Name: ________________

Pid: ________________

1. (10 points) Let S be an infinite enumerable set. Show that there is an infinite decidable set $T \subseteq S$.

Solution: Since S is enumerable, there is a total computable function $f : \mathbb{N} \to \mathbb{N}$ such that Im $f = S$. Let $g(x) = \max_{y \leq x} f(x)$. It is easy to see that g is also total and computable. Moreover, g is nondecreasing and Im$g \subseteq S$. Let us prove that Im g is infinite. Indeed, assume that Im $g \subseteq [N]$, this implies that $f(x) \leq N$ for all x which contradictions to the assumption that S is infinite. Therefore Im g is an infinite decidable subset of S.

2. (10 points) Let $S \subseteq \mathbb{N}$ be decidable and let

$$D = \{p \;:\; p \text{ is prime and } p \text{ divides some } n \in S\}.$$

Is the set D always decidable?

Solution: Let $p_1, \ldots, p_n, \ldots$ be all the prime numbers in the increasing order. We define the set S such that $p_n^t \in S$ if $U(n, n)$ terminates after t steps. It is clear that S is decidable. However, $p_n \in S$ iff $U(n, n)$ terminates. Therefore S is not decidable.