Math 307 – PS4 **Name: Alexander Powell**
Due: 5pm on 10/27/2014

(**1**) Suppose that H and K are subgroups of G and there are elements a and b in G such that $aH \subseteq bK$. Prove that $H \subseteq K$.

Proof. First take $h \in H$. Then, $\exists k \in K$ such that $ah = bk$. Therefore, we can write that $h \in a^{-1}bk$. We know that $a \in aH$. let $i \in K$, then $a = bi$. Therefore, we have that $a^{-1} = i^{-1}b^{-1}$. Substitution into our equation from earlier, we can write that $h = (i^{-1}b^{-1})bk = i^{-1}k \in K$. Therefore, $H \subseteq K$. □

(**2**) Let H and K be subgroups of a finite group G with $H \subseteq K \subseteq G$. Prove that

$$|G : H| = |G : K||K : H|.$$

Proof. We can rewrite $|G : H|$ as $\frac{|G|}{|H|}$. This is equivalent to $\frac{|G|}{|K|}\frac{|K|}{|H|}$. Changing the fraction back to their original representation gives us $|G : K||K : H|$. Therefore, $|G : H| = |G : K||K : H|$. □

(**3**) Let $\mathbf{G} = GL_n(\mathbb{R})$ and $\mathbf{H} = \{H \in \mathbf{G} : \det(H) = \pm 1\}$.

(**a**) Prove that $\mathbf{H} \leq \mathbf{G}$.

Proof. First we must show that H is nonempty. This is clear from the fact that the $\det(I_n) = 1$ and $I_n \in H$. Next, let A be some $n \times n$ invertible matrix $\in H$. Then $\det(A^{-1}) = (\det(A))^{-1}$. Therefore, $A^{-1} \in H$. Hence, $\mathbf{H} \leq \mathbf{G}$. □

(**b**) Given A, $B \in \mathbf{G}$, prove that $A\mathbf{H} = B\mathbf{H}$ if and only if $\det(A) = \pm\det(B)$.

Proof. First, we should state that $A^{-1}B \in H iff \det(A) = \pm\det(B)$. This can be rewritten $\det(A^{-1})\det(B) = \pm 1$ or $(\det(A))^{-1}det(B) = \pm 1$ or $\det(B) = \pm\det(A)$. Therefore, $A\mathbf{H} = B\mathbf{H}$ if and only if $\det(A) = \pm\det(B)$. □