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1. (10 points) Show that  $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$  for all integers  $n \geq 1$ .

**Solution:** We prove the statement using induction by  $n$ . The base case for  $n = 1$  is true since  $1^2 = 1$  and  $\frac{1(1+1)(2+1)}{6} = 1$ .

Now we need to prove the induction step from  $n$  to  $n + 1$ . Assume that  $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$ . By the hypothesis,  $1^2 + 2^2 + 3^2 + \cdots + n^2 + (n + 1)^2 = \frac{n(n+1)(2n+1)}{6} + (n + 1)^2$ . Note that

$$\begin{aligned} \frac{n(n+1)(2n+1)}{6} + (n+1)^2 &= \frac{n(n+1)(2n+1) + 6n^2 + 12n + 6}{6} = \\ &= \frac{n^3 + 2n^2 + n^2 + n + 6n^2 + 12n + 6}{6} = \frac{n^3 + 9n^2 + 13n + 6}{6} = \\ &= \frac{(n+1)(n+2)(2n+3)}{6}. \end{aligned}$$

Hence, the induction step is true and by the induction principle,  $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$  for all positive integers  $n$ .

2. (10 points) Let  $a_0 = 2$ ,  $a_1 = 5$ , and  $a_n = 5a_{n-1} - 6a_{n-2}$  for all integers  $n \geq 2$ . Show that  $a_n = 3^n + 2^n$  for all integers  $n \geq 0$ .

**Solution:** We prove the statement using induction by  $n$  for  $n \geq 0$ . The base cases for  $n = 0$  and  $n = 1$  are true since  $3^0 + 2^0 = 1 + 1 = 2 = a_0$  and  $3^1 + 2^1 = 3 + 2 = 5 = a_1$ .

Let us now prove the induction step from  $n$  and  $n - 1$  to  $n + 1$ . Assume that  $a_{n-1} = 3^{n-1} + 2^{n-1}$  and  $a_n = 3^n + 2^n$ . Note that  $a_{n+1} = 5a_n - 6a_{n-1}$ ; hence, by the induction hypothesis,  $a_{n+1} = 5(3^n + 2^n) - 6(3^{n-1} + 2^{n-1}) = (5 \cdot 3 - 6)3^{n-1} + (5 \cdot 2 - 6)2^{n-1} = 9 \cdot 3^{n-1} + 4 \cdot 2^{n-1} = 3^{n+1} + 2^{n+1}$ .

As a result, the induction step is true and by the induction principle,  $a_n = 3^n + 2^n$  for all integers  $n \geq 0$ . (Note that we proved a stronger statement than it was asked in the problem.)

3. (10 points) Let  $n$  be a positive integer and  $A_1, \dots, A_n$  be some sets. Let us define union of these sets as follows:

1.  $\cap_{i=1}^1 A_i = A_1$ ,
2.  $\cap_{i=1}^{k+1} A_i = (\cap_{i=1}^k A_i) \cap A_{k+1}$ .

Show that  $\cap_{i=1}^n \{x \in \mathbb{N} : i \leq x \leq n\} = \{n\}$ .

**Solution:** We prove using induction by  $m$  that  $\cap_{i=1}^m \{x \in \mathbb{N} : i \leq x \leq n\} = \{m, m+1, \dots, n\}$ .

The base case is for  $m = 1$  is true since

$$\cap_{i=1}^1 \{x \in \mathbb{N} : i \leq x \leq n\} = [n] = \{1, 2, \dots, n\}.$$

Let us now prove the induction step from  $m$  to  $m+1$ . Assume that  $\cap_{i=1}^m \{x \in \mathbb{N} : i \leq x \leq n\} = \{m, m+1, \dots, n\}$ . Note that

$$\cap_{i=1}^{m+1} \{x \in \mathbb{N} : i \leq x \leq n\} = (\cap_{i=1}^m \{x \in \mathbb{N} : i \leq x \leq n\}) \cap \{x \in \mathbb{N} : m+1 \leq x \leq n\}.$$

Therefore

$$\cap_{i=1}^{m+1} \{x \in \mathbb{N} : i \leq x \leq n\} = \{m, m+1, \dots, n\} \cap \{m+1, \dots, n\} = \{m+1, \dots, n\}.$$

Hence, by the induction principle, the statement is true for all  $m$ . As a result, we proved for  $m = n$  that  $\cap_{i=1}^n \{x \in \mathbb{N} : i \leq x \leq n\} = \{n\}$ .

4. (10 points) Let  $U$  be the set of sequences of the following symbols: “+”, “.”, “ $x_1$ ”, ..., “ $x_n$ ”. Let  $B = \{x_i : i \in [n]\}$ ; i.e.,  $B$  is the set of sequences consisting of only one symbol  $x_i$ . Let  $\mathcal{F} = \{f_+, f.\}$ , where  $f_+(F_1, F_2) = (F_1 + F_2)$  and  $f.(F_1, F_2) = (F_1 \cdot F_2)$  (by  $(F_1 \# F_2)$  we denote the sequence obtained by concatenating “(”,  $F_1$ , “#”,  $F_2$ , and “)”). Let  $S$  be the set generated by  $\mathcal{F}$  from  $B$ .

For  $s : [n] \rightarrow \{0, 1\}$  and  $F \in S$ , we define the function  $\text{val}(F, s)$  using structural recursion as follows.

1.  $\text{val}(x_i, s) = s(i)$ ,
2.  $\text{val}((F_1 + F_2), s) = \text{val}(F_1, s) + \text{val}(F_2, s)$ ,
3.  $\text{val}((F_1 \cdot F_2), s) = \text{val}(F_1, s) \cdot \text{val}(F_2, s)$ .

Let  $F_1, \dots, F_n \in S$ . Let us define the sum of these formulas as follows:

1.  $\sum_{i=j}^j F_i = F_j$ ,
2.  $\sum_{i=j}^{j+k} F_i = f_+(\sum_{i=j}^{j+k-1} F_i, F_{j+k})$  for  $k \geq 1$ .

Show that  $\text{val}(\sum_{i=1}^n x_i, s) = \text{val}(\sum_{i=1}^n x_{n-i+1}, s)$  for any  $s$ .

**Solution:** Before we start working with the arithmetic formulas, let us prove several statements for real number. Let  $a_1, \dots, a_n$  be some real numbers. We show that  $\sum_{i=m}^{m+n} a_i = a_m + \sum_{i=m+1}^{m+n} a_i$  for  $n \geq 1$  using induction by  $n$ . The base case is true for  $n = 1$  since  $\sum_{i=m}^{m+1} a_i = a_m + a_{m+1} = a_m + \sum_{i=m+1}^{m+1} a_i$ .

Let us now prove the induction step from  $n$  to  $n + 1$ . Assume that  $\sum_{i=m}^{m+n} a_i = a_m + \sum_{i=m+1}^{m+n} a_i$ . Note that by the induction hypothesis,

$$\sum_{i=m}^{m+n+1} a_i = \sum_{i=m}^{m+n} a_i + a_{m+n+1} = a_m + \sum_{i=m+1}^{m+n} a_i + a_{m+n+1} = a_m + \sum_{i=m+1}^{m+n+1} a_i.$$

Using this statement we may show that  $\sum_{i=1}^m a_i = \sum_{i=n-m+1}^n a_{n-i+1}$  for  $m \geq 1$  using induction by  $m$ . The base case is true since  $\sum_{i=1}^1 a_i = a_1 = \sum_{i=n}^n a_{n-i+1}$ . To prove the induction step from  $m$  to  $m + 1$ ; assume  $\sum_{i=1}^m a_i = \sum_{i=n-m+1}^n a_{n-i+1}$ . Note that the hypothesis implies that

$$\sum_{i=1}^{m+1} a_i = \sum_{i=1}^m a_i + a_{m+1} = \sum_{i=n-m+1}^n a_{n-i+1} + a_{m+1} = \sum_{i=n-m}^n a_{n-i+1}.$$

Therefore by the induction hypothesis,  $\sum_{i=1}^m a_i = \sum_{i=n-m+1}^n a_{n-i+1}$  for  $m \geq 1$ . If we consider  $m = n$ , we get  $\sum_{i=1}^n a_i = \sum_{i=1}^n a_{n-i+1}$ .

Let us now explain how to get this statement for arithmetic formulas. Let  $F_1, \dots, F_m$  be some arithmetic formulas. Then we may show that  $\text{val}(\sum_{i=1}^m F_i, s) = \sum_{i=1}^m \text{val}(F_i, s)$  for all  $s$ . Fix some  $s$ ; we prove this statement also using induction. The base case for  $m = 1$  is true since  $\sum_{i=1}^1 F_i = F_1$  and  $\sum_{i=1}^1 \text{val}(F_i, s) = \text{val}(F_1, s)$ . To prove the induction step from  $m$  to  $m + 1$ ; assume  $\text{val}(\sum_{i=1}^m F_i, s) = \sum_{i=1}^m \text{val}(F_i, s)$ . Note that  $\sum_{i=1}^{m+1} F_i = f_+(\sum_{i=1}^m F_i, F_{m+1})$ , and  $\text{val}(f_+(\sum_{i=1}^m F_i, F_{m+1}), s) = \text{val}(\sum_{i=1}^m F_i, s) + \text{val}(F_{m+1}, s)$ . Hence,

$$\text{val}(\sum_{i=1}^{m+1} F_i, s) = \text{val}(\sum_{i=1}^m F_i, s) + \text{val}(F_{m+1}, s) = \sum_{i=1}^m \text{val}(F_i, s) + \text{val}(F_{m+1}, s) = \sum_{i=1}^{m+1} \text{val}(F_i, s).$$

Using all these statement, we may notice that

$$\text{val}(\sum_{i=1}^n x_i, s) = \sum_{i=1}^n \text{val}(x_i, s) = \sum_{i=1}^n \text{val}(x_{n-i+1}, s) = \text{val}(\sum_{i=1}^n x_{n-i+1}, s).$$