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1. (a) (10 points) Let ϕ , ψ , and χ be propositional formulas on Ω . Show that $(\phi \vee (\psi \wedge \chi))|_{\rho} = ((\phi \vee \psi) \wedge (\phi \vee \chi))|_{\rho}$ for any assignment ρ to the variables Ω .

Solution: Let us fix some propositional assignment ρ to Ω . Note that by the definition

$$(\phi \vee (\psi \wedge \chi))|_{\rho} = \phi|_{\rho} \vee (\psi \wedge \chi)|_{\rho} = \phi|_{\rho} \vee (\psi|_{\rho} \wedge \chi|_{\rho})$$

and

$$((\phi \vee \psi) \wedge (\phi \vee \chi))|_{\rho} = (\phi \vee \psi)|_{\rho} \wedge (\phi \vee \chi)|_{\rho} = (\phi|_{\rho} \vee \psi|_{\rho}) \wedge (\phi|_{\rho} \vee \chi|_{\rho}).$$

However, $\phi|_{\rho} \vee (\psi|_{\rho} \wedge \chi|_{\rho}) = (\phi|_{\rho} \vee \psi|_{\rho}) \wedge (\phi|_{\rho} \vee \chi|_{\rho})$ by the distributivity of disjunction and conjunction.

- (b) (10 points) Let $\psi_{1,1}, \dots, \psi_{1,n}, \psi_{2,1}, \dots, \psi_{2,m}$ be propositional formulas on Ω . Let $\phi_1 = \bigwedge_{i=1}^n \psi_{1,i}$ and $\phi_2 = \bigwedge_{j=1}^m \psi_{2,j}$. Show that $(\phi_1 \vee \phi_2)|_{\rho} = (\bigwedge_{i=1}^n \bigwedge_{j=1}^m (\psi_{1,i} \vee \psi_{2,j}))|_{\rho}$ for any assignment ρ to the variables Ω .

Solution: Let us again fix some propositional assignment ρ to Ω .

We prove the statement in two steps. In the first one we prove that

$$(\phi_1 \vee \chi)|_{\rho} = \left(\bigwedge_{i=1}^n (\psi_{1,i} \vee \chi) \right) \Big|_{\rho}$$

using induction by n .

The base case for $n = 1$ is clear. Let us now prove the induction step from k to $k + 1$. Note that

$$\left(\bigwedge_{i=1}^{k+1} \psi_{1,i} \right) \vee \chi = \left(\left(\bigwedge_{i=1}^k \psi_{1,i} \right) \wedge \psi_{1,k+1} \right) \vee \chi.$$

By the previous problem, this implies that

$$\left(\left(\bigwedge_{i=1}^{k+1} \psi_{1,i} \right) \vee \chi \right) \Big|_{\rho} = \left(\left(\left(\bigwedge_{i=1}^k \psi_{1,i} \right) \vee \chi \right) \wedge (\psi_{1,k+1} \vee \chi) \right) \Big|_{\rho}.$$

The induction hypothesis, implies that

$$\left(\left(\bigwedge_{i=1}^{k+1} \psi_{1,i} \right) \vee \chi \right) \Big|_{\rho} = \left(\left(\bigwedge_{i=1}^k (\psi_{1,i} \vee \chi) \right) \wedge (\psi_{1,k+1} \vee \chi) \right) \Big|_{\rho}.$$

Therefore, using the definition of the conjunction of several formulas, we proved that

$$\left(\left(\bigwedge_{i=1}^{k+1} \psi_{1,i} \right) \vee \chi \right) \Big|_{\rho} = \left(\bigwedge_{i=1}^{k+1} (\psi_{1,i} \vee \chi) \right) \Big|_{\rho}.$$

On the second step we prove the statement of the problem using induction by m . The base case follows from the result we just proved. Let us now prove the induction step from k to $k + 1$. Note that

$$\left(\bigwedge_{i=1}^m \psi_{1,i} \right) \vee \left(\bigwedge_{i=1}^{k+1} \psi_{2,i} \right) = \left(\bigwedge_{i=1}^m \psi_{1,i} \right) \vee \left(\left(\bigwedge_{i=1}^k \psi_{2,i} \right) \wedge \psi_{2,k+1} \right).$$

By the previous problem,

$$\left(\left(\bigwedge_{i=1}^m \psi_{1,i} \right) \vee \left(\bigwedge_{i=1}^{k+1} \psi_{2,i} \right) \right) \Big|_{\rho} = \left(\left(\left(\bigwedge_{i=1}^m \psi_{1,i} \right) \vee \psi_{2,k+1} \right) \wedge \left(\left(\bigwedge_{i=1}^m \psi_{1,i} \right) \vee \left(\bigwedge_{i=1}^k \psi_{2,i} \right) \right) \right) \Big|_{\rho}.$$

Therefore by the previous statement,

$$\left(\left(\bigwedge_{i=1}^m \psi_{1,i} \right) \vee \left(\bigwedge_{i=1}^{k+1} \psi_{2,i} \right) \right) \Big|_{\rho} = \left(\left(\bigwedge_{i=1}^m (\psi_{1,i} \vee \psi_{2,k+1}) \right) \wedge \left(\left(\bigwedge_{i=1}^m \psi_{1,i} \right) \vee \left(\bigwedge_{i=1}^k \psi_{2,i} \right) \right) \right) \Big|_{\rho}.$$

Finally, using the induction hypothesis,

$$\left(\left(\bigwedge_{i=1}^m \psi_{1,i} \right) \vee \left(\bigwedge_{i=1}^{k+1} \psi_{2,i} \right) \right) \Big|_{\rho} = \left(\left(\bigwedge_{i=1}^m (\psi_{1,i} \vee \psi_{2,k+1}) \right) \wedge \left(\bigwedge_{i=1}^m \bigwedge_{i=1}^k (\psi_{1,i} \vee \psi_{2,i}) \right) \right) \Big|_{\rho}.$$

- (c) (10 points) Let Ω be a set of variables. We say that a propositional formula is a literal if the formula is equal to x or $\neg x$ for $x \in \Omega$.

We say that a propositional formula on Ω is in conjunctive normal form if it is equal to $\bigwedge_{i=1}^n \bigvee_{j=1}^{m_i} \psi_{i,j}$, where $\psi_{i,j}$ is a literal.

Let ϕ be a propositional formula on Ω . Show using structural induction that there is a propositional formula ψ on Ω in conjunctive normal form such that $\psi|_{\rho} = \phi|_{\rho}$ for any assignment ρ to Ω .

Solution: Before we prove the statement of the problem, we need to show several equalities.

Let $\chi_{1,1}, \dots, \chi_{1,n_1}, \chi_{2,1}, \dots, \chi_{2,n_2}, \chi_1, \dots, \chi_{n_1+n_2}$ be propositional formulas over the variables from Ω such that $\chi_i = \chi_{1,i}$ for $1 \leq i \leq n_1$ and $\chi_i = \chi_{2,i-n_1}$ for $n_1 < i \leq n_1 + n_2$. Then for any propositional assignment ρ to Ω ,

$$\left(\left(\bigwedge_{i=1}^{n_1} \chi_{1,i} \right) \wedge \left(\bigwedge_{i=1}^{n_2} \chi_{2,i} \right) \right) \Big|_{\rho} = \left(\bigwedge_{i=1}^{n_1+n_2} \chi_i \right) \Big|_{\rho}.$$

We can prove the statement using induction by n_2 . The base case for $n_2 = 1$ follows from the definition of the long conjunction. Let us prove the induction step from k to $k+1$. By the definition of the long conjunction,

$$\left(\bigwedge_{i=1}^{n_1} \chi_{1,i} \right) \wedge \left(\bigwedge_{i=1}^{k+1} \chi_{2,i} \right) = \left(\bigwedge_{i=1}^{n_1} \chi_{1,i} \right) \wedge \left(\left(\bigwedge_{i=1}^k \chi_{2,i} \right) \wedge \chi_{2,k+1} \right)$$

Note that we proved in class the following

$$\begin{aligned} \left(\left(\bigwedge_{i=1}^{n_1} \chi_{1,i} \right) \wedge \left(\bigwedge_{i=1}^{k+1} \chi_{2,i} \right) \right) \Big|_{\rho} &= \left(\left(\bigwedge_{i=1}^{n_1} \chi_{1,i} \right) \wedge \left(\left(\bigwedge_{i=1}^k \chi_{2,i} \right) \wedge \chi_{2,k+1} \right) \right) \Big|_{\rho} = \\ &= \left(\left(\left(\bigwedge_{i=1}^{n_1} \chi_{1,i} \right) \wedge \left(\bigwedge_{i=1}^k \chi_{2,i} \right) \right) \wedge \chi_{2,k+1} \right) \Big|_{\rho}. \end{aligned}$$

By the induction hypothesis,

$$\left(\left(\bigwedge_{i=1}^{n_1} \chi_{1,i} \right) \wedge \left(\bigwedge_{i=1}^{k+1} \chi_{2,i} \right) \right) \Big|_{\rho} = \left(\left(\bigwedge_{i=1}^{n_1+k} \chi_i \right) \wedge \chi_{2,k+1} \right) \Big|_{\rho}.$$

Which implies the statement by the definition of the long conjunction.

Consider $e_{n,m} : \{0, \dots, m^n - 1\} \rightarrow \{0, \dots, m - 1\}^n$ be a bijection such that

- $e_{n,m}(i + m \cdot r, 0) = i$ and
- $e_{n,m}(i + m \cdot r, j) = e_{n-1,m}(r, j - 1)$,

for any $0 \leq i < m$, $0 \leq r < m^{n-1}$, and $0 \leq j < n$. We also show that $\bigvee_{j=0}^{n-1} \bigwedge_{i=0}^{m-1} \chi_{j,i} = \bigwedge_{q=0}^{m^n-1} \bigvee_{j=0}^{n-1} \chi_{j,e_{n,m}(q,j)}$. We prove the statement using induction by n . The case of $n = 1$ is clear. We prove now the induction step from k to $k + 1$. Note that

$$\bigvee_{j=0}^k \bigwedge_{i=0}^{m-1} \chi_{j,i} = \left(\bigvee_{j=0}^{k-1} \bigwedge_{i=0}^{m-1} \chi_{j,i} \right) \vee \bigwedge_{i=0}^{m-1} \chi_{k,i}.$$

By the induction hypothesis,

$$\left(\bigvee_{j=0}^k \bigwedge_{i=0}^{m-1} \chi_{j,i} \right) \Big|_{\rho} = \left(\left(\bigwedge_{q=0}^{m^k-1} \bigvee_{j=0}^{k-1} \chi_{j,e_{k,m}(q,j)} \right) \vee \bigwedge_{i=0}^{m-1} \chi_{k,i} \right) \Big|_{\rho}.$$

Therefore,

$$\left(\bigvee_{j=0}^k \bigwedge_{i=0}^{m-1} \chi_{j,i} \right) \Big|_{\rho} = \left(\bigwedge_{q=0}^{m^k-1} \left(\left(\bigvee_{j=0}^{k-1} \chi_{j,e_{k,m}(q,j)} \right) \vee \bigwedge_{i=0}^{m-1} \chi_{k,i} \right) \right) \Big|_{\rho}.$$

So

$$\left(\bigvee_{j=0}^k \bigwedge_{i=0}^{m-1} \chi_{j,i} \right) \Big|_{\rho} = \left(\bigwedge_{q=0}^{m^k-1} \bigwedge_{i=0}^{m-1} \left(\left(\bigvee_{j=0}^{k-1} \chi_{j,e_{k,m}(q,j)} \right) \vee \chi_{k,i} \right) \right) \Big|_{\rho}.$$

As a result,

$$\left(\bigvee_{j=0}^k \bigwedge_{i=0}^{m-1} \chi_{j,i} \right) \Big|_{\rho} = \left(\bigwedge_{q=0}^{m^{k+1}-1} \left(\bigvee_{j=0}^k \chi_{j,e_{k+1,m}(q,j)} \right) \right) \Big|_{\rho}.$$

Finally, we are ready to prove the statement of the problem. Let us consider the following cases.

- The first case is when $\phi = x_i$. In this case we can choose $n = 1$, $m_1 = 1$, and $\psi_{1,1} = x_i$.
- The second case is when $\phi = \phi_1 \wedge \phi_2$. Note that by the induction hypothesis, $\phi_1 = \bigwedge_{j=1}^{n_1} \bigvee_{k=1}^{m_{1,i}} \psi_{1,i,j}$ and $\phi_2 = \bigwedge_{j=1}^{n_2} \bigvee_{k=1}^{m_{2,i}} \psi_{2,i,j}$. Using the statement of the previous problem, $\phi = \bigwedge_{j=1}^{n_1+n_2} \bigvee_{k=1}^{m_i} \psi_{i,j}$.
- The third case is when $\phi = \phi_1 \vee \phi_2$. Note that by the induction hypothesis, $\phi_1 = \bigwedge_{i=1}^{n_1} \bigvee_{j=1}^{m_{1,i}} \psi_{1,i,j}$ and $\phi_2 = \bigwedge_{i=1}^{n_2} \bigvee_{j=1}^{m_{2,i}} \psi_{2,i,j}$. Using the just proved statement we may conclude that $\phi = \bigwedge_{i=1}^{n_1+n_2} \bigvee_{j=1}^{m_i} \psi_{i,j}$, where $m_i = m_{1,i}$ for $1 \leq i \leq n_1$ and $m_{2,i-n_1}$ for $n_1 < i \leq n_1 + n_2$, $\psi_{i,j} = \psi_{1,i,j}$ for $1 \leq i \leq n_1$ and $\psi_{i,j} = \psi_{2,i-n_1,j}$ for $n_1 < i \leq n_1 + n_2$.
- Finally, the last case is when $\phi = \neg \phi'$. Note that by the induction hypothesis, $\phi' = \bigwedge_{j=1}^{n'} \bigvee_{k=1}^{m'_i} \psi'_{i,j}$.

Hence, $\phi|_{\rho} = \bigvee_{j=1}^{n'} \bigwedge_{k=1}^{m'_i} \neg \psi'_{i,j}$. And using the second proven observation in this problem we can present a CNF representation of ϕ .