioms 1 and 3. We discuss this below and disclose some advantages of the premetric-based method.

1. The Cook–Seiford distance is a metric because it satisfies a metric's requirements of *nonnegativity*, *symmetry* and *triangle inequality*. However, the consensus gap is just a premetric (an 'almost' metric) because it just satisfies two of the requirements for a metric, namely, the non-negativity and the symmetry requirement.
2. The Cook–Seiford distance between two CSVs is 0 if, and only if, the two CSVs are identical. Mathematically, for two CSVs $CS^{(1)} = (v_i^{(1)})_{n\times1}$ and $CS^{(2)} = (v_i^{(2)})_{n\times1}$, we have $d(CS^{(1)}, CS^{(2)}) = 0$ if and only if $\forall i(v_i^{(1)} - v_i^{(2)} = 0)$.
The consensus gap between two PMs is 0 if, and only if, the two PMs are in consensus (Definition 4). Mathematically, for two PMs $T^{(1)} = (T_i^{(1)})_{n\times1}$ and $T^{(2)} = (T_i^{(2)})_{n\times1}$, we have $\Delta(T^{(1)}, T^{(2)}) = 0$ if and only if $\forall i(T_i^{(1)} \cap T_i^{(2)} \neq \emptyset)$.
3. The Cook–Seiford distance satisfies the triangle inequality, while the triangle inequality does NOT always hold for the consensus gap. For instance, we consider the three rankings, $A_1 \succ A_2$, $A_1 \sim A_2$ and $A_2 \succ A_1$. Their corresponding PMs are $T^{(1)} = (\{1\}, \{2\})$, $T^{(2)} = (\{1,2\}, \{1,2\})$ and $T^{(3)} = (\{2\}, \{1\})$, respectively. Using Formula (11), we have $\Delta(T^{(1)}, T^{(3)}) = 2$, $\Delta(T^{(1)}, T^{(2)}) = 0$ and $\Delta(T^{(2)}, T^{(3)}) = 0$. Therefore,

$$\Delta(T^{(1)}, T^{(3)}) > \Delta(T^{(1)}, T^{(2)}) + \Delta(T^{(2)}, T^{(3)}),$$

which indicates that the triangle inequality does not hold for the considered instance. However, not always satisfying the triangle inequality is not a disadvantage of the consensus gap. Indeed, it is in accordance with the consensus intransitivity in GDM. This will be well illustrated next by an example.

It is the above differences that guarantee an important fact that the premetric-based approach is superior to the distance-based approach as shown below:

1. The consensus gap is an indicator capable of describing consensus in GDM: if the experts rank an alternative in some common positions, then the experts have consensus with regard to this alternative. That is, no gap happens to the experts' preferences; otherwise, a consensus gap would be observed. Definition 4 describes this kind of consensus by examining whether the intersection of the components in the same location of the experts' PMs is empty or not.
For example, let us consider three persons, Ann, Bob and Cathy, whose preferences over two alternatives, coffee (A_1) and tea (A_2), are provided as follows:
   - Ann prefers coffee to tea.
   - Bob prefers both coffee and tea.
   - Cathy prefers tea to coffee.

   Obviously, Ann and Bob have consensus, Bob and Cathy also have consensus, while Ann and Cathy do not have consensus.
Now we use the notion of the consensus gap to describe consensus. The persons' preferences can be represented by PMs as follows.
   - Ann's preference is represented by a PM as $T^{(1)} = (\{1\}, \{2\})$.
   - Bob's preference is represented by a PM as $T^{(2)} = (\{1,2\}, \{1,2\})$.
   - Cathy's preference is represented by a PM as $T^{(3)} = (\{2\}, \{1\})$.

   According to Formula (11), we obtain the consensus gaps between all the pairs of the previous three hypothetical persons: $\Delta(T^{(1)}, T^{(3)}) = 2$, $\Delta(T^{(1)}, T^{(2)}) = 0$ and $\Delta(T^{(2)}, T^{(3)}) = 0$. The gap values show that the consensus gap reflects the consensus effectively: Ann and Bob have consensus and thus their preferences have no gap, Bob and Cathy also have consensus and thus their preferences have no gap either, while Ann and Cathy do not have consensus and accordingly their preferences have a consensus gap which can be calculated to be equal to 2.
On the other hand, if one uses the Cook–Seiford vectors and the Cook–Seiford distances for the discussion of the above example, then every pair of the persons' preferences will be measured as some positive distance value as a result of the fact that the persons' preferences are not identical with each other (Axiom 1 for the Cook–Seiford distance).
2. As aforementioned, the consensus gap does not necessarily satisfy the triangle inequality. Not always satisfying the triangle inequality is not a disadvantage of the consensus gap; rather, it is a desirable trait because it reflects the consensus intransitivity in GDM. Consensus intransitivity in GDM refers to a well-known fact that 'if A and B have consensus, and if also B and C have consensus, then A and C may not have consensus'.
For instance, we again consider the 'coffee–tea' example investigated above. As already mentioned, Ann and Bob have

consensus, Bob and Cathy also have consensus, but Ann and Cathy do not have consensus; meanwhile, the gap between Ann and Bob is 0 ($\Delta(T^{(1)}, T^{(2)}) = 0$), the gap between Bob and Cathy is also 0 ($\Delta(T^{(2)}, T^{(3)}) = 0$), but, the gap between Ann and Cathy is a positive value of 2 ($\Delta(T^{(1)}, T^{(3)}) = 2$). Here the triangle inequality is not satisfied since $\Delta(T^{(1)}, T^{(3)}) > \Delta(T^{(1)}, T^{(2)}) + \Delta(T^{(2)}, T^{(3)})$. However, we know that the consensus between the three persons is not transitive. Therefore, the consensus gap not satisfying the triangle inequality is not a disadvantage but a desirable trait because by not satisfying the triangle inequality it coincides well with the consensus intransitivity in GDM.

7. Conclusion

This paper has investigated the consensus and disagreements concepts when a group of experts is given the task to rank a finite set of alternatives. The main contributions of this paper can be summarized as follows:

1. A set of indicators are introduced which reflect the level of expert consensus and capture the notion of expert disagreements. A very interesting and useful contribution is the introduction of the concept of the consensus evaluation sequence (CES). The CES concept is useful because it can be utilized to characterize the overall consensus level of the group, recognize which pair of the experts have disagreements, and identify on which alternatives the experts have the highest dispute. Therefore, it can provide guidance to the group of experts on how to adjust their preferences in order to get a better consensus;
2. It proposed a novel and highly flexible iterative approach for achieving consensus in group decision problems where the experts' preferences are provided by ties-permitted ordinal rankings. The proposed approach is based on an assignment formulation which may be combined with a number of alternative consensus improving strategies. To improve consensus, one may focus on experts' pairwise disagreements only or on alternatives' disputes only or on both. Also, one may choose the maximum, the minimum or just a random value from the existing disagreements to decide which way to re-evaluate the experts preferences. This provides a much needed degree of flexibility and effectiveness in addressing the highly important problem of achieving consensus when ranking a finite set of alternatives by a group of experts;
3. The premetric-based approach is shown, from an axiomatic point of view, to be superior to the traditional distance-based approaches for describing experts' consensus or disagreements. The premetric approach fits the task of expressing expert agreements or disagreements more naturally in that the triangular property may not hold (something that cannot be achieved with traditional metric-based approaches).

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