Math 307 – PS5 **Name: Alexander Powell**

Printed-copy due: 5pm on 11/12/2014

(**1**) Let $M_2(\mathbb{R})$ be the group of all real 2×2 matrices under addition. Prove that $M_2(\mathbb{R}) \cong \mathbb{R}^4$, where $\mathbb{R}^4$ is considered as a group under vector addition.

Proof. To prove that $M_2(\mathbb{R}) \cong \mathbb{R}^4$ we need to show there exists an isomorphism. This is clear when you define the relation $\omega : M_2(\mathbb{R}) \cong \mathbb{R}^4$ as $\omega(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$. □

(**2**) Let $\mathbb{P}_n$ be the set of all polynomial functions of degree at most n, i.e.,

$$\mathbb{P}_n = \left\{ p(t) = a_0 + a_1 t + \cdots + a_n t^n = \sum_{k=0}^{n} a_k t^k : a_k \in \mathbb{R},\ k = 0, \ldots, n \right\}.$$

(**a**) Prove that $\mathbb{P}_n$ is a group under *function addition*, i.e., for $p,\ q \in \mathbb{P}_n$, $(p+q)(t) := p(t) + q(t)$.

Proof. To prove $\mathbb{P}_n$ is a group we need to establish closure, associativity, identity, and inverse.
To establish closure, take 2 elements $p, q \in \mathbb{P}_n$. We have that $(p+q)t = p(t)+q(t) = \sum_{k=0}^{n} a_k t^k + \sum_{k=0}^{n} a_k t^k = \sum_{k=0}^{n} (a_k + b_k) t^k$.
For associativity, it is clear that $(pq)t = (pt)q = (qt)p$.
To establish identity, it can be shown that when $m = 0 + 0t + ... + 0t^n = 0$. Therefore, we can say that $m \in \mathbb{P}_n$ and $m + p = p$.
Finally, to establish an inverse it can be shown that

$$-p(t) = \sum_{k=0}^{n} (-a_k) t^k \in \mathbb{P}_n$$

□

(**b**) Prove that $\mathbb{P}_n \cong \mathbb{R}^{n+1}$.

Proof. Similarly from Problem 1, we need to show an isomorphism between the relation $\phi : \mathbb{P}_n \cong \mathbb{R}^{n+1}$. This relationship can be defined as: $\phi : (p(t) = \sum_{k=0}^{n} a_k t^k) = \begin{bmatrix} a_0 \\ a_1 \\ . \\ . \\ . \\ a_n \end{bmatrix}$ □

(**3**) If $G = H_1 \times \cdots \times H_n$, show that $H_i \cap H_j = \{e\}$ for all $1 \leq i, j \leq n$, $i \neq j$.

Proof. Assume, to the contrary, there exists some $m \in H_i \cap H_j$ and that $m \neq e$. The we have that $(H_1 ... H_i ... H_{j-1}) \cap H_j = \{e\}$. However, it is the case that $m = e ... m ... e \in H_1 ... H_i ... H_{j-1}$. Also, $m \in H_j$ so $x \in (H_1 ... H_i ... H_{j-1}) \cap H_j$. However, this is a contradiction, so the original statement holds true: $H_i \cap H_j = \{e\}$. □

(**4**) If $\varphi : G \longrightarrow \overline{G}$ is an isomorphism and $H \lhd G$, prove that $\overline{H} := \varphi(H) \lhd \overline{G}$.

Proof. Since H is non-empty and $\varphi(e) = \bar{e}$, we know that $\bar{H}$ is non-empty as well. Lets start by taking elements $\bar{h}$ and $\bar{g}$ in $\bar{H}$ and $\bar{G}$, respectively. Then there exists elements h and g in H and G, respectively. Also, there exists $\bar{h} \in \bar{H}$ such that $\bar{h} = ghg^{-1}$ and since φ is an isomorphism it can be shown that $\varphi(\bar{h}) = \varphi(ghg^{-1}) = \varphi(g)\varphi(h)\varphi(g^{-1}) = \bar{g}\bar{h}\bar{g}^{-1}$ and $\bar{g}\bar{h}\bar{g}^{-1} \in \bar{H}$. □