Math 214 – Foundations of Mathematics
Homework 3

Due February 5, 2014

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Solve the following problems. For proof problems, please remember to use complete sentences and good grammar.

1. *(4 Points)* For parts (a) - (c), first, write the statements so that there are no $\sim$ symbols. Then, rewrite the statements in words so that there are no $\forall, \exists, \in$ or $=$ symbols. Finally determine whether the statement is true or false.

   (a) $\sim (\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, xy = 1)$;

   Solution: $(\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, xy \neq 1)$
   There exists some real number x, such that for all real numbers y, the product of x and y does not equal 1. This statement is **True**.

   (b) $\sim (\exists y \in \mathbb{R}, \forall x \in \mathbb{R}, xy = 0)$;

   Solution: $(\forall y \in \mathbb{R}, \exists x \in \mathbb{R}, xy \neq 0)$
   For all real numbers y, there exists some real number x such that the product of x and y does not equal 0. This statement is **False**.

   (c) $\sim (\exists n \in \mathbb{Z}, \exists m \in \mathbb{Z}, m \leq n)$.

   Solution: $(\forall n \in \mathbb{Z}, \forall m \in \mathbb{Z}, m > n)$
   For all integers n, and for all integers m, $m > n$. This statement is **False**.

2. *(4 Points)*

   (a) "Some birds sing all the time; All birds sing sometimes."
   The first part of sentence can be quantified as: "$\exists$ bird b such that, $\forall$ time t, b sings." Use a similar quantified statement for the second part.

   Solution: $\exists$ time t such that, $\forall$ bird b, b sings.

   (b) Consider the sentence, "For every integer $n > 0$ there exists some real number $x > 0$ such that $x < 1/n$." Without using words of negation, write a complete sentence that negates the sentence. Which sentence (the original or the negation) is true?

   Solution: There exists an integer $n \leq 0$ such that for all real numbers $x \leq 0$, $x \geq 1/n$. The original sentence is **True**.

3. *(2 Points)* Prove directly that if x is an even integer, then $7x + 5$ is an odd integer.

Solution:

Proof. Let x be an even number, then there exists $m \in \mathbb{Z}$ such that $x = 2m$. So,
$7x + 5 = 7(2m) + 5$
$= 14m + 5$
$= 2(7m + 2) + 1$
Which is an odd integer.

$\square$

4. *(2 Points)* Prove by contrapositive that if $3x + 5$ is even then $x + 2$ is odd.

Solution:

Proof. First, prove contrapositive: If $x + 2$ is even, then $3x + 5$ is odd.
Let $x + 2$ be even, then $\exists n \in \mathbb{Z}$ such that
$x + 2 = 2n$ or $x = 2n - 2$
So, $3x + 5 = 3(2n - 2) + 5$
$= 6n - 6 + 5$
$= 6n - 1$
$= 2(3n) - 1$
Which is an odd number. Hence the contrapositive is true then the original is also true.

$\square$

5. *(4 Points)* Let $x \in \mathbb{Z}$. Use a lemma to prove that if $7x - 4$ is even, then $3x - 11$ is odd. (Hint: x should be even or odd?)

Solution:
Step 1: If $7x - 4$ is even, then x is even.

Proof. **Use contrapositive:** If x is odd, then $7x - 4$ is odd.
Let x be odd, then $x = 2n + 1$ for some $n \in \mathbb{Z}$. So,
$7x - 4 = 7(2n + 1) - 4$
$= 14n + 7 - 4$
$= 14n + 3$
$= 2(7n + 1) + 1$
Which is an odd number. Hence the contrapositive is true, the original is also true.

$\square$

Step 2: If x is even, then $3x - 11$ is odd.

Proof. Let x be even, then $x = 2n$ for some $n \in \mathbb{Z}$.
$3x - 11 = 3(2n) - 11$
$= 6n - 11$
$= 6n - 10 - 1$
$= 2(3n - 5) - 1$
Which is an odd number. Therefore, it is proven that if $7x - 4$ is even, then $3x - 11$ is odd. $\square$

6. *(4 Points)* Prove that if $n \in \mathbb{Z}$, then $n^3 - n$ is even. (Hint: use proof by cases)

Solution:
There are two cases to this proof: either n is even or odd.
Case 1: If n is even.

Proof. Let n be even, then $n = 2x$ for some $x \in \mathbb{Z}$.
$n^3 - n = (2x)^3 - 2x$
$= 8x^3 - 2x$
$= 2(4x^3 - x)$
Which is an even number. $\square$

Case 2: If n is odd.

Proof. Let n be odd, then $n = 2x + 1$ for some $x \in \mathbb{Z}$.
$n^3 - n = (2x + 1)^3 - (2x + 1)$
$= (4x^2 + 4x + 1)(2x + 1) - (2x + 1)$
$= 8x^3 + 8x^2 + 2x + 4x^2 + 4x + 1 - 2x - 1$
$= 8x^3 + 12x^2 + 4x$
$= 2(4x^3 + 6x^2 + 2x)$
Which is an even number. $\square$

Therefore, it is proven that if $n \in \mathbb{Z}$, then $n^3 - n$ is even.

7. *(4 Points)* Let $a, b \in \mathbb{N}$. Prove that if $ab = 4$, then $(a - b)^3 - 9(a - b) = 0$.

Solution:
This is a proof by cases.
Case 1: $a = 2, b = 2$

Proof. Let $a = b = 2$, then
$(2 - 2)^3 - 9(2 - 2) = 0$
$0 - 9(0) = 0$
This satisfies the hypothesis. $\square$

Case 2: $a = 1, b = 4$

Proof. Let $a = 1$ and $b = 4$, then

$(1 - 4)^3 - 9(1 - 4) = 0$

$-27 - 9(-3) = 0$

This satisfies the hypothesis. $\square$

Case 3: $a = 4, b = 1$

Proof. Let $a = 4$ and $b = 1$, then

$(4 - 1)^3 - 9(4 - 1) = 0$

$27 - 9(3) = 0$

This satisfies the hypothesis. $\square$

Therefore, it is proven that if $ab = 4$, then $(a - b)^3 - 9(a - b) = 0$.

8. *(extra 4 Points)* It was proved by Andrew Wiles and Richard Taylor that Fermat Theorem is true: for every $x, y, z \in \mathbb{N}$ and $n \in \mathbb{N}$, $n \geq 3$, $x^n + y^n \neq z^n$. Here we consider the case when n is a negative integer.

   - Show that there are infinitely many $x, y, z \in \mathbb{N}$ such that
   $$\frac{1}{x} + \frac{1}{y} = \frac{1}{z}.$$
   - Prove or disprove that there are infinitely many $x, y, z \in \mathbb{N}$ such that
   $$\frac{1}{x^2} + \frac{1}{y^2} = \frac{1}{z^2}.$$
   - Assuming the Fermat Theorem, show that there is no $x, y, z \in \mathbb{N}$ and $n \geq 3$ such that
   $$\frac{1}{x^n} + \frac{1}{y^n} = \frac{1}{z^n}.$$