Digital Signal Processing for Music

Part 7: Fourier Series

alexander lerch

Georgia Tech Center for Music Technology

Fourier analysis
overview

1. **Fourier series**:
   periodic signals as sum of sinusoidals
2. **Fourier transform** (next slide deck):
   frequency content of any signal
   - Fourier series to transform
   - properties
   - windowed Fourier transform

Fourier series
introduction

- periodic signals are **superposition of sinusoidals**
- **properties**
  - amplitude a_k
  - frequency as integer multiple k of fundamental f_0
  - phase Φ_k

$$x(t) = \sum_{k=0}^{\infty} a_k \sin(k\omega_0 t + \Phi_k)$$

- **observations**
  - time domain is continuous (t)
  - frequency domain is discrete ($\sum$)

Fourier series
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Fourier series
complex representation 1/2

$$x(t) = \sum_{k=0}^{\infty} a_k \sin(k\omega_0 t + \Phi_k)$$

- trigonometric identity $\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$

$\Rightarrow$

Fourier series

complex representation 1/2

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$$x(t) = \sum_{k=0}^{\infty} a_k \sin(\Phi_k)\cdot\cos(k\omega_0 t) + a_k\cos(\Phi_k)\cdot\sin(k\omega_0 t)$$

Fourier series

complex representation 1/2

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Fourier series
complex representation 2/2

$$
\begin{aligned}
e^{\mathrm{j}\omega t} &= \cos(\omega t) + \mathrm{j}\sin(\omega t) \\
\mathrm{j} &= \sqrt{-1}
\end{aligned}
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phasor representation in complex plane

Fourier series
complex representation 2/2

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phasor representation in complex plane

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\begin{aligned}
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\sin(\omega t) &= ?
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Fourier series

complex representation 2/2

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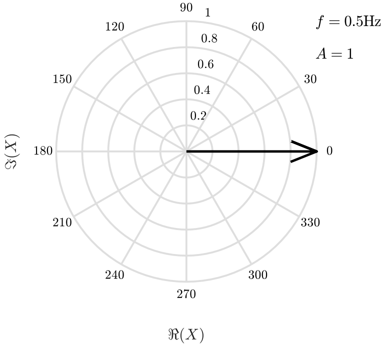
phasor representation in complex plane

$$
\begin{aligned}
\cos(\omega t) &= \frac{1}{2}\left(e^{\mathrm{j}\omega t} + e^{-\mathrm{j}\omega t}\right) \\
\sin(\omega t) &= \frac{1}{2\mathrm{j}}\left(e^{\mathrm{j}\omega t} - e^{-\mathrm{j}\omega t}\right)
\end{aligned}
$$

Fourier series

phasor representation



- phasor representation:
  1. sine value is defined by magnitude and phase
  2. decreasing the amplitude $\Rightarrow$ shorter vector
  3. increasing the frequency $\Rightarrow$ increasing speed

matlab source: matlab/animatePhasor.m

fundamentals

conjugate complex multiplication

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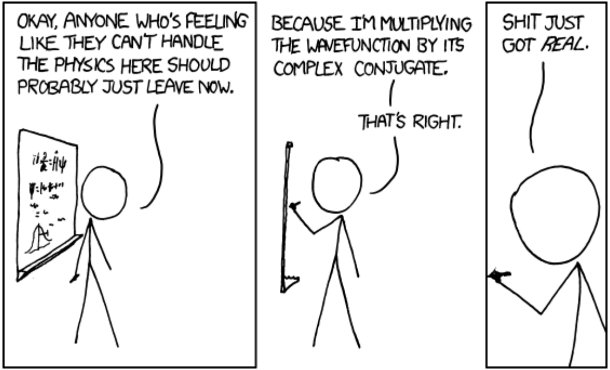

https://xkcd.com/849

Fourier series
real to complex

$$\cos(\omega t) = \frac{1}{2}\left(e^{\mathrm{j}\omega t} + e^{-\mathrm{j}\omega t}\right) \qquad \sin(\omega t) = \frac{1}{2\mathrm{j}}\left(e^{\mathrm{j}\omega t} - e^{-\mathrm{j}\omega t}\right)$$

$$\begin{aligned}
x(t) &= \sum_{k=0}^{\infty} A_k \cos(k\omega t) + B_k \sin(k\omega t) \\
&= \sum_{k=0}^{\infty} \frac{A_k}{2}\left(e^{\mathrm{j}\omega kt} + e^{-\mathrm{j}\omega kt}\right) - \mathrm{j}\frac{B_k}{2}\left(e^{\mathrm{j}\omega kt} - e^{-\mathrm{j}\omega kt}\right) \\
&= \sum_{k=0}^{\infty} \frac{1}{2}\left(A_k - \mathrm{j}B_k\right) e^{\mathrm{j}\omega kt} + \frac{1}{2}\left(A_k + \mathrm{j}B_k\right) e^{-\mathrm{j}\omega kt} \\
&= \sum_{k=0}^{\infty} \underbrace{\frac{1}{2}\left(A_k - \mathrm{j}B_k\right)}_{c_k} e^{\mathrm{j}\omega kt} + \frac{1}{2}\left(A_k + \mathrm{j}B_k\right) e^{-\mathrm{j}\omega kt}
\end{aligned}$$

$$\text{with } c_{-k} := c_k^* \Rightarrow \quad x(t) = \sum_{k=-\infty}^{\infty} c_k e^{\mathrm{j}\omega_0 kt}$$

Fourier series
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Fourier series

coefficients

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$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{\mathrm{j}\omega_0 kt}$$

1. multiply both sides with $e^{-\mathrm{j}\omega_0 nt}$: $x(t) \cdot e^{-\mathrm{j}\omega_0 nt} = \sum\limits_{k=-\infty}^{\infty} c_k e^{\mathrm{j}\omega_0 (k-n)t}$
2. integrate both sides: $\int\limits_0^{T_0} x(t) \cdot e^{-\mathrm{j}\omega_0 nt} dt = \int\limits_0^{T_0} \sum\limits_{k=-\infty}^{\infty} c_k e^{\mathrm{j}\omega_0 (k-n)t} dt$
3. flip sum and integral: $\int\limits_0^{T_0} x(t) \cdot e^{-\mathrm{j}\omega_0 nt} dt = \sum\limits_{k=-\infty}^{\infty} c_k \int\limits_0^{T_0} e^{\mathrm{j}\omega_0 (k-n)t} dt$

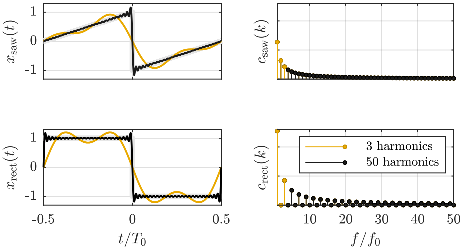
$$\int_0^{T_0} e^{\mathrm{j}\omega_0 (k-n)t} dt \quad = 0 \quad k \neq n \qquad\qquad \int_0^{T_0} e^{\mathrm{j}\omega_0 (k-n)t} dt \quad = T_0 \quad k = n$$

$$\Rightarrow \qquad \int_0^{T_0} x(t) \cdot e^{-\mathrm{j}\omega_0 nt} dt = c_n T_0$$

Fourier series
limited number of coefficients

reconstruction of periodic signals with a limited number of sinusoidals:

$$\hat{x}(t) = \sum_{k=-\mathcal{K}}^{\mathcal{K}} c_k e^{j\omega_0 kt}$$



matlab source: plotFourierSeries.m

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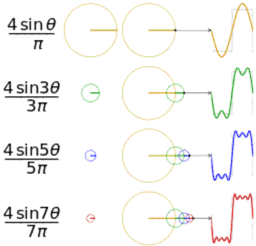

commons.wikimedia.org/wiki/File:Fourier_series_square_wave_circles_animation.gif

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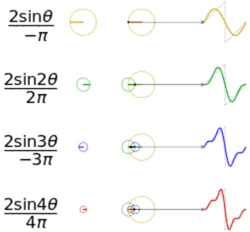

commons.wikimedia.org/wiki/File:Fourier_series_sawtooth_wave_circles_animation.gif

Fourier series
summary 1/2

- any periodic signal $\Rightarrow$ representation in **Fourier Series**

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{\mathrm{j}\omega_0 kt}$$

  - $\omega_0 = 2\pi \cdot f_0$
  - $e^{\mathrm{j}\omega_0 kt} = \cos(\omega_0 kt) + \mathrm{j}\sin(\omega_0 kt)$
- Fourier series coefficients c_k

$$c_k = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)e^{-\mathrm{j}\omega_0 kt}\,dt$$

Fourier series

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Fourier series

summary 2/2

- complex coefficients are just a tool to represent the addition of sines neatly
- to derive the coefficients from a signal we need
  - fundamental frequency
  - functional description
- "frequency domain" of Fourier series is discrete (integer multiples)
- "time domain" can be continuous or discrete (discrete may be a pain to integrate, though)

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