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1. (10 points) Let us formulate the pigeonhole principle using propositional formulas.  $\Omega = \{x_{1,1}, \dots, x_{n+1,1}, x_{1,2}, \dots, x_{n+1,n}\}$  (informally  $x_{i,j}$  is true iff the  $i$ th pigeon is in the  $j$ th hole). Consider the following propositional formulas on the variables from  $\Omega$ .

- $L_i$  ( $i \in [n+1]$ ) is equal to  $\bigvee_{j=1}^n x_{i,j}$ . (Informally this formula says that the  $i$ th pigeon is in a hole.)
- $R_j$  ( $j \in [n]$ ) is equal to  $\bigvee_{i_1=1}^{n+1} \bigvee_{i_2=i_1+1}^{n+1} (x_{i_1,j} \wedge x_{i_2,j})$ . (Informally this formula says that there are two pigeons in the  $j$ th hole.)

Show that there is a natural deduction proof of  $\left(\bigwedge_{i=1}^{n+1} L_i\right) \implies \left(\bigvee_{i=1}^n R_i\right)$ .

**Solution:** Let  $\phi = \left(\bigwedge_{i=1}^{n+1} L_i\right) \implies \left(\bigvee_{i=1}^n R_i\right)$ . Note that if we prove that  $\phi|_\rho$  is true for any assignment  $\rho$ , then by completeness theorem there is a natural deduction proof of  $\phi$ .

Therefore it is enough to show that  $\phi|_\rho$  is true for any assignment  $\rho$ . Let us fix some  $\rho$ .

- If  $\left(\bigwedge_{i=1}^{n+1} L_i\right)|_\rho$  is false, then  $\phi|_\rho$  is true.
- If  $\left(\bigvee_{i=1}^n R_i\right)|_\rho$  is false, then  $\phi|_\rho$  is true.
- Hence, to finish the proof we need to show that any other case is impossible. Assume that  $\left(\bigvee_{i=1}^n R_i\right)|_\rho$  and  $\left(\bigwedge_{i=1}^{n+1} L_i\right)|_\rho$  are true.

Note that the fact that the first formula is true guarantees that  $\sum_{i=1}^{n+1} \sum_j 1\rho(x_{i,j}) \geq n+1$ . However, the second formula says that  $\sum_{i=1}^{n+1} \sum_j 1\rho(x_{i,j}) \leq n$ , which leads us to a contradiction.

2. (10 points) Let  $\phi = \bigvee_{i=1}^m \lambda_i$  be a clause; we say that the width of the clause is equal to  $m$ . Let  $\phi = \bigwedge_{i=1}^\ell \chi_i$  be a formula in CNF; we say that the width of  $\phi$  is equal to the maximal width of  $\chi_i$  for  $i \in [\ell]$ .

Let  $p_n : \{T, F\}^n \rightarrow \{T, F\}$  such that  $p_n(x_1, \dots, x_n) = T$  iff the set  $\{i : x_i = T\}$  has odd number of elements. Show that any CNF representation of  $p_n$  has width at least  $n$ .

**Solution:** Let us assume the opposite; i.e., that there is a formula  $\phi = \bigwedge_{i=1}^m C_i$  CNF such that  $\phi|_{x_1=v_1, \dots, x_n=v_n} = p_n(v_1, \dots, v_n)$  for all  $v_1, \dots, v_n \in \{T, F\}^n$  and width of each  $C_i$  is less than  $n$

Without loss of generality we may assume that  $C_1 = x_1 \vee \dots \vee x_k$  for  $k < n$ . Let us consider a

propositional assignement  $\rho$  to the variables  $x_1, \dots, x_n$  such that  $\rho_u(x_i) = \begin{cases} F & \text{if } i \leq k \\ u & \text{if } i = k+1 \\ F & \text{otherwise.} \end{cases}$  It is

clear that  $\phi|_{\rho_T} = \phi|_{\rho_F} = F$ . However,  $p_n(\rho_T(x_1), \dots, \rho_T(x_n)) \neq p_n(\rho_F(x_1), \dots, \rho_F(x_n))$ ; therefore,  $\phi|_{\rho_T} \neq \phi|_{\rho_F}$ . As a result, the assumption is wrong.

3. (10 points) Write a natural deduction derivation of  $(W \vee Y) \Rightarrow (X \vee Z)$  from hypotheses  $W \Rightarrow X$  and  $Y \Rightarrow Z$ .

**Solution:**

1		$W \Rightarrow X$	
2		$Y \Rightarrow Z$	
3		$W \vee Y$	
4			$W$
5			$X$ $\Rightarrow$ E, 1, 4
6			$X \vee Z$ $\vee$ I, 5
7			$Y$
8			$Z$ $\Rightarrow$ E, 2, 7
9			$X \vee Z$ $\vee$ I, 8
10		$X \vee Z$	$\vee$ E, 3, 4–6, 7–9
11		$(W \vee Y) \Rightarrow (X \vee Z)$	$\Rightarrow$ I, 3–10

4. (10 points) We say that a clause  $C$  can be obtained from clauses  $A$  and  $B$  using the *resolution* rule if  $C = A' \vee B'$ ,  $A = x \vee A'$ , and  $B = \neg x \vee B'$ , for some variable  $x$ .

We say that a clause  $C$  can be derived from clauses  $A_1, \dots, A_m$  using resolutions if there is a sequence of clauses  $D_1, \dots, D_\ell = C$  such that each  $D_i$

- is either obtained from clauses  $D_j$  and  $D_k$  for  $j, k < i$  using the *resolution* rule, or
- is equal to  $A_j$  for some  $j \in [m]$ , or
- is equal to  $D_j \vee E$  for some  $j < i$  and a clause  $E$ .

Show that if  $A_1, \dots, A_m$  semantically imply  $C$ , then  $C$  can be derived from clauses  $A_1, \dots, A_m$  using resolutions.

**Solution:** First we prove that if clauses  $A_1, \dots, A_n$  semantically imply  $\perp$ , then there is a derivation of  $\perp$  from  $A_1, \dots, A_n$  using the resolution rule.

We prove this using the induction on the number  $k$  of variables used in clauses  $A_1, \dots, A_n$ . The base case for  $k = 0$  is clear since in this case  $A_i = \perp$  for all  $i \in [n]$ .

Let us now prove the induction step from  $k$  to  $k + 1$ . We fix a variable  $x$  that is used by clauses  $A_1, \dots, A_n$ . Let us split set of clauses  $A_1, \dots, A_n$  into three groups:

- the clauses  $x \vee B_1, \dots, x \vee B_p$  (i.e., the clauses that contain  $x$ ),
- the clauses  $\neg x \vee C_1, \dots, \neg x \vee C_q$  (i.e., the clauses that contain  $\neg x$ ),
- the clauses  $D_1, \dots, D_r$  (i.e., the clauses that neither contain  $x$ , nor  $\neg x$ ).

Note that any assignment that sets  $x$  to be equal to  $T$  cannot make all  $A_i$  to be true since  $A_1, \dots, A_n$  semantically imply  $\perp$ . Therefore,  $C_1, \dots, C_q, D_1, \dots, D_r$  semantically imply  $\perp$ . However, by the induction hypothesis, there is a derivation of  $\perp$  from  $C_1, \dots, C_q, D_1, \dots, D_r$  using the resolution rule. It is easy to see that this imply that there is a derivation of  $\neg x$  from  $\neg x \vee C_1, \dots, \neg x \vee C_q, D_1, \dots, D_r$ . Similarly, there is a derivation of  $x$  from  $x \vee B_1, \dots, x \vee B_p, D_1, \dots, D_r$ . Hence, using the resolution rule we can derive  $\perp$  from  $A_1, \dots, A_n$ .