Math 307 – PS6 **Name: Alexander Powell**

Printed-copy due: 5pm on 11/24/2014

(**1**) Let ϕ be a homomorphism from G to $\bar{G}$ and σ be a homomorphism from $\bar{G}$ to $\tilde{G}$.

(**a**) Prove that $\psi := \sigma \circ \phi$ is a homomorphism from G to $\tilde{G}$.

Proof. Take p and $q \in G$. Then we have that $\psi(pq) = \sigma(\phi(pq)) = \sigma(\phi(p)\phi(q)) = \sigma(\phi(p))\sigma(\phi(q)) = \psi(p)\psi(q)$. Therefore, $\psi := \sigma \circ \phi$ is a homomorphism from G to $\tilde{G}$. □

(**b**) Prove that $\ker(\phi) \leq \ker(\psi)$.

Proof. To prove that $\ker(\phi) \leq \ker(\psi)$ we need to establish that $\ker(\phi) \subseteq \ker(\psi)$. By taking some $p \in \ker(\phi)$ then we have that $\psi(p) = \sigma(\phi(p)) = \sigma(\bar{e}) = \tilde{e}$. Since $p \in \ker(\psi)$ we can say that $\ker(\phi) \subseteq \ker(\psi)$ and thus $\ker(\phi) \leq \ker(\psi)$ is established. □

(**2**) Show that a homomorphism defined on a cyclic group is completely determined by its action on a generator of the group.

Proof. Take some homomorphism $h : G \to \bar{G}$. If $\langle a \rangle = G$, then $h(a^k) = h(a)^k$, and since every element of G is in the form a^k their images are all determined by the image of a and the product in the image group. □

(**3**) Let $\phi : \mathbb{R} \to SL_2(\mathbb{R})$ be defined by

$$\phi(x) = \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix}.$$

(**a**) Prove that ϕ is a homomorphism.

Proof. For any $x, y \in R$ then we can define

$$\phi(x+y) = \begin{bmatrix} \cos(x+y) & \sin(x+y) \\ -\sin(x+y) & \cos(x+y) \end{bmatrix}$$

Then using basic trig properties we can rewrite the matrix as

$$\begin{bmatrix} \cos x \cos y - \sin x \sin y & \cos x \sin y + \sin x \cos y \\ -\sin x \cos y - \cos x \sin y & -\sin x \sin y + \cos x \cos y \end{bmatrix} = \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix} \times \begin{bmatrix} \cos y & \sin y \\ -\sin y & \cos y \end{bmatrix}$$

So we have that $\phi(x+y) = \phi(x)\phi(y)$, therefore ϕ is a homomorphism. □

(**b**) Prove that $\ker(\phi) = \langle 2\pi \rangle$.

Proof. Any element $p \in \ker(\phi)$ if and only if

$$\begin{bmatrix} \cos p & \sin p \\ -\sin p & \cos p \end{bmatrix}$$

is equal to the 2×2 identity matrix. Since $\cos p = 1$ and $\sin p = 0$ only holds true if $p = 2\pi k$, $k \in \mathbb{Z}$. Therefore, it is established that $\ker(\phi) = \langle 2\pi \rangle$. □

(**4**) Let $\phi : \mathbb{C}^\times \longrightarrow \mathbb{C}^\times$ be defined by $\phi(z) = z^n$.

(**a**) Prove that ϕ is a homomorphism.

Proof. Take some z and $y \in \mathbb{C}^\times$. Then we can write $z = a(\cos\theta + \mathrm{i}\sin\theta)$ and $y = b(\cos\psi + \mathrm{i}\sin\psi)$ and $a, b > 0$ with $\theta, \psi \in [0, 2\pi)$. Via DeMoivre's theorem we have that

$$\phi(zy) = (zy)^n = [a(\cos\theta + \mathrm{i}\sin\theta)b(\cos\psi + \mathrm{i}\sin\psi)]^n = a^n b^n[\cos(\theta+\psi) + \mathrm{i}\sin(\theta+\psi)]^n$$

$$= a^n b^n[\cos(n\theta + n\psi) + \mathrm{i}\sin(n\theta + n\psi)]$$

$$= a^n[\cos(n\theta) + \mathrm{i}\sin(n\theta)]b^n[\cos(n\psi) + \mathrm{i}\sin(n\psi)] = z^n y^n = \phi(z)\phi(y)$$

Therefore, it is established that ϕ is a homomorphism. □

(**b**) Prove that $\ker(\phi) = \Omega_n := \{\exp(2k\pi i/n) : k = 0, 1, \ldots, n-1\}$.

Proof. Again, take $z = a(\cos\theta + \mathrm{i}\sin\theta)$ and $y = b(\cos\psi + \mathrm{i}\sin\psi)$ where both $x, y \in \mathbb{C}$. Then the only case when $z = y$ is when $a = b$ and $\theta = 2\pi k, k \in \mathbb{Z}$. So, $z = a(\cos\theta + \mathrm{i}\sin\theta) \in \ker(\phi)$ iff $z^n = 1$ so we can write $r^n(\cos(n\theta) + \mathrm{i}\sin(n\theta)) = (\cos(2\pi k) + \mathrm{i}\sin(2\pi k)), k \in \mathbb{Z}$. This is only the case when $a^n = 1$ and $n\theta = 2\pi k$. So, $z \in \ker(\phi)$ only when $z = \cos(2\pi k/n) + \mathrm{i}\sin(2\pi k/n) = (\cos(2\pi/n) + \mathrm{i}\sin(2\pi/n))^k, k \in \mathbb{Z}$, however $z = (\cos(2\pi/n) + \mathrm{i}\sin(2\pi/n))^{k \bmod n}$. So we have that $\ker(\phi) = \Omega_n := \{\exp(2k\pi i/n) : k = 0, 1, \ldots, n-1\}$. $\square$

(**c**) Prove that $\mathbb{C}^\times/\Omega_n \cong \mathbb{C}^\times$.

Proof. If $z = a(\cos\theta + \mathrm{i}\sin\theta)$ and $y = a^{1/n}(\cos(n\theta) + \mathrm{i}\sin(n\theta))$ and both $z, y \in \mathbb{C}^\times$, then $\phi(w) = z$. Therefore $\phi(\mathbb{C}^\times) = \mathbb{C}^\times$ and $\mathbb{C}^\times/\Omega_n \cong \mathbb{C}^\times$. $\square$