Math 214 – Foundations of Mathematics
Homework 7

Due March 20, 2014

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1. *(4 Points)* Use induction to prove that, for all $n \in \mathbb{N}$,

$$\sum_{i=1}^{n} i(i+1) = \frac{n(n+1)(n+2)}{3}.$$

Proof. **Solution:** We prove by mathematical induction:

Step 1: When $n = 1$, the left hand side of the equation can be written as $1(1+1) = 2$ and the right hand side of the equation can be written as $\frac{1(1+1)(1+2)}{3} = \frac{1\cdot 2\cdot 3}{3} = \frac{6}{3} = 2$, which is the same as the left. Hence, it is true when $n = 1$ (base step).

Step 2: Next we prove if it is true for $n = k$ for $k \in \mathbb{N}$, then it is also true for $n = k+1$. Since it is true for $n = k$, then:

$$\sum_{i=1}^{k} i(i+1) = \frac{k(k+1)(k+2)}{3}$$

For $n = k+1$:

$$\sum_{i=1}^{k+1} i(i+1) = \sum_{i=1}^{k+1} i^2 + i = \frac{k(k+1)(k+2)}{3} + \frac{3(k+1)^2}{3} + \frac{3(k+1)}{3}$$

$$= \frac{k(k+1)(k+2) + 3(k+1)^2 + 3(k+1)}{3} = \frac{k^3 + 2k^2 + k^2 + 2k + 3k^2 + 6k + 3 + 3k + 3}{3}$$

$$= \frac{k^3 + 6k^2 + 11k + 6}{3} = \frac{(k^2 + 3k + 2)(k+3)}{3} = \frac{(k+1)(k+2)(k+3)}{3}$$

So, it is true for $n = k+1$.

Step 3: By the principle of mathematical induction, we have prove that $n \in \mathbb{N}$,

$$\sum_{i=1}^{n} i(i+1) = \frac{n(n+1)(n+2)}{3}.$$

□

2. *(4 Points)* Use induction to prove that for all integers $n \geq 3$, $n^3 \leq 3^n$.

Proof. **Solution:** We prove by PMI:

Step 1: When $n = 3$, $3^3 = 3^3$. Hence, it is true for $n = 3$.

Step 2: Suppose for $k \in \mathbb{N}$ and $k \geq 3$, we have $k^3 \leq 3^k$. We need to prove: $(k+1)^3 \leq 3^{k+1}$. Lets start by expanding the left side of the equation: $(k+1)^3 = k^3 + 3k^2 + 3k + 1$. Now, from inductive assumption:

$$3^{k+1} = 3 \cdot 3^k > 3k^3 = k^3 + k^3 + k^3 \leq k^3 + 3k^2 + 3k^2 > k^3 + 3k^2 + 9k$$

$$= k^3 + 3k^2 + 3k + 6k > k^3 + 3k^2 + 3k + 1$$

because $\forall k \geq 3, 6k > 1$.

Step 3: By the strong principle of mathematical induction, $\forall n \in \mathbb{Z}$ and $n \geq 3, n^3 \leq 3^n$.

□

3. *(4 Points)* Use induction to prove that for every positive integer n,

$$1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n}.$$

Proof. **Solution:** This can be restated as:

$$\sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - \frac{1}{n}$$

Step 1: When $n = 1$: The left side of the eqn equals $\frac{1}{1^2} = 1$ and the right side of the eqn equals $2 - \frac{1}{1} = 1$. They are equal, hence it is true when $n = 1$.

Step 2: Now suppose it is true for $n = k$. Consider

$$\sum_{i=1}^{k+1} \frac{1}{i^2} = \sum_{i=1}^{k} \frac{1}{i^2} + \frac{1}{(k+1)^2} \leq 2 - \frac{1}{k} + \frac{1}{(k+1)^2}$$

We need to prove that $2 - \frac{1}{k} + \frac{1}{(k+1)^2} \leq 2 - \frac{1}{k+1)}$. This is equivalent to $\frac{-1}{k} + \frac{1}{(k+1)^2} \leq \frac{-1}{k+1}$. Multiplying by a common denominator, we get $-(k+1)^2 + k \leq -k(k+1)$. Now let's prove the lemma $\forall k \in \mathbb{N}, \frac{-1}{k} + \frac{1}{(k+1)^2} \leq \frac{-1}{k+1}$. To do this, we take something we know to be true, such as: $-1 \leq 0$. With a little algebra, we get $-(k^2+2k+1)+k \leq -k-k^2$. This can be rewritten as $-(k+1)^2+k \leq -k(k+1)$, which is the same as the expression we had before when we expressed everything with a common denominator. Hence, it is true for $n = k+1$.

Step 3: By the PMI, we have proven that for every positive integer n,

$$1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n}.$$

□

4. *(4 Points)* Prove that $7|(3^{2n} - 2^n)$ for every nonnegative integer n. (You can use induction or other ways)

Proof. **Solution:**

Step 1: When $n = 1$: $3^2 - 2 = 7$ and $7|7$, hence it is true, when $n = 1$.

Step 2: Suppose it is true for $n = k$. Now consider $3^{2(k+1)} - 2(k+1) = 3^{2k+2} - 2k - 2$. Since $7|(3^{2k} - 2k)$, then $\exists m \in \mathbb{Z}$ such that $3^{2k} - 2k = 7m$ or $3^{2k} = 7m + 2k$ and $2k = 3^{2k} - 7m$. Now the equation can be restated as:

$$3^{2(k+1)} - 2(k+1) = 3^2 \cdot 3^{2k} - 2k \cdot 2$$

$$= 9(7m + 2k) - 2(3^{2k} - 7m = 18k + 63m - 6^{2k} - 14m = 18^k + 49m - 2(2^k - 7m)$$

$$= 18k - 4k + 49m + 7m = 14^k + 63m = 7(2^k + 9m)$$

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Hence, $7|(2^k + 9m)$

Step 3: By the PMI, it is proven that $7|(3^{2n} - 2^n)$ for every nonnegative integer n.

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5. *(4 Points)* We need to put n cents of stamps on an envelop, but we have only (an unlimited supply of) 5 cents and 12 cents stamps. Prove that we can perform the task if $n \geq 44$. (You can use Strong Principle of Mathematical Induction or other methods.)

Solution: The above statement is equivalent to: $\forall n \in \mathbb{N}$ and $n \geq 44$, $\exists x, y \in \mathbb{N} \cup \{0\}$ such that $n = 5x + 12y$. So, we can reach the following calculations:

$n = 44 = 5 \times 4 + 12 \times 2$

$n = 45 = 5 \times 9 + 12 \times 0$

$n = 46 = 5 \times 2 + 12 \times 3$

$n = 47 = 5 \times 7 + 12 \times 1$

$n = 48 = 5 \times 0 + 12 \times 4$

$n = 49 = (n = 44) + 5$

$n = 50 = (n = 45) + 5$

$n = 51 = (n = 46) + 5$

Proof. **Step 1:** $n = 44$

Let $x = 4$ and $y = 2$, then $44 = 5 \times 4 + 12 \times 2$ (base step).

Step 2: Suppose for: $44 \leq n \leq k, \exists x, y \in \mathbb{N} \cup \{0\}$ such that $n = 5x + 12y$. We need to prove $k + 1 = 5m + 12p$ for some $m, n \in \mathbb{N} \cup \{0\}$. This can be expressed as $(k + 1) = (k - 4) + 5$.

Case 1: $k - 4 \geq 44$ then $44 \leq k - 4 \leq k$. From assumption, $\exists x, y \in \mathbb{N} \cup \{0\}$ such that $k - 4 = 5x + 12y$ then $k + 1 = (k - 4) + 5$ so $k + 1 = 5(x + 1) + 12y$. So, $m = x + 1$ and $p = y$.

Case 2: If $k - 4 < 44$, then there are 4 scenarios:

a) $k - 4 = 43$ so $k = 47$ and $k + 1 = 48 = 5 \times 0 + 12 \times 4$. So, $m = 0$ and $p = 4$.

b) $k - 4 = 42$ so $k = 46$ and $k + 1 = 47 = 5 \times 7 + 12 \times 1$. So, $m = 7$ and $p = 1$.

c) $k - 4 = 41$ so $k = 45$ and $k + 1 = 46 = 5 \times 2 + 12 \times 3$. So, $m = 2$ and $p = 3$.

d) $k - 4 = 40$ so $k = 44$ and $k + 1 = 45 = 5 \times 9 + 12 \times 0$. So, $m = 9$ and $p = 0$.

Step 3: From the strong principle of mathematical induction, any postage greater than or equal to 44 cents can be given by a combination of 5 and 12 cent stamps.

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6. *(4 Points)*

(a) Prove that for $n \in \mathbb{N}$, $2013^n \equiv 3^n \pmod{10}$. (Use 4.11 and induction)

Proof. **Solution:** If $2013^n \equiv 3^n \pmod{10}$, then $10|(2013^n - 3^n)$
Step 1: When $n = 1$ (base step), $2013^1 - 3^1 = 2010$ and $10|2010$, hence it is true when $n = 1$.
Step 2: Suppose it is true for $n = k$, so $10|(2013^k - 3^k)$. Now consider $n = k + 1$.

$$2013^{k+1} - 3^{k+1} = 2013 \cdot 2013^k - 3 \cdot 3^k$$

Now, this means that $\exists m \in \mathbb{N}$ such that $2013^k - 3^k = 10m$ or $2013^k = 10m + 3^k$. Substituting in this expression, we get:

$$2013 \cdot (10m + 3^k) - 3 \cdot 3^k$$

which can be rewritten as

$$10m + 2013 \cdot 3^k - 3 \cdot 3^k = 10m + 2010 \cdot 3^k = 10(m + 201 \cdot 3^k)$$

Therefore, $10|(2013^k - 3^k)$
Step 3: By the principle of mathematical induction, we have proven that for $n \in \mathbb{N}$, $2013^n \equiv 3^n \pmod{10}$.

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(b) Prove that for $k \in \mathbb{N}$, $3^{4k-3} \equiv 3 \pmod{10}$, $3^{4k-2} \equiv 9 \pmod{10}$, $3^{4k-1} \equiv 7 \pmod{10}$, $3^{4k} \equiv 1 \pmod{10}$. (Use induction)

Proof. **Solution:**
Step 1: When $n = 1$:
$3^{4\cdot1-3} = 3^1 = 3 \equiv 3 \pmod{10}$
$3^{4\cdot1-2} = 3^2 = 9 \equiv 9 \pmod{10}$
$3^{4\cdot1-1} = 3^3 = 27 \equiv 7 \pmod{10}$
$3^{4\cdot1} = 3^4 = 81 \equiv 1 \pmod{10}$
Step 2: Suppose it is true for $n = k$, $k \geq 1$.
$3^{4(k+1)-3} = 3^{4k+4-3} = 3^{4k+1} \equiv 3^4 \cdot 3 = 243 \equiv 3 \pmod{10}$
$3^{4(k+1)-2} = 3^{4k+4-2} = 3^{4k+2} \equiv 3^4 \cdot 9 = 729 \equiv 9 \pmod{10}$
$3^{4(k+1)-1} = 3^{4k+4-1} = 3^{4k+3} \equiv 3^4 \cdot 7 = 567 \equiv 7 \pmod{10}$
$3^{4(k+1)} = 3^{4k+4} = 3^{4k+4} \equiv 3^4 \cdot 1 = 81 \equiv 1 \pmod{10}$
Step 3: By the PMI, we have proven that $k \in \mathbb{N}$, $3^{4k-3} \equiv 3 \pmod{10}$, $3^{4k-2} \equiv 9 \pmod{10}$, $3^{4k-1} \equiv 7 \pmod{10}$, $3^{4k} \equiv 1 \pmod{10}$.

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(c) Find the last digit of 2013^{2010} by using (a) and (b).

Solution: We have already proven that the last digit of 2013^2010 will be the same as the last digit of 3^2010. Since we have already proven (b), we just need to determine whether 2010 falls under $4k - 3$, $4k - 2$, $4k - 1$, or $4k$. Dividing 2010 by 4 leaves us with a remainder of 2, so that means the 2010 has to be expressed using $4k - 2$. From part (b), $3^{4k-2} \equiv 9 \pmod{10}$, so the last digit of 2013^{2010} has to be 9.

7. *(extra 2 Points)* Find the last three digits of 7^{9999}.