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1. Show that  $\sum_{i=1}^n (i+1)2^i = n2^{n+1}$  for all positive integers  $n$ .

**Solution:** We prove the statement using induction by  $n$ . The base case for  $n = 1$  is true since  $4 = (1+1) \cdot 2 = 1 \cdot 2^{1+1} = 4$ .

Now we need to prove the induction step. Let us assume that  $\sum_{i=1}^n (i+1)2^i = n2^{n+1}$ . Note that  $\sum_{i=1}^{n+1} (i+1)2^i = \sum_{i=1}^n (i+1)2^i + (n+2)2^{n+1} = n2^{n+1} + (n+2)2^{n+1} = (2n+2)2^{n+1} = (n+1)2^{n+2}$ . Hence, by the induction principle  $\sum_{i=1}^n (i+1)2^i = n2^{n+1}$  for any positive integer  $n$ .

2. Let  $n$  be a positive integer and  $A_1, \dots, A_n$  be some sets. Let us define union of these sets as follows:

1.  $\cup_{i=1}^1 A_i = A_1$ ,
2.  $\cup_{i=1}^{k+1} A_i = (\cup_{i=1}^k A_i) \cup A_{k+1}$ .

Show that  $\cup_{i=1}^n [i] = [n]$ .

**Solution:** We prove the statement using induction by  $n$ . If  $n = 1$ , the statement is true since  $\cup_{i=1}^1 [i] = [1] = [1]$ .

To prove the induction step we assume that  $\cup_{i=1}^n [i] = [n]$ . By the definition of the union  $\cup_{i=1}^{n+1} [i] = (\cup_{i=1}^n [i]) \cup [n+1]$ . Hence,  $\cup_{i=1}^{n+1} [i] = [n] \cup [n+1] = [n+1]$ .

Therefore by the induction principle,  $\cup_{i=1}^n [i] = [n]$  for any positive integer  $n$ .

3. (10 points) Let  $\Omega$  be some set. Consider  $A_1, \dots, A_n \subseteq \Omega$ . Show that  $\cup_{i=1}^n A_i = \{x \in \Omega : \exists i \in [n] x \in A_i\}$ .

**Solution:** We prove the statement using induction by  $n$ . For  $n = 1$  the statement is clearly true. Let us now prove the induction step from  $n$  to  $n + 1$ . Assume that  $\cup_{i=1}^n A_i = \{x \in \Omega : \exists i \in [n] x \in A_i\}$ . Note that  $\cup_{i=1}^{n+1} A_i = (\cup_{i=1}^n A_i) \cup A_{n+1} = \{x \in \Omega : \exists i \in [n] x \in A_i\} \cup A_{n+1}$ . We denote  $\{x \in \Omega : \exists i \in [n] x \in A_i\}$  by  $B$ . By the definition of union,  $B \cup A$  is the set of all  $x$  such that either  $x \in B$  or  $x \in A_{n+1}$ ; therefore

$$B \cup A = \{x \in \Omega : (\exists i \in [n] x \in A_i) \text{ or } x \in A_{n+1}\} = \{x \in \Omega : \exists i \in [n+1] x \in A_i\}.$$

Which finishes the proof.