Name: ______________________________

Pid: ______________________________

1. (10 points) Show that $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$ for all integers $n \geq 1$.

2. (10 points) Let $a_0 = 2$, $a_1 = 5$, and $a_n = 5a_{n-1} - 6a_{n-2}$ for all integers $n \geq 2$. Show that $a_n = 3^n + 2^n$ for all integers $n \geq 0$.

3. (10 points) Let n be a positive integer and $A_1, \ldots, A_n$ be some sets. Let us define union of these sets as follows:

   1. $\cap_{i=1}^{1} A_i = A_1$,
   2. $\cap_{i=1}^{k+1} A_i = (\cap_{i=1}^{k} A_i) \cap A_{k+1}$.

   Show that $\cap_{i=1}^{n}\{x \in \mathbb{N} \,:\, i \leq x \leq n\} = \{n\}$.

4. (10 points) Let U be the set of sequences of the following symbols: "+", "·", "x_1", …, "x_n". Let $B = \{x_i \ : \ i \in [n]\}$; i.e., B is the set of sequences consisting of only one symbol x_i. Let $\mathcal{F} = \{f_+, f_\cdot\}$, where $f_+(F_1, F_2) = (F_1 + F_2)$ and $f_\cdot(F_1, F_2) = (F_1 \cdot F_2)$ (by $(F_1 \# F_2)$ we denote the sequence obtained by concatenating "(", F_1, "#", F_2, and ")"). Let S be the set generated by $\mathcal{F}$ from B.

   For $s : [n] \to \{0, 1\}$ and $F \in S$, we define the function $\mathrm{val}(F, s)$ using structural recursion as follows.

   1. $\mathrm{val}(x_i, s) = s(i)$,
   2. $\mathrm{val}((F_1 + F_2), s) = \mathrm{val}(F_1, s) + \mathrm{val}(F_2, s)$,
   3. $\mathrm{val}((F_1 \cdot F_2), s) = \mathrm{val}(F_1, s) \cdot \mathrm{val}(F_2, s)$.

   Let $F_1, \ldots, F_n \in S$. Let us define the sum of these formulas as follows:

   1. $\sum_{i=j}^{j} F_i = F_j$,
   2. $\sum_{i=j}^{j+k} F_i = f_+(\sum_{i=j}^{j+k-1} F_i, F_{j+k})$ for $k \geq 1$.

   Show that $\mathrm{val}(\sum_{i=1}^{n} x_i, s) = \mathrm{val}(\sum_{i=1}^{n} x_{n-i+1}, s)$ for any s.