Finite Automata Homework 9

Due: Thursday, Nov 12
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1. Show that the collection of Turing recognizable languages is closed under the operation of:

   (b) Concatenation **Solution:**

   Proof. Let A and B be two Turing recognizable languages and let M_A and M_B be two turing machines that recognize A and B respectively. We must construct a non-deterministic turing machine M_{AB} that recognizes the language AB. The machine M_{AB} will begin by non-deterministically partitioning an input w into w_1 and w_2. It will then run w_1 through M_A. If M_A halts and rejects, then M_{AB} rejects as well. If it accepts, then it proceeds to run w_2 through M_B. If M_B halts and rejects then M_{AB} rejects as well. If M_B accepts (and thus M_A has already accepted) then M_{AB} accepts as well.

   Because we have constructed a Turing machine that recognizes the concatenation of two TRLs, we have proven that TRLs are closed under concatenation. □

   (d) Intersection **Solution:**

   Proof. Let A and B be two Turing recognizable languages and let M_A and M_B be two turing machines that recognize A and B respectively. We need to construct a machine M_{AB} that recognizes $A \cap B$. This new machine will take in an input string w. It will first run w through M_A. If M_A halts and rejects, then M_{AB}. If M_A accepts then it proceeds to run w through M_B. If M_B halts and rejects then M_{AB} rejects. If M_B accepts (and thus M_A has already accepted) then M_{AB} also accepts.

   Therefore, because we could construct a Turing machine to recognize the intersection of two recognizable languages, we have proven that TRLs are closed under intersection. □

2. Let B be the set of all infinite sequences over $\Sigma = \{0, 1\}$. Show that B is uncountable using a proof by diagonalization. **Solution:**

   Proof. Let's assume, to the contrary, that B is countable. The all the infinite sequences can be represented by $n = 1, 2, 3 \ldots$. We can also define a function $f(n) = (w_{n1}, w_{n2}, w_{n3}, \ldots)$. This can be represented in the table shown below:

n	sequence
1	$(w_{11}, w_{12}, w_{13}, w_{14}, \ldots)$
2	$(w_{21}, w_{22}, w_{23}, w_{24}, \ldots)$
3	$(w_{31}, w_{32}, w_{33}, w_{34}, \ldots)$
$\vdots$	$\vdots$

   Using the principle of diagonalization, we can find a sequence over the alphabet $\{0, 1\}$ that is not in $f(n)$, which means that sequence is not an element of B. This is a contradiction to our originial statement that we had already listed all infinite sequences in the table, so our original statement is false, and we have proven that B is uncountable.

   □

3. In class, we have learned that A_D is non-TR, A_{TM} and $HALT_{TM}$ are TR but non-TD. What can you say about their complements? Are they non-TR, TR but non-TD, or TD? Justify your answers.

   Solution:

   We can say that $\overline{A}_{TM}$ and $\overline{HALT}_{TM}$ are both non-TR because we already know that A_{TM} and $HALT_{TM}$ are TR. Also, just to be clear, they are certainly non-TD because if something's not recognizable it has no hope of being decidable.

   As far as $\overline{A}_D$, we know that it is non-TD because TDLs are closed under complementation and we know that A_D is non-TD. However, when $A_D = \{w \mid w \notin L(M)\}$ then $\overline{A}_D = \{w \mid w \in L(M)\}$. We can construct a TM R that takes in w and M, which is then passed into A_{TM} which will either accept or reject. Therefore, $\overline{A}_D$ is TR.