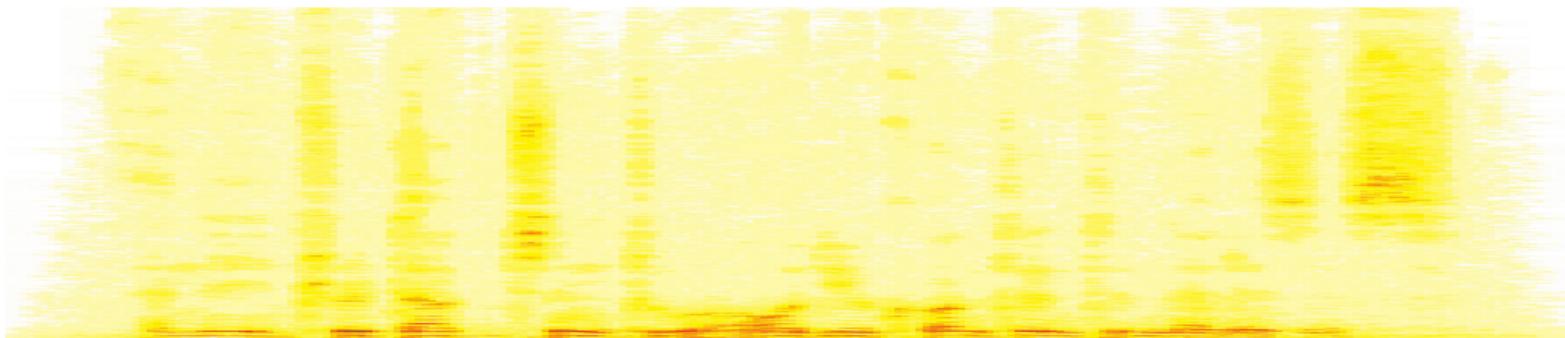


Introduction to Audio Content Analysis

Module 2.5: Fundamentals — Fourier Transform

alexander lerch



introduction

overview

corresponding textbook section

[Chapter 2 — Fundamentals](#): pp. 20–23

[Appendix B — Fourier Transform](#): pp. 185–197

● lecture content

- FT of continuous signals
- FT properties
- FT of sample signals
- Short Time FT (STFT)
- DFT

● learning objectives

- name and explain definition and properties of the FT



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[Chapter 2 — Fundamentals](#): pp. 20–23

[Appendix B — Fourier Transform](#): pp. 185–197

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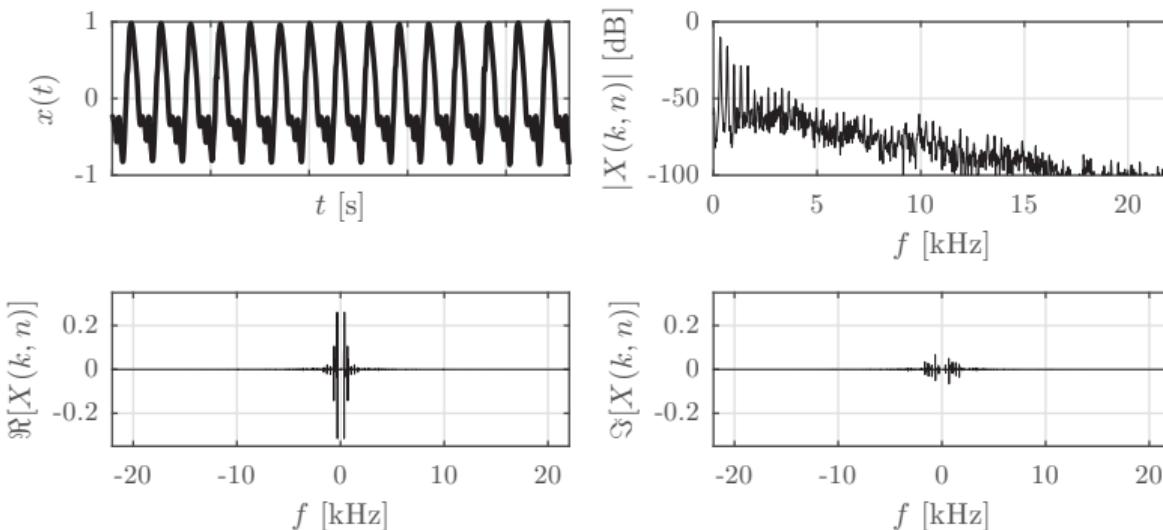
● learning objectives

- name and explain definition and properties of the FT



fourier transform

introduction



top

bottom

time domain signal

real spectrum

magnitude spectrum in dB

imaginary spectrum

fourier transform

definition (continuous)

$$X(j\omega) = \mathfrak{F}[x(t)] = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt$$

sidenote: Fourier series coefficients

$$a_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t)e^{-j\omega_0 kt} dt$$

- $T_0 \rightarrow \infty$ to allow the analysis of aperiodic functions
⇒ $k\omega_0 \rightarrow \omega$

fourier transform

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fourier transform representations

$$\begin{aligned} X(j\omega) &= \Re[X(j\omega)] + \Im[X(j\omega)] \\ &= \underbrace{|X(j\omega)|}_{\text{magnitude}} \cdot \underbrace{\Phi_X(\omega)}_{\text{phase}} \end{aligned}$$

$$\begin{aligned} |X(j\omega)| &= \sqrt{\Re[X(j\omega)]^2 + \Im[X(j\omega)]^2} \\ \Phi_X(\omega) &= \text{atan} 2 \left(\frac{\Im[X(j\omega)]}{\Re[X(j\omega)]} \right) \end{aligned}$$

- complex spectrum either represented as magnitude & phase or as real & imaginary
- magnitude spectrum has no negative values

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fourier transform

property 1: invertibility

$$\begin{aligned}x(t) &= \mathfrak{F}^{-1}[X(j\omega)] \\&= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega\end{aligned}$$

- time domain signal can be **perfectly reconstructed** — no information loss
- FT and IFT are very similar, largely equivalent

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property 2: superposition

$$y(t) = c_1 \cdot x_1(t) + c_2 \cdot x_2(t)$$

↪

$$Y(j\omega) = c_1 \cdot X_1(j\omega) + c_2 \cdot X_2(j\omega)$$

- FT is a *linear* transform

fourier transform

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fourier transform

property 3: convolution and multiplication

$$\begin{aligned}y(t) &= \int_{-\infty}^{\infty} h(\tau) \cdot x(t - \tau) d\tau \\&\mapsto \\Y(j\omega) &= H(j\omega) \cdot X(j\omega)\end{aligned}$$

- convolution in time domain means multiplication in frequency domain
- convolution in frequency domain means multiplication in time domain

fourier transform

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fourier transform

property 4: Parseval's theorem

$$\int_{-\infty}^{\infty} x^2(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$$

- energy of the signal is preserved when switching between time and frequency domains

fourier transform

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fourier transform

property 5: time & frequency shift

- **time shift**

$$x(t - t_0) \mapsto X(j\omega)e^{-j\omega t_0}$$

- **frequency shift**

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega - \omega_0) e^{j\omega t} d\omega = e^{j\omega_0 t} \cdot x(t)$$

- time shift results in phase shift
- frequency shift results in modulation of time signal

fourier transform

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fourier transform

property 6: symmetry

$$\begin{aligned}|X(j\omega)| &= |X(-j\omega)| \\ \Phi_X(\omega) &= -\Phi_X(-\omega)\end{aligned}$$

- spectrum of (real) signal is conjugate complex
 - magnitude spectrum is symmetric to ordinate
 - phase spectrum is symmetric to origin
- even signals have no imaginary spectrum
- odd signals have no real spectrum

fourier transform

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property 7: time & frequency scaling

$$\begin{aligned}y(t) &= x(c \cdot t) \\&\mapsto \\Y(j\omega) &= \frac{1}{|c|} X\left(j\frac{\omega}{c}\right)\end{aligned}$$

- scaling of abscissa in one domain leads to inverse scaling in the other domain

fourier transform

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fourier transform

sampled time signals 1/2

- sampled time signal can be modeled as multiplication of original signal with delta pulse $\delta_T(t)$
- multiplication in time domain \mapsto convolution in frequency domain

$$\begin{aligned}\mathfrak{F}[x(i)] &= \mathfrak{F}[x(t) \cdot \delta_T(t)] \\ &= \mathfrak{F}[x(t)] * \mathfrak{F}[\delta_T(t)] \\ &= X(j\omega) * \Delta_T(j\omega)\end{aligned}$$

note

- even if time domain signal is discrete, its Fourier transform is *still continuous*
- spectrum is *repeated periodically*

fourier transform

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STFT
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DFT
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fourier transform

sampled time signals 2/2



matlab source: [matlab/animateSampling.m](#)



fourier transform

STFT 1/2

short time Fourier transform (STFT):
Fourier transform of a short time segment

- **reasons:**
 - remember block based processing
 - segment can be seen as quasi-periodic or stationary
- **implementation:**
 - pretend signal is 0 outside of the segment
 - ⇒ multiplication of signal and *window function*

fourier transform

STFT 1/2

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STFT 2/2



matlab source: [matlab/animateStft.m](https://matlab.animateStft.m)



fourier transform

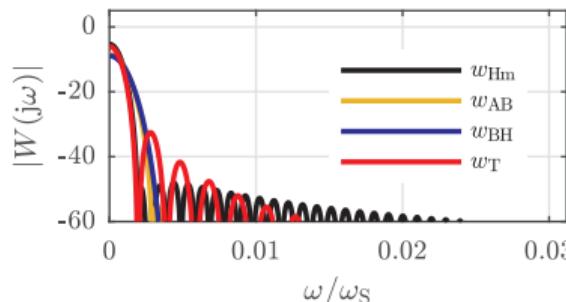
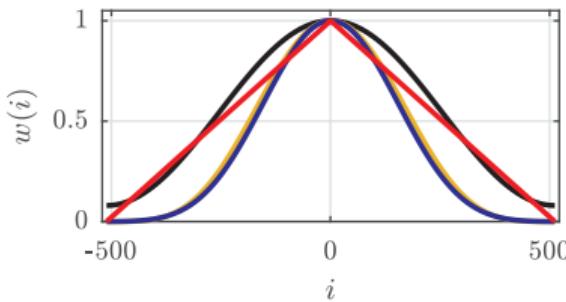
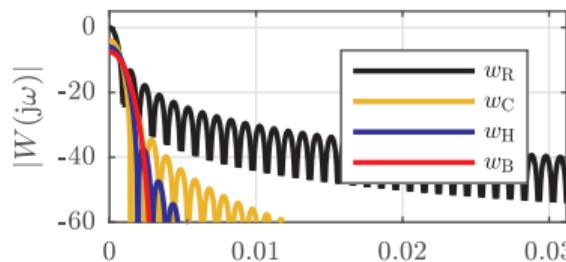
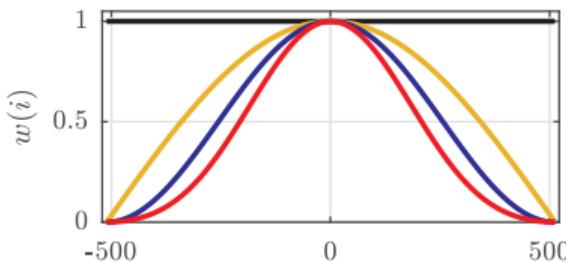
STFT: window functions

- time domain multiplication \mapsto frequency domain convolution
- time domain shape determines frequency domain shape of the window

fourier transform

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fourier transform

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spectral leakage characterization

- main lobe width
- side lobe height
- side lobe attenuation

fourier transform

DFT

digital domain: requires discrete frequency values:
⇒ discrete Fourier transform

$$X(k) = \sum_{i=0}^{\mathcal{K}-1} x(i) e^{-jki \frac{2\pi}{\mathcal{K}}}$$

with

$$\Delta\Omega = \frac{2\pi}{\mathcal{K} T_S} = \frac{2\pi f_S}{\mathcal{K}}$$

2 interpretations:

- sampled continuous Fourier transform
- continuous Fourier transform of periodically extended time domain segment

fourier transform

DFT

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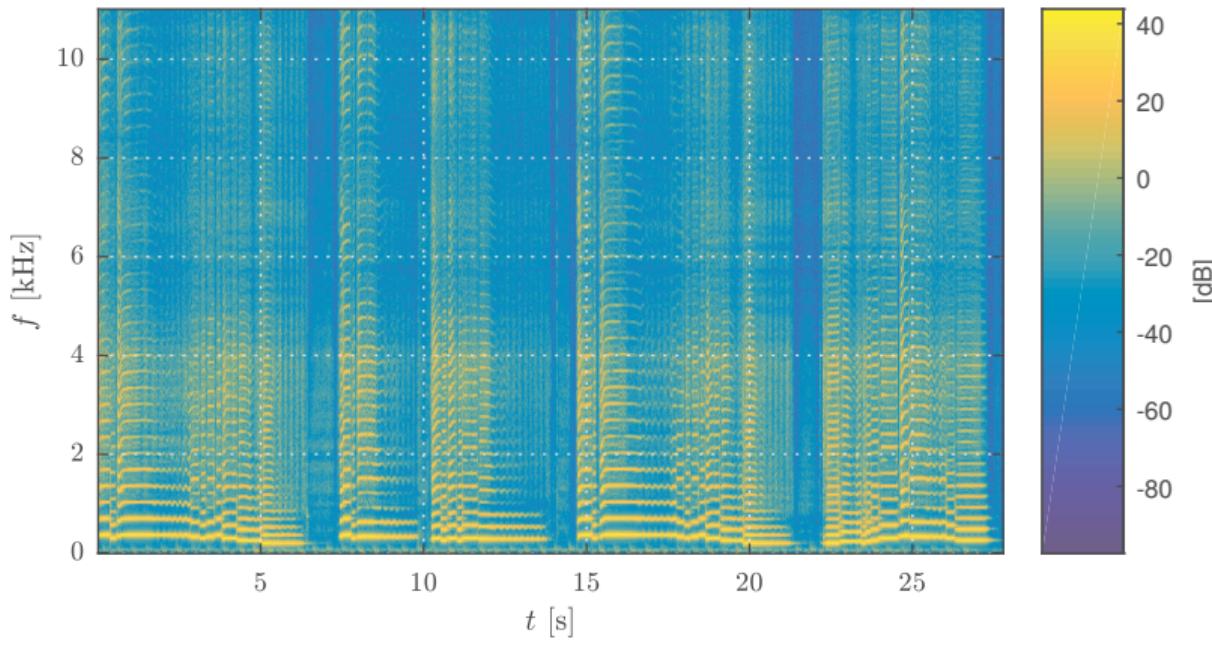
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fourier transform spectrogram

- spectrogram allows to visualize temporal changes in the spectrum
- displays the *magnitude spectrum* only



summary

lecture content

● Fourier Transform (of a real signal)

- is conjugate complex
- often represented as magnitude + phase
- invertible
- linear
- convolution in time domain is multiplication in frequency domain
- energy preserving
- time shift result in phase shift, frequency shift results in amplitude modulation
- symmetric
- time scaling result in inverse frequency scaling

● FT of sampled signals:

- is periodic with sample rate

● STFT

- window results in spectral leakage (convolution in freq domain)

● DFT

- discrete in both time and freq domain

