Math 214 – Foundations of Mathematics
Homework 4

Due February 13, 2014

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Solve the following problems. Please remember to use complete sentences and good grammar.

1. *(4 Points)* Let $x, y \in \mathbb{Z}$. Prove that if $3 \nmid x$ and $3 \nmid y$, then $3|(x^2 - y^2)$.

Solution:

Proof. First, it should be stated that $(x^2 - y^2) = (x + y)(x - y)$. Now, if $3 \nmid x$ and $3 \nmid y$, then there are 4 cases that must be proven:

$$\begin{gathered}
\textbf{Case 1: } x = 3n + 1 \text{ and } y = 3m + 1 \\
\text{Let } 3|(3n + 1)^2 - (3m + 1)^2 \\
\text{Then, } ((3n + 1) + (3m + 1))((3n + 1) - (3m + 1)) \\
= (3n + 1 + 3m + 1)(3n + 1 - 3m - 1) \\
= 9n^2 + 9mn + 6n - 9mn - 9m^2 - 6m \\
= 3(3n^2 - 3m^2 + 2n - 2m) \\
\text{Therefore, 3 divides } x^2 - y^2 \text{ in this case.}
\end{gathered}$$

$$\begin{gathered}
\textbf{Case 2: } x = 3n + 1 \text{ and } y = 3m + 2 \\
\text{Let } 3|(3n + 1)^2 - (3m + 2)^2 \\
\text{Then, } ((3n + 1) + (3m + 2))((3n + 1) - (3m + 2)) \\
= (3n + 3m + 3)(3n + 1 - 3m - 2) \\
= 9n^2 + 9mn + 9n - 9mn - 9m^2 - 9m - 3n - 3m - 3 \\
= 3(3n^2 + 3n - 3m^2 - 3m - n - m - 1) \\
\text{Therefore, 3 divides } (x^2 - y^2) \text{ in this case.}
\end{gathered}$$

$$\begin{gathered}
\textbf{Case 3: } x = 3n + 2 \text{ and } y = 3m + 1 \\
\text{Let } 3|(3n + 2)^2 - (3m + 1)^2 \\
\text{Then, } ((3n + 2) + (3m + 1))((3n + 2) - (3m + 1)) \\
= (3n + 3m + 3)(3n - 3m + 1) \\
= 9n^2 + 9mn + 9n - 9mn - 9m^2 - 9m + 3n + 3m + 3 \\
= 3(3n^2 + 3n - 3m^2 - 3m + n + m + 1) \\
\text{Therefore, 3 divides } (x^2 - y^2) \text{ in this case.}
\end{gathered}$$

$$\textbf{Case 4: } x = 3n + 2 \text{ and } y = 3m + 2$$
$$\text{Let } 3|(3n + 2)^2 - (3m + 2)^2$$
$$\text{Then, } ((3n + 2) + (3m + 2))((3n + 2) - (3m + 2))$$
$$= (3n + 3m + 4)(3n - 3m)$$
$$= 9n^2 + 9mn + 12n - 9mn - 9m^2 - 12m$$
$$= 3(3n^2 + 4n - 3m^2 - 4m)$$
$$\text{Therefore, 3 divides } (x^2 - y^2) \text{ in this case.}$$

Therefore, it is proven that if $3 \nmid x$ and $3 \nmid y$, then $3|(x^2 - y^2)$.

□

2. *(4 Points)* Let $n \in \mathbb{Z}$. Prove that $2|(n^4 - 3)$ if and only if $4|(n^2 + 3)$. (Hint: prove n is odd)

Solution:

Proof. Because of the "if and only if" in the question, it is necessary to prove that if $2|(n^4 - 3)$ then $4|(n^2 + 3)$ and if $4|(n^2 + 3)$ then $2|(n^4 - 3)$. This proof will be divided into two parts.

Part 1: Prove that if $2|(n^4 - 3)$ then $4|(n^2 + 3)$. To do this we will introduce the lemma that n is odd.

Step 1: Prove that if $2|(n^4 - 3)$, then n is odd. Use contrapositive: If n is even, then $2 \nmid (n^4 - 3)$.

$$\text{Let } n \text{ be even, then } n = 2x, x \in \mathbb{Z}.$$
$$\text{So, } ((2x)^4 - 3)$$
$$= 16x^4 - 3$$
$$= 2(8x^4 - 3/2)$$
$$\text{Therefore, } 2 \nmid (n^4 - 3).$$

$$\textbf{Step 2: } \text{If } n \text{ is odd, then } 4|(n^2 + 3).$$
$$\text{Let } n \text{ be odd, then } n = 2x + 1, x \in \mathbb{Z}.$$
$$\text{So, } (2x + 1)^2 + 3$$
$$= (2x + 1)(2x + 1) + 3$$
$$= 4x^2 + 4x + 4$$
$$= 4(x^2 + x + 1)$$
$$\text{Therefore, if } n \text{ is an odd number, then } 4|(n^2 + 3).$$

Part 2: Next, we need to prove that if $4|(n^2 + 3)$ then $2|(n^4 - 3)$. Again, we will use a lemma, n is odd.

$$\textbf{Step 1: } \text{Use contrapositive: If } n \text{ is even, then } 4 \nmid (n^2 + 3)$$
$$\text{Let } n \text{ be even, then } n = 2x, x \in \mathbb{Z}.$$
$$\text{So, } (2x)^2 + 3$$
$$= 4x^2 + 3$$
$$= 4(x^2 + 3/4)$$
$$\text{Therefore, 4 does not divide } (n^2 + 3).$$

Step 2: If n is odd then $2|(n^4-3)$.
Let n be odd, then $n = 2x+1, x \in \mathbb{Z}$.
So, $(2x+1)^4 - 3$
$= 16x^4 + 32x^3 + 24x^2 + 8x - 2$
$= 2(8x^4 + 16x^3 + 12x^2 + 4x - 1)$
Therefore, if n is odd then $2|(n^4-3)$.

Therefore, following parts 1 and 2, we have finally proven that $2|(n^4-3)$ if and only if $4|(n^2+3)$.

$\square$

3. *(4 Points)* Prove that if x is a real number such that $x^2 + x > 2$, then either $x < -2$ or $x > 1$. (Hint: use axioms and Theorems 1-2 in Notes 1)

Solution:

Proof. Prove by contrapositive: If $x \geq -2$ and $x \leq 1$, then $x^2 + x \leq 2$.
Since $x \geq -2$, then $x + 2 \geq 0$. Since $x \leq 1$, then $x - 1 \leq 0$.
Recall the real number axiom: if $x \geq 0$ and $y \leq 0$, then $xy \leq 0$. Thus $x + 2 \geq 0$ and $x - 1 \leq 0$.
This implies that $(x+2)(x-1) \leq 0$. Expanding $(x+2)(x-1)$, we get:

$$(x+2)(x-1) = x^2 + x - 2 \leq 0$$

$$x^2 + x \leq 2$$

Hence the contrapositive is true, then the original is also true.

$\square$

4. *(4 Points)* Prove that for every two positive real numbers a and b that

$$(a+b) \cdot \left(\frac{1}{a} + \frac{1}{b}\right) \geq 4.$$

(Hint: use axioms and Theorems 1-2 in Notes 1)

Solution:

Proof. First of all, lets rearrange the left side of the inequality:

$$\frac{a}{a} + \frac{a}{b} + \frac{b}{a} + \frac{b}{b} \geq 4$$

Let a and b be any positive real numbers. Then, the inequality can be rewritten as:

$$\frac{a}{b} + \frac{b}{a} + 2 \geq 4$$

or

$$\frac{a}{b} + \frac{b}{a} \geq 2$$

We know $\frac{a}{b} > 0$ and $\frac{b}{a} > 0$, so $\frac{a}{b} + \frac{b}{a} \geq 0$. Then the equivalent inequality

$$\frac{a^2 + b^2}{ab} \geq 0$$

.

Following from this:

$$\frac{(a^2 + b^2 + 2ab) - 2ab}{ab} \geq 0$$

$$\frac{(a+b)^2}{ab} - \frac{2ab}{ab} \geq 0$$

$$\frac{(a+b)^2}{ab} \geq 2$$

Hence, $\frac{a}{b} + \frac{b}{a} \geq 2$

Therefore, for every two positive real numbers a and b that $(a+b) \cdot \left(\frac{1}{a} + \frac{1}{b}\right) \geq 4$. □

5. *(4 Points)* Let A, B, C be sets. Prove that $(A - B) \cup (A - C) = A - (B \cap C)$.

Solution:

Proof. **Step 1:** $(A - B) \cup (A - C) \subseteq A - (B \cap C)$

Let $x \in (A - B) \cup (A - C)$, then either $x \in (A - B)$ or $x \in (A - C)$.

Case 1: $x \in (A - B)$

Then $x \in A$ and $x \notin B$

Since $x \notin B$ then $x \notin (B \cap C)$

Since $x \in A$ and $x \notin (B \cap C)$

Then $x \in A - (B \cap C)$

Case 2: $x \in (A - C)$

Then $x \in A$ and $x \notin C$

Since $x \notin C$ then $x \notin (B \cap C)$

Since $x \in A$ and $x \notin (B \cap C)$

Then $x \in A - (B \cap C)$.

Step 2: $A - (B \cap C) \subseteq (A - B) \cup (A - C)$

Let $x \in A - (B \cap C)$

Then $x \in A$ and $x \notin (B \cap C)$

Since, $x \notin (B \cap C)$

Then $x \notin B$ and $x \notin C$

So, $x \in A \subseteq (A - B) \cup (A - C)$

Therefore, $(A - B) \cup (A - C) = A - (B \cap C)$.

□

6. *(4 Points)* Let A and B be sets. Prove that $A = (A - B) \cup (A \cap B)$. (Hint: if $x \in A$, then there are two cases: $x \in B$ or $x \notin B$.)

Solution:

Proof. **Step 1:** Prove that $A \subseteq (A - B) \cup (A \cap B)$

Let $x \in A$, then $x \in B$ or $x \notin B$.

Case 1: $x \in B$

Since $x \in A$ and $x \in B$,

Then, $x \in (A \cap B) \subseteq (A - B) \cup (A \cap B)$

Case 2: $x \notin B$

Since $x \in A$ and $x \notin B$,

Then, $x \in (A - B) \subseteq (A - B) \cup (A \cap B)$

Step 2: Prove that $(A - B) \cup (A \cap B) \subseteq A$

Let $x \in (A - B) \cup (A \cap B)$.

Then either $x \in (A - B)$ or $x \in (A \cap B)$.

Case 1: $x \in (A - B)$

Then $x \in A$ and $x \notin B$

and $x \in (A - B) \subseteq A$

Case 2: $x \in (A \cap B)$

then $x \in A$ and $x \in B$

So, $x \in (A \cap B) \subseteq A$.

Therefore, it is proven that $A = (A - B) \cup (A \cap B)$.

□

7. *(extra 2 Points)* Prove that for every three positive real numbers a, b and c that

$$(a + b + c) \cdot \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) \geq 9.$$

8. *(extra 2 Points)* Prove that for every three positive real numbers a, b and c that

$$a^2 + b^2 + c^2 \geq ab + bc + ac.$$

Note: For problem 4, 7 and 8, you can only use axioms and Theorems 1-2 in Notes 1, but not other more advanced theorems or known inequalities.