Name: ____________________

Pid: ____________________

1. (10 points) Let us consider four-lines geometry, it is a theory with undefined terms: "point", "line", "is on", and axioms:

   1. there exist exactly four lines,
   2. any two distinct lines have exactly one point on both of them, and
   3. each point is on exactly two lines.

   Show that every line has exactly three points on it. (Be careful with the terms you use and axioms you use.)

Solution: Lets denote the lines as $l_1, \ldots, l_4$ (all of them exist and different by Axiom 1). Due to symmetry of the problem it is enough to prove that l_4 has exactly three points on it.

Let p_i ($1 \leq i \leq 3$) be the point that is on l_i and l_4 (they exist by Axiom 2). Let us now prove that p_1, p_2, and p_3 are all different. Assume that $p_i = p_j$ for $i \neq j$ ($1 \leq i, j \leq 3$) for the sake of contradiction. In this case p_i is on l_i, l_j, and l_4 which contradicts Axiom 3.

Let us now prove that there are no other points on l_4. Assume that it is not true and there is p_4 in additon to p_1, p_2, and p_3 on l_4. By Axiom 3, there is i ($1 \leq i \leq 3$) such that p_4 is on l_i. Hence, p_i and p_4 are on l_i which contradicts to Axiom 2.

2. (10 points) In Euclidean (standard) geometry, prove: If two lines share a common perpendicular, then the lines are parallel. (You do not need to use axioms of Euclidean geometry in this exercsise, you can use all the standard knowledge about geometry.)

Solution: Let us denote by AB the common perpendicular. Assume that the lines are not parallel (note that these lines are different) i.e. that there is an intersection C of these lines.

Note that the angles CAB and CBA are right, hence, the angle ACB is equal to 0 degrees. So the lines are the same, which is a contradiction.

Hence, the assumption was incorrect i.e. the lines are parallel.