Name: ______________________

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1. (10 points) Give a natural deduction derivation of $\exists x\ (A(x) \lor B(x))$ from $\exists x\ A(x) \lor \exists x\ B(x)$.

2. (10 points) Let us consider the following formulas on the variables from the set $\{x_0, \ldots, x_n\}$.

   1. The formula I_n is equal to x_0.
   2. The formula $S_{n,i}$ is equal to $x_{i-1} \implies x_i$.
   3. The formula T_n is equal to x_n.

   Show that there is a natural deduction derivation of T_n from $I_n \wedge \bigwedge_{i=1}^{n} S_{n,i}$.

3. (10 points) Let $\phi = \bigvee_{i=1}^{m} \lambda_i$ be a clause; we say that the width of the clause is equal to m. Let $\phi = \bigwedge_{i=1}^{\ell} \chi_i$ be a formula in CNF; we say that the width of ϕ is equal to the maximal width of χ_i for $i \in [\ell]$.

   Let $m_n : \{T, F\}^n \to \{T, F\}$ such that $m_n(x_1, \ldots, x_n) = T$ iff the number of elements in the set $\{i : x_i = T\}$ is divisible by 3.

   Show that any CNF representation of m_n has width at least $n - 2$.

4. (10 points) Let $A\Delta B = (A \cup B) \setminus (A \cap B)$; we say that $A\Delta B$ is the symmetric difference of A and B. Let Ω, and $A_1, \ldots, A_n \subseteq \Omega$ be some sets We say that $\Delta_{i=1}^{1} A_i = A_1$ and $\Delta_{i=1}^{k+1} A_i = (\Delta_{i=1}^{k} A_i)\Delta A_{k+1}$. Show that
$$\Delta_{i=1}^{n} A_i = \{x \in \Omega \;:\; x \in A_i \text{ for odd number of } i \in [n]\}.$$

5. (10 points) Let $\mathcal{S}$ be a signature with two predicate symbols $=$ and S such that the first is binary and the last is ternary.

   Let us consider the structure $\mathfrak{M}$ such that it corresponds to the points on a two-dimmnesional plane, $=$ is a standard equality, and $S(x, y, z)$ is true iff $|xz| = |yz|$.

   Let R be a relation such that $(A, B, C) \in R$ iff A, B, and C lay on the same line. Show that R is representable in $\mathfrak{M}$.

6. (10 points) Let us define the set S defined as follows:

   - $3 \in S$ and
   - if $x \in S$ and $y \in S$, then $(x + y) \in S$.

   Show that $S = \{3k \ : \ k \in \mathbb{N}\}$.

7. (10 points) Let $f, g_1, \ldots, g_n : \mathbb{R}^\ell \to \mathbb{R}$ We say that the equation $f(x) = 0$ can be derived from the equations $g_1(x) = 0, \ldots, g_n(x) = 0$ iff there is a sequence of functions $h_1, \ldots, h_m : \mathbb{R}^\ell \to \mathbb{R}$ such that $h_m = f$ and for each $i \in [m]$,

   - either h_i is equal to g_j for some $j \in [n]$, or
   - $h_i = h_j + h_k$ for some $1 \leq j, k < i$, or
   - $h_i = c \cdot h_j$ for some $1 \leq j < i$ and some $c \in \mathbb{R}$.

   Show that if the equation $f(x) = 0$ can be derived from the equations $g_1(x) = 0, \ldots, g_n(x) = 0$, then for any $v \in \mathbb{R}^\ell$, $f(v) = 0$ provided that $g_1(v) = \cdots = g_n(v) = 0$.