Digital Signal Processing for Music

Part 4: Signal Description

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introduction

description of (random) signals

- ergodic signals do not have a functional description

$\Rightarrow$ other ways of describing these signals have to be found

- ergodic signal characteristics are not time variant

$\Rightarrow$ we are looking for **time-independent descriptions**

- these descriptions might also be convenient to use for some deterministic signals

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audio signal description
probability and occurrence

N: number of overall observations

$N(x_i)$: number of occurrences of symbol x_i

- relative number of occurrences:

$$\hat{p}_i = \frac{N(x_i)}{N}$$

- probability:

$$p_i = \lim_{N \to \infty} \frac{N(x_i)}{N}$$

properties

$$\sum_i p_i = 1$$

$$0 \leq p_i \leq 1$$

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audio signal description
probability distribution example

- roll of a die

value	1	2	3	4	5	6
p(x)	$^1/_6$	$^1/_6$	$^1/_6$	$^1/_6$	$^1/_6$	$^1/_6$

audio signal description
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probability distribution for the roll of two dice

audio signal description

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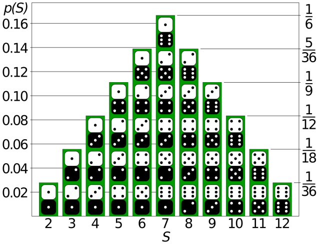
probability distribution for the roll of two dice

value	2	3	4	5	6	7	8	9	10	11	12
p(x)	$^1/_{36}$	$^1/_{18}$	$^1/_{12}$	$^1/_9$	$^5/_{36}$	$^1/_6$	$^5/_{36}$	$^1/_9$	$^1/_{12}$	$^1/_{18}$	$^1/_{36}$



audio signal description

continuous probability density distribution

$i \rightarrow$ continuous $\Rightarrow$ **PDF**

$$\int\limits_{-\infty}^{\infty} p_X(x)dx = 1$$

$$0 \leq p_X(x)$$

probability of x being a value smaller than or equal x_c

$$\int\limits_{-\infty}^{x_c} p_X(x)dx$$

audio signal description

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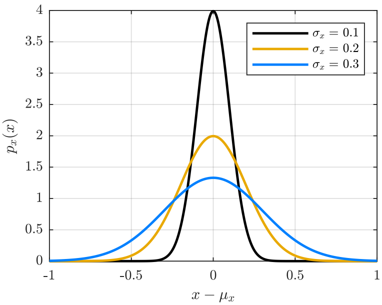
example PDF: Gaussian

$$p_X(x) = \frac{1}{\sqrt{2\pi}\sigma_X} e^{-\frac{(x-\mu_X)^2}{2\sigma_X^2}}$$



matlab source: plotGaussian.m

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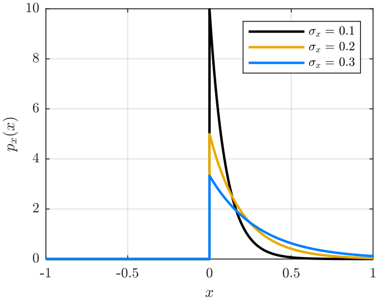
example PDF: Exponential

$$p_X(x) = \begin{cases} \frac{1}{\sigma_X} e^{-\frac{x}{\sigma_X}} & x > 0 \\ 0 & \text{else} \end{cases}$$



matlab source: plotExponential.m

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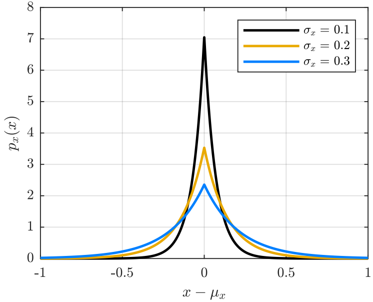
example PDF: Laplace (2-sided exp)

$$p_X(x) = \frac{1}{\sqrt{2}\sigma_X} e^{-\sqrt{2}\frac{|x-\mu_X|}{\sigma_X}}$$



matlab source: plotLaplace.m

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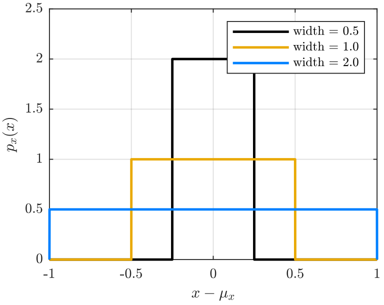
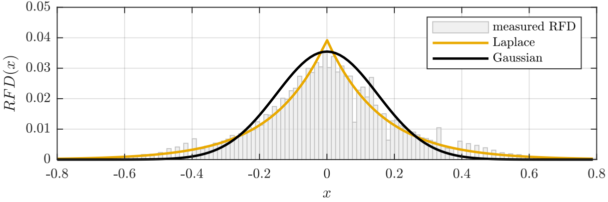
example PDF: Laplace (2-sided exp)

$$p_X(x) = \begin{cases} \frac{1}{width} & |x - mu_x| < width/2 \\ 0 & \text{else} \end{cases}$$



matlab source: plotUniform.m

audio signal description
measured RDF



matlab source: plotRfd.m

audio signal description

PDFs of generated signals 1/2

describe the shape of the following PDFs

audio signal description

PDFs of generated signals 1/2

describe the shape of the following PDFs

- white noise (uniform)
- white noise (Gaussian)
- DC
- square
- sinusoidal
- sawtooth

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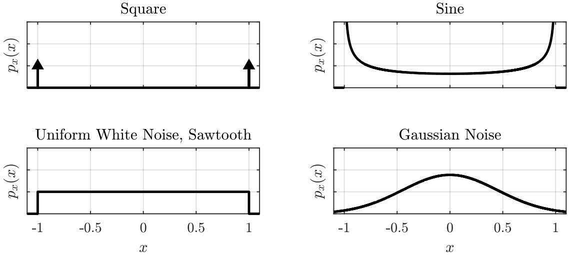
PDFs of generated signals 1/2

describe the the shape of the following PDFs



matlab source: plotPdfExamples.m

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expected value 1/3

Example: average grade, five students, grades: 1, 2, 1, 3, 5

$$\hat{\mu}_X = \frac{1+2+1+3+5}{5} = 2.4$$

Grade	# occurrences	relative frequency
1	2	2/5
2	1	1/5
3	1	1/5
4	0	0/5
5	1	1/5

audio signal description

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audio signal description

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$$\mu_X = 1 \cdot \frac{2}{5} + 2 \cdot \frac{1}{5} + 3 \cdot \frac{1}{5} + 4 \cdot \frac{0}{5} + 5 \cdot \frac{1}{5} = 2.4$$

$$\mu_X = \sum_{\forall x} x \cdot p_X(x)$$

$$\mu_X = \mathcal{E}\{X\} = \int_{-\infty}^{+\infty} x \cdot p_X(x) dx$$

audio signal description

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expected value 3/3

generalization:

$$\mathcal{E}\{f(X)\} = \sum_i f(x)p(x)$$

examples:

- mean: $f(x) = x$
- quad. mean: $f(x) = x^2$

audio signal description
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audio signal description

(central) moments 1/2

- kth moment

$$\mathcal{E}\{X^k\} = \int\limits_{-\infty}^{+\infty} x^k p_X(x)dx$$

- kth central moment

$$\mathcal{E}\{(X-\mu_X)^k\} = \int\limits_{-\infty}^{+\infty} (x-\mu_X)^k p_X(x)dx$$

- example: 2nd order central moment: **Variance**

$$\sigma_X^2 = \mathcal{E}\{(X-\mu_X)^2\} = \int\limits_{-\infty}^{+\infty} (x-\mu_X)^2 p_X(x)dx$$

audio signal description

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audio signal description

(central) moments 2/2

calculation of moments

(central) moments (mean, power, variance, etc.) can be computed from

- the signal
- the signal's PDF

audio signal description
arithmetic mean

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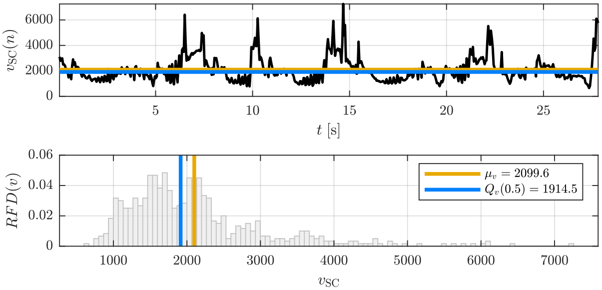
- from time series x:

$$\mu_x(n) = \frac{1}{\mathcal{K}} \sum_{i=i_\mathrm{s}(n)}^{i_\mathrm{e}(n)} x(i)$$

- from distribution p_x:

$$\mu_x(n) = \sum_{x=-\infty}^{\infty} x \cdot p_x(x)$$



matlab source: plotArithmeticMean.m

audio signal description
variance & standard deviation

measure of *spread* of the signal around its mean

- **variance**
  - from signal block:

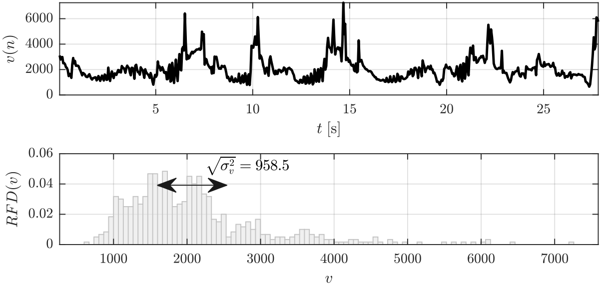
$$\sigma_x^2(n) = \frac{1}{\mathcal{K}} \sum_{i=i_s(n)}^{i_e(n)} \left(x(i) - \mu_x(n)\right)^2$$

  - from distribution:

$$\sigma_x^2(n) = \sum_{x=-\infty}^{\infty} (x - \mu_x)^2 \cdot p_x(x)$$

- **standard deviation**

$$\sigma_x(n) = \sqrt{\sigma_x^2(n)}$$



matlab source: plotStandardDeviation.m

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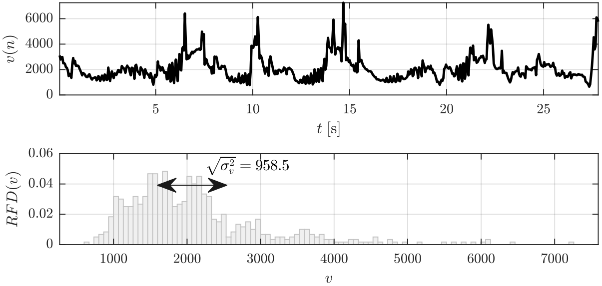

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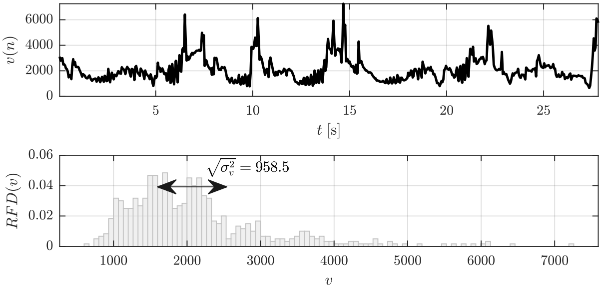

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$$\sigma_x(n) = \sqrt{\sigma_x^2(n)}$$

matlab source: plotStandardDeviation.m

audio signal description

central moments summary

order	name	obs (cont)	pdf (cont)
1	μ_X	$\frac{1}{T}\int\limits_{-T/2}^{T/2} x(t)dt$	$\int\limits_{-\infty}^{\infty} x p_X(x)dx$
2	σ_X^2	$\frac{1}{T}\int\limits_{-T/2}^{T/2} (x(t)-\mu_X)^2 dt$	$\int\limits_{-\infty}^{\infty} (x-\mu_X)^2 p_X(x)dx$

order	name	obs (disc)	pdf (disc)
1	μ_X	$\frac{1}{N}\sum\limits_{i=0}^{N} x(i)$	$\sum\limits_{\forall x} x p(x)$
2	σ_X^2	$\frac{1}{N}\sum\limits_{i=0}^{N}(x(i)-\mu_X)^2$	$\sum\limits_{\forall x}(x-\mu_X)^2 p(x)$
3	skewness: $\mathcal{E}\{(X-\mu_X)^3\}$	$\frac{1}{N}\sum\limits_{i=0}^{N}(x(i)-\mu_X)^3$	$\sum\limits_{\forall x}(x-\mu_X)^3 p(x)$
4	kurtosis: $\mathcal{E}\{(X-\mu_X)^4\}$	$\frac{1}{N}\sum\limits_{i=0}^{N}(x(i)-\mu_X)^4$	$\sum\limits_{\forall x}(x-\mu_X)^4 p(x)$

audio signal description
summary

1. PDF and derived metrics are often an **effective description or the signal**
2. statistical measures can be used to describe PDF and signal properties, but **cannot be used to reconstruct the signal**
3. statistical measures **can be derived from both the time domain signal and its PDF**
4. often-used measures are:
   - mean and median
   - variance and standard deviation
   - higher order moments less frequently (skewness, kurtosis)
   - other pdf descriptions (quartile-distances etc.)

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