Introduction to **Audio Content Analysis**

Module 10.1: Alignment — Dynamic Time Warping

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corresponding textbook section

section 10.1

- **lecture content**
  - Dynamic Time Warping (DTW):
    synchronization of two sequences with similar content

- **learning objectives**
  - explain the standard DTW algorithm
  - discuss disadvantages of and modifications to the standard DTW algorithm
  - implement DTW

introduction
overview

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- **lecture content**
  - Dynamic Time Warping (DTW):
    synchronization of two sequences with similar content

- **learning objectives**
  - explain the standard DTW algorithm
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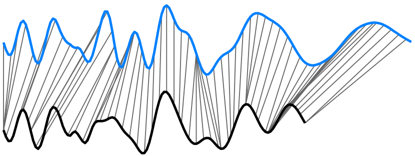
dynamic time warping
problem statement

- **synchronize two sequences**
  - *similar* musical content
  - *different* tempo and timing

$$A(n_\mathrm{A}) \quad n_\mathrm{A} \in [0; \mathcal{N}_\mathrm{A} - 1]$$

$$B(n_\mathrm{B}) \quad n_\mathrm{B} \in [0; \mathcal{N}_\mathrm{B} - 1]$$

$\Rightarrow$ find alignment path
- minimizing pairwise distance between sequences
- covering whole sequence
- moving only forward in time



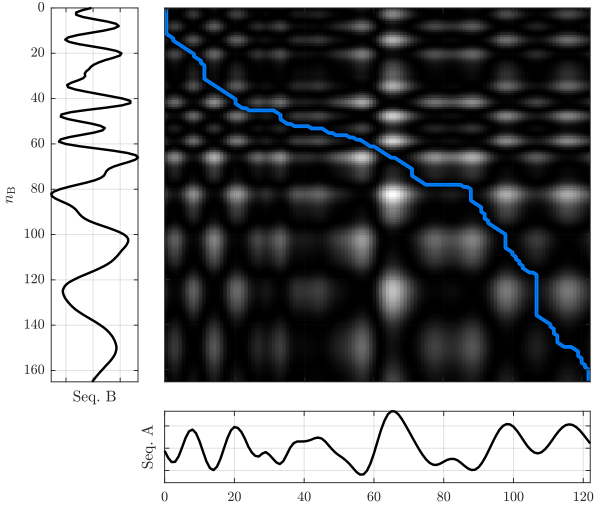
matlab source: plotSequenceAlignment.m

dynamic time warping
overview

- dynamic programming technique
- time is warped non-linearly to match sequences
- finds optimal match between two sequences given a cost function
- the overall cost indicates the overall distance between the sequences

dynamic time warping
processing steps

1. extract suitable **features**
   $\Rightarrow$ two series of feature vectors
2. compute **distance matrix** $\boldsymbol{D}_{\mathrm{AB}}(n_{\mathrm{A}}, n_{\mathrm{B}})$
3. compute **alignment path** $\boldsymbol{p}(n_{\mathrm{P}})$ with $n_{\mathrm{P}} \in [0; \mathcal{N}_{\mathrm{P}} - 1]$
   $\Rightarrow$ minimal *overall* distance
4. (align sequences using dynamic time stretching)

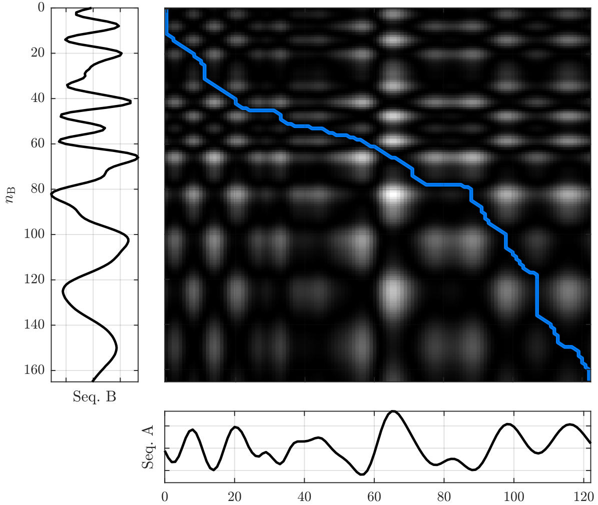


matlab source: plotDtwPath.m

dynamic time warping

distance matrix computation

- given 2 sequences of vectors, compute the distance between all pairs of observations
- compute distance matrix $\boldsymbol{D}_{\mathrm{AB}}(n_{\mathrm{A}}, n_{\mathrm{B}})$
  - example $\boldsymbol{D}_{\mathrm{AB}}(1, n_{\mathrm{B}})$ is the distance of the first vector in Seq. A to all vectors in Seq. B



matlab source: plotDtwPath.m

dynamic time warping

path properties 1/2

- **boundaries**: covers both A, B from beginning to end

$$\begin{aligned}\boldsymbol{p}(0) &= [0,0]\\ \boldsymbol{p}(\mathcal{N}_\mathrm{P}-1) &= [\mathcal{N}_\mathrm{A}-1, \mathcal{N}_\mathrm{B}-1]\end{aligned}$$

- **causality**: only forward movement

$$n_\mathrm{A}\big|_{\boldsymbol{p}(n_\mathrm{P})} \leq n_\mathrm{A}\big|_{\boldsymbol{p}(n_\mathrm{P}+1)}$$
$$n_\mathrm{B}\big|_{\boldsymbol{p}(n_\mathrm{P})} \leq n_\mathrm{B}\big|_{\boldsymbol{p}(n_\mathrm{P}+1)}$$

- **continuity**: no jumps

$$n_\mathrm{A}\big|_{\boldsymbol{p}(n_\mathrm{P}+1)} \leq (n_\mathrm{A}+1)\big|_{\boldsymbol{p}(n_\mathrm{P})}$$
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dynamic time warping

path properties 1/2

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- **continuity**: no jumps

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dynamic time warping

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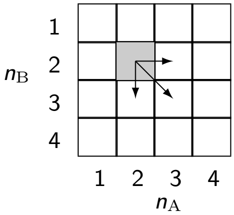
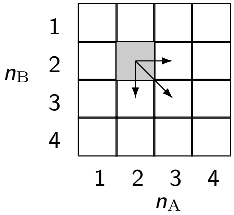
alignment
path properties 2/2



what is the minimum/maximum path length

alignment

path properties 2/2



what is the minimum/maximum path length

$$\mathcal{N}_{\mathrm{P,min}} = \max(\mathcal{N}_{\mathrm{A}}, \mathcal{N}_{\mathrm{B}})$$

$$\mathcal{N}_{\mathrm{P,max}} = \mathcal{N}_{\mathrm{A}} + \mathcal{N}_{\mathrm{B}} - 2$$

alignment
DTW: overall cost

- every path has an *overall cost*

$$\mathfrak{C}_{\mathrm{AB}}(j) = \sum_{n_\mathrm{P}=0}^{\mathcal{N}_\mathrm{P}-1} \boldsymbol{D}\left(\boldsymbol{p}_j(n_\mathrm{P})\right)$$

- *optimal* path minimizes the overall cost

$$\mathfrak{C}_{\mathrm{AB},min} = \min_{\forall j}\left(\mathfrak{C}_{\mathrm{AB}}(j)\right)$$

$$j_\mathrm{opt} = \operatorname*{argmin}_{\forall j}\left(\mathfrak{C}_{\mathrm{AB}}(j)\right)$$

⇒ stay in the 'valleys' of distance matrix

how to determine the optimal path

alignment
DTW: overall cost

- every path has an *overall cost*

$$\mathfrak{C}_{\mathrm{AB}}(j) = \sum_{n_{\mathrm{P}}=0}^{\mathcal{N}_{\mathrm{P}}-1} \boldsymbol{D}\left(\boldsymbol{p}_j(n_{\mathrm{P}})\right)$$

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$\Rightarrow$ stay in the 'valleys' of distance matrix

how to determine the optimal path

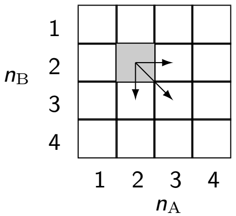
alignment
DTW: accumulated cost 1/2

accumulated cost: *cost matrix*

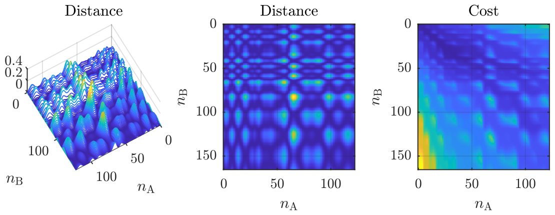
$$\boldsymbol{C}_{\mathrm{AB}}(n_\mathrm{A}, n_\mathrm{B}) = \boldsymbol{D}_{\mathrm{AB}}(n_\mathrm{A}, n_\mathrm{B}) + \min \begin{cases} \boldsymbol{C}_{\mathrm{AB}}(n_\mathrm{A}-1, n_\mathrm{B}-1) \\ \boldsymbol{C}_{\mathrm{AB}}(n_\mathrm{A}-1, n_\mathrm{B}) \\ \boldsymbol{C}_{\mathrm{AB}}(n_\mathrm{A}, n_\mathrm{B}-1) \end{cases}$$

- initialization

$$\boldsymbol{C}_{\mathrm{AB}}(0,0) = \boldsymbol{D}_{\mathrm{AB}}(0,0)$$



alignment
DTW: accumulated cost 2/2



matlab source: plotDtwCost.m

alignment
DTW: algorithm description 1/2

- **initialization**:

$$C_{\mathrm{AB}}(0,0) = D_{\mathrm{AB}}(0,0),\ C_{\mathrm{AB}}(n_{\mathrm{A}},-1) = \infty,\ C_{\mathrm{AB}}(-1,n_{\mathrm{B}}) = \infty$$

- **recursion**:

$$\begin{aligned}
C_{\mathrm{AB}}(n_{\mathrm{A}},n_{\mathrm{B}}) &= D_{\mathrm{AB}}(n_{\mathrm{A}},n_{\mathrm{B}}) + \min\begin{cases} C_{\mathrm{AB}}(n_{\mathrm{A}}-1,n_{\mathrm{B}}-1)\\ C_{\mathrm{AB}}(n_{\mathrm{A}}-1,n_{\mathrm{B}})\\ C_{\mathrm{AB}}(n_{\mathrm{A}},n_{\mathrm{B}}-1)\end{cases}\\
j &= \operatorname{argmin}\begin{cases} C_{\mathrm{AB}}(n_{\mathrm{A}}-1,n_{\mathrm{B}}-1)\\ C_{\mathrm{AB}}(n_{\mathrm{A}}-1,n_{\mathrm{B}})\\ C_{\mathrm{AB}}(n_{\mathrm{A}},n_{\mathrm{B}}-1)\end{cases}\\
\Delta p(n_{\mathrm{A}},n_{\mathrm{B}}) &= \begin{cases} [-1,-1] & \text{if } j=0\\ [-1,0] & \text{if } j=1\\ [0,-1] & \text{if } j=2\end{cases}
\end{aligned}$$

alignment

DTW: algorithm description 1/2

- **initialization**:

$$C_{\mathrm{AB}}(0,0) = D_{\mathrm{AB}}(0,0), C_{\mathrm{AB}}(n_{\mathrm{A}}, -1) = \infty, C_{\mathrm{AB}}(-1, n_{\mathrm{B}}) = \infty$$

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\Delta \boldsymbol{p}(n_{\mathrm{A}}, n_{\mathrm{B}}) &= \begin{cases} [-1, -1] & \text{if } j = 0 \\ [-1, 0] & \text{if } j = 1 \\ [0, -1] & \text{if } j = 2 \end{cases}
\end{aligned}$$

alignment

DTW: algorithm description 2/2

- **termination**:

$$n_\mathrm{A} = \mathcal{N}_\mathrm{A} - 1 \wedge n_\mathrm{B} = \mathcal{N}_\mathrm{B} - 1$$

- **path backtracking**:

$$\boldsymbol{p}(n_\mathrm{P}) = \boldsymbol{p}(n_\mathrm{P}+1) + \Delta\boldsymbol{p}\big(\boldsymbol{p}(n_\mathrm{P}+1)\big), \; n_\mathrm{P} = \mathcal{N}_\mathrm{P} - 2, \mathcal{N}_\mathrm{P} - 3, \ldots, 0$$

alignment

DTW: algorithm description 2/2

- **termination**:

$$n_\mathrm{A} = \mathcal{N}_\mathrm{A} - 1 \wedge n_\mathrm{B} = \mathcal{N}_\mathrm{B} - 1$$

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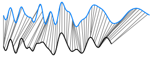
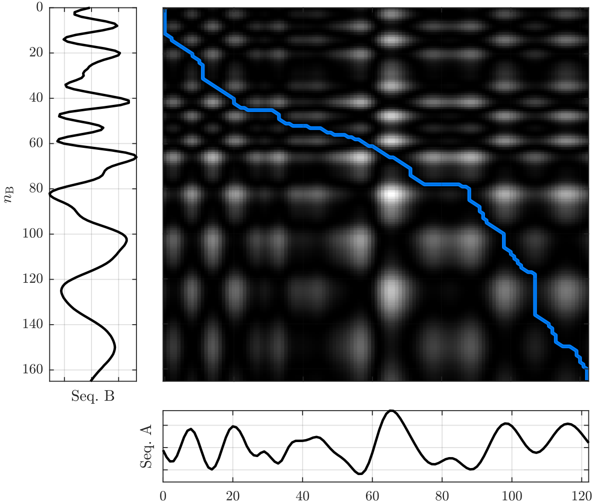
dynamic time warping
DTW: example





matlab source: plotDtwPath.m

dynamic time warping
example

$$\begin{aligned} A &= [1,\ 2,\ 3,\ 0], \\ B &= [1,\ 0,\ 2,\ 3,\ 1], \end{aligned}$$

compute distance matrix, cost matrix, and DTW path

dynamic time warping

example

$$
\begin{aligned}
A &= [1,\ 2,\ 3,\ 0], \\
B &= [1,\ 0,\ 2,\ 3,\ 1],
\end{aligned}
$$

compute distance matrix, cost matrix, and DTW path

$$
\boldsymbol{D}_{\mathrm{AB}} = \begin{bmatrix} 0 & 1 & 2 & 1 \\ 1 & 2 & 3 & 0 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 3 \\ 0 & 1 & 2 & 1 \end{bmatrix}
$$

dynamic time warping

example

$$A = [1,\ 2,\ 3,\ 0],$$
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compute distance matrix, cost matrix, and DTW path

$$\boldsymbol{D}_{\mathrm{AB}} = \begin{bmatrix} 0 & 1 & 2 & 1 \\ 1 & 2 & 3 & 0 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 3 \\ 0 & 1 & 2 & 1 \end{bmatrix} \qquad \boldsymbol{C}_{\mathrm{AB}} = \begin{bmatrix} 0 & \leftarrow 1 & \leftarrow 3 & \leftarrow 4 \\ \uparrow 1 & \nwarrow 2 & \nwarrow 4 & \nwarrow 3 \\ \uparrow 2 & \nwarrow 1 & \leftarrow 2 & \leftarrow 4 \\ \uparrow 4 & \uparrow 2 & \nwarrow 1 & \leftarrow 4 \\ \uparrow 4 & \uparrow 3 & \uparrow 3 & \nwarrow 2 \end{bmatrix}$$

dynamic time warping

example

$$A = [1,\ 2,\ 3,\ 0],$$
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compute distance matrix, cost matrix, and DTW path

$$\boldsymbol{D}_{\mathrm{AB}} = \begin{bmatrix} 0 & 1 & 2 & 1 \\ 1 & 2 & 3 & 0 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 3 \\ 0 & 1 & 2 & 1 \end{bmatrix} \qquad \boldsymbol{C}_{\mathrm{AB}} = \begin{bmatrix} 0 & \leftarrow 1 & \leftarrow 3 & \leftarrow 4 \\ \uparrow 1 & \nwarrow 2 & \nwarrow 4 & \nwarrow 3 \\ \uparrow 2 & \nwarrow 1 & \leftarrow 2 & \leftarrow 4 \\ \uparrow 4 & \uparrow 2 & \nwarrow 1 & \leftarrow 4 \\ \uparrow 4 & \uparrow 3 & \uparrow 3 & \nwarrow 2 \end{bmatrix}$$

dynamic time warping
variants

- transition weights: favor specific path directions

$$C_{\mathrm{AB}}(n_{\mathrm{A}}, n_{\mathrm{B}}) = \min \begin{cases} C_{\mathrm{AB}}(n_{\mathrm{A}}-1, n_{\mathrm{B}}-1) & + \quad \lambda_{\mathrm{d}} \cdot D_{\mathrm{AB}}(n_{\mathrm{A}}, n_{\mathrm{B}}) \\ C_{\mathrm{AB}}(n_{\mathrm{A}}-1, n_{\mathrm{B}}) & + \quad \lambda_{\mathrm{v}} \cdot D_{\mathrm{AB}}(n_{\mathrm{A}}, n_{\mathrm{B}}) \\ C_{\mathrm{AB}}(n_{\mathrm{A}}, n_{\mathrm{B}}-1) & + \quad \lambda_{\mathrm{h}} \cdot D_{\mathrm{AB}}(n_{\mathrm{A}}, n_{\mathrm{B}}) \end{cases}$$

- step types

dynamic time warping
variants

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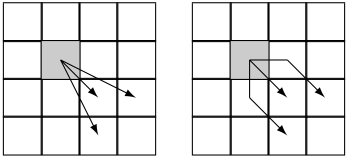
- step types



dynamic time warping
optimization

- **challenge**: distance matrix dimensions $\mathcal{N}_{\mathrm{A}} \cdot \mathcal{N}_{\mathrm{B}}$

$\Rightarrow$ DTW *inefficient* for long sequences
  - high memory requirements
  - large number of operations

optimizations:

1. maximum time and tempo deviation
2. sliding window
3. multi-scale DTW (several downsampled iterations)

dynamic time warping
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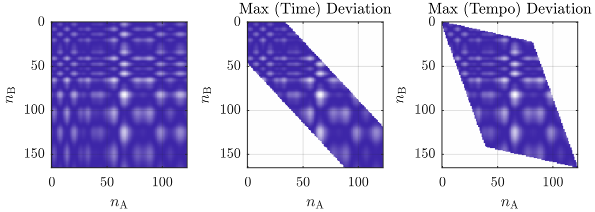

matlab source: plotDtwConstraints.m

dynamic time warping
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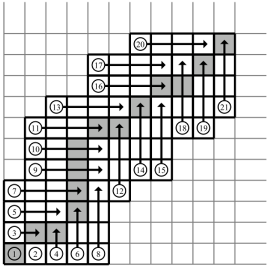

1

[1] S. Dixon and G. Widmer, "MATCH: A Music Alignment Tool Chest," in *Proceedings of the 6th International Conference on Music Information Retrieval (ISMIR)*, London, Sep. 2005.

dynamic time warping
optimization

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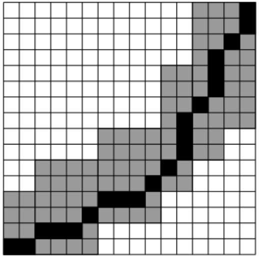

1

[1] M. Müller, H. Mattes, and F. Kurth, "An Efficient Multiscale Approach to Audio Synchronization," in *Proceedings of the International Society for Music Information Retrieval Conference (ISMIR)*, Victoria, 2006.

dynamic time warping

DTW vs. Viterbi

similarities and differences of DTW and the Viterbi algorithm

dynamic time warping
DTW vs. Viterbi

similarities and differences of DTW and the Viterbi algorithm

- **commonalities**
  - find path through matrix
  - maximizes overall probability/minimizes overall cost
  - based on dynamic programming principles
- **differences**
  - DTW has more constraints: start/end in corner, move only to neighbor
  - DTW is not usually parametrized by training data (transition probs, construction of distance/emission prob matrix)
  - Viterbi path length is predefined, DTW path length is not

summary
lecture content

- **dynamic time warping**
  - find globally optimal alignment path between two sequences

- **processing steps**
  1. compute distance matrix
  2. compute cost matrix
  3. back-track path