

Name: \_\_\_\_\_

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1. (10 points) Let us consider four-lines geometry, it is a theory with undefined terms: “point”, “line”, “is on”, and axioms:

1. there exist exactly four lines,
2. any two distinct lines have exactly one point on both of them, and
3. each point is on exactly two lines.

Show that every line has exactly three points on it. (Be careful with the terms you use and axioms you use.)

**Solution:** Lets denote the lines as  $l_1, \dots, l_4$  (all of them exist and different by Axiom 1). Due to symmetry of the problem it is enough to prove that  $l_4$  has exactly three points on it.

Let  $p_i$  ( $1 \leq i \leq 3$ ) be the point that is on  $l_i$  and  $l_4$  (they exist by Axiom 2). Let us now prove that  $p_1$ ,  $p_2$ , and  $p_3$  are all different. Assume that  $p_i = p_j$  for  $i \neq j$  ( $1 \leq i, j \leq 3$ ) for the sake of contradiction. In this case  $p_i$  is on  $l_i$ ,  $l_j$ , and  $l_4$  which contradicts Axiom 3.

Let us now prove that there are no other points on  $l_4$ . Assume that it is not true and there is  $p_4$  in addition to  $p_1$ ,  $p_2$ , and  $p_3$  on  $l_4$ . By Axiom 3, there is  $i$  ( $1 \leq i \leq 3$ ) such that  $p_4$  is on  $l_i$ . Hence,  $p_i$  and  $p_4$  are on  $l_i$  which contradicts to Axiom 2.

2. (10 points) In Euclidean (standard) geometry, prove: If two lines share a common perpendicular, then the lines are parallel. (You do not need to use axioms of Euclidean geometry in this exercise, you can use all the standard knowledge about geometry.)

**Solution:** Let us denote by  $AB$  the common perpendicular. Assume that the lines are not parallel (note that these lines are different) i.e. that there is an intersection  $C$  of these lines.

Note that the angles  $CAB$  and  $CBA$  are right, hence, the angle  $ACB$  is equal to 0 degrees. So the lines are the same, which is a contradiction.

Hence, the assumption was incorrect i.e. the lines are parallel.