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1. (10 points) Let  $S \subseteq \mathbb{N}$  be a nonempty set. Show that  $S$  is decidable iff there is a function  $f : \mathbb{N} \rightarrow \mathbb{N}$  such that  $f$  is computable,  $f$  is nondecreasing, and  $\text{Im } f = S$ .

**Solution:** Let us assume that  $S$  is decidable and  $\mathcal{A}$  decides  $S$ . Let  $x_{\min}$  be the minimal element of  $S$ . Consider the following algorithm.

```

1: function  $\mathcal{F}(n)$ 
2:   if  $n < x_{\min}$  then
3:     return  $x_{\min}$ 
4:   end if
5:   Let  $x \leftarrow n$ 
6:   while  $\neg \mathcal{A}(x)$  and  $x > x_{\min}$  do
7:      $x \leftarrow x - 1$ 
8:   end while
9:   return  $x$ 
10: end function

```

It is clear that this algorithm computes the total function  $f : \mathbb{N} \rightarrow \mathbb{N}$  such that

$$f(x) = \begin{cases} x_{\min} & \text{if } x < x_{\min} \\ \max\{y \in S : y \leq x\} & \text{otherwise} \end{cases}.$$

Therefore  $f$  is nondecreasing. We need to prove now that  $\text{Im } f = S$ . To prove this first we note that it is easy to see that  $\text{Im } f \subseteq S$ . In addition, if  $x \in S$ , then  $f(x) = x$ , which implies that  $\text{Im } f = S$ .

Let us now prove the statement in the opposite direction. Assume that there is a total nondecreasing function  $f : \mathbb{N} \rightarrow \mathbb{N}$  such that  $\text{Im } f = S$ .

- If  $S$  is finite, then we prove in class that it is decidable.
- Let  $S$  be an infinite set, and let  $\mathcal{F}$  be an algorithm computing  $f$ . Consider the following algorithm

```

1: function  $\mathcal{A}(x)$ 
2:   while  $\mathcal{F}(n) < x$  do
3:      $n \leftarrow n + 1$ 
4:   end while
5:   if  $\mathcal{F}(n) = x$  then
6:     return 1
7:   else
8:     return 0
9:   end if
10: end function

```

In lines 2-4 this algorithm finds the minimal  $n$  such that  $f(n) \geq x$  (it always exists since  $S$  is infinite). If  $x \in S$ , then such a minimal  $n$  is equal  $x$ , and the algorithm returns 1. Otherwise  $f(n) \neq x$  and the algorithm returns 0.

2. (10 points) Let  $A, B \subseteq \mathbb{N}$  be enumerable sets. Show that  $A \times B$  is enumerable.

**Solution:** Let  $\mathcal{A}$  and  $\mathcal{B}$  be semideciding algorithms for  $A$  and  $B$ , respectively. Consider the following algorithm.

```
1: function  $\mathcal{C}(x, y)$   
2:    $\mathcal{A}(x)$   
3:    $\mathcal{B}(y)$   
4:   return 1  
5: end function
```

Note that if  $x \in A$  and  $y \in B$ , then the algorithm return 1. Otherwise the algorithm never terminates. Therefore, if  $(x, y) \in A \times B$ , then the algorithm return 1. Otherwise the algorithm never terminates.