Name: ____________________

Pid: ____________________

Note that since this class is about proofs, every statement in the midterm should be proved. The only exceptions are statements that were proven in previous homework or midterms and statements proven earlier in the class.

1. (10 points) Show that $F(\mathbb{N}, \{0,1\})$ is equipotent to $F(\mathbb{N}, \{0,1,2\})$

Solution: We are going to prove it using Cantor-Bernstein theorem. Hence, the proof consists of two parts.

1. In the first part we prove that $|F(\mathbb{N}, \{0,1\})| \leq F(\mathbb{N}, \{0,1,2\})$. To prove this we need to show that there is an injection from $F(\mathbb{N}, \{0,1\})$ to $F(\mathbb{N}, \{0,1,2\})$ and it is clear that $\mathcal{F} : F(\mathbb{N}, \{0,1\}) \to F(\mathbb{N}, \{0,1,2\})$ such that $\mathcal{F}(f) = f$ is an injection.

2. In the second part we prove that $|F(\mathbb{N}, \{0,1,2\})| \leq F(\mathbb{N}, \{0,1\})$. To prove this we need to show that there is an injection from $F(\mathbb{N}, \{0,1,2\})$ to $F(\mathbb{N}, \{0,1\})$. Let us consider $\mathcal{G}$: $F(\mathbb{N}, \{0,1,2\}) \to F(\mathbb{N}, \{0,1\})$ such that $\mathcal{F}(f) = f'$, where $h'(2n-1) = \begin{cases} 0 & \text{if } h(n) \neq 2 \\ 1 & \text{otherwise} \end{cases}$ and $h'(2n) = \begin{cases} 0 & \text{if } h(n) \neq 1 \\ 1 & \text{otherwise.} \end{cases}$ We need to prove that this is an injection. Assume the opposite i.e. that $\mathcal{G}(h_1) = \mathcal{G}(h_2)$ but $h_1 \neq h_2$. There is n such that $h_1(n) \neq h_2(n)$ since $h_1 \neq h_2$. WLOG it is enough to consider the following cases.

   - If $h_1(n) = 0$ and $h_2(n) = 1$, then $\mathcal{G}(h_1)(2n) = 0$ and $\mathcal{G}(h_2)(2n) = 1$ which is a contradiction.
   - If $h_1(n) = 0$ and $h_2(n) = 2$, then $\mathcal{G}(h_1)(2n-1) = 0$ and $\mathcal{G}(h_2)(2n-1) = 1$ which is a contradiction.
   - If $h_1(n) = 1$ and $h_2(n) = 2$, then $\mathcal{G}(h_1)(2n-1) = 0$ and $\mathcal{G}(h_2)(2n-1) = 1$ which is a contradiction.

2. (10 points) Let $\oplus$ be a Boolean operation such that for any $a, b \in \{0,1\}$, $a \oplus b = 1$ iff $a \neq b$. A function $f : \{0,1\}^n \to \{0,1\}$ can be represented as a monomial iff $f(x_1, \dots, x_n) = x_{i_1} \cdot x_{i_2} \cdot \dots \cdot x_{i_k}$ for all $x_1, \dots, x_n \in \{0,1\}$, where $k \geq 0$ and $i_1, \dots, i_k \in [n]$. Finally we say that a function $f : \{0,1\}^n \to \{0,1\}$ can be represented as a Zhegalkin polynomial iff $f(x_1, \dots, x_n) = g_1(x_1, \dots, x_n) \oplus \dots \oplus g_\ell(x_1, \dots, x_n)$ for all $x_1, \dots, x_n \in \{0,1\}$, where $g_1, \dots, g_\ell : \{0,1\}^n \to \{0,1\}$ are different functions representable as monomials.

   Show that any function $f : \{0,1\}^n \to \{0,1\}$ is representable as a Zhegalkin polynomial.

   Solution: Let us compute the number of Zhegalkin polynomials: there are 2^n different monomials and as a result there are 2^{2^n} Zhegalkin polynomials. Let us also compute the number of functions from $\{0,1\}^n$ to $\{0,1\}$, we proved in class that it is equal to $|\{0,1\}|^{|\{0,1\}^n|} = 2^{2^n}$.

   Let P_n be the set of Zhegalkin polynomials on n variables and $f : P_n \to F(\{0,1\}^n, \{0,1\})$ be the function such that $f(p)$ is equal to the function corrsponding to the Zhegalkin polynomial p. First we need to show that f is an injection. Assume that it is not i.e. there are $p_1, p_2 \in P_n$ such that $f(p_1) = f(p_2)$. Let p be a polinomial consisting of the symmetirc difference of sets of monomials of p_1 and p_2. Note that $f(p) = 0$ since $f(p_1) = f(p_2)$

   Since $|P_n| = |F(\{0,1\}^n, \{0,1\})|$ f is a bijection. Hence, every function has a unique representation as a Zhegalkin polynomial.

3. Let us consider the following theory, it is a theory with undefined terms: element, plus, and times (if a and b are elements, we denote a times b by $a \cdot b$ and a plus b by $a + b$), and axioms:

Associativity: for all elements a, b, and c, $a + (b + c) = (a + b) + c$ and $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.

Commutativity: for all elements a and b, $a + b = b + a$ and $a \cdot b = b \cdot a$.

Identity elements: there exist two different elements 0 and 1 such that $a + 0 = a$ and $a \cdot 1 = a$ for every element a.

Inverses: for every element a, there exists an element, denoted $-a$, called the additive inverse of a, such that $a + (-a) = 0$ and moreover, for every $a \neq 0$, there exists an element, denoted by a^{-1} or $\frac{1}{a}$, called the multiplicative inverse of a, such that $a \cdot a^{-1} = 1$.

Distributivity: for all elements a, b, and c, $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$.

Let $k \in \mathbb{N}$ and a be an element. Then we denote $a + a + \cdots + a$ (k times) by $k \cdot a$.

We say that $p_a \in \mathbb{N}$ is a characteristic of an element a if $p_a \cdot a = 0$ but $q \cdot a \neq 0$ for all $q < p_a$.

(a) (5 points) Show that if a and b are nonzero elements and p_a exists, then their characteristics are euqal.

Solution: It is enough to prove that $p_a = p_1$ for all $a \neq 0$.
We start from proving several statements. Let a, b, and c be some elements.

1. If $a \neq b$, then $a \cdot c \neq b \cdot c$. Indeed, assume that $a \cdot c = b \cdot c$. In this case $a \cdot c\frac{1}{c} = b \cdot c \cdot \frac{1}{c}$, which implies that $a = a \cdot 1 = b \cdot 1 = b$.
2. If $a + a = a$, then $a = 0$. Indeed, $a + a + (-a) = a + (-a)$, hence, $a = 0$.
3. $a \cdot 0 = 0$ since $a \cdot 0 + a \cdot 0 = a \cdot (0 + 0) = a \cdot 0$.
4. if $a \neq 0$ and $a \cdot b = 0$, then $b = 0$. Indeed, $b = \frac{1}{a} \cdot a \cdot b = \frac{1}{a} \cdot 0 = 0$

Finally we are ready to prove the statement. First note that if p_a exists, then $\underbrace{a + \cdots + a}_{p_a \text{ times}} = 0$ which implies that $a \cdot \underbrace{(1 + \cdots + 1)}_{p_a \text{ times}} = 0$. Hence, $\underbrace{1 + \cdots + 1}_{p_a \text{ times}} = 0$ since $a \neq 0$. As a result $p_1 \leq p_a$.
Similarly $\underbrace{1 + \cdots + 1}_{p_1 \text{ times}} = 0$, hence, $\underbrace{a + \cdots + a}_{p_a \text{ times}} = 0$.

(b) (5 points) Show that if a is a nonzero element, then p_a is prime.

Solution:

4. (10 points) Show that $\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$.

Solution:

5. (10 points) Let $f_0 = 1$, $f_1 = 1$, and $f_{n+2} = f_{n+1} + f_n$ for all $n \in \mathbb{N}$. Show that $f_n \geq \left(\frac{3}{2}\right)^{n-2}$.

Solution:

6. (10 points) Sasha is training for a triathlon. Over a 30 day period, he pledges to train at least once per day, and 45 times in all. Then there will be a period of consecutive days where he trains exactly 14 times.

Solution: