Name: ____________________

Pid: ____________________

1. (10 points) Give a natural deduction derivation of $\exists x\ (A(x) \vee B(x))$ from $\exists x\ A(x) \vee \exists x\ B(x)$.

Solution:

1	$\exists x\ A(x) \vee \exists x\ B(x)$	
2	$\quad \exists x\ A(x)$	
3	$\quad\quad A(y)$	
4	$\quad\quad A(y) \vee B(y)$	$\vee$I, 3
5	$\quad\quad \exists x\ (A(x) \vee B(x))$	$\exists$I, 3
6	$\quad \exists x\ (A(x) \vee B(x))$	$\exists$E, 3
7	$\quad \exists x\ B(x)$	
8	$\quad\quad B(y)$	
9	$\quad\quad A(y) \vee B(y)$	$\vee$I, 3
10	$\quad\quad \exists x\ (A(x) \vee B(x))$	$\exists$I, 3
11	$\quad \exists x\ (A(x) \vee B(x))$	$\exists$E, 3
12	$(\exists x\ (A(x) \vee B(x))$	$\vee$E, 1, 2–5, 6–9

2. (10 points) Let us consider the following formulas on the variables from the set $\{x_0, \ldots, x_n\}$.

   1. The formula I_n is equal to x_0.
   2. The formula $S_{n,i}$ is equal to $x_{i-1} \implies x_i$.
   3. The formula T_n is equal to x_n.

   Show that there is a natural deduction derivation of T_n from $I_n \wedge \bigwedge_{i=1}^n S_{n,i}$.

Solution: By completeness theorem, it is enough to show that $I_n \wedge \bigwedge_{i=1}^n S_{n,i} \models T_n$.

Consider an assignement ρ such that $(I_n \wedge \bigwedge_{i=1}^n S_{n,i})|_\rho = T$. Note that this implies that $\rho(x_0)$ is true, and $\rho(x_{i-1}) \implies \rho(x_i)$ is true for all $i \in [n]$. We prove using induction that for any k, $\rho(x_k)$ is true. The base case is clear since $\rho(x_0)$ is true. Let us now prove the induction step. Assyme that $\rho(x_k)$ is true. So x_{k+1} is true since $\rho(x_k) \implies \rho(x_{k+1})$ is true.

As a result we proved that $\rho(x_n)$ is true, which implies that $T|_\rho$ is true.

3. (10 points) Let $\phi = \bigvee_{i=1}^{m} \lambda_i$ be a clause; we say that the width of the clause is equal to m. Let $\phi = \bigwedge_{i=1}^{\ell} \chi_i$ be a formula in CNF; we say that the width of ϕ is equal to the maximal width of χ_i for $i \in [\ell]$.

   Let $m_n : \{T, F\}^n \to \{T, F\}$ such that $m_n(x_1, \ldots, x_n) = T$ iff the number of elements in the set $\{i \,:\, x_i = T\}$ is divisible by 3.

   Show that any CNF representation of m_n has width at least $n - 2$.

Solution: Assume that there is a CNF ϕ of width at most $n-3$ representing m_n. Let $C = x_{i_1}^{u_1} \vee \ldots x_{i_\ell}^{u_\ell}$ be some clause of ϕ. Let us consider the variables y_1 and y_2 be the variables not equal to $x_{i_1}, \ldots, x_{i_\ell}$. We define ρ_{v_1,v_2} such that

$$\rho_{v_1,v_2}(x) = \begin{cases} \neg u_j & \text{if } x = x_{i_j} \text{ for } j \in [\ell] \\ v_j & \text{if } x = y_j \text{ for } j \in [2] \\ F & \text{otherwise} \end{cases}$$

Note that $C|_{\rho_{v_1,v_2}}$ is false and as a result, $\phi|_{\rho_{v_1,v_2}}$ is also false. Let $N = |\{j \,:\, u_j = F\}|$. We need to consider the following cases.

- If $N \equiv 0 \pmod 3$, then $m_n(\rho_{F,F}(x_1), \ldots, \rho_{F,F}(x_n)) = T$ which contradicts to $\phi|_{\rho_{v_1,v_2}} = F$.
- If $N \equiv 1 \pmod 3$, then $m_n(\rho_{T,T}(x_1), \ldots, \rho_{T,T}(x_n)) = T$ which contradicts to $\phi|_{\rho_{v_1,v_2}} = F$.
- If $N \equiv 2 \pmod 3$, then $m_n(\rho_{T,F}(x_1), \ldots, \rho_{T,F}(x_n)) = T$ which contradicts to $\phi|_{\rho_{v_1,v_2}} = F$.

4. (10 points) Let $A\Delta B = (A \cup B) \setminus (A \cap B)$; we say that $A\Delta B$ is the symmetric difference of A and B. Let Ω, and $A_1, \ldots, A_n \subseteq \Omega$ be some sets We say that $\Delta_{i=1}^{1} A_i = A_1$ and $\Delta_{i=1}^{k+1} A_i = (\Delta_{i=1}^{k} A_i)\Delta A_{k+1}$. Show that

$$\Delta_{i=1}^{n} A_i = \{x \in \Omega \ : \ x \in A_i \text{ for odd number of } i \in [n]\}.$$

Solution: We prove the statement using induction by n. The base case for $n = 1$ is obvious. Let us prove the induction step. Assume that

$$\Delta_{i=1}^{k} A_i = \{x \in \Omega \ : \ x \in A_i \text{ for odd number of } i \in [k]\}.$$

Note that

$$\Delta_{i=1}^{k+1} A_i = (\{x \in \Omega \ : \ x \in A_i \text{ for odd number of } i \in [k]\})\Delta A_{k+1}.$$

Let us fix some $x \in \Omega$ and consider the following cases.

- If $x \in \{x \in \Omega \ : \ x \in A_i \text{ for odd number of } i \in [k]\}$ and $x \in A_{k+1}$, then $x \notin \Delta_{i=1}^{k+1} A_i$ and for an even number of $i \in [k+1]$, $x \in A_i$.
- If $x \notin \{x \in \Omega \ : \ x \in A_i \text{ for odd number of } i \in [k]\}$ and $x \in A_{k+1}$, then $x \in \Delta_{i=1}^{k+1} A_i$ and for an odd number of $i \in [k+1]$, $x \in A_i$.
- If $x \in \{x \in \Omega \ : \ x \in A_i \text{ for odd number of } i \in [k]\}$ and $x \notin A_{k+1}$, then $x \in \Delta_{i=1}^{k+1} A_i$ and for an odd number of $i \in [k+1]$, $x \in A_i$.
- If $x \notin \{x \in \Omega \ : \ x \in A_i \text{ for odd number of } i \in [k]\}$ and $x \notin A_{k+1}$, then $x \notin \Delta_{i=1}^{k+1} A_i$ and for an even number of $i \in [k+1]$, $x \in A_i$.

5. (10 points) Let $\mathcal{S}$ be a signature with two predicate symbols $=$ and S such that the first is binary and the last is ternary.

   Let us consider the structure $\mathfrak{M}$ such that it corresponds to the points on a two-dimmnesional plane, $=$ is a standard equality, and $S(x, y, z)$ is true iff $|xz| = |yz|$.

   Let R be a relation such that $(A, B, C) \in R$ iff A, B, and C lay on the same line. Show that R is representable in $\mathfrak{M}$.

   Solution: Let us fix some points A, B, and C. Note that they lay on a line iff C is the only point such that the distance between A and C is $|AC|$ and the distance between B and C is $|BC|$. Hence, the following formula represents R: $\forall C' \ (S(C, C', A) \wedge S(C, C', B)) \implies C' = C$.

6. (10 points) Let us define the set S defined as follows:

   - $3 \in S$ and
   - if $x \in S$ and $y \in S$, then $(x + y) \in S$.

   Show that $S = \{3k \ : \ k \in \mathbb{N}\}$.

Solution: The statement consists of two parts:

- $S \subseteq \{3k \ : \ k \in \mathbb{N}\}$ and
- $S \supseteq \{3k \ : \ k \in \mathbb{N}\}$.

We prove the first one using structural induction.

(base case)) Note that $3 \in \{3k \ : \ k \in \mathbb{N}\}$.

(induction step) Let $z = x + y$ and $\{3k \ : \ k \in \mathbb{N}\}$. It is clear that $z \in \{3k \ : \ k \in \mathbb{N}\}$.

Hence, by the structural induction theorem, $S \subseteq \{3k \ : \ k \in \mathbb{N}\}$.
The second statement can be proved using induction by k. The base case for $k = 1$ is true since $3 \in S$. Let us now prove the induction step. By the induction hypothesis, $3k \in S$; hence, $3k + 3 = 3(k + 1) \in S$.

7. (10 points) Let $f, g_1, \dots, g_n : \mathbb{R}^\ell \to \mathbb{R}$ We say that the equation $f(x) = 0$ can be derived from the equations $g_1(x) = 0, \dots, g_n(x) = 0$ iff there is a sequence of functions $h_1, \dots, h_m : \mathbb{R}^\ell \to \mathbb{R}$ such that $h_m = f$ and for each $i \in [m]$,

   - either h_i is equal to g_j for some $j \in [n]$, or
   - $h_i = h_j + h_k$ for some $1 \le j, k < i$, or
   - $h_i = c \cdot h_j$ for some $1 \le j < i$ and some $c \in \mathbb{R}$.

   Show that if the equation $f(x) = 0$ can be derived from the equations $g_1(x) = 0, \dots, g_n(x) = 0$, then for any $v \in \mathbb{R}^\ell$, $f(v) = 0$ provided that $g_1(v) = \cdots = g_n(v) = 0$.

Solution: Let us fix some $f, g_1, \dots, g_n : \mathbb{R}^\ell \to \mathbb{R}$. Let $h_1, \dots, h_m : \mathbb{R}^\ell \to \mathbb{R}$ be the derivation of the equation $f(x) = 0$ from the equations $g_1(x) = 0, \dots, g_n(x) = 0$. Consider v such that $g_1(v) = \cdots = g_n(v) = 0$.

We prove using induction by k that $h_k(v) = 0$. The base case for $k = 1$ is clear since h_1 should be equal to g_j for some $j \in [n]$ so $h_1(v) = 0$. Let us now prove the induction step form $1, \dots, k-1$ to k. By the induction hypothesis, $h_1(v) = \cdots = h_{k-1}(v) = 0$.

- If $h_k = g_j$ for some $j \in [n]$, then $h_k(v) = 0$ since $g_j(v) = 0$.
- If $h_k = h_i + h_j$ for some $i, j < k$, then $h_k(v) = 0$ since $h_i(v) = h_j(v) = 0$.
- If $h_k = c \cdot h_i$ for some $i < k$, then $h_k(v) = 0$ since $c \cdot h_i(v) = c \cdot 0$.