Math 307 – PS3 **Name: Alexander Powell**

Due: 5pm on 10/01/2014

(**1**) Prove that

$$H := \left\{ \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} : n \in \mathbb{Z} \right\}$$

is a cyclic subgroup of $GL_2(\mathbb{R})$.

Proof. Let $n \in \mathbb{N}$. Then we prove by induction that when $n = 1$, then $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$. Next let $n = k+1$, then we can shoe that $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{k+1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^k \cdot \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & k+1 \\ 0 & 1 \end{bmatrix}$. Thus, via the principle of mathematical induction, it is proven that $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$ if $n > 0$.

Next, if $n < 0$, we can write that $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-n} = (\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^n)^{-1} = (\begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix})^{-1} = \begin{bmatrix} 1 & -n \\ 0 & 1 \end{bmatrix}$.

Therefore, we have proven that

$$H := \left\{ \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} : n \in \mathbb{Z} \right\}$$

is a cyclic subgroup of $GL_2(\mathbb{R})$. □

(**2**) Let G be a group and $H \leq G$. For any fixed $x \in G$, let $xHx^{-1} := \{xhx^{-1} : h \in H\}$. Prove that:

(**a**) $xHx^{-1} \leq G$.

Proof. We must begin by proving that xHx^{-1} is nonempty. This is easy to show because $e \in H$ and $xex^{-1} = e \in xHx^{-1}$. Next, we have that $xax^{-1} \cdot (xbx^{-1})^{-1} = xax^{-1} \cdot (xb^{-1}x^{-1}) = xab^{-1}x^{-1}$. □

(**b**) if H is cyclic, then xHx^{-1} is cyclic.

Proof. To prove xHx^{-1} is cyclic we must show that $xHx^{-1} = \langle xhx^{-1} \rangle$. We prove by induction that $(xhx^{-1})^n = xh^nx^{-1}$. If $n = 1$, then $(xhx^{-1}) = xhx^{-1}$, so the base step is true. Then, let $n = k+1$, so $(xhx^{-1})^{k+1} = xh^{k+1}x^{-1}$ can be rewritten as $(xhx^{-1})^k(xhx^{-1}) = xh^kkx^{-1}$. Also, if $n < 0$ the same inductive proof can be used but the $n's$ in the exponents will be negated. Therefore, it is proven that xHx^{-1} is cyclic if H is cyclic. □

(**c**) if H is Abelian, then xHx^{-1} is Abelian.

Proof. Let $A, B \in xHx^{-1}$, then $A = xax^{-1}$ and $B = xbx^{-1}$ for some a and b in H. Since $AB = (xax^{-1})(xbx^{-1}) = xax^{-1}xbx^{-1} = xabx^{-1} = xbx^{-1}xax^{-1} = BA$. Thus, we have that xHx^{-1} is Abelian. □

(**3**) Given a group G, prove that

$$Z(G) = \bigcap_{a \in G} C(a).$$

Proof. Let some element $b \in Z(G)$. Then we have that $ba = ab, \forall a \in G$. Also, $b \in C(a), \forall a \in G$, and finally, $b \in \bigcap_{a \in G} C(a)$. Thus, $Z(G) = \bigcap_{a \in G} C(a)$. □

(**4**) If a is an element of G, prove that $|gag^{-1}| = |a|$ for any $g \in G$.

Proof. There are two cases to this proof: the cardinality of a, or $|a|$ is finite or infinite.

Case 1: $|a| < \infty$

Then $(gag^{-1})^n = ga^n g^{-1} = geg^{-1} = e$. Next, we use proof by contradiction: we make the assumption that $|gag^{-1}| = m < n$. Then $(gag^{-1})^m = ga^m g^{-1} = e$. However, this leads to $a^m = g^{-1}eg = e$, which is a contradiction. Therefore, $|a| = |gag^{-1}|$.

Case 2: $|a| = \infty$

Then make the assumtion that $|gag^{-1}| = m < \infty$. If this is the case, then $(gag^{-1})^m = ga^m g^{-1} = e$. However, this implies that $a^m = g^{-1}eg = e$, which is a contradiction. Therefore, $|gag^{-1}| = |a|$. $\square$