Name: ______________________

Pid: ______________________

1. (10 points) Check all the correct statements.

   ◯ The inverse of the permutation $(1, 2)(2, 4, 5)(2, 3)$ is 51234.

   ◯ There are 27 different strings of length 3 over the alphabet with 3 letters.

   ◯ Product of the permutations 13254 and 12354 is 23154.

   ◯ There are 60 different surjective functions from [5] to [4].

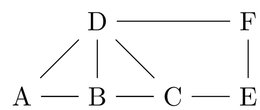
   ◯ There are 3 ways to put 4 identical balls into 3 different boxes such that all the boxes are not empty.

   ◯ A graph on 4 vertices has at most 6 edges.

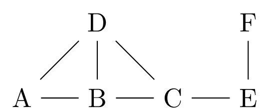
   ◯ A connected graph on 5 vertices has at most 5 edges.

   ◯ If a grpah on 5 vertices has 3 edges it should be disconnected.

   ◯ The following graph has an Eulerian path.



   ◯ The following graph has a Hamiltonian cycle.



Solution:

1. It is easy to see that $(1, 2)(2, 4, 5)(2, 3) = 23451$; hence, the inverse of it is equal to 51234.
2. By the multiplicative law, there are 3^3 words.
3. Product of these two permuations is equal to 13245.
4. In class we proved that there are $5 \cdot 4!$ different surjections from 5 to 4.
5. We proved that the answer is $\binom{3}{2} = 3$.
6. A graph on 4 vertices has at most $4 \cdot 3/2 = 6$ edges.
7. K_5 has 5 vertices but 10 edges.
8. Since $4 > 3$, the statement is true.
9. The graph has exactly two vertices with odd degree. Hence, there is an Eulerian graph.
10. No, there is no Hamiltonian cycle since if we go via the edge CE, we cannot return back.

2. (10 points) What is the maximal number of edges of a simple graph G on $[n]$ if it is not connected?

Solution: We prove that any graph with more than $\frac{(n-1)(n-2)}{2}+1$ edges is connected. Let us assume the opposite; i.e., that there is a graph on n vertices such that it has more than $\frac{(n-1)(n-2)}{2}+1$ edges but it is discinnected. Consider such a graph G with the maximal number of edges. It is easy to see that G has at most two connected components. Indeed if there are three connected components we can add an edge between two of them and the resulting graph is still not connected.

Assume that the components have $m > 0$ and $n-m > 0$ vertices, respectively. In this case the graph has at most $m(m-1)/2 + (n-m)(n-m-1)/2$. It is easy to see that the maximum reaches when $m = 1$ or $n - m = 1$. Hence, the graph has less than $\frac{(n-1)(n-2)}{2} + 1$ edges, which is a contradiction.

3. (10 points) Show that for any $n \in \mathbb{N}$, there are two permuations $p, q \in S_n$ such that any permuation from S_n can be expressed as their product (we can use each permuation multiple times).

Solution: First, let us note that any permuation can be obtained as a product of the permuations $(1, 2)$, ..., $(n-1, n)$. It is enough to prove that any permutation (i, j) $(i < j)$ can be obtained from them. Note that $(i+1, j) \ldots (j-2, j)(j-1, j)(i, i+1) \ldots (j-1, j) = (i, j)$.

So to prove the statement we need to came up with two permutations such that we can obtain $(i, i+1)$ as their product for any $i \in [n-1]$. Consider $(1, 2)$ and $(1, 2, \ldots, n)$. Note that $(1, 2, \ldots, n)^n = 12 \ldots n$. Hence, $(1, 2, \ldots, n)^{n-1} = (1, 2, \ldots, n)^{-1}$. It is also easy to see that $(i, i+1) = (1, 2, \ldots, n)^{-i}(1, 2)(1, 2, \ldots, n)^i$.

4. (10 points) Let $a_n = 2a_{n-1} - a_{n-2}$ for $n \geq 2$, $a_1 = 2$, and $a_0 = 1$. Find a closed formula (no summation signs) for the recurrent sequence a_n.

Solution: Let us, as usual, define F to be the generating function of a_n. Then $F(x) - a_1 x - a_0 = 2x(F(x) - a_0) - x^2 F(x)$. Therefore, $F(x)(1 - 2x + x^2) = 2x + 1 - 2x$. As a result, $F(x) = \frac{1}{(1-x)^2}$. Therefore, $F(x) = \sum_{n \geq 0} (n+1)x^n$. Hence, $a_n = (n+1)$.

5. (10 points) Show that among any $n+1$ positive integers not exceeding $2n$ there must be an integer that divides one of the other integers.

Solution: Let $R = \{1, 3, \ldots, 2n-1\}$ (note that $|R| = n$). Assume that there is a set L that contradicts to the statement of the problem ($|L| = n+1$). It is easy to see that any number $x \in [2n]$ is equal to $2^k \ell$, where ℓ is odd. We define the function $f : L \to R$ such that $f(2^k \ell) = \ell$. By the pigeonhole principle, there are $x < y$ such that $f(x) = f(y)$. But it implies that x divides y.

6. (10 points) Give a simple closed form expression for the sum

$$\sum_{\substack{a+b+c=7\\a,b,c\geq 0}} \binom{7}{a,b,c}.$$

Solution: We proved in class that $(x+y+z)^n = \sum_{\substack{a+b+c=n\\a,b,c\geq 0}} \binom{n}{a,b,c} x^a y^b z^c$. Therefore $\sum_{\substack{a+b+c=7\\a,b,c\geq 0}} \binom{7}{a,b,c} = 3^7$.