

Name: \_\_\_\_\_

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1. (10 points) Let us formulate the pigeonhole principle using propositional formulas.  $\Omega = \{x_{1,1}, \dots, x_{n+1,1}, x_{1,2}, \dots, x_{n+1,n}\}$  (informally  $x_{i,j}$  is true iff the  $i$ th pigeon is in the  $j$ th hole). Consider the following propositional formulas on the variables from  $\Omega$ .
- $L_i$  ( $i \in [n+1]$ ) is equal to  $\bigvee_{j=1}^n x_{i,j}$ . (Informally this formula says that the  $i$ th pigeon is in a hole.)
  - $R_j$  ( $j \in [n]$ ) is equal to  $\bigvee_{i_1=1}^{n+1} \bigvee_{i_2=i_1+1}^{n+1} (x_{i_1,j} \wedge x_{i_2,j})$ . (Informally this formula says that there are two pigeons in the  $j$ th hole.)

Show that there is a natural deduction proof of  $\left(\bigwedge_{i=1}^{n+1} L_i\right) \implies \left(\bigvee_{j=1}^n R_j\right)$ .

2. (10 points) Let  $\phi = \bigvee_{i=1}^m \lambda_i$  be a clause; we say that the width of the clause is equal to  $m$ . Let  $\phi = \bigwedge_{i=1}^{\ell} \chi_i$  be a formula in CNF; we say that the width of  $\phi$  is equal to the maximal width of  $\chi_i$  for  $i \in [\ell]$ .

Let  $p_n : \{T, F\}^n \rightarrow \{T, F\}$  such that  $p_n(x_1, \dots, x_n) = T$  iff the set  $\{i : x_i = T\}$  has odd number of elements. Show that any CNF representation of  $p_n$  has width at least  $n$ .

3. (10 points) Write a natural deduction derivation of  $(W \vee Y) \implies (X \vee Z)$  from hypotheses  $W \implies X$  and  $Y \implies Z$ .

4. (10 points) We say that a clause  $C$  can be obtained from clauses  $A$  and  $B$  using the *resolution* rule if  $C = A' \vee B'$ ,  $A = x \vee A'$ , and  $B = \neg x \vee B'$ , for some variable  $x$ .

We say that a clause  $C$  can be derived from clauses  $A_1, \dots, A_m$  using resolutions if there is a sequence of clauses  $D_1, \dots, D_\ell = C$  such that each  $D_i$

- is either obtained from clauses  $D_j$  and  $D_k$  for  $j, k < i$  using the *resolution* rule, or
- is equal to  $A_j$  for some  $j \in [m]$ , or
- is equal to  $D_j \vee E$  for some  $j < i$  and a clause  $E$ .

Show that if  $A_1, \dots, A_m$  semantically imply  $C$ , then  $C$  can be derived from clauses  $A_1, \dots, A_m$  using resolutions.