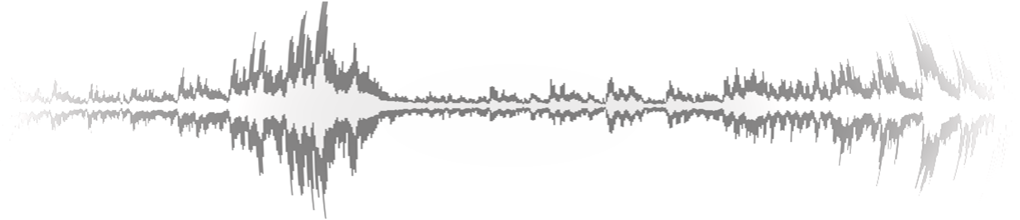
Digital Signal Processing for Music

Part 27: Denoising

alexander lerch



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denoising
introduction

- **problem**: signal y is noisy

$$y(i) = x(i) + n(i)$$

- **assumptions**:
  - noise is uncorrelated
  - noise is stationary
- **objective**: estimate $\hat{x}$ which minimizes the error

$$e(i) = x(i) - \hat{x}(i)$$

- **approach**: filter the noisy signal

$$\hat{x}(i) = \sum_{j=0}^{\mathcal{O}-1} w(j) \cdot y(i-j)$$

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introduction 2/2

here: only presenting the simplest approach to noise reduction

the **Wiener Filter**

denoising
introduction 2/2

here: only presenting the simplest approach to noise reduction

the **Wiener Filter**

denoising
Wiener filter 1/2

$$\hat{x}(i) = \sum_{j=0}^{\mathcal{O}-1} w(j) \cdot y(i-j)$$

$$\hat{X}(\mathrm{j}\omega) = W(\mathrm{j}\omega) \cdot Y(\mathrm{j}\omega)$$

$$E(\mathrm{j}\omega) = X(\mathrm{j}\omega) - W(\mathrm{j}\omega) \cdot Y(\mathrm{j}\omega)$$

$$\frac{\partial \mathcal{E}\{|E(\mathrm{j}\omega)|^2\}}{\partial W(\mathrm{j}\omega)} = 0$$

$$\frac{\partial \mathcal{E}\{(X(\mathrm{j}\omega) - W(\mathrm{j}\omega) \cdot Y(\mathrm{j}\omega))^*(X(\mathrm{j}\omega) - W(\mathrm{j}\omega) \cdot Y(\mathrm{j}\omega))\}}{\partial W(\mathrm{j}\omega)} = 0$$

$$2W(\mathrm{j}\omega)S_{\mathrm{YY}}(\mathrm{j}\omega) - 2S_{\mathrm{XY}}(\mathrm{j}\omega) = 0$$

$$\Rightarrow W(\mathrm{j}\omega) = \frac{S_{\mathrm{XY}}(\mathrm{j}\omega)}{S_{\mathrm{YY}}(\mathrm{j}\omega)}$$

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Wiener filter 1/2

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$$
\begin{aligned}
\hat{x}(i) &= \sum_{j=0}^{\mathcal{O}-1} w(j)\cdot y(i-j)\\
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\end{aligned}
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Wiener filter 1/2

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Wiener filter 1/2

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denoising

Wiener filter 2/2

reminder: signal and noise are uncorrelated $\rightarrow r_{\mathrm{XN}}(i) = 0$

$$
\begin{aligned}
\boldsymbol{R}_{\mathrm{YY}} &= \boldsymbol{R}_{\mathrm{XX}} + \boldsymbol{R}_{\mathrm{NN}} \\
\boldsymbol{r}_{\mathrm{XY}} &= \boldsymbol{r}_{\mathrm{XX}} \\
\Rightarrow S_{\mathrm{YY}}(\mathrm{j}\omega) &= S_{\mathrm{XX}}(\mathrm{j}\omega) + S_{\mathrm{NN}}(\mathrm{j}\omega) \\
\Rightarrow S_{\mathrm{XY}}(\mathrm{j}\omega) &= S_{\mathrm{XX}}(\mathrm{j}\omega) \\
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denoising
Wiener filter 2/2

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Wiener filter discussion 1/2

$$W(\mathrm{j}\omega) = \frac{S_{\mathrm{XX}}(\mathrm{j}\omega)}{S_{\mathrm{XX}}(\mathrm{j}\omega) + S_{\mathrm{NN}}(\mathrm{j}\omega)}$$

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Wiener filter discussion 1/2

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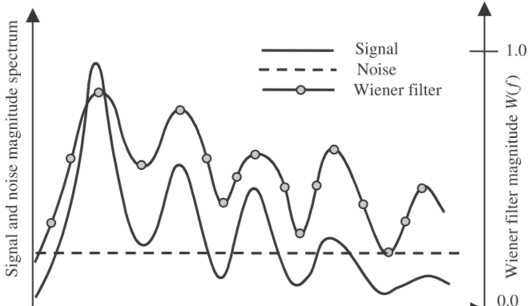
$$
\begin{aligned}
W(j\omega) &= \frac{S_{XX}(j\omega)}{S_{XX}(j\omega) + S_{NN}(j\omega)} \\
&= \frac{SNR(\omega)}{SNR(\omega) + 1}
\end{aligned}
$$

$\Rightarrow$ attenuates noisy components *in proportion to SNR*



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Wiener filter discussion 2/2

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$$W(j\omega) = \frac{S_{XX}(j\omega)}{S_{XX}(j\omega) + S_{NN}(j\omega)} = \frac{SNR(\omega)}{SNR(\omega)+1}$$

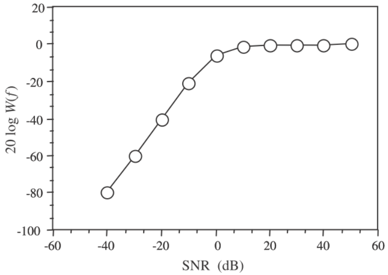
$$0 \leq W(j\omega) \leq 1$$

- limiting case 1: noise free

$$SNR(\omega) = \infty$$
$$\Rightarrow W(j\omega) \rightarrow 1$$

- limiting case 2: extr. noisy

$$SNR(\omega) = 0$$
$$\Rightarrow W(j\omega) \rightarrow 0$$



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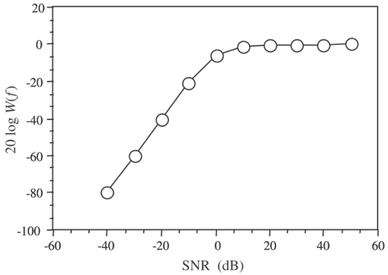

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Wiener filter discussion 2/2

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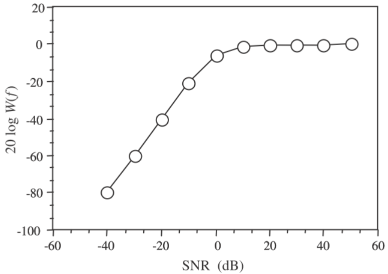

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Wiener filter discussion 2/2

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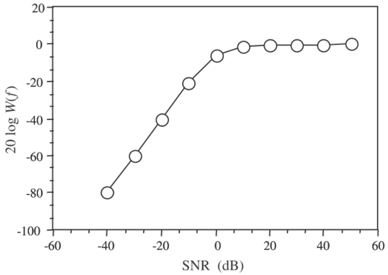

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Wiener filter question

How to estimate the noise spectrum

How to estimate the noise spectrum

- user input: noise fingerprint
- estimate from signal through, e.g.,
  - non-real-time: pause detection and automatic noise fingerprint selection
  - real-time: prediction error with smoothing constraints

idea: Why not subtract the noise spectrum from the signal spectrum?

$$
\begin{aligned}
|\hat{X}(\mathrm{j}\omega)|^2 &= |Y(\mathrm{j}\omega)|^2 - |N(\mathrm{j}\omega)|^2 \\
&= H(\mathrm{j}\omega) \cdot |Y(\mathrm{j}\omega)|^2
\end{aligned}
$$

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Spectral Subtraction

idea: Why not subtract the noise spectrum from the signal spectrum?

$$\begin{aligned} |\hat{X}(\mathrm{j}\omega)|^2 &= |Y(\mathrm{j}\omega)|^2 - |N(\mathrm{j}\omega)|^2 \\ &= H(\mathrm{j}\omega) \cdot |Y(\mathrm{j}\omega)|^2 \end{aligned}$$

$$\begin{aligned} \Rightarrow H &= 1 - \frac{|N(\mathrm{j}\omega)|^2}{|Y(\mathrm{j}\omega)|^2} \\ &= \frac{|Y(\mathrm{j}\omega)|^2 - |N(\mathrm{j}\omega)|^2}{|Y(\mathrm{j}\omega)|^2} \end{aligned}$$

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idea: Why not subtract the noise spectrum from the signal spectrum?

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spectral subtraction identical to Wiener filter when the power density spectrum estimates approach the ensemble means