Digital Signal Processing for Music

Part 16: z-transform

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z-transform
introduction

the z-transform is

- a generalization of DFT,
- widely used in DSP as analysis tool,
- a useful tool to characterize systems (also recursive systems!),
- a useful tool to check for system stability and causality,
- the discrete-time counterpart of the Laplace transform.

z-transform
definition

$$X(z) = \sum_{i=-\infty}^{\infty} x(i)z^{-i}, \quad z \in \mathfrak{C}$$

- $X(z)$: complex function of a complex number
- compare Fourier transform $X(\mathrm{j}\omega)$: complex function of real-valued ω

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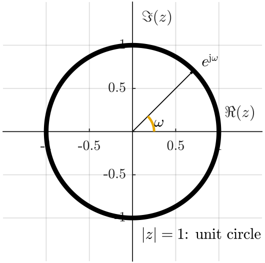
$$X(\mathrm{j}\omega) = \sum_{i=-\infty}^{\infty} x(i) e^{-\mathrm{j}\omega i} \Rightarrow X(\mathrm{j}\omega) = X(z) \text{ at } z = e^{\mathrm{j}\omega}$$

z-transform

zplane

- $X(z)$ defined on complex plane
- $X(\mathrm{j}\omega)$ defined on unit circle

- observation: $X(\mathrm{j}\omega)$ is periodic with 2π



matlab source: plotZplaneIntro.m

z-transform
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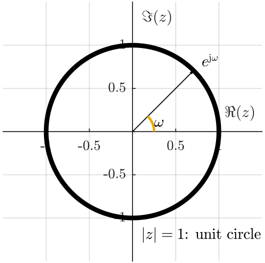

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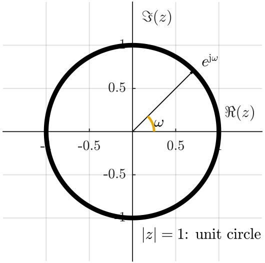

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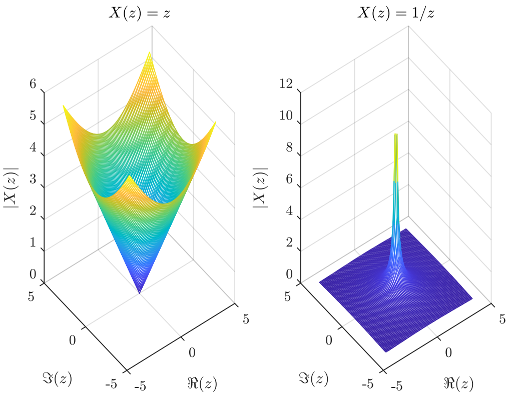
z-transform

trivial examples

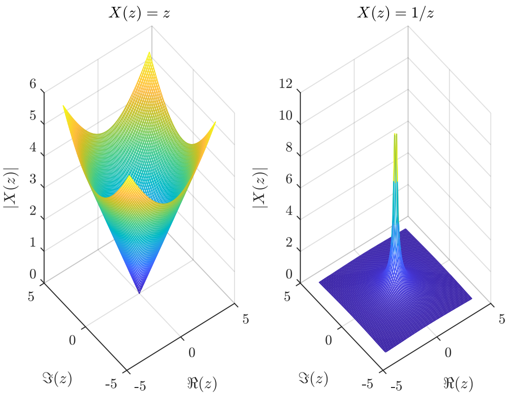


what is the magnitude for $X(z) = 1/(z - 0.5)$

matlab source: plotZplaneExample.m

z-transform

trivial examples



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matlab source: plotZplaneExample.m

z-transform

system description

Fourier transform and z-transform have largely similar properties, most importantly

- **linearity**

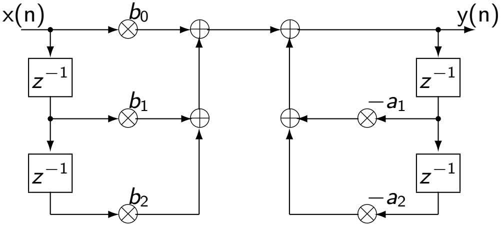
$$\begin{aligned} y(i) = c_1 x_1(i) + c_2 x_2(i) \quad &\Rightarrow \quad Y(\mathrm{j}\omega) = c_1 X_1(\mathrm{j}\omega) + c_2 X_2(\mathrm{j}\omega) \\ &\Rightarrow \quad Y(z) = c_1 X_1(z) + c_2 X_2(z) \end{aligned}$$

- **time shift**

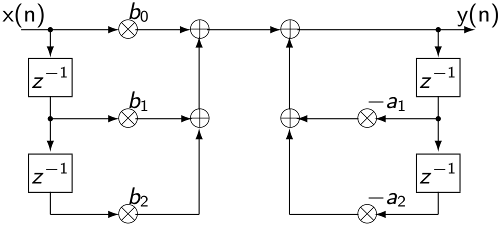
$$\begin{aligned} y(i) = x(i-n) \quad &\Rightarrow \quad Y(\mathrm{j}\omega) = e^{-\mathrm{j}\omega n} X(\mathrm{j}\omega) \\ &\Rightarrow \quad Y(z) = z^{-n} X(z) \end{aligned}$$

z-transform

biquad: difference equation



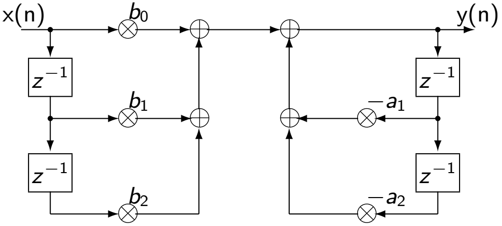
z-transform
biquad: difference equation



$$y(i) = \sum_{j=0}^{2} b_j x(i-j) - \sum_{k=1}^{2} a_j y(i-j)$$

z-transform

biquad: difference equation



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z-transform

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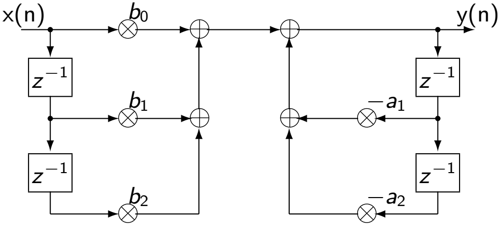

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$$Y(z)\left(1 + \sum^{2} a_j z^{-j}\right) = X(z) \sum^{2} b_j z^{-j}$$

biquad

transfer function

$$H(z) = \frac{Y(z)}{X(z)}$$

$$= \frac{\sum\limits_{j=0}^{2} b_j z^{-j}}{1 + \sum\limits_{j=1}^{2} a_j z^{-j}}$$

$$= \frac{b_0 + b_1 \cdot z^{-1} + b_2 \cdot z^{-2}}{1 + a_1 \cdot z^{-1} + a_2 \cdot z^{-2}}$$

$$= \frac{\text{numerator polynomial}}{\text{denominator polynomial}}$$

biquad

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biquad
poles and zeros

- numerator $\rightarrow 0$: zero
- denominator $\rightarrow 0$: pole

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$$\begin{aligned} 1 + a_1 \cdot z^{-1} + a_2 \cdot z^{-2} &= 0 \\ \Longrightarrow z_{\infty 1,2} &= \frac{a_1}{2} \pm \frac{1}{2}\sqrt{a_1^2 - 4a_2} \end{aligned}$$

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biquad
poles and zeros

- numerator $\rightarrow 0$: zero
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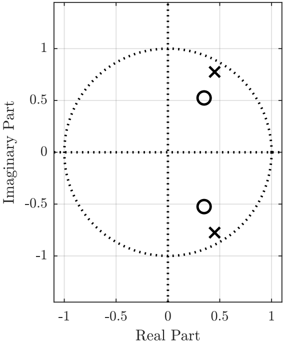
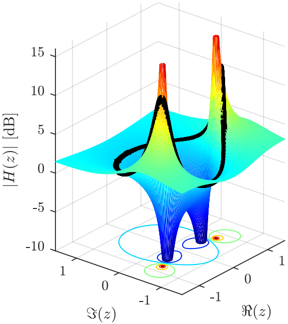
- zeros and poles are a simplified way of visualizing filter properties in the zplane

biquad

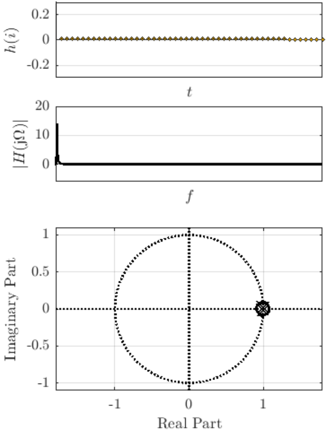
zplane example

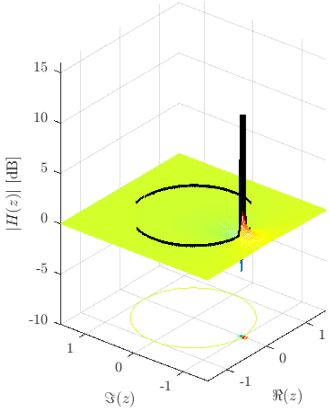




matlab source: plotBiquadZplane.m

biquad
animation





matlab source: matlab/animateBiquadZplane.m

filters
z-plane characteristics

- **stability**:
  poles within unit circle
- **zero points and poles**
  are either real or complex conjugate
- **minimal phase systems**:
  no zero points outside of unit circle
- **all pass system**:
  poles and zeros symmetric wrt unit circle
- **linear phase**:
  zero points within and outside unit circle symmetric wrt unit circle

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- **impulse invariance**: sample impulse response
  - if continuous system is band-limited, frequency response will be approximately equal (below $f_S/2$)
  - special case: no filter definition available → FIR coefficients
- **bi-linear transform**
  - map filter from (analogue) Laplace-plane to (digital) z-plane
  - introduces frequency warping (increasing towards Nyquist frequency)
- **frequency transformation**
  - transform a (low-pass) prototype filter
  - usually via all-pass mapping filter
- **iterative approximation** of the magnitude response
- **intuitive methods**
  - manually move zeros and poles in z-plane
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$$s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}$$
$$z = \frac{1+s T_\mathrm{S}/2}{1-s T_\mathrm{S}/2}$$

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- quantization of filter coefficients can lead to problems
- effects depend on filter type and structure:
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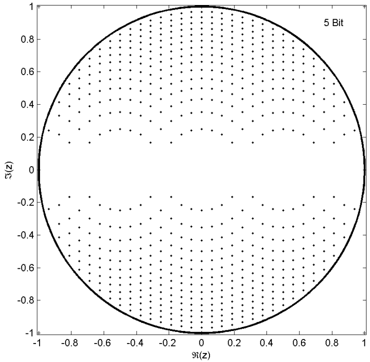

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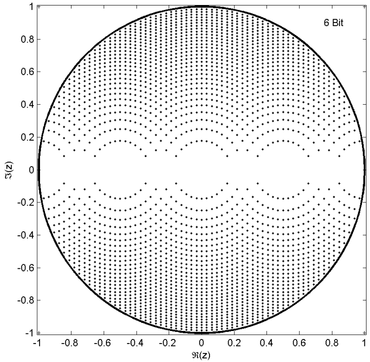

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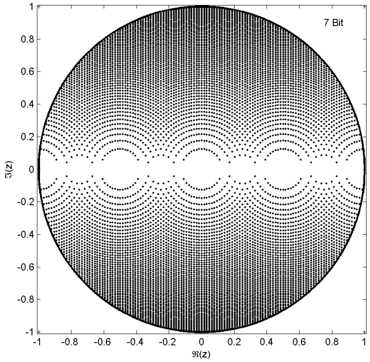

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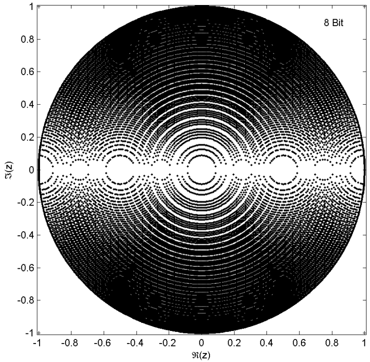

filter summary
FIR & IIR

	FIR	**IIR**
IR length	finite	infinite
structure	non-recursive	recursive
phase linearity	possible	impossible
ratio steepness/workload	low	high
stability	guaranteed	possibly unstable

- every LTI system is **completely described** either by
  - its complex transfer function,
  - its impulse response, or
  - its pole and zero positions in the z-plane