Name: ______________________

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1. (a) (10 points) Let ϕ, ψ, and χ be propositional formulas on Ω. Show that $\big(\phi\vee(\psi\wedge\chi)\big)\big|_\rho = \big((\phi\vee\psi)\wedge(\phi\vee\chi)\big)\big|_\rho$ for any assignment ρ to the variables Ω.

Solution: Let us fix some propositional assignment ρ to Ω. Note that by the definition

$$\big(\phi\vee(\psi\wedge\chi)\big)\big|_\rho = \phi\big|_\rho \vee (\psi\wedge\chi)\big|_\rho = \phi\big|_\rho \vee (\psi\big|_\rho \wedge \chi\big|_\rho)$$

and

$$\big((\phi\vee\psi)\wedge(\phi\vee\chi)\big)\big|_\rho = (\phi\vee\psi)\big|_\rho \wedge (\phi\vee\chi)\big|_\rho = (\phi\big|_\rho\vee\psi\big|_\rho)\wedge(\phi\big|_\rho\vee\chi\big|_\rho).$$

However, $\phi\big|_\rho \vee (\psi\big|_\rho \wedge \chi\big|_\rho) = (\phi\big|_\rho\vee\psi\big|_\rho)\wedge(\phi\big|_\rho\vee\chi\big|_\rho)$ by the distributivity of disjunction and conjunction.

(b) (10 points) Let $\psi_{1,1}, \ldots, \psi_{1,n}, \psi_{2,1}, \ldots, \psi_{2,m}$ be propositional formulas on Ω. Let $\phi_1 = \bigwedge_{i=1}^n \psi_{1,i}$ and $\phi_2 = \bigwedge_{j=1}^m \psi_{2,j}$.
Show that $\big(\phi_1\vee\phi_2\big)\big|_\rho = \big(\bigwedge_{i=1}^n\bigwedge_{j=1}^m(\psi_{1,i}\vee\psi_{2,j})\big)\big|_\rho$ for any assignment ρ to the variables Ω.

Solution: Let us again fix some propositional assignment ρ to Ω.
We prove the statement in two steps. In the first one we prove that

$$(\phi_1\vee\chi)|_\rho = \left(\bigwedge_{i=1}^n(\psi_{1,i}\vee\chi)\right)\Bigg|_\rho$$

using induction by n.
The base case for $n=1$ is clear. Let us now prove the induction step from k to $k+1$. Note that

$$\left(\bigwedge_{i=1}^{k+1}\psi_{1,i}\right)\vee\chi = \left(\left(\bigwedge_{i=1}^{k}\psi_{1,i}\right)\wedge\psi_{1,k+1}\right)\vee\chi.$$

By the previous problem, this implies that

$$\left(\left(\bigwedge_{i=1}^{k+1}\psi_{1,i}\right)\vee\chi\right)\Bigg|_\rho = \left(\left((\bigwedge_{i=1}^{k}\psi_{1,i})\vee\chi\right)\wedge(\psi_{1,k+1}\vee\chi)\right)\Bigg|_\rho.$$

The induction hypothesis, implies that

$$\left(\left(\bigwedge_{i=1}^{k+1}\psi_{1,i}\right)\vee\chi\right)\Bigg|_\rho = \left(\left(\bigwedge_{i=1}^{k}(\psi_{1,i}\vee\chi)\right)\wedge(\psi_{1,k+1}\vee\chi)\right)\Bigg|_\rho.$$

Therefore, using the definition of the conjunction of several formulas, we proved that

$$\left(\left(\bigwedge_{i=1}^{k+1}\psi_{1,i}\right)\vee\chi\right)\Bigg|_\rho = \left(\bigwedge_{i=1}^{k+1}(\psi_{1,i}\vee\chi)\right)\Bigg|_\rho.$$

On the second step we prove the statement of the problem using induction by m. The base case follows from the result we just proved. Let us now prove the induction step from k to $k+1$. Note that

$$\left(\bigwedge_{i=1}^{m}\psi_{1,i}\right)\vee\left(\bigwedge_{i=1}^{k+1}\psi_{2,i}\right) = \left(\bigwedge_{i=1}^{m}\psi_{1,i}\right)\vee\left(\left(\bigwedge_{i=1}^{k}\psi_{2,i}\right)\wedge\psi_{2,k+1}\right).$$

By the previous problem,

$$\left(\left(\bigwedge_{i=1}^{m}\psi_{1,i}\right)\vee\left(\bigwedge_{i=1}^{k+1}\psi_{2,i}\right)\right)\Bigg|_{\rho}=\left(\left(\left(\bigwedge_{i=1}^{m}\psi_{1,i}\right)\vee\psi_{2,k+1}\right)\wedge\left(\left(\bigwedge_{i=1}^{m}\psi_{1,i}\right)\vee\left(\bigwedge_{i=1}^{k}\psi_{2,i}\right)\right)\right)\Bigg|_{\rho}.$$

Therefore by the previous statement,

$$\left(\left(\bigwedge_{i=1}^{m}\psi_{1,i}\right)\vee\left(\bigwedge_{i=1}^{k+1}\psi_{2,i}\right)\right)\Bigg|_{\rho}=\left(\left(\bigwedge_{i=1}^{m}(\psi_{1,i}\vee\psi_{2,k+1})\right)\wedge\left(\left(\bigwedge_{i=1}^{m}\psi_{1,i}\right)\vee\left(\bigwedge_{i=1}^{k}\psi_{2,i}\right)\right)\right)\Bigg|_{\rho}.$$

Finally, using the induction hypothesis,

$$\left(\left(\bigwedge_{i=1}^{m}\psi_{1,i}\right)\vee\left(\bigwedge_{i=1}^{k+1}\psi_{2,i}\right)\right)\Bigg|_{\rho}=\left(\left(\bigwedge_{i=1}^{m}(\psi_{1,i}\vee\psi_{2,k+1})\right)\wedge\left(\bigwedge_{i=1}^{m}\bigwedge_{i=1}^{k}(\psi_{1,i}\vee\psi_{2,i})\right)\right)\Bigg|_{\rho}.$$

(c) (10 points) Let Ω be a set of variables. We say that a propositional formula is a literal if the formula is equal to x or $\neg x$ for $x\in\Omega$.

We say that a propositional formula on Ω is in conjunctive normal form if it is equal to $\bigwedge_{i=1}^{n}\bigvee_{j=1}^{m_i}\psi_{i,j}$, where $\psi_{i,j}$ is a literal.

Let ϕ be a propositional formula on Ω. Show using structural induction that there is a propositional formula ψ on Ω in conjunctive normal form such that $\psi\big|_{\rho}=\phi\big|_{\rho}$ for any assignment ρ to Ω.

Solution: Before we prove the statement of the problem, we need to show several equalities. Let $\chi_{1,1},\ldots,\chi_{1,n_1},\chi_{2,1},\ldots,\chi_{2,n_2},\chi_1,\ldots,\chi_{n_1+n_2}$ be propositional formulas over the variables from Ω such that $\chi_i=\chi_{1,i}$ for $1\le i\le n_1$ and $\chi_i=\chi_{2,i-n_1}$ for $n_1<i\le n_1+n_2$. Then for any propositional assignment ρ to Ω,

$$\left(\left(\bigwedge_{i=1}^{n_1}\chi_{1,i}\right)\wedge\left(\bigwedge_{i=1}^{n_2}\chi_{2,i}\right)\right)\Bigg|_{\rho}=\left(\bigwedge_{i=1}^{n_1+n_2}\chi_i\right)\Bigg|_{\rho}.$$

We can prove the statement using induction by n_2. The base case for $n_2=1$ follows from the definition of the long conjunction. Let us prove the induction step from k to $k+1$. By the definition of the long conjunction,

$$\left(\bigwedge_{i=1}^{n_1}\chi_{1,i}\right)\wedge\left(\bigwedge_{i=1}^{k+1}\chi_{2,i}\right)=\left(\bigwedge_{i=1}^{n_1}\chi_{1,i}\right)\wedge\left(\left(\bigwedge_{i=1}^{k}\chi_{2,i}\right)\wedge\chi_{2,k+1}\right)$$

Note that we proved in class the following

$$\left(\left(\bigwedge_{i=1}^{n_1}\chi_{1,i}\right)\wedge\left(\bigwedge_{i=1}^{k+1}\chi_{2,i}\right)\right)\Bigg|_{\rho}=\left(\left(\bigwedge_{i=1}^{n_1}\chi_{1,i}\right)\wedge\left(\left(\bigwedge_{i=1}^{k}\chi_{2,i}\right)\wedge\chi_{2,k+1}\right)\right)\Bigg|_{\rho}=$$
$$\left(\left(\left(\bigwedge_{i=1}^{n_1}\chi_{1,i}\right)\wedge\left(\bigwedge_{i=1}^{k}\chi_{2,i}\right)\right)\wedge\chi_{2,k+1}\right)\Bigg|_{\rho}.$$

By the induction hypothesis,

$$\left(\left(\bigwedge_{i=1}^{n_1}\chi_{1,i}\right)\wedge\left(\bigwedge_{i=1}^{k+1}\chi_{2,i}\right)\right)\Bigg|_{\rho}=\left(\left(\bigwedge_{i=1}^{n_1+k}\chi_i\right)\wedge\chi_{2,k+1}\right)\Bigg|_{\rho}.$$

Which implies the statement by the definition of the long conjunction.
Consider $e_{n,m} : \{0, \dots, m^n - 1\} \to \{0, \dots, m-1\}^n$ be a bijection such that

- $e_{n,m}(i + m \cdot r, 0) = i$ and
- $e_{n,m}(i + m \cdot r, j) = e_{n-1,m}(r, j-1)$,

for any $0 \le i < m$, $0 \le r < m^{n-1}$, and $0 \le j < n$. We also show that $\bigvee_{j=0}^{n-1} \bigwedge_{i=0}^{m-1} \chi_{j,i} = \bigwedge_{q=0}^{m^n-1} \bigvee_{j=0}^{n-1} \chi_{j,e_{n,m}(q,j)}$. We prove the statement using induction by n. The case of $n = 1$ is clear. We prove now the induction step from k to $k+1$. Note that

$$\bigvee_{j=0}^{k} \bigwedge_{i=0}^{m-1} \chi_{j,i} = \left(\bigvee_{j=0}^{k-1} \bigwedge_{i=0}^{m-1} \chi_{j,i} \right) \vee \bigwedge_{i=0}^{m-1} \chi_{k,i}.$$

By the induction hypothesis,

$$\left(\bigvee_{j=0}^{k} \bigwedge_{i=0}^{m-1} \chi_{j,i} \right) \Bigg|_\rho = \left(\left(\bigwedge_{q=0}^{m^k-1} \bigvee_{j=0}^{k-1} \chi_{j,e_{k,m}(q,j)} \right) \vee \bigwedge_{i=0}^{m-1} \chi_{k,i} \right) \Bigg|_\rho.$$

Therefore,

$$\left(\bigvee_{j=0}^{k} \bigwedge_{i=0}^{m-1} \chi_{j,i} \right) \Bigg|_\rho = \left(\bigwedge_{q=0}^{m^k-1} \left(\left(\bigvee_{j=0}^{k-1} \chi_{j,e_{k,m}(q,j)} \right) \vee \bigwedge_{i=0}^{m-1} \chi_{k,i} \right) \right) \Bigg|_\rho.$$

So

$$\left(\bigvee_{j=0}^{k} \bigwedge_{i=0}^{m-1} \chi_{j,i} \right) \Bigg|_\rho = \left(\bigwedge_{q=0}^{m^k-1} \bigwedge_{i=0}^{m-1} \left(\left(\bigvee_{j=0}^{k-1} \chi_{j,e_{k,m}(q,j)} \right) \vee \chi_{k,i} \right) \right) \Bigg|_\rho.$$

As a result,

$$\left(\bigvee_{j=0}^{k} \bigwedge_{i=0}^{m-1} \chi_{j,i} \right) \Bigg|_\rho = \left(\bigwedge_{q=0}^{m^{k+1}-1} \left(\bigvee_{j=0}^{k} \chi_{j,e_{k+1,m}(q,j)} \right) \right) \Bigg|_\rho.$$

Finally, we are ready to prove the statement of the problem. Let us consider the following cases.

- The first case is when $\phi = x_i$. In this case we can choose $n = 1$, $m_1 = 1$, and $\psi_{1,1} = x_i$.
- The second case is when $\phi = \phi_1 \wedge \phi_2$. Note that by the induction hypothesis, $\phi_1 = \bigwedge_{j=1}^{n_1} \bigvee_{k=1}^{m_{1,i}} \psi_{1,i,j}$ and $\phi_2 = \bigwedge_{j=1}^{n_2} \bigvee_{k=1}^{m_{2,i}} \psi_{2,i,j}$. Using the statement of the previous problem, $\phi = \bigwedge_{j=1}^{n_1+n_2} \bigvee_{k=1}^{m_i} \psi_{i,j}$.
- The third case is when $\phi = \phi_1 \vee \phi_2$. Note that by the induction hypothesis, $\phi_1 = \bigwedge_{i=1}^{n_1} \bigvee_{j=1}^{m_{1,i}} \psi_{1,i,j}$ and $\phi_2 = \bigwedge_{i=1}^{n_2} \bigvee_{j=1}^{m_{2,i}} \psi_{2,i,j}$. Using the just proved statement we may conclude that $\phi = \bigwedge_{i=1}^{n_1+n_2} \bigvee_{j=1}^{m_i} \psi_{i,j}$, where $m_i = m_{1,i}$ for $1 \le i \le n_1$ and $m_{2,i-n_1}$ for $n_1 < i \le n_1 + n_2$, $\psi_{i,j} = \psi_{1,i,j}$ for $1 \le i \le n_1$ and $\psi_{i,j} = \psi_{2,i-n_1,j}$ for $n_1 < i \le n_1 + n_2$.
- Finally, the last case is when $\phi = \neg\phi'$. Note that by the induction hypothesis, $\phi' = \bigwedge_{j=1}^{n'} \bigvee_{k=1}^{m'_i} \psi'_{i,j}$.

  Hence, $\phi|_\rho = \bigvee_{j=1}^{n'} \bigwedge_{k=1}^{m'_i} \neg\psi'_{i,j}$. And using the second proven observation in this problem we can present a CNF representation of ϕ.