Finite Automata Homework 4

Due: Tuesday, Oct 6
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1. Prove by the pumping lemma that the language A of strings of 0s and 1s whose length is a power of two is not regular.

   Proof. Let's begin by assuming that A is a regular language so that we can apply the pumping lemma to A. If we let P be the pumping length we can choose our value of $s = 1^P$. Clearly we have that $|1^P| \geq P$, so it satisfies the length requirement. We can also express $|s| = |x| + |y| + |z| = 2^n, n \geq 0$.

   From here we have two cases:

   (a) $|s| = |y|$ (or $x = \varepsilon$)
   In this case, we know that $|y| = 2^n$. By choosing $i = 3$ then $|y^3| = |3 \times 2^n|$ and 3×2^n is clearly not a power of 2. This is a contradiction to our assumption.

   (b) $|s| > |y|$ (or $x \neq \varepsilon$)
   In the cases where x is not the empty string then $|xy^2z| = |xyz| + |y|$. This can be rewritten to equal $2^n + |y|$. Because the next power of 2 after 2^n is $2^n + 2^n$ and $|y| < 2^n$ then $2^n + |y| < 2^n + 2^n$ so it cannot be a power of 2. Again, this gives us a contradiction to our assumption.

   Because we reached a contradiction to the pumping lemma in both cases after assuming that A is regular, we can conclude that the language A is not regular.

   □

2. Are the following languages regular? Prove you answers.

   (a) $A = \{uww^R v | u, v, w \in 0, 1^+\}$

   Proof. To prove this is a regular language we simply need to come up with a DFA, NFA, or regular expression that will generate the language. The following regular expression generates every member of the language A.

   $$(0 \cup 1)^+ (00 \cup 11)^+ (0 \cup 1)^+$$

   Therefore, this proves that A is regular. □

   (b) $B = \{uww^R v | u, v, w \in 0, 1^+, |u| \geq |v|\}$
   The language B is not a regular language. We use a proof by contradiction to establish this.

   Proof. Let's assume to the contrary that B is a regular language. Then the pumping lemma applies to B. This leads to two different cases: the pumping length P is either even or odd.

   i. Case 1: P is even
   We take our s to be $s = (01)^{P/2} 00 (10)^{P/2}$ so we can deduce the length of s is $|s| = 2p + 2$. In this case, w goes up through the first 0 in the 00 and w^R begins at the second 0 in 00. In this case we know that u and v should be of equal length. When we examine the string resulting from fixing $i = 0$ we know $|y|$ has to be at least 1 and therefore y^0 will put one more character after w^R than before w. This is a contradiction to the rule that $|u| = |v|$.

ii. Case 2: P is odd
In the second case, our choice of s can be written as $s = (01)^{(p-1)/2}100(01)^{(p-1)/2}$ and again $|s| = 2p+2$. Similarly to the first case, when we fix $i = 0$ we get another contradiction to the rule stating $u| \geq |v|$.
Therefore, after seeing in both cases that we arrive at a contradiction to our originial assumption, we can conclude that B is not a regular language.

$\square$

Collaborators: Derek O'Connell

3. Problem 1.54 (a) on page 91. Use closure properties.

We are given the language $F = \{a^i b^j c^k | i, j, k \geq 0 \text{ and if } i = 1, \text{ then } j = k\}$. Prove that F is not regular:

Proof. To prove F is not regular, let's assume to the contrary that F is regular and thus the pumping lemma applies. Now, if we come up with a new language $L = \{ab^i c^j | i, j \geq 0\}$, then L is clearly regular because it can be described by the regular expression ab^*c^* and therefore the intersection of these two languages which we can call $G = F \cap L$ is regular as well, according to the closure properties of regular languages. The intersection, G, of these languages can further be written as $G = \{ab^i c^i | i \geq 0\}$. Now, using the pumping lemma with P being the pumping length we have $s = ab^P c^P \in G$ and $|s| > P$ so there exists x, y, and z such that $xy^i z \in G, \forall i \geq 0$ and $|y| > 0$ and $|xy| \leq P$. Now, let's fix $i = 2$ and from here there are two cases to consider:

(a) Case 1: y includes a (so $x = \varepsilon$)
If y includes a then our s looks like $xyyz$ which contains two as, giving us a contradiction.

(b) Case 2: y does not include a (so $x \neq \varepsilon$)
If y does not include a then it either contains both b and c or either b or c. In these two subcases then there are either at least 2 bcs in $xyyz$ or the number of bs and cs in $xyyz$ is not equal, respectively. Again, this contradicts our assumtion.
In all cases we eventually arrive at a contradiction to the statement that $G = F \cap L$ was regular. Since we know L is definitely a regular language then we can conclude that F is not regular.

$\square$