Math 214 – Foundations of Mathematics
Homework 8

Due March 27

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Solve the following problems. Please remember to use complete sentences and good grammar.

1. *(4 Points)* A sequence $\{a_n\}$ is defined recursively by $a_1 = 1, a_2 = 4, a_3 = 9$, and

$$a_n = a_{n-1} - a_{n-2} + a_{n-3} + 2(2n - 3)$$

for $n \geq 4$. Conjecture a formula for a_n and prove that your conjecture is correct using Strong Principle of Mathematical Induction.

Solution: First, let's examine the first several entries of the recursive function:
$a_4 = a_3 - a_2 + a_1 + 2(2(4) - 3) = 16$
$a_5 = a_4 - a_3 + a_2 + 2(2(5) - 3) = 25$
$a_6 = a_5 - a_4 + a_3 + 2(2(6) - 3) = 36$
$a_7 = a_6 - a_5 + a_4 + 2(2(7) - 3) = 49$
From these results, we can form the conjectures that $a_n = n^2, \forall n \in \mathbb{N}$

Proof. We will prove this with strong PMI:
If $a_n = a_{n-1} - a_{n-2} + a_{n-3} + 2(2n - 3)$ then $\forall n \in \mathbb{N}, a_n = n^2$.
Step 1: When $n = 4$, $a_1 = 4^2 = 16$, which matches our result from above.
Step 2: Suppose $a_n = n^2$ is true for $1 \leq n \leq k$, then for $n = k + 1$:

$$a_{k+1} = a_{(k=1)-1} - a_{(k+1)-2} + a_{(k+1)-3} + 2(2(k + 1) - 3)$$

$$= a_k - a_{k-1} + a_{k-2} + 4k - 2 = k^2 - (k - 1)^2 + (k - 2)^2 + 4k - 2$$

$$= k^2 - k^2 + 2k - 1 + k^2 - 4k + 4 + 4k - 2 = k^2 + 2k + 1 = (k + 1)(k + 1) = (k + 1)^2$$

Hence, it is true for $n = k + 1$.
Step 3: From the strong PMI, we have proven that $\forall n \in \mathbb{N}, a_n = n^2$.

□

2. (12 points) Use Strong Mathematical Induction Principle to prove that if $a_1 = 1$, $a_2 = 2$, and $a_{n+1} = a_n + a_{n-1}$ for $n \geq 2$, then $a_n < 2^n$ for $n \in \mathbb{N}$.

Solution:

The above statement can be rewritten as:

$$a_n = a_{n-1} + a_{n-2}, n \geq 3$$

Proof. $\forall n \in \mathbb{N}, a_n < 2^n$ **Step 1:**

$n = 1, a_1 = 1 < 2^1$ and

$n = 2, a_2 = 2 < 2^2 = 4$ so it holds for the base step.

Step 2: Suppose $a_n < 2^n$ is true for $n = k$ then for $n = k + 1$:

$$a_{k+1} = a_{(k+1)-1} + a_{(k+1)-2} = a_k + a_{k-1} < 2^k + 2^{k-1}$$

$$= 2 \cdot 2^{k-1} + 2^{k-1} = 2^{k-1} \cdot (2 + 1) = 3 \cdot 2^{k-1} < 2^k + 2^{k-1} < 2^k + 2 \cdot 2^{k-1} = 2^k + 2^k = 2 \cdot 2^k = 2^{k+1}$$

Step 3: From strong PMI, we have proven that if $a_1 = 1$, $a_2 = 2$, and $a_{n+1} = a_n + a_{n-1}$ for $n \geq 2$, then $a_n < 2^n$ for $n \in \mathbb{N}$.

□

3. *(4 Points)* A relation R is defined on $\mathbb{Z}$ by $(x, y) \in R$ if $x|y$. Prove or disprove the following: (a) R is reflexive, (b) R is symmetric, (c) R is transitive.

   Solution:

   a) R is reflexive: R is reflexive if $x|x$, which we know is always true since any number can divide into itself. So, R is reflexive.

   b) R is symmetric: R is symmetric if $x|y$ and $y|x$. This is not true because if we let $x = 3$ and $y = 6$, we get that $3|6$ which is true because $\exists x \in \mathbb{Z}$, in this case 2 such that $6 = 3 \cdot 2$. However, $6 \nmid 3$ because there is no integer that solves the following equation: $3 = 6 \cdot m$. Therefore, R is not symmetric.

   c) R is transitive: R is transitive if the following holds true: If $x|y$ and $y|z$, then $x|z$. From result 4.1, if $a, b, c \in \mathbb{Z}$ and $a \neq 0$ and $b \neq 0$, then if $a|b$ and $b|c$, then $a|c$. So, we can conclude that R is transitive.

4. *(4 Points)* A relation R is defined on $\mathbb{Z}$ by $(a, b) \in R$ if $3|(x + y)$. Prove or disprove the following: (a) R is reflexive, (b) R is symmetric, (c) R is transitive. (In the case of disprove, provide a concrete example.)

   Solution:

   a) R is reflexive: R is not reflexive because if it was, then $3|(x + x)$. If we choose x to be, say, 2, then $2 + 2 = 4$ and $3 \nmid 4$. So, R is not reflexive.

   b) R is symmetric: If $3|(x + y)$ then $\exists m \in \mathbb{Z}$ such that $x + y = 3m$. Now, $y + x = 3m$ and it is true that $3|3m$, hence yRx.

   c) R is transitive: Suppose xRy and yRz. Let $x = 2, y = 1, and z = 8$. Then we have that $3|(2+1)$ and $3|(1+8)$ but $3 \nmid (2 + 8)$ because there does not exist an integer, m, such that $3m = 10$. So, R is not transitive.

5. *(4 Points)* Let S be a nonempty subset of $\mathbb{Z}$, and let R be a relation defined on S by xRy if $3|(x + 2y)$.

   (a) Prove that R is an equivalence relation.

   Solution: To prove that R is an equivalence relation, we have to prove 3 cases: that R is reflexive, symmetric, and transitive:
   1) R is reflexive: This is true, since $\forall x \in \mathbb{Z}, x + 2x = 3x$ and $3|3x$. So, xRx.
   2) R is symmetric: Suppose xRy, then $3|(x + 2y)$, thus $\exists m \in \mathbb{Z}$ such that $x + 2y = 3m$. This leaves us with:

   $$y + 2x = 3(x + y) - (x + 2y) = 3(x + y) - 3m = 3(x + y - m)$$

   Hence, $3|(y + 2x)$ so yRx.
   3) R is transitive: If xRy and yRz, then xRz.

   $$x + 2y = 3m, y + 2z = 3n, m, n \in \mathbb{Z}$$

   Adding the two, we get:
   $x + 3y + 2z = 3(m + n)$, so $x + 2z = 3(m + n - y)$. Hence, R is transitive, so xRz.

(b) If $S = \{-7, -6, -2, 0, 1, 4, 5, 7\}$, then what are the distinct equivalence classes in this case?

Solution: The equivalence classes are as follows:
$[0] = \{0, -6\} = [-6]$
$[1] = \{1, 4, -7\} = [4]$
$[5] = \{5, -7\} = [-7]$
$[7] = \{7, -2, 1, 4\} = [-2]$

6. *(4 Points)* Let $S = \mathbb{R}^2$. Define a relation R by $(x_1, y_1)R(x_2, y_2)$ if $|x_1| + |y_1| = |x_2| + |y_2|$. Prove that R is an equivalence relation. Determine the equivalence classes of R and describe the geometric property of each equivalence class. What is the geometric shape of the equivalent class $(x, y) = (1, 1)$ belonging to?

Solution: To prove that R is an equivalence relation, we have to prove 3 cases: that R is reflexive, symmetric, and transitive:

1) Reflexive: $\forall (x_1, y_1) \in \mathbb{R}^2$, $|x_1| + |y_1| = |x_1| + |y_1|$. This is clearly the case, so $(x_1, y_1)R(x_1, y_1)$.

2) Symmetric: If $|x_1| + |y_1| = |x_2| + |y_2|$ then $|x_2| + |y_2| = |x_1| + |y_1|$. Again, this is intuitively clear, so $(x_1, y_1)R(x_2, y_2)$.

3) Transitive: If $|x_1| + |y_1| = |x_2| + |y_2|$ and $|x_2| + |y_2| = |x_3| + |y_3|$ then If $|x_1| + |y_1| = |x_3| + |y_3|$. Once again, this is true so $(x_1, y_1)R(x_3, y_3)$.

From the three steps above, we have proven that R meets the criteria for being an equivalence relation. Its equivalence classes can be expressed as

$$[(x_1, y_1)] = \{(x_2, y_2) \in \mathbb{R}^2 : |x_2| + |y_2| = |x_1| + |y_1|\}$$

If we let $x = y = 1$, we have $[(1, 1)] = \{(x, y) \in |R^2 : |x| + |y| = 2\}$. So, every equivalence class is a graph of a square centered at the origin, or $(0, 0)$ and a side length of $2 \cdot (|x| + |y|)$. The only exception is $[(0, 0)] = \{(0, 0)\}$, which can be thought of as a point in $\mathbb{R}^2$ or a square with side length equal to 0.

So, the equivalence classes are concentric squares centered at $(0, 0)$ with side length $R \in [0, \infty)$ and there are infinitely many distinct equivalence classes.

7. *(extra 2 Points)* A sequence (a_n) is defined by $a_0 = -1$, $a_1 = 0$, and $a_{n+1} = a_n^2 - (n + 1)^2 a_{n-1} - 1$, for all positive integers n. Find an explicit formula for a_n and use mathematical induction to prove it.