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1. (10 points) Let us formulate the pigeonhole principle using propositional formulas. $\Omega = \{x_{1,1}, \dots, x_{n+1,1}, x_{1,2}, \dots, x_{n+1,n}\}$ (informally $x_{i,j}$ is true iff the i th pigeon is in the j th hole). Consider the following propositional formulas on the variables from Ω .

- L_i ($i \in [n+1]$) is equal to $\bigvee_{j=1}^n x_{i,j}$. (Informally this formula says that the i th pigeon is in a hole.)
- R_j ($j \in [n]$) is equal to $\bigvee_{i_1=1}^{n+1} \bigvee_{i_2=i_1+1}^{n+1} (x_{i_1,j} \wedge x_{i_2,j})$. (Informally this formula says that there are two pigeons in the j th hole.)

Show that there is a natural deduction proof of $\left(\bigwedge_{i=1}^{n+1} L_i\right) \implies \left(\bigvee_{i=1}^n R_i\right)$.

Solution: Let $\phi = \left(\bigwedge_{i=1}^{n+1} L_i\right) \implies \left(\bigvee_{i=1}^n R_i\right)$. Note that if we prove that $\phi|_\rho$ is true for any assignment ρ , then by completeness theorem there is a natural deduction proof of ϕ .

Therefore it is enough to show that $\phi|_\rho$ is true for any assignment ρ . Let us fix some ρ .

- If $\left(\bigwedge_{i=1}^{n+1} L_i\right)|_\rho$ is false, then $\phi|_\rho$ is true.
- If $\left(\bigvee_{i=1}^n R_i\right)|_\rho$ is false, then $\phi|_\rho$ is true.
- Hence, to finish the proof we need to show that any other case is impossible. Assume that $\left(\bigvee_{i=1}^n R_i\right)|_\rho$ and $\left(\bigwedge_{i=1}^{n+1} L_i\right)|_\rho$ are true.

Note that the fact that the first formula is true guarantees that $\sum_{i=1}^{n+1} \sum_j 1\rho(x_{i,j}) \geq n+1$. However, the second formula says that $\sum_{i=1}^{n+1} \sum_j 1\rho(x_{i,j}) \leq n$, which leads us to a contradiction.

2. (10 points) Let $\phi = \bigvee_{i=1}^m \lambda_i$ be a clause; we say that the width of the clause is equal to m . Let $\phi = \bigwedge_{i=1}^\ell \chi_i$ be a formula in CNF; we say that the width of ϕ is equal to the maximal width of χ_i for $i \in [\ell]$.

Let $p_n : \{T, F\}^n \rightarrow \{T, F\}$ such that $p_n(x_1, \dots, x_n) = T$ iff the set $\{i : x_i = T\}$ has odd number of elements. Show that any CNF representation of p_n has width at least n .

Solution: Let us assume the opposite; i.e., that there is a formula $\phi = \bigwedge_{i=1}^m C_i$ CNF such that $\phi|_{x_1=v_1, \dots, x_n=v_n} = p_n(v_1, \dots, v_n)$ for all $v_1, \dots, v_n \in \{T, F\}^n$ and width of each C_i is less than n

Without loss of generality we may assume that $C_1 = x_1 \vee \dots \vee x_k$ for $k < n$. Let us consider a

propositional assignement ρ to the variables x_1, \dots, x_n such that $\rho_u(x_i) = \begin{cases} F & \text{if } i \leq k \\ u & \text{if } i = k+1 \\ F & \text{otherwise.} \end{cases}$ It is

clear that $\phi|_{\rho_T} = \phi|_{\rho_F} = F$. However, $p_n(\rho_T(x_1), \dots, \rho_T(x_n)) \neq p_n(\rho_F(x_1), \dots, \rho_F(x_n))$; therefore, $\phi|_{\rho_T} \neq \phi|_{\rho_F}$. As a result, the assumption is wrong.

3. (10 points) Write a natural deduction derivation of $(W \vee Y) \Rightarrow (X \vee Z)$ from hypotheses $W \Rightarrow X$ and $Y \Rightarrow Z$.

Solution:

1		$W \Rightarrow X$	
2		$Y \Rightarrow Z$	
3		$W \vee Y$	
4			W
5			X \Rightarrow E, 1, 4
6			$X \vee Z$ \vee I, 5
7			Y
8			Z \Rightarrow E, 2, 7
9			$X \vee Z$ \vee I, 8
10		$X \vee Z$	\vee E, 3, 4-6, 7-9
11		$(W \vee Y) \Rightarrow (X \vee Z)$	\Rightarrow I, 3-10

4. (10 points) We say that a clause C can be obtained from clauses A and B using the *resolution* rule if $C = A' \vee B'$, $A = x \vee A'$, and $B = \neg x \vee B'$, for some variable x .

We say that a clause C can be derived from clauses A_1, \dots, A_m using resolutions if there is a sequence of clauses $D_1, \dots, D_\ell = C$ such that each D_i

- is either obtained from clauses D_j and D_k for $j, k < i$ using the *resolution* rule, or
- is equal to A_j for some $j \in [m]$, or
- is equal to $D_j \vee E$ for some $j < i$ and a clause E .

Show that if A_1, \dots, A_m semantically imply C , then C can be derived from clauses A_1, \dots, A_m using resolutions.

Solution: First we prove that if clauses A_1, \dots, A_n semantically imply \perp , then there is a derivation of \perp from A_1, \dots, A_n using the resolution rule.

We prove this using the induction on the number k of variables used in clauses A_1, \dots, A_n . The base case for $k = 0$ is clear since in this case $A_i = \perp$ for all $i \in [n]$.

Let us now prove the induction step from k to $k + 1$. We fix a variable x that is used by clauses A_1, \dots, A_n . Let us split set of clauses A_1, \dots, A_n into three groups:

- the clauses $x \vee B_1, \dots, x \vee B_p$ (i.e., the clauses that contain x),
- the clauses $\neg x \vee C_1, \dots, \neg x \vee C_q$ (i.e., the clauses that contain $\neg x$),
- the clauses D_1, \dots, D_r (i.e., the clauses that neither contain x , nor $\neg x$).

Note that any assignment that sets x to be equal to T cannot make all A_i to be true since A_1, \dots, A_n semantically imply \perp . Therefore, $C_1, \dots, C_q, D_1, \dots, D_r$ semantically imply \perp . However, by the induction hypothesis, there is a derivation of \perp from $C_1, \dots, C_q, D_1, \dots, D_r$ using the resolution rule. It is easy to see that this imply that there is a derivation of $\neg x$ from $\neg x \vee C_1, \dots, \neg x \vee C_q, D_1, \dots, D_r$. Similarly, there is a derivation of x from $x \vee B_1, \dots, x \vee B_p, D_1, \dots, D_r$. Hence, using the resolution rule we can derive \perp from A_1, \dots, A_n .