Math 214 – Foundations of Mathematics
Homework 9

Due April 3rd

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Solve the following problems. Please remember to use complete sentences and good grammar.

1. *(4 Points)* Solve the following problems in $\mathbb{Z}_n$.

   (a) In $\mathbb{Z}_8$, express the following sums and products as $[r]$, where $0 \leq r < 8$:

   $$[3] + [6], [3] \cdot [6], [-13] + [138], [-13] \cdot [138]$$

   Solution: First, we know that $\mathbb{Z}_8 = \{[0], [1], [2], [3], [4], [5], [6], [7]\}$. So it follows that:
   $[3] + [6] = [1]$,
   $[3] \cdot [6] = [2]$,
   $[-13] + [138] = [5]$, and
   $[-13] \cdot [138] = [6]$

   (b) Let $[a], [b] \in \mathbb{Z}_8$. If $[a] \cdot [b] = [0]$, does it follow that $[a] = [0]$ or $[b] = [0]$?
   Solution: No, because if, for example, you let $[a] = [2]$ and $[b] = [4]$, it is evident that $[a] \cdot [b] = [0]$ while neither$[a]$ not $[b]$ equal $[0]$.

2. *(4 Points)* Prove that for any prime p, if $[a], [b] \in \mathbb{Z}_p$, then $[a] \cdot [b] = [0]$ implies $[a] = [0]$ or $[b] = [0]$.
   Solution:

   Proof. If $[a], [b] \in \mathbb{Z}_p$, then $\exists k \in \mathbb{Z}$ such that $a \cdot b = P \cdot k$. Then it follows that $P|(a \cdot b)$. So now we just need to prove that $P|a$ or $P|b$. From theorem 11.14, we have that if $a|bc$ then either $a|b$ or $a|c$. Hence, it follows that for any prime p, if $[a], [b] \in \mathbb{Z}_p$, then $[a] \cdot [b] = [0]$ implies $[a] = [0]$ or $[b] = [0]$. □

3. *(4 Points)* Prove that the multiplication in Z_n, $n \geq 2$, defined by $[a] \cdot [b] = [ab]$ is well-defined. (See Theorem 8.9 for the case of addition, and Result 4.11)
   Solution:

   Proof. Let $[a], [b], [c], [d] \in \mathbb{Z}_n$, where $[a] = [b]$ and $[c] = [d]$. Now we prove that [ac]=[bd]. It follows that $a \equiv c$ (mod n) and $b \equiv d$ (mod n). Thus, $ab \equiv cd$ (mod n) from result 4.11. So, we can conclude that $abRcd$ which implies that $[ab] = [cd]$. Hence, multiplication in $\mathbb{Z}_n$ is well-defined. □

4. *(4 Points)* Let p be a positive prime number and let $f : \mathbb{Z}_p \to \mathbb{Z}_p$ be defined as $f([x]) = [x^2]$. Show that f is a function. Give examples to show that it is not necessarily injective or surjective.
   Solution:

   Proof. First we prove that f is a function: i) $\forall [x] \in \mathbb{Z}_p, \exists [y] \in \mathbb{Z}_p$ such that $([x], [y]) \in f$. It is clear that $([x], [x^2]) \in f$, so f passes the first condition of being a function. ii) Next, if $[x] = [y]$, then $[x^2] = [y^2]$.

   Then, $x = y + kp$ and $x^2 = (y + kp)2 = (y + kp)(y + kp) = y^2 + 2ykp + (kp)^2$, which equals $y^2 + pm$ for some $m \in \mathbb{Z}$. Hence, f is a well-defined function.

   Next we show that f is not necessarily injective or surjective.

First, an example to show f is not injective: Let $p = 5$, then,

$$f([0]) = [0]$$

and

$$f([5]) = [5^2] = [25] = [0]$$

, so f is not injective.

No we show that f is not surjective: This can be demonstrated by letting $y = -5$. There is no x such that $[x^2] = [-5]$. $\square$

5. *(4 Points)* Consider the function $f(x) = \dfrac{3x-5}{x+2}$.

   (a) Determine the domain $D(f) \subseteq \mathbb{R}$ and range $R(f) \subseteq \mathbb{R}$ of the function f.
   Solution: The only restrictions on the domain of f are that the denominator of the functions cannot equal to 0, or $x + 2 \neq 0$, or $x \neq -2$. So, the domain of f is all $x \in \mathbb{R}$ not equal to 2. The range of the function is all real numbers, or $\mathbb{R}$.

   (b) Prove that the function $f : D(f) \to R(f)$ is bijective.

Solution: To prove that f is bijective we must prove that it is both injective and surjective. To prove it is injective we examine the equation:

$$\frac{3x-5}{x+2} = \frac{3y-5}{y+2}$$

Which can be rewritten as $(3x-5)(y+2) = (3y-5)(x+2)$,

which multiplies out to $3xy + 6x - 5y - 10 = 3xy + 6y - 5x - 10$

$3xy + 6x - 5y = 3xy + 6y - 5x$

$6x - 5y = 6y - 5x$

$11x = 11y$

$x = y$, so f is injective.

Next, we prove that f is surjective. If $f(x) = \frac{3x-5}{x+2}$, let $x \in \mathbb{R}$. So, $\forall y \in \mathbb{R}$, we try to find $x \in \mathbb{R}$ such that $f(x) = y$. Set $\frac{3x-5}{x+2} = y$. Then:

$3x - 5 = y(x+2)$

$-5 = y(x+2) - 3x$

$-5 = xy + 2y - 3x$

$-5 - 2y = xy - 3x$

$\frac{-5-2y}{y-3} = x$ or $x = \frac{2y+5}{3-y}$

So, $\forall y \in \mathbb{R}$, let $x = \frac{2y+5}{3-y}$, proving f is surjective. And since we have shown that f is surjective and injective, we can say that f is bijective.

6. *(4 Points)*

   (a) Prove that the function $f(x) = x^2 - 2x + 3$, with domain $x \in \mathbb{R}$, is neither injective nor surjective.
   Solution:

   Proof. First we prove that f is not injective:
   Suppose: $f(x) = f(y)$ or $x^2 - 2x + 3 = y^2 - 2y + 3$
   $x^2 - 2x = y^2 - 2y$
   $x^2 - y^2 - 2x + 2y = 0$
   $(x^2 - y^2) + (-2x + 2y) = 0$
   $(x+y)(x-y) - 2(x-y) = 0$
   $(x-y)(x+y-2) = 0$

Now, let $x = 0$ and $x = 2$ so that:
$f(0) = 0^2 - 2(0) + 3$ and $f(2) = 2^2 - 2(2) + 3 = 3$. So, f is not injective since $f(0) = f(2) = 3$.
Next we prove that f is not surjective.
Set $x^2 - 2x + 3 = y$
$x^2 - 2x + 1 - 1 + 3 = y$
$(x - 1)^2 = y - 2$
So, $\forall x \in \mathbb{R}$, $(x - 1)^2 \geq 0$. Now, choose $y = -15$, then $y - 2 = -15 - 2 = -17 < 0$.
This implies that for $y = -15$, $\forall x \in \mathbb{R}, x^2 - 2x + 3 \neq -15$. So, $f : \mathbb{R} \to \mathbb{R}$ is not surjective.

$\square$

(b) Prove that the function $f(x) = x^2 - 2x + 3$, with domain $x \in (-\infty, 0)$, is a bijection from $(-\infty, 0)$ to its range. What is the range of $f(x)$? Determine the inverse function $f^{-1}(x)$, its domain and range.

Solution: To prove something is bijective, we must prove it is injective and surjective. To prove it is injective:
Suppose $f(x) = f(y), x, y \in A$. Then
$x^2 - 2x + 3 = y^2 - 2y + 3$
$(x - y)(x + y - 2) = 0$
Since $x > 0$ and $y > 0$, then either $x - y > 0$ or $x + y - 2 \geq -2$. Now, recall the axiom: $a, b \in \mathbb{R}, a \cdot b = 0 \implies a = 0 \vee b = 0$. Then $x - y = 0$ so $x = y$. So, f is injective from $(-\infty, 0)$.
Next we prove surjectivity:
Solving $x^2 - 2x + 3 = y$, we get $(x - 1)^2 = y - 2$ and $y - 2 > 0$ so $\sqrt{y - 2} \in \mathbb{R}$, or $x = 1 \pm \sqrt{y - 2}$.
Choose $x = 1 \pm \sqrt{y - 2} \in A$, then

$$f(1 \pm \sqrt{y - 2}) = (1 \pm \sqrt{y - 2})^2 - 2(1 \pm \sqrt{y - 2}) + 3 = y$$

, so f is surjective.
Also, the range of $f(x)$ is $[2, \infty)$.
To find the inverse of $f(x)$, we solve for the variable x, and then switch the variables, as shown below.
$x = y^2 - 2y + 3$
$y^2 - 2y = x - 3$
$y^2 - 2y + 1 - 1 = x - 3$
$\sqrt{(y - 1)^2} = \pm\sqrt{x - 2}$
$y = 1 \pm \sqrt{x - 2}$
So, $f^{-1}(x) = 1 \pm \sqrt{x - 2}$ and the domain of $f^{-1}(x)$ is $[2, \infty)$ and the range is the set of all real numbers.