

Name: _____

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1. (10 points) Show that $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$ for all integers $n \geq 1$.

2. (10 points) Let $a_0 = 2$, $a_1 = 5$, and $a_n = 5a_{n-1} - 6a_{n-2}$ for all integers $n \geq 2$. Show that $a_n = 3^n + 2^n$ for all integers $n \geq 0$.

3. (10 points) Let n be a positive integer and A_1, \dots, A_n be some sets. Let us define union of these sets as follows:

1. $\cap_{i=1}^1 A_i = A_1,$

2. $\cap_{i=1}^{k+1} A_i = (\cap_{i=1}^k A_i) \cap A_{k+1}.$

Show that $\cap_{i=1}^n \{x \in \mathbb{N} : i \leq x \leq n\} = \{n\}.$

4. (10 points) Let U be the set of sequences of the following symbols: “+”, “.”, “ x_1 ”, \dots , “ x_n ”. Let $B = \{x_i : i \in [n]\}$; i.e., B is the set of sequences consisting of only one symbol x_i . Let $\mathcal{F} = \{f_+, f.\}$, where $f_+(F_1, F_2) = (F_1 + F_2)$ and $f.(F_1, F_2) = (F_1 \cdot F_2)$ (by $(F_1 \# F_2)$ we denote the sequence obtained by concatenating “(”, F_1 , “#”, F_2 , and “)”). Let S be the set generated by \mathcal{F} from B .

For $s : [n] \rightarrow \{0, 1\}$ and $F \in S$, we define the function $\text{val}(F, s)$ using structural recursion as follows.

1. $\text{val}(x_i, s) = s(i)$,
2. $\text{val}((F_1 + F_2), s) = \text{val}(F_1, s) + \text{val}(F_2, s)$,
3. $\text{val}((F_1 \cdot F_2), s) = \text{val}(F_1, s) \cdot \text{val}(F_2, s)$.

Let $F_1, \dots, F_n \in S$. Let us define the sum of these formulas as follows:

1. $\sum_{i=j}^j F_i = F_j$,
2. $\sum_{i=j}^{j+k} F_i = f_+(\sum_{i=j}^{j+k-1} F_i, F_{j+k})$ for $k \geq 1$.

Show that $\text{val}(\sum_{i=1}^n x_i, s) = \text{val}(\sum_{i=1}^n x_{n-i+1}, s)$ for any s .