

Name: _____

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1. (10 points) Give a natural deduction derivation of $\exists x (A(x) \vee B(x))$ from $\exists x A(x) \vee \exists x B(x)$.

2. (10 points) Let us consider the following formulas on the variables from the set $\{x_0, \dots, x_n\}$.
1. The formula I_n is equal to x_0 .
 2. The formula $S_{n,i}$ is equal to $x_{i-1} \implies x_i$.
 3. The formula T_n is equal to x_n .

Show that there is a natural deduction derivation of T_n from $I_n \wedge \bigwedge_{i=1}^n S_{n,i}$.

3. (10 points) Let $\phi = \bigvee_{i=1}^m \lambda_i$ be a clause; we say that the width of the clause is equal to m . Let $\phi = \bigwedge_{i=1}^\ell \chi_i$ be a formula in CNF; we say that the width of ϕ is equal to the maximal width of χ_i for $i \in [\ell]$.

Let $m_n : \{T, F\}^n \rightarrow \{T, F\}$ such that $m_n(x_1, \dots, x_n) = T$ iff the number of elements in the set $\{i : x_i = T\}$ is divisible by 3.

Show that any CNF representation of m_n has width at least $n - 2$.

4. (10 points) Let $A\Delta B = (A \cup B) \setminus (A \cap B)$; we say that $A\Delta B$ is the symmetric difference of A and B . Let Ω , and $A_1, \dots, A_n \subseteq \Omega$ be some sets. We say that $\Delta_{i=1}^1 A_i = A_1$ and $\Delta_{i=1}^{k+1} A_i = (\Delta_{i=1}^k A_i) \Delta A_{k+1}$. Show that

$$\Delta_{i=1}^n A_i = \{x \in \Omega : x \in A_i \text{ for odd number of } i \in [n]\}.$$

5. (10 points) Let \mathcal{S} be a signature with two predicate symbols $=$ and S such that the first is binary and the last is ternary.

Let us consider the structure \mathfrak{M} such that it corresponds to the points on a two-dimensional plane, $=$ is a standard equality, and $S(x, y, z)$ is true iff $|xz| = |yz|$.

Let R be a relation such that $(A, B, C) \in R$ iff A , B , and C lay on the same line. Show that R is representable in \mathfrak{M} .

6. (10 points) Let us define the set S defined as follows:

- $3 \in S$ and
- if $x \in S$ and $y \in S$, then $(x + y) \in S$.

Show that $S = \{3k : k \in \mathbb{N}\}$.

7. (10 points) Let $f, g_1, \dots, g_n : \mathbb{R}^\ell \rightarrow \mathbb{R}$. We say that the equation $f(x) = 0$ can be derived from the equations $g_1(x) = 0, \dots, g_n(x) = 0$ iff there is a sequence of functions $h_1, \dots, h_m : \mathbb{R}^\ell \rightarrow \mathbb{R}$ such that $h_m = f$ and for each $i \in [m]$,
- either h_i is equal to g_j for some $j \in [n]$, or
 - $h_i = h_j + h_k$ for some $1 \leq j, k < i$, or
 - $h_i = c \cdot h_j$ for some $1 \leq j < i$ and some positive $c \in \mathbb{R}$.

Show that if the equation $f(x) = 0$ can be derived from the equations $g_1(x) = 0, \dots, g_n(x) = 0$, then for any $v \in \mathbb{R}^\ell$, $f(v) = 0$ provided that $g_1(v) = \dots = g_n(v) = 0$.