Math 214 – Foundations of Mathematics
Homework 10

Due April 10th

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Solve the following problems. Please remember to use complete sentences and good grammar.

1. *(4 Points)* Let $A = \mathbb{R} - \{1\}$ and define a function $f : A \to A$ by $f(x) = \dfrac{x}{x-1}$ for all $x \in A$.

   (a) Prove that f is bijective.
   Solution: First we prove f is injective. Suppose $x, y \in A$ and $f(x) = f(y)$ then

   $$\frac{x}{x-1} = \frac{y}{y-1} \to x(y-1) = y(x-1)$$

   $xy - x = xy - y \to -x = -y \to x = y$, so f is injective.
   Now we prove f is surjective. Take $y \in A$. Solve $y = \frac{x}{x-1} \to y(x-1) = x$.
   $xy - y - x = 0$
   $xy - x = y \to x(y-1) = y \to x = \frac{y}{y-1}$. So. $\forall y \in A$, we define $x = \frac{y}{y-1}$. Then

   $$f(\frac{y}{y-1}) = \frac{(\frac{y}{y-1})}{(\frac{y}{y-1}) - 1} = y$$

   So f is surjective. Because f is both injective and surjective, we can say f is bijective.

   (b) Determine f^{-1}.
   Solution $\frac{x}{x-1} \to xy - y - x = 0 \to x(y-1) = y \to x = \frac{y}{y-1}$. So the inverse can be defined by $f^{-1}(x) = \frac{x}{x-1}$.

   (c) Determine $f \circ f \circ f$.

   Solution: First, let's determine $f \circ f = \dfrac{(\dfrac{x}{x-1})}{(\dfrac{x}{x-1}) - 1}$. Then $f \circ f \circ f =$

   $$\frac{\dfrac{(\dfrac{x}{x-1})}{(\dfrac{x}{x-1}) - 1}}{\dfrac{(\dfrac{x}{x-1})}{(\dfrac{x}{x-1}) - 1} - 1}$$

2. *(4 Points)* Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be defined by

   $$f(x) = 2x + 3, \quad g(x) = -3x + 5.$$

   One can prove that f and g are both bijective.

   (a) Determine the composition $g \circ f$.
   Solution: $g \circ f = 3(2x + 3) + 5$ or $y = -6x - 4$

(b) Determine the inverse functions f^{-1} and g^{-1}.
Solution: $f^{-1}(x) = \frac{x-1}{2}$ and $g^{-1}(x) = \frac{x-5}{-3}$

(c) Determine the inverse function $(g \circ f)^{-1}$ of $g \circ f$ and the composition $f^{-1} \circ g^{-1}$. What conclusion can you obtain here?
Solution: $(g \circ f)^{-1}(x) = \frac{x+4}{-6}$ and $f^{-1} \circ g^{-1} = \frac{(\frac{x-5}{-3})-3}{2}$. Because f anf g are both bijective, then so is $(g \circ f)$.

3. *(4 Points)* Define $h : \mathbb{Z}_4 \to \mathbb{Z}_6$ by $h([a]) = [3a]$ for each $a \in \mathbb{Z}$.

(a) Prove that h is well-defined thus it is a function.
Solution: h is well defined because if $(x, y) \in h$, and $(x, z) \in h$, then $y = z$. This is clear because the equivalence classes for $\mathbb{Z}_4 = \{[0], [1], [2], [3]\}$ and $\mathbb{Z}_6 = \{[0], [1], [2], [3], [4], [5]\}$. Because of the following:
$[3 \cdot 0] = [0] \in \mathbb{Z}_6$, and
$[3 \cdot 1] = [3] \in \mathbb{Z}_6$, and
$[3 \cdot 2] = [6] = [0] \in \mathbb{Z}_6$, and
$[3 \cdot 3] = [9] = [3] \in \mathbb{Z}_6$. So, h is well defined.

(b) Prove h is neither injective nor surjective.
Solution: h is not injective because if you take $a = 0$ and $a = 2$, then $[3 \cdot 0] = [0] \in \mathbb{Z}_6$, and $[3 \cdot 2] = [6] = [0] \in \mathbb{Z}_6$.
h is not surjective because if you take $a = 5$, then $[3 \cdot 5] = [15] = [3]$

4. *(4 Points)* Prove that the function $f : [0, \infty) \to [0, \infty)$ defined by $f(x) = \dfrac{x^2}{2x+1}$ is a bijection, and determine the inverse function $f^{-1}(x)$ for $x \in [0, \infty)$.

Solution: First, we must prove it is injective. Suppose $f(x) = f(y)$, then

$$\frac{x^2}{2x+1} = \frac{y^2}{2y+1} \to x^2(2y+1) = y^2(2x+1)$$

$$2yx^2 - 2xy^2 + x^2 - y^2 = 0 \to (2xy + x + y)(x - y) = 0$$

Since $x > 0$ and $y > 0$, then $2xy + x + y > 0$, so $x - y = 0$ or $x = y$, so f is injective.

Now we prove surjectivity: Let $y = \frac{x^2}{2x+1} \to y(2x+1) = x^2 \to y = x^2 - 2xy$

So, for any x, let $x = y + sqrty^2 + y$

Also, the inverse function can be expressed as $f^{-1}(x) = x + sqrtx^2 + x$.

5. *(4 Points)* Give an example of a function $f : \mathbb{Z} \to \mathbb{N}$ that is

(a) surjective but not injective;
Solution: The function $f(x) = x^2 + 1$ defined by $f : \mathbb{Z} \to \mathbb{N}$ is surjective but not injective.

(b) injective but not surjective.
Solution: The function $f(x) = e^x + 5$ defined by $f : \mathbb{Z} \to \mathbb{N}$ is injective but not surjective.

6. *(4 Points)* For nonempty sets A and B and functions $f : A \to B$ and $g : B \to A$ suppose that $g \circ f = i_A$, the identity function on A. Prove that f is injective and g is surjective.

Solution: Suppose $f(x) = f(y)$ for some $x, y \in A$.

Since $f(x) = f(y)$, then $g(f(x)) = g(f(y))$ or $(g \circ f)(x) = (g \circ f)(y)$. Since $(g \circ f) = i_A$, then $(g \circ f)(x) = x$ and $(g \circ f)(y) = y$, hence $x = (g \circ f)(x) = (g \circ f)(y) = y$ So, f is injective.

Now we prove surjectivity: $\forall y \in A$, we need to find $x \in A$ such that $f(x) = y$. Since, $(g \circ f) = i_A$, then $(g \circ f)(y) = i_A(y) = y$. let $x = f(y)$, then $f(x) = g(f(xy)) = y$, so f is surjective.

7. *(extra 4 Points)* For nonempty sets A and B and functions $f : A \to B$ and $g : B \to A$ suppose that $g \circ f = i_A$, the identity function on A.

(a) *(1 Points)* Show that f is not necessarily surjective.
(b) *(1 Points)* Show that g is not necessarily injective.
(c) *(2 Points)* Prove: f is surjective if and only if g is injective.