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1. (a) (10 points) Let  $\phi$ ,  $\psi$ , and  $\chi$  be propositional formulas on  $\Omega$ . Show that  $(\phi \vee (\psi \wedge \chi))|_{\rho} = ((\phi \vee \psi) \wedge (\phi \vee \chi))|_{\rho}$  for any assignment  $\rho$  to the variables  $\Omega$ .

**Solution:** Let us fix some propositional assignment  $\rho$  to  $\Omega$ . Note that by the definition

$$(\phi \vee (\psi \wedge \chi))|_{\rho} = \phi|_{\rho} \vee (\psi \wedge \chi)|_{\rho} = \phi|_{\rho} \vee (\psi|_{\rho} \wedge \chi|_{\rho})$$

and

$$((\phi \vee \psi) \wedge (\phi \vee \chi))|_{\rho} = (\phi \vee \psi)|_{\rho} \wedge (\phi \vee \chi)|_{\rho} = (\phi|_{\rho} \vee \psi|_{\rho}) \wedge (\phi|_{\rho} \vee \chi|_{\rho}).$$

However,  $\phi|_{\rho} \vee (\psi|_{\rho} \wedge \chi|_{\rho}) = (\phi|_{\rho} \vee \psi|_{\rho}) \wedge (\phi|_{\rho} \vee \chi|_{\rho})$  by the distributivity of disjunction and conjunction.

- (b) (10 points) Let  $\psi_{1,1}, \dots, \psi_{1,n}, \psi_{2,1}, \dots, \psi_{2,m}$  be propositional formulas on  $\Omega$ . Let  $\phi_1 = \bigwedge_{i=1}^n \psi_{1,i}$  and  $\phi_2 = \bigwedge_{j=1}^m \psi_{2,j}$ . Show that  $(\phi_1 \vee \phi_2)|_{\rho} = (\bigwedge_{i=1}^n \bigwedge_{j=1}^m (\psi_{1,i} \vee \psi_{2,j}))|_{\rho}$  for any assignment  $\rho$  to the variables  $\Omega$ .

**Solution:** Let us again fix some propositional assignment  $\rho$  to  $\Omega$ .

We prove the statement in two steps. In the first one we prove that

$$(\phi_1 \vee \chi)|_{\rho} = \left( \bigwedge_{i=1}^n (\psi_{1,i} \vee \chi) \right) \Big|_{\rho}$$

using induction by  $n$ .

The base case for  $n = 1$  is clear. Let us now prove the induction step from  $k$  to  $k + 1$ . Note that

$$\left( \bigwedge_{i=1}^{k+1} \psi_{1,i} \right) \vee \chi = \left( \left( \bigwedge_{i=1}^k \psi_{1,i} \right) \wedge \psi_{1,k+1} \right) \vee \chi.$$

By the previous problem, this implies that

$$\left( \left( \bigwedge_{i=1}^{k+1} \psi_{1,i} \right) \vee \chi \right) \Big|_{\rho} = \left( \left( \left( \bigwedge_{i=1}^k \psi_{1,i} \right) \vee \chi \right) \wedge (\psi_{1,k+1} \vee \chi) \right) \Big|_{\rho}.$$

The induction hypothesis, implies that

$$\left( \left( \bigwedge_{i=1}^{k+1} \psi_{1,i} \right) \vee \chi \right) \Big|_{\rho} = \left( \left( \bigwedge_{i=1}^k (\psi_{1,i} \vee \chi) \right) \wedge (\psi_{1,k+1} \vee \chi) \right) \Big|_{\rho}.$$

Therefore, using the definition of the conjunction of several formulas, we proved that

$$\left( \left( \bigwedge_{i=1}^{k+1} \psi_{1,i} \right) \vee \chi \right) \Big|_{\rho} = \left( \bigwedge_{i=1}^{k+1} (\psi_{1,i} \vee \chi) \right) \Big|_{\rho}.$$

On the second step we prove the statement of the problem using induction by  $m$ . The base case follows from the result we just proved. Let us now prove the induction step from  $k$  to  $k + 1$ . Note that

$$\left( \bigwedge_{i=1}^m \psi_{1,i} \right) \vee \left( \bigwedge_{i=1}^{k+1} \psi_{2,i} \right) = \left( \bigwedge_{i=1}^m \psi_{1,i} \right) \vee \left( \left( \bigwedge_{i=1}^k \psi_{2,i} \right) \wedge \psi_{2,k+1} \right).$$

By the previous problem,

$$\left( \left( \bigwedge_{i=1}^m \psi_{1,i} \right) \vee \left( \bigwedge_{i=1}^{k+1} \psi_{2,i} \right) \right) \Big|_{\rho} = \left( \left( \left( \bigwedge_{i=1}^m \psi_{1,i} \right) \vee \psi_{2,k+1} \right) \wedge \left( \left( \bigwedge_{i=1}^m \psi_{1,i} \right) \vee \left( \bigwedge_{i=1}^k \psi_{2,i} \right) \right) \right) \Big|_{\rho}.$$

Therefore by the previous statement,

$$\left( \left( \bigwedge_{i=1}^m \psi_{1,i} \right) \vee \left( \bigwedge_{i=1}^{k+1} \psi_{2,i} \right) \right) \Big|_{\rho} = \left( \left( \bigwedge_{i=1}^m (\psi_{1,i} \vee \psi_{2,k+1}) \right) \wedge \left( \left( \bigwedge_{i=1}^m \psi_{1,i} \right) \vee \left( \bigwedge_{i=1}^k \psi_{2,i} \right) \right) \right) \Big|_{\rho}.$$

Finally, using the induction hypothesis,

$$\left( \left( \bigwedge_{i=1}^m \psi_{1,i} \right) \vee \left( \bigwedge_{i=1}^{k+1} \psi_{2,i} \right) \right) \Big|_{\rho} = \left( \left( \bigwedge_{i=1}^m (\psi_{1,i} \vee \psi_{2,k+1}) \right) \wedge \left( \bigwedge_{i=1}^m \bigwedge_{i=1}^k (\psi_{1,i} \vee \psi_{2,i}) \right) \right) \Big|_{\rho}.$$

- (c) (10 points) Let  $\Omega$  be a set of variables. We say that a propositional formula is a literal if the formula is equal to  $x$  or  $\neg x$  for  $x \in \Omega$ .

We say that a propositional formula on  $\Omega$  is in conjunctive normal form if it is equal to  $\bigwedge_{i=1}^n \bigvee_{j=1}^{m_i} \psi_{i,j}$ , where  $\psi_{i,j}$  is a literal.

Let  $\phi$  be a propositional formula on  $\Omega$ . Show using structural induction that there is a propositional formula  $\psi$  on  $\Omega$  in conjunctive normal form such that  $\psi|_{\rho} = \phi|_{\rho}$  for any assignment  $\rho$  to  $\Omega$ .

**Solution:** Before we prove the statement of the problem, we need to show several equalities.

Let  $\chi_{1,1}, \dots, \chi_{1,n_1}, \chi_{2,1}, \dots, \chi_{2,n_2}, \chi_1, \dots, \chi_{n_1+n_2}$  be propositional formulas over the variables from  $\Omega$  such that  $\chi_i = \chi_{1,i}$  for  $1 \leq i \leq n_1$  and  $\chi_i = \chi_{2,i-n_1}$  for  $n_1 < i \leq n_1 + n_2$ . Then for any propositional assignment  $\rho$  to  $\Omega$ ,

$$\left( \left( \bigwedge_{i=1}^{n_1} \chi_{1,i} \right) \wedge \left( \bigwedge_{i=1}^{n_2} \chi_{2,i} \right) \right) \Big|_{\rho} = \left( \bigwedge_{i=1}^{n_1+n_2} \chi_i \right) \Big|_{\rho}.$$

We can prove the statement using induction by  $n_2$ . The base case for  $n_2 = 1$  follows from the definition of the long conjunction. Let us prove the induction step from  $k$  to  $k+1$ . By the definition of the long conjunction,

$$\left( \bigwedge_{i=1}^{n_1} \chi_{1,i} \right) \wedge \left( \bigwedge_{i=1}^{k+1} \chi_{2,i} \right) = \left( \bigwedge_{i=1}^{n_1} \chi_{1,i} \right) \wedge \left( \left( \bigwedge_{i=1}^k \chi_{2,i} \right) \wedge \chi_{2,k+1} \right)$$

Note that we proved in class the following

$$\begin{aligned} \left( \left( \bigwedge_{i=1}^{n_1} \chi_{1,i} \right) \wedge \left( \bigwedge_{i=1}^{k+1} \chi_{2,i} \right) \right) \Big|_{\rho} &= \left( \left( \bigwedge_{i=1}^{n_1} \chi_{1,i} \right) \wedge \left( \left( \bigwedge_{i=1}^k \chi_{2,i} \right) \wedge \chi_{2,k+1} \right) \right) \Big|_{\rho} = \\ &= \left( \left( \left( \bigwedge_{i=1}^{n_1} \chi_{1,i} \right) \wedge \left( \bigwedge_{i=1}^k \chi_{2,i} \right) \right) \wedge \chi_{2,k+1} \right) \Big|_{\rho}. \end{aligned}$$

By the induction hypothesis,

$$\left( \left( \bigwedge_{i=1}^{n_1} \chi_{1,i} \right) \wedge \left( \bigwedge_{i=1}^{k+1} \chi_{2,i} \right) \right) \Big|_{\rho} = \left( \left( \bigwedge_{i=1}^{n_1+k} \chi_i \right) \wedge \chi_{2,k+1} \right) \Big|_{\rho}.$$

Which implies the statement by the definition of the long conjunction.

Consider  $e_{n,m} : \{0, \dots, m^n - 1\} \rightarrow \{0, \dots, m - 1\}^n$  be a bijection such that

- $e_{n,m}(i + m \cdot r, 0) = i$  and
- $e_{n,m}(i + m \cdot r, j) = e_{n-1,m}(r, j - 1)$ ,

for any  $0 \leq i < m$ ,  $0 \leq r < m^{n-1}$ , and  $0 \leq j < n$ . We also show that  $\bigvee_{j=0}^{n-1} \bigwedge_{i=0}^{m-1} \chi_{j,i} = \bigwedge_{q=0}^{m^n-1} \bigvee_{j=0}^{n-1} \chi_{j,e_{n,m}(q,j)}$ . We prove the statement using induction by  $n$ . The case of  $n = 1$  is clear. We prove now the induction step from  $k$  to  $k + 1$ . Note that

$$\bigvee_{j=0}^k \bigwedge_{i=0}^{m-1} \chi_{j,i} = \left( \bigvee_{j=0}^{k-1} \bigwedge_{i=0}^{m-1} \chi_{j,i} \right) \vee \bigwedge_{i=0}^{m-1} \chi_{k,i}.$$

By the induction hypothesis,

$$\left( \bigvee_{j=0}^k \bigwedge_{i=0}^{m-1} \chi_{j,i} \right) \Big|_{\rho} = \left( \left( \bigwedge_{q=0}^{m^k-1} \bigvee_{j=0}^{k-1} \chi_{j,e_{k,m}(q,j)} \right) \vee \bigwedge_{i=0}^{m-1} \chi_{k,i} \right) \Big|_{\rho}.$$

Therefore,

$$\left( \bigvee_{j=0}^k \bigwedge_{i=0}^{m-1} \chi_{j,i} \right) \Big|_{\rho} = \left( \bigwedge_{q=0}^{m^k-1} \left( \left( \bigvee_{j=0}^{k-1} \chi_{j,e_{k,m}(q,j)} \right) \vee \bigwedge_{i=0}^{m-1} \chi_{k,i} \right) \right) \Big|_{\rho}.$$

So

$$\left( \bigvee_{j=0}^k \bigwedge_{i=0}^{m-1} \chi_{j,i} \right) \Big|_{\rho} = \left( \bigwedge_{q=0}^{m^k-1} \bigwedge_{i=0}^{m-1} \left( \left( \bigvee_{j=0}^{k-1} \chi_{j,e_{k,m}(q,j)} \right) \vee \chi_{k,i} \right) \right) \Big|_{\rho}.$$

As a result,

$$\left( \bigvee_{j=0}^k \bigwedge_{i=0}^{m-1} \chi_{j,i} \right) \Big|_{\rho} = \left( \bigwedge_{q=0}^{m^{k+1}-1} \left( \bigvee_{j=0}^k \chi_{j,e_{k+1,m}(q,j)} \right) \right) \Big|_{\rho}.$$

Finally, we are ready to prove the statement of the problem. Let us consider the following cases.

- The first case is when  $\phi = x_i$ . In this case we can choose  $n = 1$ ,  $m_1 = 1$ , and  $\psi_{1,1} = x_i$ .
- The second case is when  $\phi = \phi_1 \wedge \phi_2$ . Note that by the induction hypothesis,  $\phi_1 = \bigwedge_{j=1}^{n_1} \bigvee_{k=1}^{m_{1,i}} \psi_{1,i,j}$  and  $\phi_2 = \bigwedge_{j=1}^{n_2} \bigvee_{k=1}^{m_{2,i}} \psi_{2,i,j}$ . Using the statement of the previous problem,  $\phi = \bigwedge_{j=1}^{n_1+n_2} \bigvee_{k=1}^{m_i} \psi_{i,j}$ .
- The third case is when  $\phi = \phi_1 \vee \phi_2$ . Note that by the induction hypothesis,  $\phi_1 = \bigwedge_{i=1}^{n_1} \bigvee_{j=1}^{m_{1,i}} \psi_{1,i,j}$  and  $\phi_2 = \bigwedge_{i=1}^{n_2} \bigvee_{j=1}^{m_{2,i}} \psi_{2,i,j}$ . Using the just proved statement we may conclude that  $\phi = \bigwedge_{i=1}^{n_1+n_2} \bigvee_{j=1}^{m_i} \psi_{i,j}$ , where  $m_i = m_{1,i}$  for  $1 \leq i \leq n_1$  and  $m_{2,i-n_1}$  for  $n_1 < i \leq n_1 + n_2$ ,  $\psi_{i,j} = \psi_{1,i,j}$  for  $1 \leq i \leq n_1$  and  $\psi_{i,j} = \psi_{2,i-n_1,j}$  for  $n_1 < i \leq n_1 + n_2$ .
- Finally, the last case is when  $\phi = \neg \phi'$ . Note that by the induction hypothesis,  $\phi' = \bigwedge_{j=1}^{n'} \bigvee_{k=1}^{m'_i} \psi'_{i,j}$ .

Hence,  $\phi|_{\rho} = \bigvee_{j=1}^{n'} \bigwedge_{k=1}^{m'_i} \neg \psi'_{i,j}$ . And using the second proven observation in this problem we can present a CNF representation of  $\phi$ .