

Digital Signal Processing for Music

Part 4: Signal Description

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introduction

description of (random) signals

- ergodic signals do not have a functional description
- ⇒ other ways of describing these signals have to be found

- ergodic signal characteristics are not time variant
- ⇒ we are looking for **time-independent descriptions**

- these descriptions might also be convenient to use for some deterministic signals

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audio signal description

probability and occurrence

N : number of overall observations

$N(x_i)$: number of occurrences of symbol x_i

■ relative number of occurrences:

$$\hat{p}_i = \frac{N(x_i)}{N}$$

■ probability:

$$p_i = \lim_{N \rightarrow \infty} \frac{N(x_i)}{N}$$

properties

$$\sum_i p_i = 1$$

$$0 \leq p_i \leq 1$$

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audio signal description

probability distribution example

■ roll of a die

value	1	2	3	4	5	6
p(x)	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$

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probability distribution for the roll of two dice



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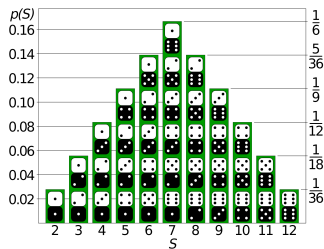
probability distribution example

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probability distribution for the roll of two dice

value	2	3	4	5	6	7	8	9	10	11	12
$p(x)$	$1/36$	$1/18$	$1/12$	$1/9$	$5/36$	$1/6$	$5/36$	$1/9$	$1/12$	$1/18$	$1/36$



audio signal description

continuous probability density distribution

$i \rightarrow \text{continuous} \Rightarrow$ **PDF**

$$\int_{-\infty}^{\infty} p_X(x) dx = 1$$
$$0 \leq p_X(x)$$

probability of x being a value smaller than or equal x_c

$$\int_{-\infty}^{x_c} p_X(x) dx$$

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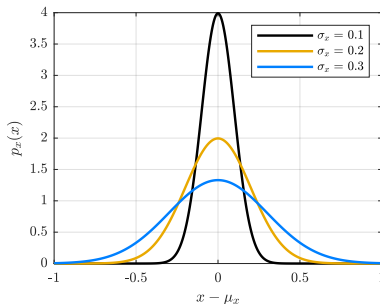
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example PDF: Gaussian

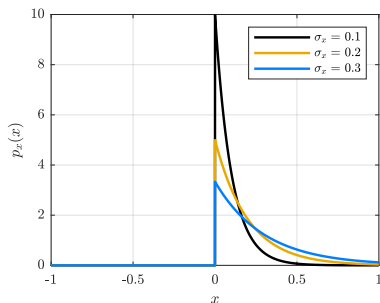
$$p_X(x) = \frac{1}{\sqrt{2\pi}\sigma_X} e^{-\frac{(x-\mu_X)^2}{2\sigma_X^2}}$$



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example PDF: Exponential

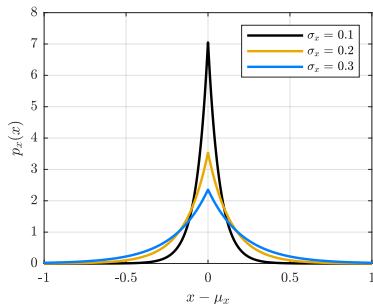
$$p_X(x) = \begin{cases} \frac{1}{\sigma_X} e^{-\frac{x}{\sigma_X}} & x > 0 \\ 0 & \text{else} \end{cases}$$



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example PDF: Laplace (2-sided exp)

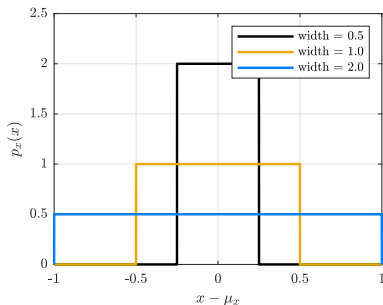
$$p_X(x) = \frac{1}{\sqrt{2}\sigma_X} e^{-\sqrt{2} \frac{|x-\mu_X|}{\sigma_X}}$$



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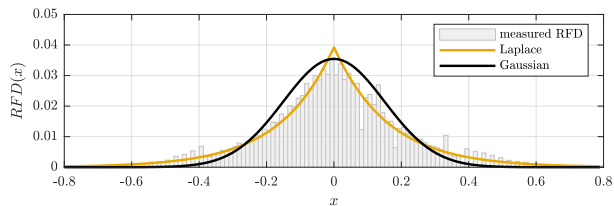
example PDF: Laplace (2-sided exp)

$$p_X(x) = \begin{cases} \frac{1}{width} & |x - \mu_x| < width/2 \\ 0 & \text{else} \end{cases}$$



audio signal description

measured RDF



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PDFs of generated signals 1/2

describe the shape of the following PDFs



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PDFs of generated signals 1/2

describe the shape of the following PDFs

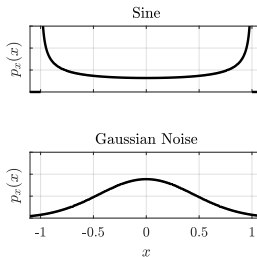
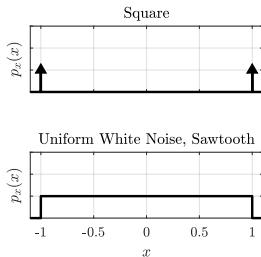
- white noise (uniform)
- white noise (Gaussian)
- DC
- square
- sinusoidal
- sawtooth



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PDFs of generated signals 1/2

describe the shape of the following PDFs



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expected value 1/3

Example: average grade, five students, grades: 1, 2, 1, 3, 5

$$\hat{\mu}_X = \frac{1 + 2 + 1 + 3 + 5}{5} = 2.4$$

Grade	# occurrences	relative frequency
1	2	2/5
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4	0	0/5
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audio signal description

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$$\mu = \frac{2}{5} \cdot 1 + \frac{1}{5} \cdot 2 + \frac{1}{5} \cdot 3 + \frac{0}{5} \cdot 4 + \frac{1}{5} \cdot 5 = 2.4$$

$$\mu_X = \sum_{\forall x} p(x) \cdot x$$

$$\mu_X = \mathcal{E}\{X\} = \int_{-\infty}^{+\infty} x p_X(x) dx$$

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expected value 3/3

generalization:

$$\mathcal{E}\{f(X)\} = \sum_i f(x)p(x)$$

examples:

- mean: $f(x) = x$
- quad. mean: $f(x) = x^2$

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audio signal description

(central) moments 1/2

■ kth moment

$$\mathcal{E}\{X^k\} = \int_{-\infty}^{+\infty} x^k p_X(x) dx$$

■ kth central moment

$$\mathcal{E}\{(X - \mu_X)^k\} = \int_{-\infty}^{+\infty} (x - \mu_X)^k p_X(x) dx$$

■ example: 2nd order central moment: **Variance**

$$\sigma_X^2 = \mathcal{E}\{(X - \mu_X)^2\} = \int_{-\infty}^{+\infty} (x - \mu_X)^2 p_X(x) dx$$

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(central) moments 2/2

calculation of moments

(central) moments (mean, power, variance, etc.) can be computed from

- the signal
- the signal's PDF



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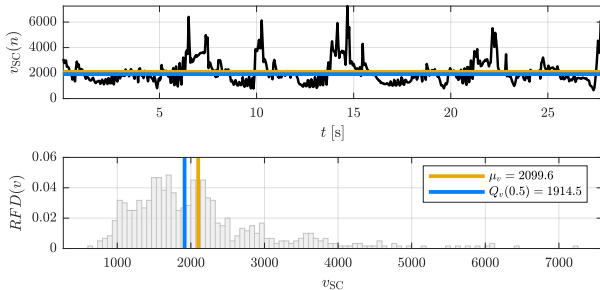
arithmetic mean

- from time series x :

$$\mu_x(n) = \frac{1}{K} \sum_{i=i_s(n)}^{i_e(n)} x(i)$$

- from distribution p_x :

$$\mu_x(n) = \sum_{x=-\infty}^{\infty} x \cdot p_x(x)$$



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variance & standard deviation

measure of *spread* of the signal around its mean

■ variance

- from signal block:

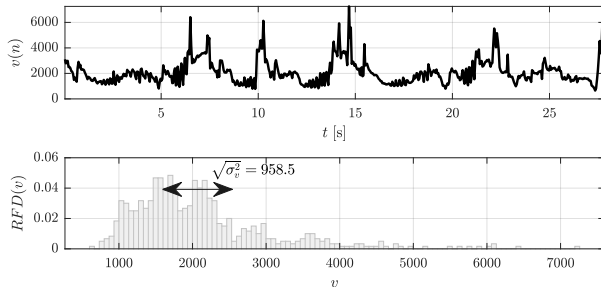
$$\sigma_x^2(n) = \frac{1}{K} \sum_{i=i_s(n)}^{i_e(n)} (x(i) - \mu_x(n))^2$$

- from distribution:

$$\sigma_x^2(n) = \sum_{x=-\infty}^{\infty} (x - \mu_x)^2 \cdot p_x(x)$$

■ standard deviation

$$\sigma_x(n) = \sqrt{\sigma_x^2(n)}$$



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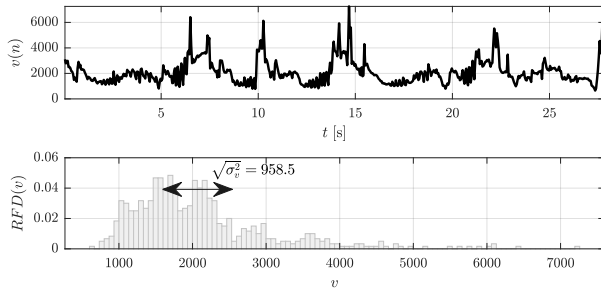
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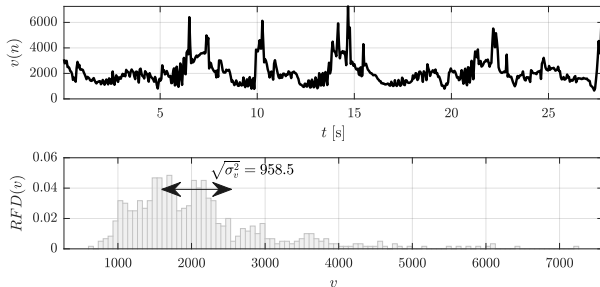
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audio signal description

central moments summary

order	name	obs (cont)	pdf (cont)
1	μ_X	$\frac{1}{T} \int_{-T/2}^{T/2} x(t) dt$	$\int_{-\infty}^{\infty} x p_X(x) dx$
2	σ_X^2	$\frac{1}{T} \int_{-T/2}^{T/2} (x(t) - \mu_X)^2 dt$	$\int_{-\infty}^{\infty} (x - \mu_X)^2 p_X(x) dx$

order	name	obs (disc)	pdf (disc)
1	μ_X	$\frac{1}{N} \sum_{i=0}^N x(i)$	$\sum_{\forall x} x p(x)$
2	σ_X^2	$\frac{1}{N} \sum_{i=0}^N (x(i) - \mu_X)^2$	$\sum_{\forall x} (x - \mu_X)^2 p(x)$

standard deviation $\sigma_X = \sqrt{\sigma_X^2}$

audio signal description

summary

- 1 PDF can tell us many important details about a signal
- 2 statistical measures can be used to describe signal properties, but cannot be used to reconstruct the signal
- 3 statistical measures can be derived from both the time domain signal and its pdf
- 4 often-used measures are:
 - mean and median
 - variance and standard deviation
 - higher order moments less frequently (skewness, kurtosis)
 - other pdf description possible (quartile-distances etc.)

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