Finite Automata Homework 7

Due: Thursday, Oct 29
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1. (a) Use the pumping lemma to show that $A = \{0^n1^n0^n1^n | n \geq 0\}$ is not a context free language.
Solution:

Proof. Assume A is a context free language. Then the pumping lemma for context free languages applies to A. If we take $s = 0^p1^p0^p1^p$ then clearly $|s| = 4p \geq p$. Also, we can describe say $s = uvxyz$ where $|vxy| \leq p$. In other words, s can be written as:

$$s = \underbrace{0\ldots0}_{\text{p}}\underbrace{1\ldots1}_{\text{p}}\underbrace{0\ldots0}_{\text{p}}\underbrace{1\ldots1}_{\text{p}}$$

This creates three cases:

i. **Case 1:** vxy is composed of all $0s$ and is therefore contained completely within the first or second string of $0s$ of length p. Because we know that $|vy| > 0$ then either v or y must have at least one 0. By setting $i = 0$ we get $s = uv^0xy^0z$, which causes one of the two strings of $0s$ in the original s to have at least one fewer $0s$ than the other. This is a contradiction to our assumption because $uv^0xy^0z \notin A$.

ii. **Case 2:** vxy is composed of all $1s$ and thus is made up of either the first or second string of $1s$ in the original s. Now, either v or y must contain at least one 1 since $|vy| > 0$. Again, we set $i = 0$ and get uv^0xy^0z which generates a string where one of the two substrings of $1s$ has a different length than the other (by at least one). Therefore $uv^0xy^0z \notin A$, which is a contradiction to our assumption.

iii. **Case 3:** The third case covers scenarios where vxy is made up of a combination of $0s$ and $1s$. We can further split this into two cases:

   A. **Subcase 3a:** vxy is made up of $0s$ followed by $1s$
   In this case, either vxy straddles the first partition of $0s$ and $1's$ or the second partition. Because $|vxy| \leq p$ then by "pumping" up or down the lengths of the substrings of $0s$ and $1s$ immediately next to vxy will be changed but not the other substring. If we fix $i = 2$, then uv^2xy^2z would give us a string like $0^{p'}1^{p'}0^p1^p$ or $0^p1^p0^{p'}1^{p'}$, where $p' \neq p$. This is a contradiction to our assumption because these strings are not elements of A.

   B. **Subcase 3b:** vxy is made up of $1s$ followed by $0s$
   The same argument can be made for the case where a string is made up of $1s$ followed by $0s$. Again, let's fix $i = 2$, then uv^2xy^2z gives us a string that looks like $0^p1^{p'}0^{p'}1^p$ which is clearly not in A because $p' \neq p$. Again, we have a contradiction.

So, because we have shown that we reach a contradiction in every possible permutation of s, we have proven that A is not a context free language.

□

(b) Use the pumping lemma to prove that

$$D = \{t_1\#t_2\#\ldots\#t_k | k \geq 2, \text{each } t_i \in \{a,b\}^* \text{ and } t_i = t_j \text{ for some } i \neq j\}$$

is not a context free language.

Solution:

Proof. Assume to the contrary that D is a context free language so the pumping lemma applies. We can take $s = a^p b^p \# a^p b^p \in D$. Then we have that $|s| = 4p > p$. Also, we know that $|vxy| \leq p$ and $|vy| > 0$.

i. **Case 1:** If vxy is contained entirely on either the left or right side of the #.
By setting $i = 2$ then uv^2xy^2z will cause s to look like:

$$a^{p'} b^p \# a^p b^p \text{ or,}$$
$$a^p b^{p'} \# a^p b^p \text{ or,}$$
$$a^p b^p \# a^{p'} b^p \text{ or,}$$
$$a^p b^p \# a^p b^{p'}$$

where $p' \neq p$. Clearly none of these strings is in the language D so we have a contradiction.

ii. **Case 2:** If vxy contains the # character.

A. **Subcase 2a:** If v or y contains the # then we can set $i = 0$ to get $uv^0xy^0z = uxz$ which doesn't contain the # character so uv^0xy^0z is not an element of the language D, which is a contradiction.

B. **Subcase 2b:** If x contains the # character then v is a substring of $b^p = underbraceb\ldots b_{\text{p}}$ and y is a substring of $a^p = underbracea\ldots a_{\text{p}}$. Again, by setting $i = 0$ then we get uv^0xy^0z which looks something like $a^p b^{p'} \# a^{p'} b^p$ where $p' \neq p$. Therefore, $t_1 \neq t_2$ so $uv^0xy^0z \notin D$, which is a contradiction to our original assumption.

Since we've ended up with contradictions at every possible case of representing s, we can conclude that D is not a context free language. $\square$

2. Let $\Sigma = \{1,2,3,4\}$ and $C = \{w \in \Sigma^* |$ in w, the number of 1*s* equals the number of 2*s* and the number of 3*s* equals the n
Show that C is not a context free language.

Solution:

Proof. Assume to the contrary that C is a context free language. Then the pumping lemma for context free languages applies to C. We can divide this proof up into the four cases shown below.

(a) Case 1: vxy contains a 1

By setting $i = 2$, then $uv^2xy^2z \notin C$ because $|vxy| \leq p$ so the number of 1*s* and 2*s* will be different. This generates a string that's not in the language C, so we have a contradiction.

(b) Case 2: vxy contains a 2

The rest of the cases are similar to the first. By setting $i = 2$, then $uv^2xy^2z \notin C$ because the number of 1*s* and 2*s* will be different. This generates a string that's not in the language C, so again we have a contradiction.

(c) Case 3: vxy contains a 3

Again, by setting $i = 2$, then $uv^2xy^2z \notin C$ because the number of 3*s* and 4*s* will be different. This generates a string that's not in the language C, so again we have a contradiction.

(d) Case 4: vxy contains a 4

Again, by setting $i = 2$, then $uv^2xy^2z \notin C$ because the number of 3*s* and 4*s* will be different. This generates a string that's not in the language C, so again we have a contradiction.

Because we end up with a contradiction in every step we have proven that C is not a context free language. $\square$