

Name: \_\_\_\_\_

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1. (10 points) Show that  $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$  for all integers  $n \geq 1$ .

2. (10 points) Let  $a_0 = 2$ ,  $a_1 = 5$ , and  $a_n = 5a_{n-1} - 6a_{n-2}$  for all integers  $n \geq 2$ . Show that  $a_n = 3^n + 2^n$  for all integers  $n \geq 0$ .

3. (10 points) Let  $n$  be a positive integer and  $A_1, \dots, A_n$  be some sets. Let us define union of these sets as follows:

1.  $\cap_{i=1}^1 A_i = A_1,$

2.  $\cap_{i=1}^{k+1} A_i = (\cap_{i=1}^k A_i) \cap A_{k+1}.$

Show that  $\cap_{i=1}^n \{x \in \mathbb{N} : i \leq x \leq n\} = \{n\}.$

4. (10 points) Let  $U$  be the set of sequences of the following symbols: “+”, “.”, “ $x_1$ ”, ..., “ $x_n$ ”. Let  $B = \{x_i : i \in [n]\}$ ; i.e.,  $B$  is the set of sequences consisting of only one symbol  $x_i$ . Let  $\mathcal{F} = \{f_+, f.\}$ , where  $f_+(F_1, F_2) = (F_1 + F_2)$  and  $f.(F_1, F_2) = (F_1 \cdot F_2)$  (by  $(F_1 \# F_2)$  we denote the sequence obtained by concatenating “(”,  $F_1$ , “#”,  $F_2$ , and “”). Let  $S$  be the set generated by  $\mathcal{F}$  from  $B$ .

For  $s : [n] \rightarrow \{0, 1\}$  and  $F \in S$ , we define the function  $\text{val}(F, s)$  using structural recursion as follows.

1.  $\text{val}(x_i, s) = s(i)$ ,
2.  $\text{val}((F_1 + F_2), s) = \text{val}(F_1, s) + \text{val}(F_2, s)$ ,
3.  $\text{val}((F_1 \cdot F_2), s) = \text{val}(F_1, s) \cdot \text{val}(F_2, s)$ .

Let  $F_1, \dots, F_n \in S$ . Let us define the sum of these formulas as follows:

1.  $\sum_{i=j}^j F_i = F_j$ ,
2.  $\sum_{i=j}^{j+k} F_i = f_+(\sum_{i=j}^{j+k-1} F_i, F_{j+k})$  for  $k \geq 1$ .

Show that  $\text{val}(\sum_{i=1}^n x_i, s) = \text{val}(\sum_{i=1}^n x_{n-i+1}, s)$  for any  $s$ .