Introduction to Audio Content Analysis

Module 3.4.1: Time-Frequency Representations — Fourier Transform

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introduction
overview

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corresponding textbook section

Section 3.4.1
Appendix C

- **lecture content**
  - FT of continuous signals
  - FT properties
  - FT of sampled signals
  - Short Time FT (STFT)
  - DFT

- **learning objectives**
  - name and explain definition and properties of the FT

introduction
overview

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corresponding textbook section

Section 3.4.1
Appendix C

- **lecture content**
  - FT of continuous signals
  - FT properties
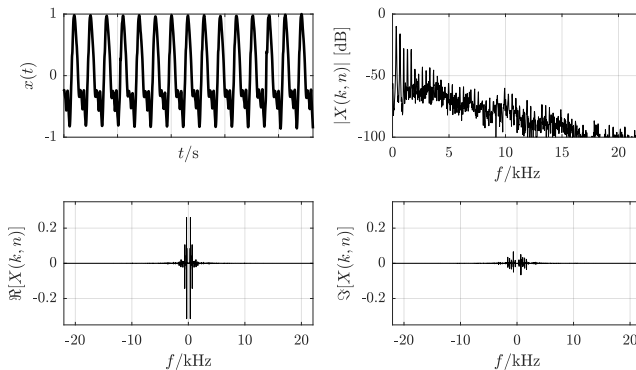 - FT of sampled signals
  - Short Time FT (STFT)
  - DFT

- **learning objectives**
  - name and explain definition and properties of the FT

fourier transform
introduction

top	time domain signal	magnitude spectrum in dB
bottom	real spectrum	imaginary spectrum

matlab source: plotFourierTransform.m

fourier transform
definition (continuous)

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$$X(\mathrm{j}\omega) = \mathfrak{F}[x(t)] = \int\limits_{-\infty}^{\infty} x(t)e^{-\mathrm{j}\omega t}\, dt$$

sidenote: Fourier series coefficients

$$a_k = \frac{1}{T_0} \int\limits_{-T_0/2}^{T_0/2} x(t)e^{-\mathrm{j}\omega_0 k t}\, dt$$

■ $T_0 \to \infty$ to allow the analysis of aperiodic functions

$\Rightarrow\ k\omega_0 \to \omega$

fourier transform
definition (continuous)

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fourier transform
representations

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$$X(j\omega) = \Re[X(j\omega)] + \Im[X(j\omega)]$$
$$= \underbrace{|X(j\omega)|}_{\text{magnitude}} \cdot \underbrace{\Phi_X(\omega)}_{\text{phase}}$$

$$|X(j\omega)| = \sqrt{\Re[X(j\omega)]^2 + \Im[X(j\omega)]^2}$$
$$\Phi_X(\omega) = \text{atan}2\left(\frac{\Im[X(j\omega)]}{\Re[X(j\omega)]}\right)$$

■ complex spectrum either represented as magnitude & phase or as real & imaginary

■ magnitude spectrum has no negative values

fourier transform
representations

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\begin{aligned}
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note:

- complex spectrum either represented as magnitude & phase or as real & imaginary
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fourier transform
property 1: invertibility

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$$
\begin{aligned}
x(t) &= \mathfrak{F}^{-1}[X(\mathrm{j}\omega)] \\
&= \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} X(\mathrm{j}\omega) e^{\mathrm{j}\omega t} \, d\omega
\end{aligned}
$$

- time domain signal can be **perfectly reconstructed** — no information loss
- FT and IFT are very similar, largely equivalent

fourier transform
property 1: invertibility

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overview   introduction   definition   **properties**   sampled   STFT   DFT   summary
○   ○   ○○   ○●○○○○○   ○○   ○○○   ○   ○

fourier transform   Georgia | Center for Music
property 2: superposition   Tech || Technology
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$$y(t) = c_1 \cdot x_1(t) + c_2 \cdot x_2(t)$$
$$\mapsto$$
$$Y(j\omega) = c_1 \cdot X_1(j\omega) + c_2 \cdot X_2(j\omega)$$

■ FT is a *linear* transform

fourier transform
property 2: superposition

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note:

- FT is a *linear* transform

fourier transform
property 3: convolution and multiplication

$$y(t) = \int_{-\infty}^{\infty} h(\tau) \cdot x(t - \tau)\, d\tau$$
$$\mapsto$$
$$Y(j\omega) = H(j\omega) \cdot X(j\omega)$$

■ convolution in time domain means multiplication in frequency domain
■ convolution in frequency domain means multiplication in time domain

fourier transform
property 3: convolution and multiplication

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fourier transform
property 4: Parseval's theorem

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$$\int\limits_{-\infty}^{\infty} x^2(t)\, dt = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} |X(\mathrm{j}\omega)|^2\, d\omega$$

- energy of the signal is preserved when switching between time and frequency domains

fourier transform
property 4: Parseval's theorem

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fourier transform
property 5: time & frequency shift

- **time shift**

$$x(t - t_0) \mapsto X(\mathrm{j}\omega)e^{-\mathrm{j}\omega t_0}$$

- **frequency shift**

$$\frac{1}{2\pi} \int\limits_{-\infty}^{\infty} X(\mathrm{j}\omega - \omega_0)e^{\mathrm{j}\omega t} \, d\omega = e^{\mathrm{j}\omega_0 t} \cdot x(t)$$

- time shift results in phase shift
- frequency shift results in modulation of time signal

fourier transform
property 5: time & frequency shift

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- **frequency shift**

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- time shift results in phase shift
- frequency shift results in modulation of time signal

fourier transform
property 6: symmetry

$$|X(j\omega)| = |X(-j\omega)|$$
$$\Phi_X(\omega) = -\Phi_X(-\omega)$$

- spectrum of (real) signal is conjugate complex
  - magnitude spectrum is symmetric to ordinate
  - phase spectrum is symmetric to origin
- even signals have no imaginary spectrum
- odd signals have no real spectrum

fourier transform
property 6: symmetry

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fourier transform
property 7: time & frequency scaling

$$y(t) = x(c \cdot t)$$
$$\mapsto$$
$$Y(j\omega) = \frac{1}{|c|}X\left(j\frac{\omega}{c}\right)$$

■ scaling of abscissa in one domain leads to inverse scaling in the other domain

fourier transform
property 7: time & frequency scaling

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fourier transform
sampled time signals 1/2

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- sampled time signal can be modeled as multiplication of original signal with delta pulse $\delta_\mathrm{T}(t)$
- multiplication in time domain $\mapsto$ convolution in frequency domain

$$
\begin{aligned}
\mathfrak{F}[x(i)] &= \mathfrak{F}[x(t) \cdot \delta_\mathrm{T}(t)] \\
&= \mathfrak{F}[x(t)] * \mathfrak{F}[\delta_\mathrm{T}(t)] \\
&= X(\mathrm{j}\omega) * \Delta_\mathrm{T}(\mathrm{j}\omega)
\end{aligned}
$$

note

- even if time domain signal is discrete, its Fourier transform is *still continuous*
- spectrum is *repeated periodically*

fourier transform
sampled time signals 1/2

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fourier transform
sampled time signals 2/2

matlab source: matlab/animateSampling.m

fourier transform
STFT 1/2

short time Fourier transform (STFT):
Fourier transform of a short time segment

- **reasons**:
    - remember block based processing
    - segment can be seen as quasi-periodic or stationary

- **implementation**:
    - pretend signal is 0 outside of the segment
    - ⇒ multiplication of signal and *window function*
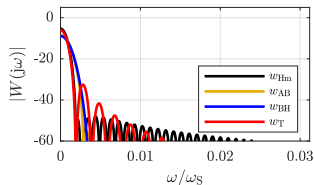
fourier transform
STFT 1/2

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overview · introduction · definition · properties · sampled · STFT · DFT · summary

fourier transform
STFT 1/2

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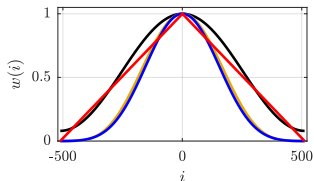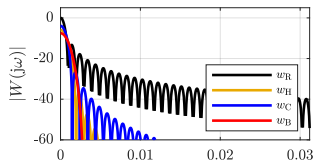 - ⇒ multiplication of signal and *window function*

fourier transform
STFT 2/2

matlab source: matlab/animateStft.m

fourier transform
STFT: window functions

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- time domain multiplication $\mapsto$ frequency domain convolution
- time domain shape determines frequency domain shape of the window
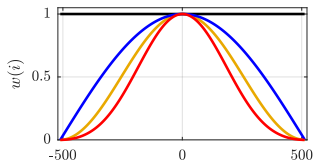
fourier transform
STFT: window functions

- time domain multiplication $\mapsto$ frequency domain convolution
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matlab source: plotSpectralWindows.m

fourier transform
STFT: window functions

- time domain multiplication $\mapsto$ frequency domain convolution
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spectral leakage characterization
- main lobe width
- side lobe height
- side lobe attenuation

overview   introduction   definition   properties   sampled   STFT   DFT   summary

fourier transform
DFT

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digital domain: requires discrete frequency values:
$\Rightarrow$ discrete Fourier transform

$$X(k) = \sum_{i=0}^{\mathcal{K}-1} x(i) e^{-\mathrm{j}ki\frac{2\pi}{\mathcal{K}}}$$

with

$$\Delta\Omega = \frac{2\pi}{\mathcal{K}T_{\mathrm{S}}} = \frac{2\pi f_{\mathrm{S}}}{\mathcal{K}}$$

2 interpretations:

- sampled continuous Fourier transform
- continuous Fourier transform of periodically extended time domain segment

overview   introduction   definition   properties   sampled   STFT   DFT   summary
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fourier transform
DFT

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2 interpretations:

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fourier transform
spectrogram

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- spectrogram allows to visualize temporal changes in the spectrum
- displays the *magnitude spectrum* only

matlab source: plotSpecgram.m
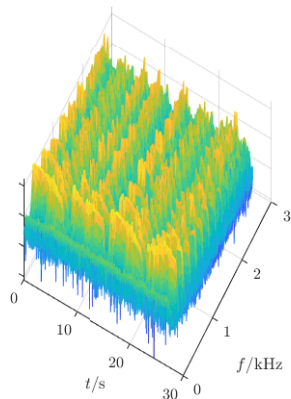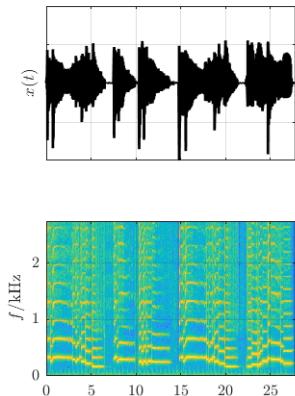
summary
lecture content

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- **Fourier Transform (of a real signal)**
  - is conjugate complex
  - often represented as magnitude $+$ phase
  - invertible
  - linear
  - convolution in time domain is multiplication in frequency domain
  - energy preserving
  - time shift result in phase shift, frequency shift results in amplitude modulation
  - symmetric
  - time scaling result in inverse frequency scaling
- **FT of sampled signals**:
  - is periodic with sample rate
- **STFT**
  - window results in spectral leakage (convolution in freq domain)
- **DFT**
  - discrete in both time and freq domain