



# Introduction to **Audio Content Analysis**

module 3.3.1: time-frequency representations — Fourier transform

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# introduction

## overview

### corresponding textbook section

section 3.3.1

appendix B

#### ■ lecture content

- FT of continuous signals
- FT properties
- FT of sampled signals
- Short Time FT (STFT)
- DFT

#### ■ learning objectives

- name and explain definition and properties of the FT



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- FT of continuous signals
- FT properties
- FT of sampled signals
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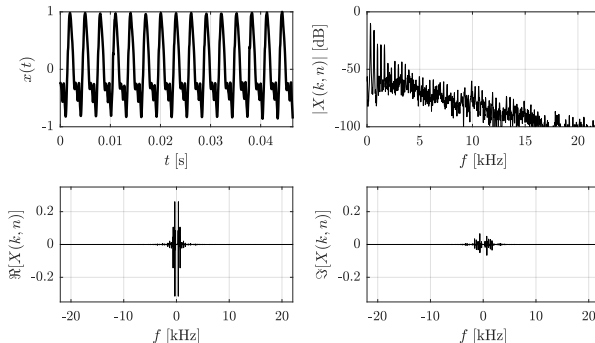
#### ■ learning objectives

- name and explain definition and properties of the FT



# fourier transform

## introduction



top      time domain signal      magnitude spectrum in dB  
bottom      real spectrum      imaginary spectrum

# fourier transform

## definition (continuous)

$$X(j\omega) = \mathfrak{F}[x(t)] = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt$$

sidenote: Fourier series coefficients

$$a_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t)e^{-j\omega_0 kt} dt$$

■  $T_0 \rightarrow \infty$  to allow the analysis of aperiodic functions

$\Rightarrow k\omega_0 \rightarrow \omega$

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# fourier transform

## representations

$$\begin{aligned}X(j\omega) &= \Re[X(j\omega)] + \Im[X(j\omega)] \\&= \underbrace{|X(j\omega)|}_{\text{magnitude}} \cdot \underbrace{\Phi_X(\omega)}_{\text{phase}}\end{aligned}$$

$$\begin{aligned}|X(j\omega)| &= \sqrt{\Re[X(j\omega)]^2 + \Im[X(j\omega)]^2} \\ \Phi_X(\omega) &= \text{atan2} \left( \frac{\Im[X(j\omega)]}{\Re[X(j\omega)]} \right)\end{aligned}$$

- complex spectrum either represented as magnitude & phase or as real & imaginary
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# fourier transform

## property 1: invertibility

$$\begin{aligned}x(t) &= \mathfrak{F}^{-1}[X(j\omega)] \\&= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega\end{aligned}$$

- time domain signal can be **perfectly reconstructed** — no information loss
- FT and IFT are very similar, largely equivalent

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# fourier transform

## property 2: superposition

$$\begin{aligned} y(t) &= c_1 \cdot x_1(t) + c_2 \cdot x_2(t) \\ &\mapsto \\ Y(j\omega) &= c_1 \cdot X_1(j\omega) + c_2 \cdot X_2(j\omega) \end{aligned}$$

- FT is a *linear* transform

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**note:**

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# fourier transform

## property 3: convolution and multiplication

$$y(t) = \int_{-\infty}^{\infty} h(\tau) \cdot x(t - \tau) d\tau$$

$$\mapsto$$

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- convolution in time domain means multiplication in frequency domain
- convolution in frequency domain means multiplication in time domain

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# fourier transform

## property 4: Parseval's theorem

$$\int_{-\infty}^{\infty} x^2(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$$

- energy of the signal is preserved when switching between time and frequency domains

# fourier transform

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# fourier transform

## property 5: time & frequency shift

### ■ time shift

$$x(t - t_0) \mapsto X(j\omega)e^{-j\omega t_0}$$

### ■ frequency shift

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega - \omega_0) e^{j\omega t} d\omega = e^{j\omega_0 t} \cdot x(t)$$

- time shift results in phase shift
- frequency shift results in modulation of time signal

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# fourier transform

## property 6: symmetry

$$\begin{aligned}|X(j\omega)| &= |X(-j\omega)| \\ \Phi_X(\omega) &= -\Phi_X(-\omega)\end{aligned}$$

- spectrum of (real) signal is conjugate complex
  - magnitude spectrum is symmetric to ordinate
  - phase spectrum is symmetric to origin
- even signals have no imaginary spectrum
- odd signals have no real spectrum

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## property 7: time & frequency scaling

$$\begin{aligned} y(t) &= x(c \cdot t) \\ &\mapsto \\ Y(j\omega) &= \frac{1}{|c|} X\left(j\frac{\omega}{c}\right) \end{aligned}$$

- scaling of abscissa in one domain leads to inverse scaling in the other domain

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## sampled time signals 1/2

- sampled time signal can be modeled as multiplication of original signal with delta pulse  $\delta_T(t)$
- multiplication in time domain  $\mapsto$  convolution in frequency domain

$$\begin{aligned}\mathfrak{F}[x(i)] &= \mathfrak{F}[x(t) \cdot \delta_T(t)] \\ &= \mathfrak{F}[x(t)] * \mathfrak{F}[\delta_T(t)] \\ &= X(j\omega) * \Delta_T(j\omega)\end{aligned}$$

### note

- even if time domain signal is discrete, its Fourier transform is *still continuous*
- spectrum is *repeated periodically*

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# fourier transform

## sampled time signals 2/2



# fourier transform

## STFT 1/2

short time Fourier transform (STFT):  
Fourier transform of a short time segment

### ■ reasons:

- remember block based processing
- segment can be seen as quasi-periodic or stationary

### ■ implementation:

- pretend signal is 0 outside of the segment
- ⇒ multiplication of signal and *window function*

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## STFT 2/2



# fourier transform

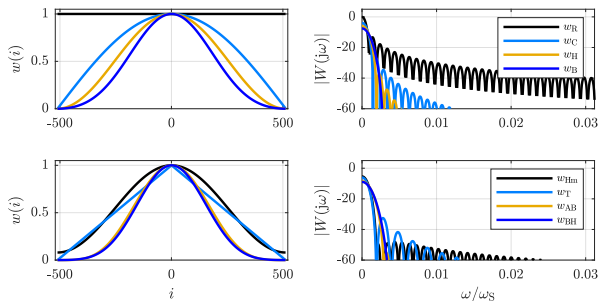
## STFT: window functions

- time domain multiplication  $\mapsto$  frequency domain convolution
- time domain shape determines frequency domain shape of the window

# fourier transform

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# fourier transform

## STFT: window functions

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### spectral leakage characterization

- main lobe width
- side lobe height
- side lobe attenuation

# fourier transform

## DFT

digital domain: requires discrete frequency values:

⇒ discrete Fourier transform

$$X(k) = \sum_{i=0}^{K-1} x(i) e^{-jki \frac{2\pi}{K}}$$

with

$$\Delta\Omega = \frac{2\pi}{KT_S} = \frac{2\pi f_S}{K}$$

2 interpretations:

- sampled continuous Fourier transform
- continuous Fourier transform of periodically extended time domain segment

# fourier transform

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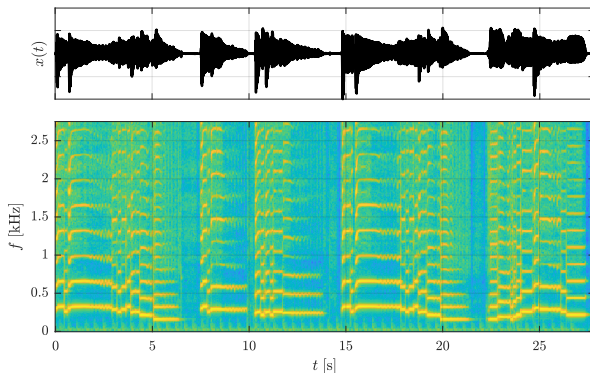
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# fourier transform

## spectrogram

- spectrogram allows to visualize temporal changes in the spectrum
- displays the *magnitude spectrum* only



# summary

## lecture content

### ■ Fourier Transform (of a real signal)

- is conjugate complex
- often represented as magnitude + phase
- invertible
- linear
- convolution in time domain is multiplication in frequency domain
- energy preserving
- time shift result in phase shift, frequency shift results in amplitude modulation
- symmetric
- time scaling result in inverse frequency scaling

### ■ FT of sampled signals:

- is periodic with sample rate

### ■ STFT

- window results in spectral leakage (convolution in freq domain)

### ■ DFT

- discrete in both time and freq domain

