Introduction to Audio Content Analysis

Module 3.4.1: Time-Frequency Representations — Fourier Transform

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Georgia Tech Center for Music Technology
College of Design

corresponding textbook section

Section 3.4.1
Appendix C

- **lecture content**
  - FT of continuous signals
  - FT properties
  - FT of sampled signals
  - Short Time FT (STFT)
  - DFT

- **learning objectives**
  - name and explain definition and properties of the FT

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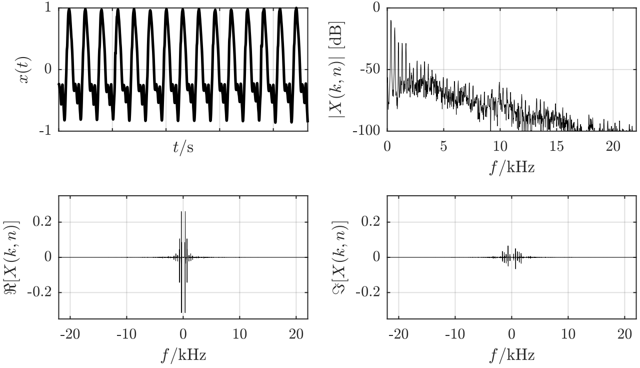
fourier transform
introduction



top	time domain signal	magnitude spectrum in dB
bottom	real spectrum	imaginary spectrum

matlab source: plotFourierTransform.m

fourier transform
definition (continuous)

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$$X(\mathrm{j}\omega) = \mathfrak{F}[x(t)] = \int_{-\infty}^{\infty} x(t) e^{-\mathrm{j}\omega t}\, dt$$

sidenote: Fourier series coefficients

$$a_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t) e^{-\mathrm{j}\omega_0 k t}\, dt$$

- $T_0 \rightarrow \infty$ to allow the analysis of aperiodic functions

$\Rightarrow$ $k\omega_0 \rightarrow \omega$

fourier transform
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fourier transform
representations

$$\begin{aligned} X(\mathrm{j}\omega) &= \Re[X(\mathrm{j}\omega)] + \Im[X(\mathrm{j}\omega)] \\ &= \underbrace{|X(\mathrm{j}\omega)|}_{\text{magnitude}} \cdot \underbrace{\Phi_{\mathrm{X}}(\omega)}_{\text{phase}} \end{aligned}$$

$$\begin{aligned} |X(\mathrm{j}\omega)| &= \sqrt{\Re[X(\mathrm{j}\omega)]^2 + \Im[X(\mathrm{j}\omega)]^2} \\ \Phi_{\mathrm{X}}(\omega) &= \operatorname{atan2}\left(\frac{\Im[X(\mathrm{j}\omega)]}{\Re[X(\mathrm{j}\omega)]}\right) \end{aligned}$$

- complex spectrum either represented as magnitude & phase or as real & imaginary
- magnitude spectrum has no negative values

fourier transform

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fourier transform

property 1: invertibility

$$x(t) = \mathfrak{F}^{-1}[X(\mathrm{j}\omega)] = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} X(\mathrm{j}\omega)e^{\mathrm{j}\omega t}\,d\omega$$

- time domain signal can be **perfectly reconstructed** — no information loss
- FT and IFT are very similar, largely equivalent

fourier transform
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fourier transform

property 2: superposition

$$\begin{aligned} y(t) &= c_1 \cdot x_1(t) + c_2 \cdot x_2(t) \\ \mapsto & \\ Y(\mathrm{j}\omega) &= c_1 \cdot X_1(\mathrm{j}\omega) + c_2 \cdot X_2(\mathrm{j}\omega) \end{aligned}$$

- FT is a *linear* transform

fourier transform

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fourier transform

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fourier transform

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fourier transform

property 4: Parseval's theorem

$$\int\limits_{-\infty}^{\infty} x^2(t)\, dt = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} |X(\mathrm{j}\omega)|^2 \; d\omega$$

- energy of the signal is preserved when switching between time and frequency domains

fourier transform

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fourier transform

property 5: time & frequency shift

- **time shift**

$$x(t - t_0) \mapsto X(\mathrm{j}\omega)e^{-\mathrm{j}\omega t_0}$$

- **frequency shift**

$$\frac{1}{2\pi}\int\limits_{-\infty}^{\infty} X(\mathrm{j}\omega - \omega_0)e^{\mathrm{j}\omega t}\, d\omega = e^{\mathrm{j}\omega_0 t} \cdot x(t)$$

- time shift results in phase shift
- frequency shift results in modulation of time signal

fourier transform

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fourier transform

property 6: symmetry

$$\begin{aligned} |X(j\omega)| &= |X(-j\omega)| \\ \Phi_X(\omega) &= -\Phi_X(-\omega) \end{aligned}$$

- spectrum of (real) signal is conjugate complex
  - magnitude spectrum is symmetric to ordinate
  - phase spectrum is symmetric to origin
- even signals have no imaginary spectrum
- odd signals have no real spectrum

fourier transform

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fourier transform

property 7: time & frequency scaling

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y(t) &= x(c \cdot t) \\
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Y(\mathrm{j}\omega) &= \frac{1}{|c|} X\left(j \frac{\omega}{c}\right)
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fourier transform

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fourier transform

sampled time signals 1/2

- sampled time signal can be modeled as multiplication of original signal with delta pulse $\delta_{\mathrm{T}}(t)$
- multiplication in time domain $\mapsto$ convolution in frequency domain

$$\begin{aligned}\mathfrak{F}[x(i)] &= \mathfrak{F}[x(t)\cdot\delta_{\mathrm{T}}(t)] \\ &= \mathfrak{F}[x(t)] * \mathfrak{F}[\delta_{\mathrm{T}}(t)] \\ &= X(\mathrm{j}\omega) * \Delta_{\mathrm{T}}(\mathrm{j}\omega)\end{aligned}$$

note

- even if time domain signal is discrete, its Fourier transform is *still continuous*
- spectrum is *repeated periodically*

fourier transform

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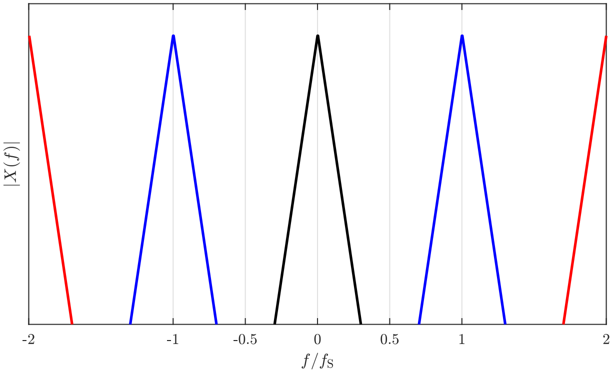
fourier transform
sampled time signals 2/2



matlab source: matlab/animateSampling.m

fourier transform
STFT 1/2

short time Fourier transform (STFT):
Fourier transform of a short time segment

- **reasons**:
  - remember block based processing
  - segment can be seen as quasi-periodic or stationary
- **implementation**:
  - pretend signal is 0 outside of the segment
  - ⇒ multiplication of signal and *window function*

fourier transform
STFT 1/2

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fourier transform
STFT 1/2

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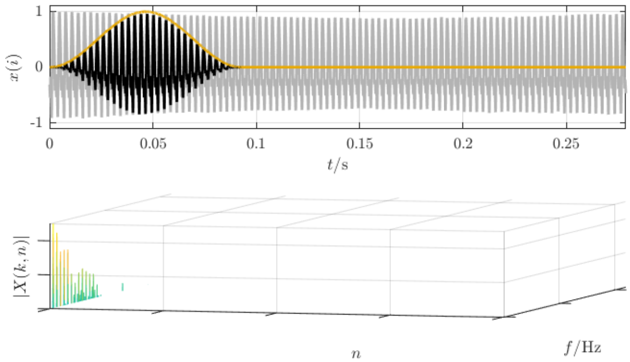
fourier transform

STFT 2/2



matlab source: matlab/animateStft.m

fourier transform
STFT: window functions

- time domain multiplication $\mapsto$ frequency domain convolution
- time domain shape determines frequency domain shape of the window

fourier transform

STFT: window functions

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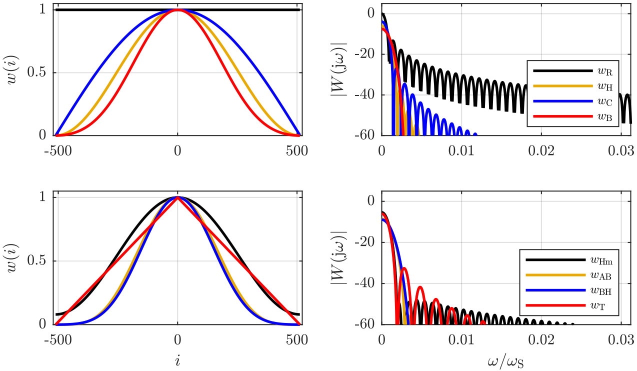

matlab source: plotSpectralWindows.m

- time domain multiplication $\mapsto$ frequency domain convolution
- time domain shape determines frequency domain shape of the window

spectral leakage characterization

- main lobe width
- side lobe height
- side lobe attenuation

fourier transform
DFT

digital domain: requires discrete frequency values:
$\Rightarrow$ discrete Fourier transform

$$X(k) = \sum_{i=0}^{\mathcal{K}-1} x(i) e^{-\mathrm{j}ki\frac{2\pi}{\mathcal{K}}}$$

with

$$\Delta\Omega = \frac{2\pi}{\mathcal{K}T_\mathrm{S}} = \frac{2\pi f_\mathrm{S}}{\mathcal{K}}$$

2 interpretations:

- sampled continuous Fourier transform
- continuous Fourier transform of periodically extended time domain segment

fourier transform
DFT

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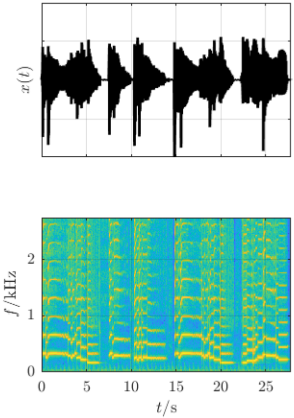
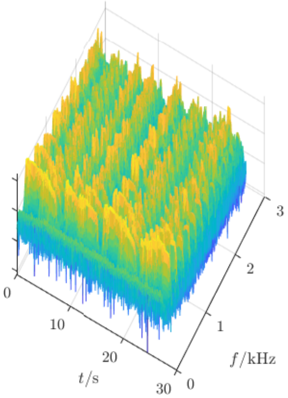
fourier transform
spectrogram

- spectrogram allows to visualize temporal changes in the spectrum
- displays the *magnitude spectrum* only





matlab source: plotSpecgram.m

summary
lecture content

- **Fourier Transform (of a real signal)**
  - is conjugate complex
  - often represented as magnitude + phase
  - invertible
  - linear
  - convolution in time domain is multiplication in frequency domain
  - energy preserving
  - time shift result in phase shift, frequency shift results in amplitude modulation
  - symmetric
  - time scaling result in inverse frequency scaling
- **FT of sampled signals**:
  - is periodic with sample rate
- **STFT**
  - window results in spectral leakage (convolution in freq domain)
- **DFT**
  - discrete in both time and freq domain