

Digital Signal Processing for Music

Part 16: z-transform

alexander lerch

z-transform

introduction

the z-transform is

- a generalization of DFT,
- widely used in DSP as analysis tool,
- a useful tool to characterize systems (also recursive systems!),
- a useful tool to check for system stability and causality,
- the discrete-time counterpart of the Laplace transform.

z-transform

definition

$$X(z) = \sum_{i=-\infty}^{\infty} x(i)z^{-i}, \quad z \in \mathbb{C}$$

- $X(z)$: complex function of a complex number
- compare Fourier transform $X(j\omega)$: complex function of real-valued ω

z-transform

definition

$$X(z) = \sum_{i=-\infty}^{\infty} x(i)z^{-i}, \quad z \in \mathbb{C}$$

- $X(z)$: complex function of a complex number
- compare Fourier transform $X(j\omega)$: complex function of real-valued ω

z-transform

definition

$$X(z) = \sum_{i=-\infty}^{\infty} x(i)z^{-i}, \quad z \in \mathbb{C}$$

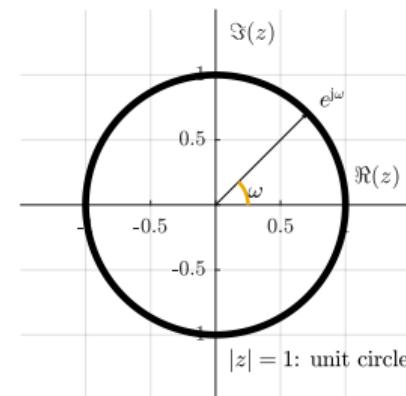
- $X(z)$: complex function of a complex number
- compare Fourier transform $X(j\omega)$: complex function of real-valued ω

$$X(j\omega) = \sum_{i=-\infty}^{\infty} x(i)e^{-j\omega i} \Rightarrow X(j\omega) = X(z) \text{ at } z = e^{j\omega}$$

z-transform

zplane

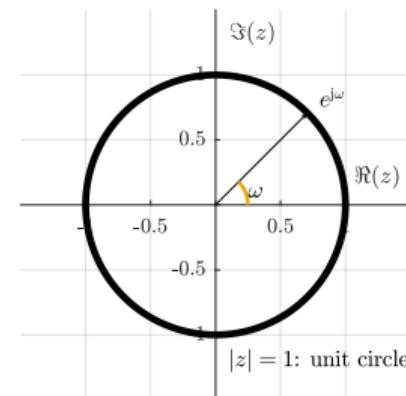
- $X(z)$ defined on complex plane
- $X(j\omega)$ defined on unit circle
- observation: $X(j\omega)$ is periodic with 2π



z-transform

zplane

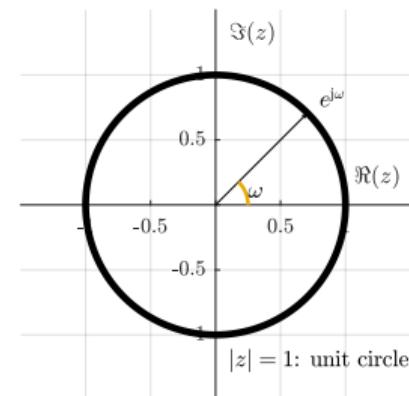
- $X(z)$ defined on complex plane
- $X(j\omega)$ defined on unit circle
- observation: $X(j\omega)$ is periodic with 2π



z-transform

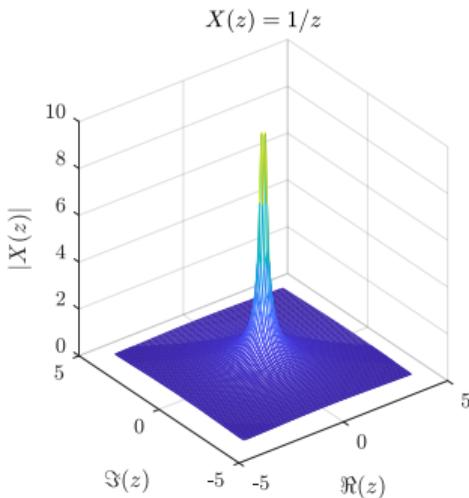
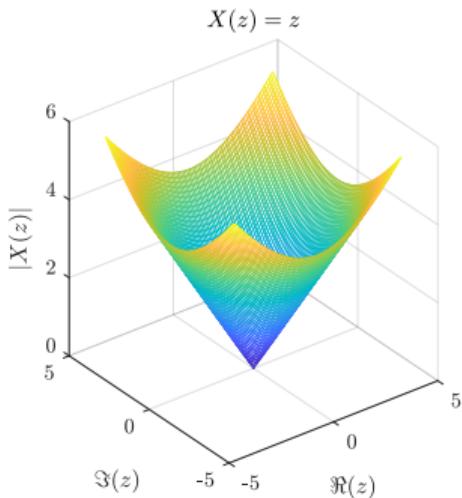
zplane

- $X(z)$ defined on complex plane
- $X(j\omega)$ defined on unit circle
- observation: $X(j\omega)$ is periodic with 2π



z-transform

trivial example

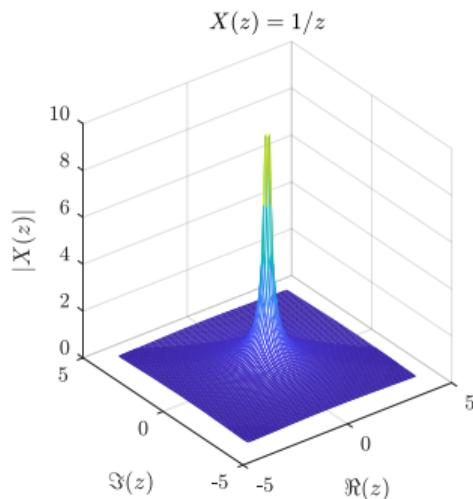
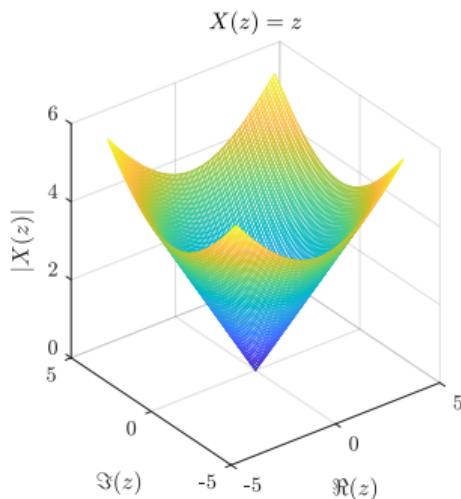


what is the magnitude for $X(z) = 1/(z - 0.5)$



z-transform

trivial example



what is the magnitude for $X(z) = 1/(z - 0.5)$

same as $1/z$ but shifted on the x-axis.



z-transform

system description

Fourier transform and z-transform have largely similar properties, most importantly

■ linearity

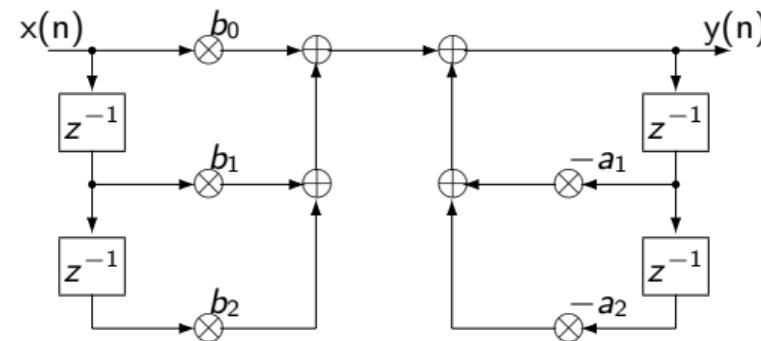
$$\begin{aligned}y(i) = c_1x_1(i) + c_2x_2(i) &\Rightarrow Y(j\omega) = c_1X_1(j\omega) + c_2X_2(j\omega) \\&\Rightarrow Y(z) = c_1X_1(z) + c_2X_2(z)\end{aligned}$$

■ time shift

$$\begin{aligned}y(i) = x(i - n) &\Rightarrow Y(j\omega) = e^{-j\omega n}X(j\omega) \\&\Rightarrow Y(z) = z^{-n}X(z)\end{aligned}$$

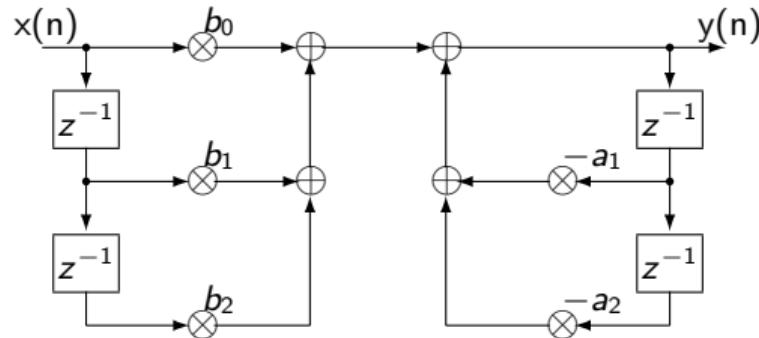
z-transform

biquad: difference equation



z-transform

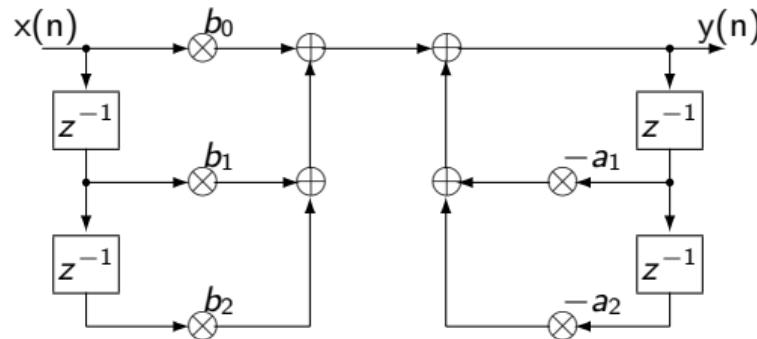
biquad: difference equation



$$y(i) = \sum_{j=0}^2 b_j x(i-j) - \sum_{k=1}^2 a_k y(i-k)$$

z-transform

biquad: difference equation

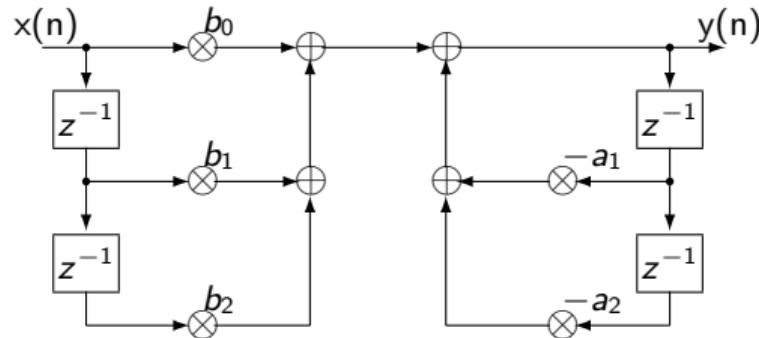


$$y(i) = \sum_{j=0}^2 b_j x(i-j) - \sum_{k=1}^2 a_k y(i-k)$$

$$Y(z) = \sum_{j=0}^2 b_j X(z) z^{-j} - \sum_{k=1}^2 a_k Y(z) z^{-k}$$

z-transform

biquad: difference equation



$$y(i) = \sum_{j=0}^2 b_j x(i-j) - \sum_{k=1}^2 a_k y(i-k)$$

$$Y(z) = \sum_{j=0}^2 b_j X(z) z^{-j} - \sum_{k=1}^2 a_k Y(z) z^{-k}$$

$$Y(z) \left(1 + \sum_{j=1}^2 a_j z^{-j} \right) = X(z) \sum_{j=0}^2 b_j z^{-j}$$

biquad

transfer function

$$\begin{aligned} H(z) &= \frac{Y(z)}{X(z)} \\ &= \frac{\sum_{j=0}^2 b_j z^{-j}}{1 + \sum_{j=1}^2 a_j z^{-j}} \\ &= \frac{b_0 + b_1 \cdot z^{-1} + b_2 \cdot z^{-2}}{1 + a_1 \cdot z^{-1} + a_2 \cdot z^{-2}} \\ &= \frac{\text{numerator polynomial}}{\text{denominator polynomial}} \end{aligned}$$

biquad

transfer function

$$\begin{aligned} H(z) &= \frac{Y(z)}{X(z)} \\ &= \frac{\sum_{j=0}^2 b_j z^{-j}}{1 + \sum_{j=1}^2 a_j z^{-j}} \\ &= \frac{b_0 + b_1 \cdot z^{-1} + b_2 \cdot z^{-2}}{1 + a_1 \cdot z^{-1} + a_2 \cdot z^{-2}} \\ &= \frac{\text{numerator polynomial}}{\text{denominator polynomial}} \end{aligned}$$

biquad

transfer function

$$\begin{aligned} H(z) &= \frac{Y(z)}{X(z)} \\ &= \frac{\sum_{j=0}^2 b_j z^{-j}}{1 + \sum_{j=1}^2 a_j z^{-j}} \\ &= \frac{b_0 + b_1 \cdot z^{-1} + b_2 \cdot z^{-2}}{1 + a_1 \cdot z^{-1} + a_2 \cdot z^{-2}} \\ &= \frac{\text{numerator polynomial}}{\text{denominator polynomial}} \end{aligned}$$

biquad

poles and zeros

- numerator $\rightarrow 0$: zero
- denominator $\rightarrow 0$: pole

biquad

poles and zeros

- numerator $\rightarrow 0$: zero
- denominator $\rightarrow 0$: pole

biquad

poles and zeros

- numerator $\rightarrow 0$: zero
- denominator $\rightarrow 0$: pole

$$\begin{aligned}1 + a_1 \cdot z^{-1} + a_2 \cdot z^{-2} &= 0 \\ \Rightarrow z_{\infty 1,2} &= \frac{a_1}{2} \pm \frac{1}{2} \sqrt{a_1^2 - 4a_2}\end{aligned}$$

biquad

poles and zeros

- numerator $\rightarrow 0$: zero
- denominator $\rightarrow 0$: pole

$$\begin{aligned}1 + a_1 \cdot z^{-1} + a_2 \cdot z^{-2} &= 0 \\ \Rightarrow z_{\infty 1,2} &= \frac{a_1}{2} \pm \frac{1}{2} \sqrt{a_1^2 - 4a_2}\end{aligned}$$

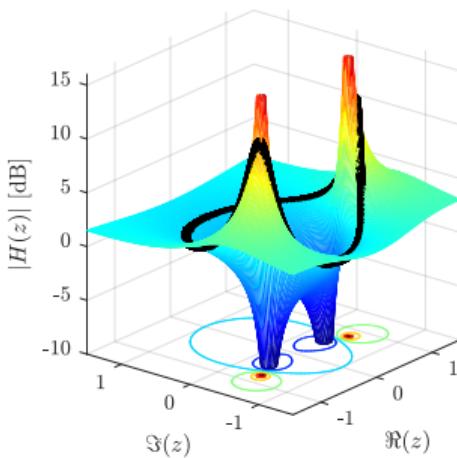
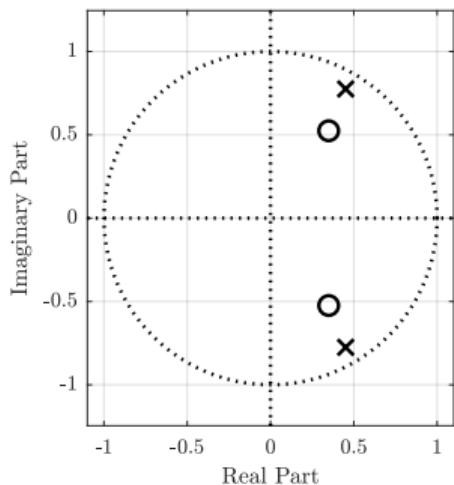
biquad

poles and zeros

- numerator $\rightarrow 0$: zero
- denominator $\rightarrow 0$: pole
- zeros and poles are a simplified way of visualizing filter properties in the zplane

biquad

zplane example



intro
o

z-transform
ooooooo●

characteristics
o

filter design
o

quantized coefficients
o

summary
o

biquad animation

Georgia Tech | Center for Music Technology
College of Design



matlab source: [matlab/animateBiquadZplane.m](#)

filters

z-plane characteristics

- **stability:**

- poles within unit circle

- **zero points and poles**

- are either real or complex conjugate

- **minimal phase systems:**

- no zero points outside of unit circle

- **all pass system:**

- poles and zeros symmetric wrt unit circle

- **linear phase:**

- zero points within and outside unit circle symmetric wrt unit circle

filters

z-plane characteristics

- **stability:**

- poles within unit circle

- **zero points and poles**

- are either real or complex conjugate

- **minimal phase systems:**

- no zero points outside of unit circle

- **all pass system:**

- poles and zeros symmetric wrt unit circle

- **linear phase:**

- zero points within and outside unit circle symmetric wrt unit circle

filters

z-plane characteristics

- **stability:**

- poles within unit circle

- **zero points and poles**

- are either real or complex conjugate

- **minimal phase systems:**

- no zero points outside of unit circle

- **all pass system:**

- poles and zeros symmetric wrt unit circle

- **linear phase:**

- zero points within and outside unit circle symmetric wrt unit circle

filters

z-plane characteristics

- **stability:**

- poles within unit circle

- **zero points and poles**

- are either real or complex conjugate

- **minimal phase systems:**

- no zero points outside of unit circle

- **all pass system:**

- poles and zeros symmetric wrt unit circle

- **linear phase:**

- zero points within and outside unit circle symmetric wrt unit circle

filters

z-plane characteristics

- **stability:**

- poles within unit circle

- **zero points and poles**

- are either real or complex conjugate

- **minimal phase systems:**

- no zero points outside of unit circle

- **all pass system:**

- poles and zeros symmetric wrt unit circle

- **linear phase:**

- zero points within and outside unit circle symmetric wrt unit circle

filters

filter design

■ impulse invariance: sample impulse response

- if continuous system is band-limited, frequency response will be approximately equal (below $f_s/2$)
- special case: no filter definition available → FIR coefficients

■ bi-linear transform

- map filter from (analogue) Laplace-plane to (digital) z-plane
- introduces frequency warping (increasing towards Nyquist frequency)

■ frequency transformation

- transform a (low-pass) prototype filter
- usually via all-pass mapping filter

■ iterative approximation of the magnitude response

■ intuitive methods

- manually move zeros and poles in z-plane
- draw magnitude response in frequency domain

filters

filter design

■ impulse invariance: sample impulse response

- if continuous system is band-limited, frequency response will be approximately equal (below $f_s/2$)
- special case: no filter definition available → FIR coefficients

■ bi-linear transform

- map filter from (analogue) Laplace-plane to (digital) z-plane

$$\begin{aligned}s &= \frac{2}{T} \frac{1 - z^{-1}}{1 + z^{-1}} \\ z &= \frac{1 + sT_s/2}{1 - sT_s/2}\end{aligned}$$

- introduces frequency warping (increasing towards Nyquist frequency)

■ frequency transformation

- transform a (low-pass) prototype filter
- usually via all-pass mapping filter

■ iterative approximation of the magnitude response

filters

filter design

■ impulse invariance: sample impulse response

- if continuous system is band-limited, frequency response will be approximately equal (below $f_s/2$)
- special case: no filter definition available → FIR coefficients

■ bi-linear transform

- map filter from (analogue) Laplace-plane to (digital) z-plane
- introduces frequency warping (increasing towards Nyquist frequency)

■ frequency transformation

- transform a (low-pass) prototype filter
- usually via all-pass mapping filter

■ iterative approximation of the magnitude response

■ intuitive methods

- manually move zeros and poles in z-plane
- draw magnitude response in frequency domain

filters

filter design

■ impulse invariance: sample impulse response

- if continuous system is band-limited, frequency response will be approximately equal (below $f_s/2$)
- special case: no filter definition available → FIR coefficients

■ bi-linear transform

- map filter from (analogue) Laplace-plane to (digital) z-plane
- introduces frequency warping (increasing towards Nyquist frequency)

■ frequency transformation

- transform a (low-pass) prototype filter
- usually via all-pass mapping filter

■ iterative approximation of the magnitude response

■ intuitive methods

- manually move zeros and poles in z-plane
- draw magnitude response in frequency domain

filters

filter design

■ impulse invariance: sample impulse response

- if continuous system is band-limited, frequency response will be approximately equal (below $f_s/2$)
- special case: no filter definition available → FIR coefficients

■ bi-linear transform

- map filter from (analogue) Laplace-plane to (digital) z-plane
- introduces frequency warping (increasing towards Nyquist frequency)

■ frequency transformation

- transform a (low-pass) prototype filter
- usually via all-pass mapping filter

■ iterative approximation of the magnitude response

■ intuitive methods

- manually move zeros and poles in z-plane
- draw magnitude response in frequency domain

filters

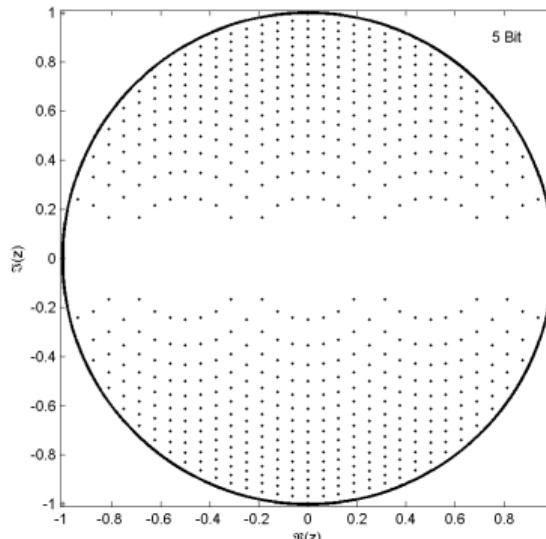
effects of word length

- quantization of filter coefficients can lead to problems
- effects depend on filter type and structure:
 - changes of transfer function
 - instability
 - quantization noise → SNR

filters

effects of word length

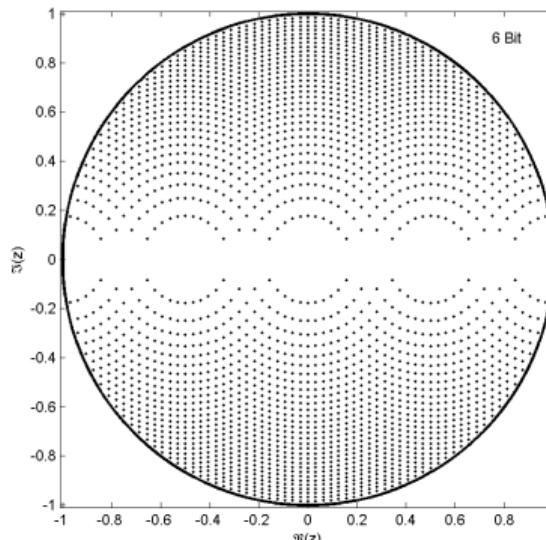
- quantization of filter coefficients can lead to problems
- effects depend on filter type and structure:
 - changes of transfer function
 - instability
 - quantization noise → SNR



filters

effects of word length

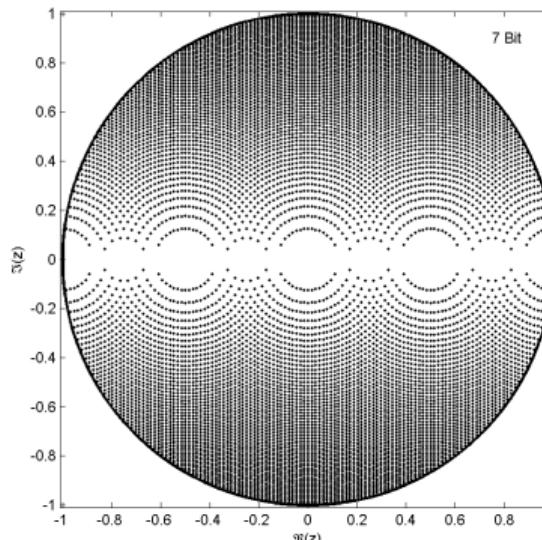
- quantization of filter coefficients can lead to problems
- effects depend on filter type and structure:
 - changes of transfer function
 - instability
 - quantization noise → SNR



filters

effects of word length

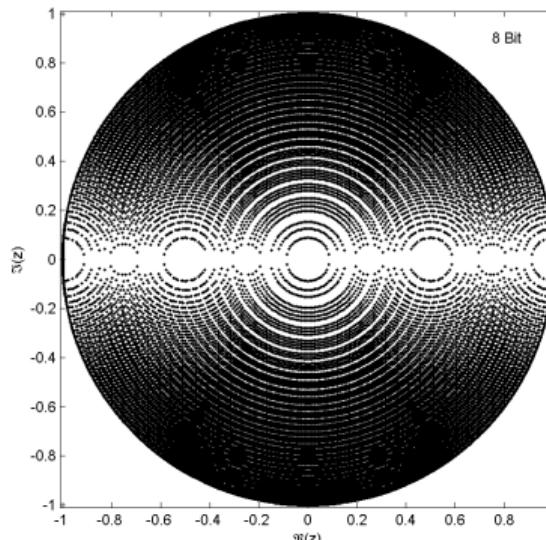
- quantization of filter coefficients can lead to problems
- effects depend on filter type and structure:
 - changes of transfer function
 - instability
 - quantization noise → SNR



filters

effects of word length

- quantization of filter coefficients can lead to problems
- effects depend on filter type and structure:
 - changes of transfer function
 - instability
 - quantization noise → SNR



filter summary

FIR & IIR

	FIR	IIR
IR length	finite	infinite
structure	non-recursive	recursive
phase linearity	possible	impossible
ratio steepness/workload	low	high
stability	guaranteed	possibly unstable

- every LTI system is **completely described** either by
 - its complex transfer function,
 - its impulse response, or
 - its pole and zero positions in the z-plane