Name: ______________________

Pid: ______________________

1. (10 points) Let $m_1, n_1, m_2, n_2 \in \mathbb{N}$, we say that $(m_1, n_1) < (m_2, n_2)$ iff either $m_1 < m_2$ or $m_1 = m_2$ and $n_1 < n_2$.

   Let $P(m, n)$ be some property of pairs of integers. Assume that we can prove the following statement for all $m, n \in \mathbb{N}$:

   if $P(x, y)$ is true for all $x, y \in \mathbb{N}$ such that $(x, y) < (m, n)$, then $P(m, n)$ is true.

   Show that we can prove that $P(m, n)$ is true for all $m, n \in \mathbb{N}$.

Solution: We prove the statement using nested induction. Let $Q(m)$ denote the statement: $P(x, y)$ is true for all $x, y \in \mathbb{N}$ such that $x \leq m$. We prove using induction by m that $Q(m)$ is true for all $m \in \mathbb{N}$.

(base case) We prove using induction that $P(1, y)$ is true for all $y \in \mathbb{N}$. Indeed, $(x', y') < (1, y)$ iff $x' = 1$ and $y' < y$; hence, by the assumption, if $P(1, y)$ is true for all $y' < y$, then $P(1, y)$ is true. Therefore, by the strong induction principle, $P(1, y)$ is true for all $y \in \mathbb{N}$. As a result, we proved $Q(1)$.

(induction step) Let us prove that if $Q(m)$ is true, then $Q(m+1)$ is also true. Assume that $Q(m)$ is true. Let us prove using induction that $P(m + 1, y)$ is true for all $y \in \mathbb{N}$.

- Note that if $(x', y') < (m + 1, 1)$ then $x' \leq m$. Therefore, by the assumption of the problem and the assumption that $Q(m)$ is true, $P(m + 1, 1)$ is true.
- Assume that $P(m+1, 1), \ldots, P(m+1, y-1)$ are true. Note that if $(x', y') < (m+1, y)$, then either $x' \leq m$ or (x', y') is equal to one of $(m+1, 1), \ldots, (m+1, y-1)$. Therefore, by the assumption of the problem and the assumption that $Q(m)$ is true, $P(m + 1, y)$ is true.

Hence, $P(m + 1, y)$ is true for all $y \in \mathbb{N}$.

As a result, by the induction principle, $Q(m)$ is true for all $m \in \mathbb{N}$.

2. (10 points) In the subtraction game where players may subtract 1, 2 or 5 chips on their turn, identify the N- and P-positions. (Please do not forget to prove correctness of your asnwer.)

Solution: Let us prove using induction that n is a P-position in this game only if $n \equiv 0 \pmod 3$. The base case for $n \leq 9$ can be verifyed using direct computations. Let us prove the induction step from n to $n+1$. Assume that $m \leq n$ is a P-position iff $m \equiv 0 \pmod 3$.

- If $n+1 \equiv 0 \pmod 3$, then we can go to $n \equiv 2 \pmod 3$, $n-1 \equiv 1 \pmod 3$, and $n-4 \equiv 1 \pmod 3$. By the induction hypothesis, all these positions are N-positions, hence, $n+1$ is a P-position.
- If $n+1 \equiv 1 \pmod 3$, then we can go to $n \equiv 0 \pmod 3$, which, by the induction hypothesis, is a P-position.
- If $n+1 \equiv 2 \pmod 3$, then we can go to $n-1 \equiv 0 \pmod 3$, which, by the induction hypothesis, is a P-position.

As a result, by the induction principle, we proved the statement.