Elliptic Curve Cryptography

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***Abstract*—This paper provides an overview of Elliptic Curve Cryptography. It also discusses its implementation over two finite fields. Additionally, we will discuss its benefits and shortcomings as well as the future potential of this type of public key cryptography.**

I. Introduction

Elliptic Curve Cryptography (ECC) is a type of public key cryptography (PKC) which uses algorithms based on mathematical problems without an efficient solution. Some of the early PKC systems relied on factoring large integers composed of two or more large prime factors. Elliptic Curve Cryptography is a method of cryptography that relies on elliptic curves to provide security. The use of elliptic curves for purposes related to cryptography was proposed by both Neal Koblitz and Victor S Miller in 1985. By 2004, ECC algorithms had gained significant popularity.

II. Overview

In general, the first assumption followed when dealing with ECC is that the equation of an ellipse is written as

$$y2 = x^3 + ax + b \text{ where } 4a^3 + 27b^2 \neq 0$$

This equation is called the Weierstrass normal form for elliptic curves. Clearly, by changing the values for a and b the curve will take on different points. Also, in general elliptic curves are symmetric about the x-axis. Additionally, a point at infinity or ideal point will be part of the curve. This point is often denoted with a 0, giving the following definition for an elliptic curve

$$\{(x,y) \in \mathbb{R}^2 \mid y^2 = x^3 + ax + b,\ 4a^3 + 27b^2 \neq 0\} \cup \{0\}$$

The next step is to define a group over elliptic curves. To prove something is a group, it must satisfy the following properties: closure, associativity, identity, and inverse. In our case, a group over elliptic curves can be established because all points on the curve make up the elements of the group, the identity element is said to be the point at infinity or ideal point. The inverse of a point in the group is the one symmetric about the x-axis. Finally, addition is given by the rule $P+Q+R=0$ where P, G, and R are three points on the same line. Just like a lot of other PKC systems rely on factoring large numbers, the security of ECC depends on the difficulty of what's called the Discrete Logarithm Problem.[1] In this problem, P and Q are two points on an elliptic curve such that $kP = Q$ where k is a scalar. Given P and Q, its incredibly difficult to solve for k if k is sufficiently large. Also, k is the discrete logarithm of Q to the base P. The discrete logarithm problem is believed to be a "hard" problem, meaning it has no known polynomial time algorithm. However, this belief has not been formally proved. This brings us to the main operation involved in the ECC system, point multiplication.

III. Point Multiplication

The basic idea behind point multiplication in ECC is multiplying a point P on an elliptic curve with a scalar k to obtain another point Q on the same elliptic curve. There are two components to point multiplication: point addition and point doubling.[2] In point addition, two points J and K are added together to obtain a new point $L(J+K=L)$. In point doubling, a point is added to itself (or multiplied by 2) to obtain a new point $(2 \times J = J + J = L)$. An example of point multiplication with a point P and $k = 23$ is given as:

$$23P = 2(2(2(2P) + P) + P) + P.$$

Here, we can see that point multiplication utilizes point addition and doubling repeatedly to find the solution. This is referred to as the double and add method, although there are many other strategies. We can also view this algebraically. First, take two points on an elliptic curve, J and K, where $J = (x_J, y_J)$ and $K = (x_k, y_k)$. Now let $L = J + K$ where $L = (x_L, y_L)$, which gives us $x_L = s^2 - x_J - x_k$ and $y_L = -y_J + s(x_J - x_L)$ and $s = (y_J - y_k)/(x_J - x_k)$, where s is the slope of the line through points J and K. We can clearly see that is $K = -J$ meaning $K = (x_J, -y_J)$ then $J + K = 0$ where 0 is the point at infinity or ideal point.[1] Also, if K = J then it follows that $J + K = 2J$ and point doubling equations can be used. Additionally, we can see that the commutative property holds and $J + K = K + J$.

IV. Finite Fields

In the previous examples, weve computed point multiplications for real numbers. These types of operations over the set of real numbers can be slow as well as inaccurate as a result of round off error. To make these operations faster and more efficient ECC is usually defined over two fields: a prime field F_P and a binary field F_2. On the prime field, the equation for the elliptical curve is slightly rewritten in its modular form:

$$y^2 \bmod p = x^3 + ax + b \bmod p, \text{ where } 4a^3 + 27b^2 \bmod p \neq 0.$$

In this case, the elements of the finite field are integers between 0 and $p-1$. When ECC is operated over the prime field, the prime p is chosen in such a way that there is a large number of

points on the elliptic curve to make the cryptography secure.[3] When generated over the prime field, the elliptic curve is not a smooth one, which causes difficulty in geometric reasoning about point multiplication. However, the algebraic rules still apply when using curves generated over F_P. When curves are generated over the binary field, the following equation is used:

$$y^2 + xy = x^3 + ax^2 + b, \text{ where } b \neq 0.$$

In this case, the elements of the field are integers with a length of at most m bits. Additionally, these numbers can be expressed as a binary polynomial (coefficients can only be 0 or 1) of degree $m-1$. Like before, it is important to choose an m such that there is a sufficiently large enough number of points on the elliptic curve.[3]

V. Key Sizes

One of the benefits to using ECC over other PKC systems is that ECC can use a smaller key size and still retain security.[4] Furthermore, this reduces storage and transmission requirements. For example, an ECC public key with 256 bits should provide an equivalent level of security to a 3072 bit RSA public key. Currently, the fastest algorithms to solve ECC codes require $O(\sqrt{n})$ iterations. This implies that the size of the field used should be twice the security parameter. For example, for a 128 bit code, an elliptic curve over the field F_q, where $q = 2^{256}$ should be used. To date, the most complex ECC system known to be broken has a 112 bit key for the prime field case and a 109 bit key for the binary field case.[2] Both of these cases took a matter of months to break.

VI. Public Key Algorithms

There are many encryption algorithms related to the field of ECC. Two of the most common are ECDH (Elliptic curve Diffie-Hellman) and ECDSA (Elliptic Curve Digital Signature Algorithm).

A. Elliptic Curve Diffie-Hellman

ECDH is a variation of the Diffie-Hellman algorithm that is used for elliptic curves. In reality, it's more of a key agreement protocol than an encryption algorithm, which means it tells us how keys should be generated and exchanges between users. The ECDH algorithm is used when there are two users who want to exchange information securely in such a way that a third user can intercept it but not decode it. To give a high level overview, there are three steps:

- To start, the two users attempting to exchange information securely, let's call them A and B both have individual public and private keys. However, these keys have been generated using the same base point on the same elliptic curve.
- Next, user A and B exchange their public keys through the third user, C, who they do not want decoding it. Note: it's assumed C can't decode the keys because to do that would mean solving the discrete logarithm problem, which doesn't have a polynomial time algorithm.
- Finally, A and B can both decode the information using their own private key and the other's public key.

It should be noted that the Diffie-Hellman is considered to be a "hard" problem. Furthermore, it's believed to be as hard as the discrete logarithm problem, although this has not been rigorously proven.[4] However, we know that it isn't any harder because solving the discrete logarithm problem would mean solving the Diffie-Hellman problem. ECDH is sometimes refered to as ECDHE, where the second E stands for Ephemeral meaning that the keys exchanged are only temporary, and not static. This is used when A and B need to generate their keys on the fly at the time the connection is established. Then the keys are later signed for authentication and exchanged.[4]

B. Elliptic Curve Digital Signature Algorithm

ECDSA attemtps to solve the problem where again there are two users A and B, and A wants to sign a message with their private key and B wants to validate the signature using A's public key. Also, only A should be able to produce valid signatures, but all parties should be able to check signatures. Here, A and B are using the same domain parameters. The ECDSA algorithm is a variation of the Digital Signature Algorithm applied to elliptic curves. ECDSA operates on the hashed message instead the actual message itself. There are an unlimited number of hash functions to choose from, but it is important that a cryptographically-secure one is chosen.

At a high level, the algorithm first generated a private key, which is then "hidden" using point multiplication. For B to verify the signature, A's public key, the hash code and the signature are needed. From this, two integers are calculated to form a new point on the elliptic curve.[4]

VII. Comparison between ECC and RSA

We've seen the ECC does a good job with secure encryptions. However, one may ask why even use ECC at all if we already have good RSA encryption algorithms. One of the most significant benefits is that ECC requires a much smaller key size to guarantee the same level of security. In the table below, we see the corresponding key sizes (in bits) between the two strategies.

RSA key size	ECC key size
1024	160
2048	224
3072	256
7680	384
15360	521

From this table, there doesn't appear to be any sort of linear relationship between the RSA and ECC key sizes, which implies that ECC uses less memory and also that key generation and signing are considerably faster.[5]

In the equations for an elliptic curve given above, some coefficient values produce stronger curves than other, from a security perspective. Generally, a random seed is used in the domain parameters to generate the elliptic curve. The

random seeds often come from digits of π, Euler's number e, or the golden ratio.[4] These numbers are random because their digits are uniformly distributed. It is not common knowlegde of where the random seeds that the NSA and NIST come from. Many have speculated that these organizations have discovered a large class of weak elliptic curves and have tries all possibly seeds until they found a vulnerable curve. However, it's important to note that random and secure are not synonymous in the field, and there's nothing to be done if the algorithm is broken no matter how long the keys are. In this case, RSA is the stronger method because it does not require special domain parameters that can be tampered with. Therefore, RSA is still probably a better choice over ECC if we cant't trust authorities or can't construct the domain parameters ourselves.[6]

VIII. Conclusion

In conclusion, we have seen that there are many benefits to the ECC strategy of public key encryption. ECC provides an incredibly fast and efficient alternative to RSA methods due to it's efficiency in the use of memory. However, for an efficient implementation of ECC, it is important for the point multiplication algorithm and the field arithmetic to be efficient. Elliptic curves have many applications including not only data encryption but also digital signatures, pseudo-random generators, integer factorization algorithms, and many other tasks. Although a relatively recent invention (1985) but only gaining real prominence in the early 2000s, the applications to ECC are endless and the benefits over other traditional approaches are huge.

References

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