Name:	

Pid:

1. (10 points) Show that $1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$ for all integers $n \ge 1$.

Solution: We prove the statement using induction by n. The base case for n=1 is true since $1^2=1$ and $\frac{1(1+1)(2+1)}{6}=1$.

Now we need to prove the induction step from n to n+1. Assume that $1^2+2^2+3^2+\cdots+n^2=\frac{n(n+1)(2n+1)}{6}$. By the hypothesis, $1^2+2^2+3^2+\cdots+n^2+(n+1)^2=\frac{n(n+1)(2n+1)}{6}+(n+1)^2$. Note that

$$\frac{n(n+1)(2n+1)}{6} + (n+1)^2 = \frac{n(n+1)(2n+1) + 6n^2 + 12n + 6}{6} = \frac{n^3 + 2n^2 + n^2 + n + 6n^2 + 12n + 6}{6} = \frac{n^3 + 9n^2 + 13n + 6}{6} = \frac{(n+1)(n+2)(2n+3)}{6}.$$

Hence, the induction step is true and by the induction pinciple, $1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$ for all positive integers n.

2. (10 points) Let $a_0 = 2$, $a_1 = 5$, and $a_n = 5a_{n-1} - 6a_{n-2}$ for all integers $n \ge 2$. Show that $a_n = 3^n + 2^n$ for all integers $n \ge 0$.

Solution: We prove the statement using induction by n for $n \ge 0$. The base cases for n = 0 and n = 1 are true since $3^0 + 2^0 = 1 + 1 = 2 = a_0$ and $3^1 + 2^1 = 3 + 2 = 5 = a_1$.

Let us now prove the induction step from n and n-1 to n+1. Assume that $a_{n-1}=3^{n-1}+2^{n-1}$ and $a_n=3^n+2^n$. Note that $a_{n+1}=5a_n-6a_{n-1}$; hence, by the induction hypothesis, $a_{n+1}=5(3^n+2^n)-6(3^{n-1}+2^{n-1})=(5\cdot 3-6)3^{n-1}+(5\cdot 2-6)2^{n-1}=9\cdot 3^{n-1}+4\cdot 2^{n-1}=3^{n+1}+2^{n+1}$.

As a result, the induction step is true and by the induction principle, $a_n = 5a_{n-1} - 6a_{n-2}$ for all integers $n \ge 0$. (Note that we proved a stronger statement than it was asked in the problem.)

- 3. (10 points) Let n be a positive integer and A_1, \ldots, A_n be some sets. Let us define union of these sets as follows:
 - 1. $\cap_{i=1}^1 A_i = A_1$,

2.
$$\bigcap_{i=1}^{k+1} A_i = (\bigcap_{i=1}^k A_i) \cap A_{k+1}$$
.

Show that $\bigcap_{i=1}^n \{x \in \mathbb{N} : i \le x \le n\} = \{n\}.$

Solution: We prove using induction by m that $\bigcap_{i=1}^m \{x \in \mathbb{N} : i \leq x \leq n\} = \{m, m+1, \dots, n\}$.

The base case is for m=1 is true since

$$\cap_{i=1}^{1} \{ x \in \mathbb{N} : i \le x \le n \} = [n] = \{1, 2, \dots, n \}.$$

Let us now prove the induction step from m to m+1. Assume that $\bigcap_{i=1}^m \{x \in \mathbb{N} : i \leq x \leq n\} = \{m, m+1, \ldots, n\}$. Note that

$$\cap_{i=1}^{m+1} \{x \in \mathbb{N} \ : \ i \le x \le n\} = \left(\cap_{i=1}^m \{x \in \mathbb{N} \ : \ i \le x \le n\} \right) \cap \{x \in \mathbb{N} \ : \ m+1 \le x \le n\}.$$

Therefore

$$\cap_{i=1}^{m+1} \{ x \in \mathbb{N} : i \le x \le n \} = \{ m, m+1, \dots, n \} \cap \{ m+1, \dots, n \} = \{ m+1, \dots, n \}.$$

Hence, by the induction principle, the statement is true for all m. As a result, we proved for m = n that $\bigcap_{i=1}^{n} \{x \in \mathbb{N} : i \leq x \leq n\} = \{n\}$.

4. (10 points) Let U be the set of sequences of the following symbols: "+", "·", " x_1 ", ..., " x_n ". Let $B = \{x_i : i \in [n]\}$; i.e., B is the set of sequences consisting of only one symbol x_i . Let $\mathcal{F} = \{f_+, f_-\}$, where $f_+(F_1, F_2) = (F_1 + F_2)$ and $f_-(F_1, F_2) = (F_1 \cdot F_2)$ (by $(F_1 \# F_2)$) we denote the sequence obtained by concatenating "(", F_1 , "#", F_2 , and ")"). Let S be the set generated by \mathcal{F} from B.

For $s:[n]\to\{0,1\}$ and $F\in S$, we define the function $\operatorname{val}(F,s)$ using structural recursion as follows.

- 1. $val(x_i, s) = s(i),$
- 2. $val((F_1 + F_2), s) = val(F_1, s) + val(F_2, s),$
- 3. $val((F_1 \cdot F_2), s) = val(F_1, s) \cdot val(F_2, s)$.

Let $F_1, \ldots, F_n \in S$. Let us define the sum of these formulas as follows:

- $1. \sum_{i=j}^{j} F_i = F_j,$
- 2. $\sum_{i=j}^{j+k} F_i = f_+(\sum_{i=j}^{j+k-1} F_i, F_{j+k})$ for $k \ge 1$.

Show that $\operatorname{val}(\sum_{i=1}^n x_i, s) = \operatorname{val}(\sum_{i=1}^n x_{n-i+1}, s)$ for any s.

Solution: Before we start working with the arythmetic formulas, let us prove several statements for real number. Let a_1, \ldots, a_n be some real numbers. We show that $\sum_{i=m}^{m+n} a_i = a_m + \sum_{i=m+1}^{m+n} a_i$ for $n \ge 1$ using induction by n. The base case is true for n = 1 since $\sum_{i=m}^{m+1} a_i = a_m + a_{m+1} = a_m + \sum_{i=m+1}^{m+1} a_i$.

Let us now prove the induction step from n to n+1. Assume that $\sum_{i=m}^{m+n} a_i = a_m + \sum_{i=m+1}^{m+n} a_i$. Note that by the induction hypothesis,

$$\sum_{i=m}^{m+n+1} a_i = \sum_{i=m}^{m+n} a_i + a_{m+n+1} = a_m + \sum_{i=m+1}^{m+n} a_i + a_{m+n+1} = a_m + \sum_{i=m+1}^{m+n+1}.$$

Using this statement we may show that $\sum_{i=1}^{m} a_i = \sum_{i=n-m+1}^{n} a_{n-i+1}$ for $m \ge 1$ using induction by m. The base case is true since $\sum_{i=1}^{1} a_i = a_1 = \sum_{i=n}^{n} a_{n-i+1}$. To prove the induction step from m to m+1; assume $\sum_{i=1}^{m} a_i = \sum_{i=n-m+1}^{n} a_{n-i+1}$. Note that the hypothesis implies that

$$\sum_{i=1}^{m+1} a_i = \sum_{i=1}^{m} a_i + a_{m+1} = \sum_{i=n-m+1}^{n} a_{n-i+1} + a_{m+1} = \sum_{i=n-m}^{n} a_{n-i+1}.$$

Therefore by the induction hypothesis, $\sum_{i=1}^{m} a_i = \sum_{i=n-m+1}^{n} a_{n-i+1}$ for $m \geq 1$. If we consider m = n, we get $\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} a_{n-i+1}$.

Let us now explain how to get this statement for arythmetic formulas. Let F_1, \ldots, F_m be some arythmetic formulas. Then we may show that $\operatorname{val}(\sum_{i=1}^m F_i, s) = \sum_{i=1}^m \operatorname{val}(F_i, s)$ for all s. Fix some s; we prove this statement also using induction. The base case for m=1 is true since $\sum_{i=1}^1 F_i = F_1$ and $\sum_{i=1}^1 \operatorname{val}(F_i, s) = \operatorname{val}(F_1, s)$. To prove the induction step from m to m+1; assume $\operatorname{val}(\sum_{i=1}^m F_i, s) = \sum_{i=1}^m \operatorname{val}(F_i, s)$. Note that $\sum_{i=1}^{m+1} F_i = f_+(\sum_{i=1}^m F_i, F_{m+1})$, and $\operatorname{val}(f_+(\sum_{i=1}^m F_i, F_{m+1}), s) = \operatorname{val}(\sum_{i=1}^m F_i, s) + \operatorname{val}(F_{m+1}, s)$. Hence,

$$\operatorname{val}(\sum_{i=1}^{m+1} F_i, s) = \operatorname{val}(\sum_{i=1}^{m} F_i, s) + \operatorname{val}(F_{m+1}, s) = \sum_{i=1}^{m} \operatorname{val}(F_i, s) + \operatorname{val}(F_{m+1}, s) = \sum_{i=1}^{m+1} \operatorname{val}(F_i, s).$$

Using all these statement, we may notice that

$$val(\sum_{i=1}^{n} x_i, s) = \sum_{i=1}^{n} val(x_i, s) = \sum_{i=1}^{n} val(x_{n-i+1}, s) = val(\sum_{i=1}^{n} x_{n-i+1}, s).$$