Math 214 – Foundations of Mathematics
Homework 11

Due April 17th

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1. *(4 Points)* Prove that $(0, 1)$ and $[0, 1]$ are numerically equivalent by defining (and proving) a bijection f between $(0, 1)$ and $[0, 1]$. (Hint: choose a sequence $\{x_n\}$ in $(0, 1)$, define $f : [0, 1] \to (0, 1)$ by mapping 0 to x_1, 1 to x_2, and x_n to x_{n+2})

   Solution: First we must prove there is an injection. Suppose $f(x) = f(y)$. Then there are 9 cases. **Case 1:** $x = 1/2, y = 1/n, n \geq 3$. Then $0 = \frac{1}{n-1} \to$ this is not possible.

   Case 2: $x = 1/n, n \geq 3, y \notin \{\frac{1}{n}\}, \frac{1}{n-2} = y$. Again, not possible.

   Case 3: $x = 1/2, y = 1/2$, so $0 = 0$. This is true.

   Case 4: $x = 1/2, y = 1/3$, so $0 = 1$. This is not possible

   Case 5: $x = 1/3, y = 1/2$, so $1 = 0$. This is not possible

   Case 6: $x = 1/n, n \geq 3, y = 1/2$ so $\frac{1}{n-2} = 0$ but this is not possible.

   Case 7: $x = 1/n, n \geq 3, y = 1/3$ so $\frac{1}{n-2} = 1$ so $n = 3$.

   Case 8: $x = 1/3, y = 1/n, n \geq 3$ so $\frac{1}{n-2} = 1$ so $n = 3$.

   Case 9: $x = 1/n, n \geq 3, y = 1/n, n \geq 3$ so $\frac{1}{n-2} = \frac{1}{m-2}$ or $m = n$.

   So, we have proven that f is injective. Now we prove surjectivity. To make f surjective the bijection can be defined as follows:

$$f(x) = \begin{cases} 0 & : x < 0 \\ 1 & : x \geq 0 \\ \frac{1}{n-1} & : x = \frac{1}{n}, n \geq 3 \\ x & : else \end{cases}$$

2. *(4 Points)* Prove that $S = \{(a, b) : a, b \in \mathbb{N}, a \geq b\}$ is denumerable.

   (Hint: you can use Theorem 10.4 and Result 10.6)

   Solution: First, we know $S \subseteq \mathbb{N} \times \mathbb{N}$. From theorem 10.4, we know every infinite subset of a denumerable set it denumerable. Also, from result 10.6, if A and B are denumerable then so is $A \times B$. And since we know $\mathbb{N}$ is denumerable, so is $\mathbb{N} \times \mathbb{N}$, which means so is S.

3. *(4 Points)*

   (a) Prove that A and B are disjoint denumerable sets, then $A \bigcup B$ is also denumerable. Here we assume that $A = \{f(n) : n \in \mathbb{N}\}$ and $B = \{g(n) : n \in \mathbb{N}\}$ where $f : \mathbb{N} \to A$ and $g : \mathbb{N} \to B$ are bijections. Define a bijection $h : \mathbb{N} \to A \bigcup B$ in terms of f and g, and prove the function h which you define is a bijection. (Hint: define $h(n)$ in cases of n is even or odd)

   Solution: Since A and B are denumerable, we have:

$$A = \{f(1), f(2), f(3), ...\}, B = \{g(1), g(2), g(3), ...\}$$

   and

$$A \cup B = \{f(1), g(1), f(2), g(2), f(3), g(3), ...\}$$

We can express the bijection of $h : \mathbb{N} \to A \cup B$ as

$$h(n) = \begin{cases} f(\frac{n+1}{2}) & : ifnisodd \\ g(\frac{n}{2}) & : ifniseven \end{cases}$$

(b) Let $A = \{3p-1 : p \in \mathbb{N}\}$ and $B = \{3p-2 : p \in \mathbb{N}\}$. Define a bijection $f : \mathbb{N} \to A$ and a bijection $g : \mathbb{N} \to B$. Then Prove that $A \bigcup B$ is denumerable by defining a bijection between $A \bigcup B$ and $\mathbb{N}$. (Hint: define $h : \mathbb{N} \to A \bigcup B$, and use problem but with specific f and g. You do not need to prove this h is a bijection again as this has been proved in part (a))

Solution: Let $f : \mathbb{N} \to A = \{3p-1 : p \in \mathbb{N}\}$, then $f(1) = 2, f(2) = 5, f(3) = 8$ and let $g : \mathbb{N} \to B = \{3p-2 : p \in \mathbb{N}\}$, then $g(1) = 1, f(2) = 4, f(3) = 7$. So, a bijection can be defined by:

$$h(n) = \begin{cases} 3(\frac{n+1}{2}) - 2 & : ifnisodd \\ 3(\frac{n}{2}) - 2 & : ifniseven \end{cases}$$

4. *(4 Points)* Define $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ by $f(i,j) = 2^{i-1}(2j-1)$. Prove f is a bijection thus $\mathbb{N} \times \mathbb{N}$ and $\mathbb{N}$ are numerically equivalent.

   (Hint: for injective, use Euclid's Lemma (11.13); for surjective, note that any positive integer n, from Theorem 11.17, n is the product of prime numbers. In particular, $n = 2^{i-1}p$ where p is the product of all prime factors of n.)

   Solution: $f(2,3) = 2^{2-1} \cdot (2 \cdot 3 - 1) = 2 \cdot 5 = 10$. This is a surjection because $\forall y \in \mathbb{N}$, you can choose some number, for example $56 = 8 \cdot 7 = 2^3 \cdot 7$, so $y = 2^{n_1} \cdot 3^{n_2} \cdot 5^{n_3}$... Now we prove injection: Suppose $f(i,j) = f(m,n)$, then

   $$2^{i-1}(2j-1) = 2^{m-1}(2n-1)$$

   From theorem 11.13, if $a, b, c \in \mathbb{Z}$ and $a \neq 0$, and if $a|bc$ and $gcd(a,b) = 1$, then $a|c$. From this, we have $2^{i-1}|2^{m-1}(2n-1)$ and we know $gcd(2^{i-1}, 2n-1) = 1$ because 2_{i-1} will always be a factor of 2 and $2n-1$ is an odd number, so we can conclude that $2^{i-1}|2^{m-1}$. Thus, f is a bijection and $\mathbb{N} \times \mathbb{N}$ and $\mathbb{N}$ are numerically equivalent.

5. *(4 Points)* Prove that the set of irrational numbers is uncountable.

   (Hint: prove by contradiction, and use problem 3 to prove $\mathbb{R}$ is denumerable, which contradicts with 10.11)

   Solution: We will prove this by contradiction. Let's suppose that the set of irrationals is countable. We know that $\mathbb{R} = \mathbb{Q} \cup \mathbb{I}$. Now, if $\mathbb{I}$ was countable then $\mathbb{R}$ would be the union of 2 countable sets, therefore making $\mathbb{R}$ countable. This is a contradiction, hence $\mathbb{I}$ is uncountable.

6. *(4 Points)* Consider the function $g : (-1,1) \to \mathbb{R}$ defined by $g(x) = \dfrac{x}{1-x^2}$. Show that $(-1,1)$ and $\mathbb{R}$ are numerically equivalent by proving (i) g is surjective; (ii) g is injective.

   Solution: (i) Prove g is surjective: Let $y = \frac{x}{1-x^2}$ and we solve for x. $y(1-x^2) = x \to y - yx^2 - x = 0$ or $-x^2y - x + y = 0$. So $x = \frac{1 \pm sqrt1+4y^2}{-2y}$, so $\forall y$, the function has a value. (ii) Prove g is injective: Suppose $g(x) = g(y)$, then

   $$\frac{x}{1-x^2} = \frac{y}{1-y^2} \to x(1-y^2) = y(1-x^2)$$

   $x - xy^2 = y - x^2y \to x - y - xy^2 + x^2y = 0$

   $(x-y)(xy-1) = 0$. Because the domain is $(-1,1)$, then $(xy-1) \neq 0$ so we know $x - y = 0$ or $x = y$. Therefore, g is injective.

7. *(extra 4 Points)* (This is same as Problem 2, but prove in a different way) Prove that $S = \{(a,b) : a, b \in \mathbb{N}, a \geq b\}$ is denumerable without using Theorem 10.4, but directly define a bijection $f : S \to \mathbb{N}$ or $g : \mathbb{N} \to S$.

8. *(extra 4 Points)* Define $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ by

$$f(i,j) = \frac{(i+j-1)(i+j-2)}{2} + i.$$

Prove that f is a bijection thus $\mathbb{N} \times \mathbb{N}$ and $\mathbb{N}$ are numerically equivalent.