High to High Dimensional Multivariate Mixture Regression

Alex White

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Idea

Goal: Correctly cluster observations & regress in high dimensional X & Y.

- $Y_{n\times q}$ Matrix of responses
- $X_{n\times p}$ Design matrix
- $A_{p\times q}$ Coefficient matrix (sparse in p)
- k clusters

$$f\left(\boldsymbol{y}_i \mid \boldsymbol{x_i};\boldsymbol{\theta}\right) = \Sigma_{k=1}^{K}\pi_k\mathcal{N}_q\left(\boldsymbol{y_i};\boldsymbol{x_i}A_k,\Sigma_k\right)$$

- Parameter space $\theta = \{\pi_k, A_k, \Sigma_k; k = 1 \dots K\}$ solved by general EM using SARRS to compute A_k.

SARRS

Algorithm 1: Subspace Assisted Regression with Row Sparsity (SARRS)

Input: Observed response matrix Y, design matrix X,
rank r, initial matrix $V_{(0)}$ and penalty function $\rho(\cdot;\lambda)$ with penalty level λ.

Output: Estimated coefficient matrix $\widehat{A}$.

1 Group penalized regression

$$B_{(1)} = \underset{B \in \mathbb{R}^{p \times r}}{\arg\min} \left\{ \|YV_{(0)} - XB\|_F^2/2 + \rho(B;\lambda) \right\},$$

2 Compute the left singular vectors of $XB_{(1)}$, denoted by $U_{(1)}$.

3 Compute the right singular vectors of $U_{(1)}U'_{(1)}Y$, denoted by $V_{(1)}$.

4 Group penalized regression

$$B_{(2)} = \underset{B \in \mathbb{R}^{p \times r}}{\arg\min} \left\{ \|YV_{(1)} - XB\|_F^2/2 + \rho(B;\lambda) \right\},$$

5 Compute the estimated coefficient matrix by $\widehat{A} = B_{(2)}V'_{(1)}$.

Figure 1: “SARRS Main Algorithm”

HTH Mixture Algorithm

- Initialize: $\pi_k^{(0)} = \frac{n_k^{(0)}}{n}$
- Randomly initialize observations into k clusters

While not converged ($m = 1, \ldots, M$) do:

- for $k = 1, \ldots, K$ apply SARRS on all observations in $C_k^{(m-1)}$ to obtain $A_k^{(m)}$, $\Sigma_k^{(m)}$
- compute $\mu_{ik}^{(m)} = \mathcal{N}_p\left(\boldsymbol{y_i}; A_k^{(m)}\boldsymbol{x_i}, \Sigma_k^{(m)}\right)$
- $C_k^{(m)} = \{i|\text{ML component k}\}$
- $\pi_k^{(m)} = \frac{n_k^{(m)}}{n}$

HTH Mixture Algorithm

- Empirically, HTH Mixture reaches local maximum quickly
- Need to determine method for efficiently exploring the likelihood space with different random initialization.

Likelihood Space Exploration Idea 1

- Mimic the idea of chains from MCMC
- In parallel, run large number of chains and select the one which reaches the largest likelihood.
- This method works well, although not the most efficient way to explore the space.

Likelihood Space Exploration Idea 2

- To more efficiently explore the space, when a local maximum is reached, perturb the initialized state by a proportion p (e.g. 20%) and rerun.

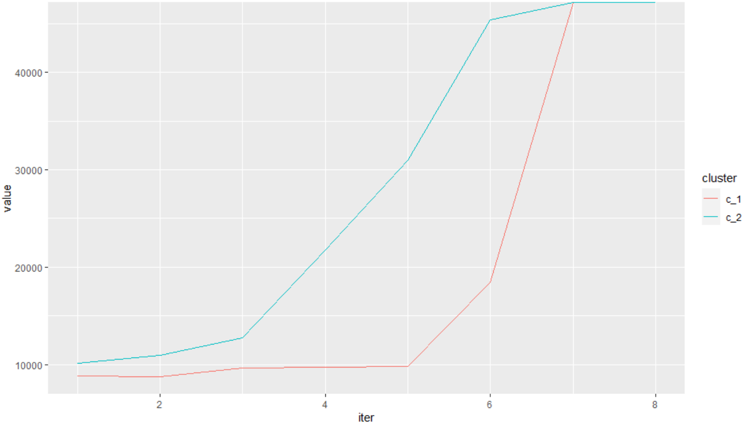
Data Simulation

- X_k consists of iid random vectors sample from $MVN(\mathbf{0}, \Sigma_k)$
- Σ_k diagonal
- Noise matrix $Z_k \in \mathbb{R}^{n \times q}$ has iid $N(0, \sigma^2)$ entries
- $A_k = \begin{pmatrix} b_k B_{0_k} B_{1_k} \\ 0 \end{pmatrix}$
  - with $b > 0$, $B_0 \in \mathbb{R}^{s \times r}$, $B_1 \in \mathbb{R}^{r \times q}$
- $Y_k = X_k A_k + Z_k$

Finally, combine X & Y

Results

- HTH algorithm consistently reaches local maximum



Results

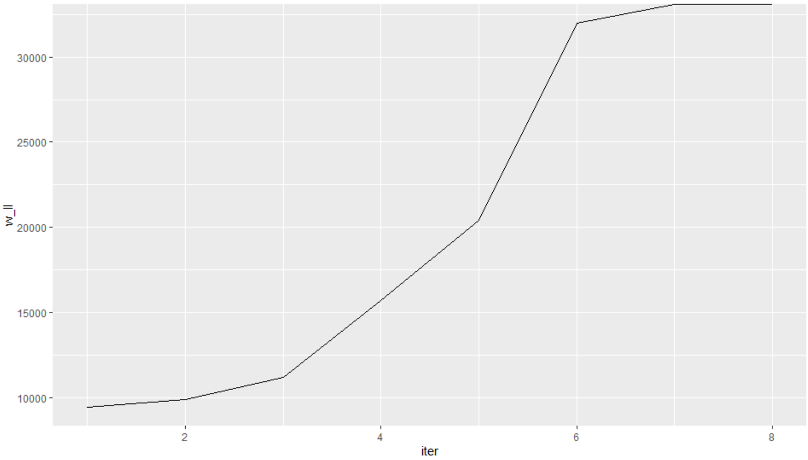
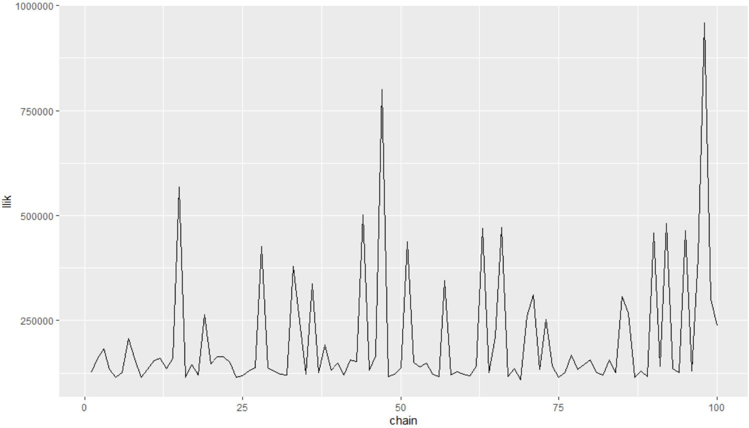


Figure 2: Weighted Likelihood

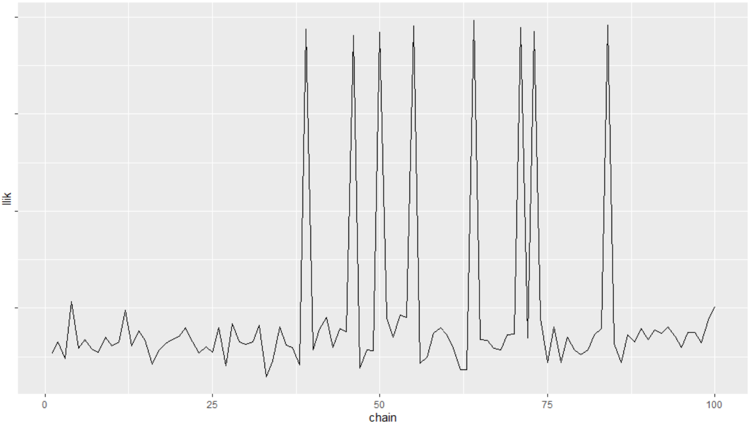
Results

- Idea 1 $\rightarrow$ random initialization each time



Results

- Idea 2 $\rightarrow$ perturb local maximum
- Appears to be more efficient



Results

- Hello?

Performance

- In simulated data, current algorithm clusters well (perfectly in most cases given enough chains)
- Challenges:
  - Likelihood space is explored inefficiently, sometimes requires large number of chains
  - Computationally cumbersome/inefficient