Digital Signal Processing for Music

Part 16: z-transform

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z-transform
introduction

the z-transform is

- a generalization of DFT,
- widely used in DSP as analysis tool,
- a useful tool to characterize systems (also recursive systems!),
- a useful tool to check for system stability and causality,
- the discrete-time counterpart of the Laplace transform.

z-transform
definition

$$X(z) = \sum_{i=-\infty}^{\infty} x(i) z^{-i}, \quad z \in \mathfrak{C}$$

- $X(z)$: complex function of a complex number
- compare Fourier transform $X(\mathrm{j}\omega)$: complex function of real-valued ω

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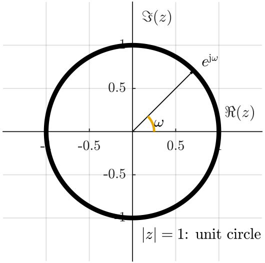
$$X(\mathrm{j}\omega) = \sum_{i=-\infty}^{\infty} x(i) e^{-\mathrm{j}\omega i} \Rightarrow X(\mathrm{j}\omega) = X(z) \text{ at } z = e^{\mathrm{j}\omega}$$

z-transform

zplane

- $X(z)$ defined on complex plane
- $X(\mathrm{j}\omega)$ defined on unit circle

- observation: $X(\mathrm{j}\omega)$ is periodic with 2π



matlab source: plotZplaneIntro.m

z-transform
zplane

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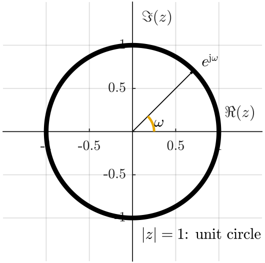

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z-transform

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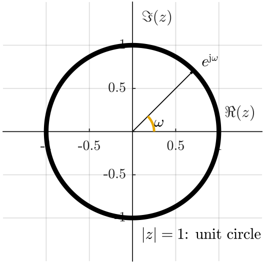

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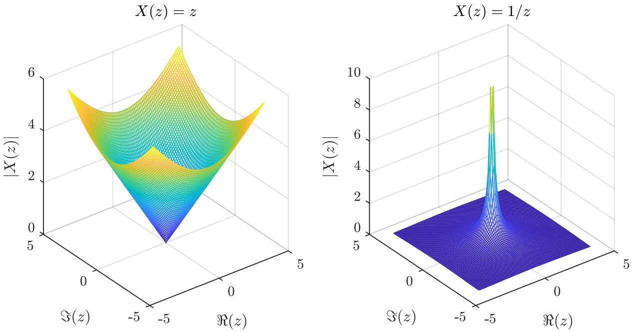
z-transform

trivial example

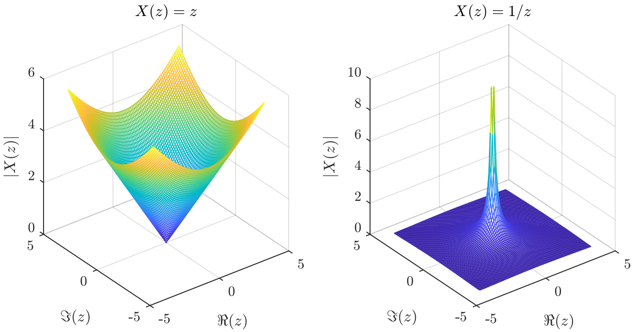


what is the magnitude for $X(z) = 1/(z - 0.5)$

matlab source: plotZplaneExample.m

z-transform

trivial example



what is the magnitude for $X(z) = 1/(z - 0.5)$

same as $1/z$ but shifted on the x-axis.

matlab source: plotZplaneExample.m

z-transform

system description

Fourier transform and z-transform have largely similar properties, most importantly

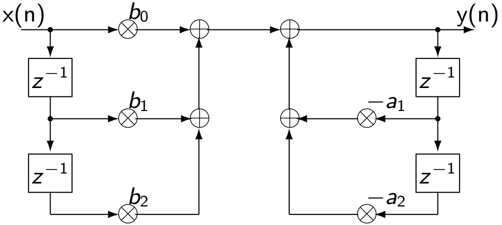
- **linearity**

$$y(i) = c_1 x_1(i) + c_2 x_2(i) \quad \Rightarrow \quad Y(\mathrm{j}\omega) = c_1 X_1(\mathrm{j}\omega) + c_2 X_2(\mathrm{j}\omega)$$
$$\Rightarrow \quad Y(z) = c_1 X_1(z) + c_2 X_2(z)$$

- **time shift**

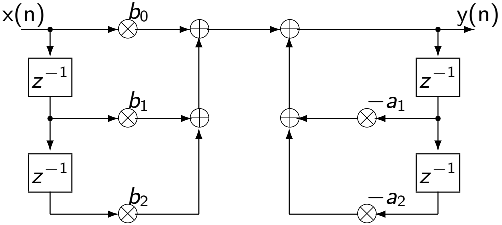
$$y(i) = x(i-n) \quad \Rightarrow \quad Y(\mathrm{j}\omega) = e^{-\mathrm{j}\omega n} X(\mathrm{j}\omega)$$
$$\Rightarrow \quad Y(z) = z^{-n} X(z)$$

z-transform
biquad: difference equation



z-transform

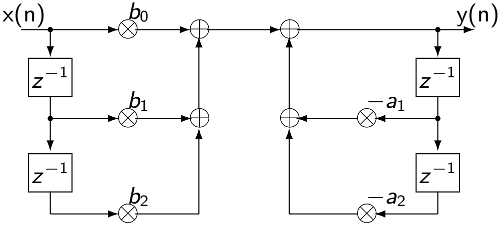
biquad: difference equation



$$y(i) = \sum_{j=0}^{2} b_j x(i-j) - \sum_{k=1}^{2} a_j y(i-j)$$

z-transform

biquad: difference equation



$$y(i) = \sum_{j=0}^{2} b_j x(i-j) - \sum_{k=1}^{2} a_j y(i-j)$$

$$Y(z) = \sum_{j=0}^{2} b_j X(z) z^{-j} - \sum_{k=1}^{2} a_j Y(z) z^{-j}$$

z-transform

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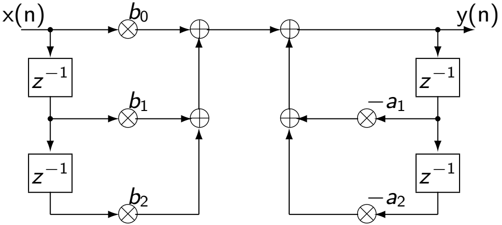

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$$Y(z) = \sum_{j=0}^{2} b_j X(z) z^{-j} - \sum_{k=1}^{2} a_j Y(z) z^{-j}$$

$$Y(z)\left(1 + \sum^{2} a_j z^{-j}\right) = X(z) \sum^{2} b_j z^{-j}$$

biquad

transfer function

$$
\begin{aligned}
H(z) &= \frac{Y(z)}{X(z)} \\
&= \frac{\sum\limits_{j=0}^{2} b_j z^{-j}}{1 + \sum\limits_{j=1}^{2} a_j z^{-j}} \\
&= \frac{b_0 + b_1 \cdot z^{-1} + b_2 \cdot z^{-2}}{1 + a_1 \cdot z^{-1} + a_2 \cdot z^{-2}} \\
&= \frac{\text{numerator polynomial}}{\text{denominator polynomial}}
\end{aligned}
$$

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biquad

poles and zeros

- numerator $\rightarrow 0$: zero
- denominator $\rightarrow 0$: pole

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$$
\begin{aligned}
1 + a_1 \cdot z^{-1} + a_2 \cdot z^{-2} &= 0 \\
\Longrightarrow z_{\infty 1,2} &= \frac{a_1}{2} \pm \frac{1}{2}\sqrt{a_1^2 - 4a_2}
\end{aligned}
$$

biquad

poles and zeros

- numerator $\rightarrow$ 0: zero
- denominator $\rightarrow$ 0: pole

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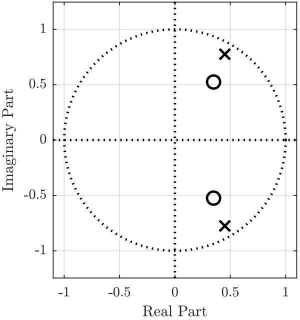
biquad
poles and zeros

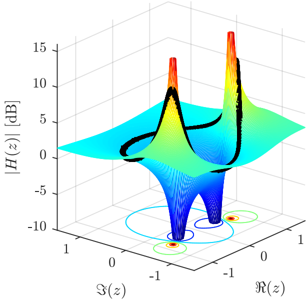
- numerator $\rightarrow 0$: zero
- denominator $\rightarrow 0$: pole

- zeros and poles are a simplified way of visualizing filter properties in the zplane

biquad

zplane example

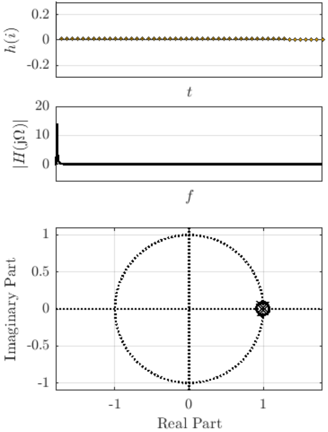


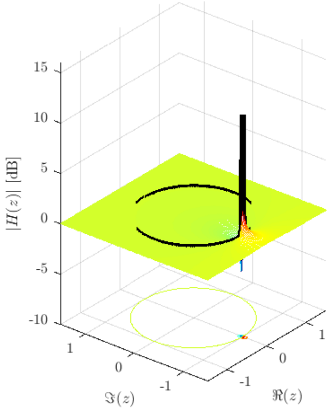


matlab source: plotBiquadZplane.m

biquad

animation

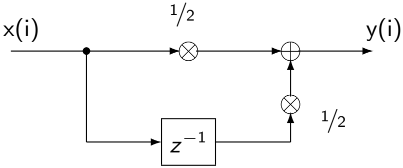




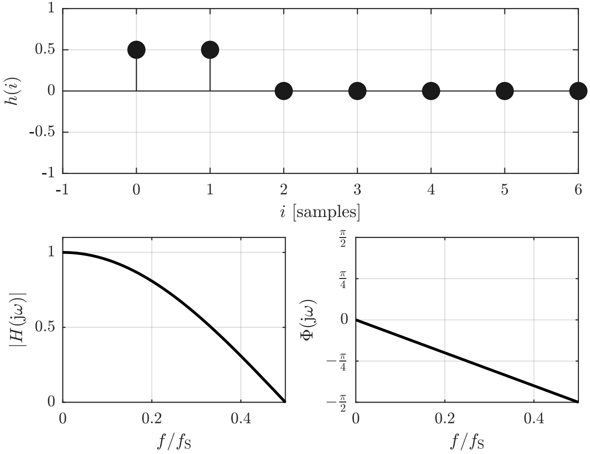
matlab source: matlab/animateBiquadZplane.m

filters

example 1 revisited



$$y(i) = 0.5 \cdot x(i) + 0.5 \cdot x(i-1)$$



matlab source: plotSimpleFilter.m

filters

example 1 revisited: zplane

$$
\begin{aligned}
H(z) &= 0.5 + 0.5 \cdot z^{-1} \\
&\Rightarrow \\
H(-1) &= 0 \\
H(0) &= \infty
\end{aligned}
$$

filters

example 1 revisited: zplane

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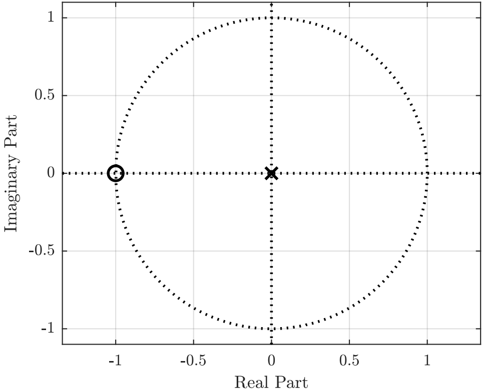
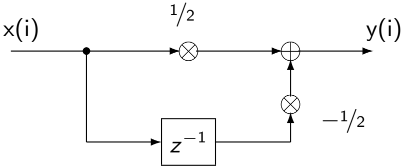


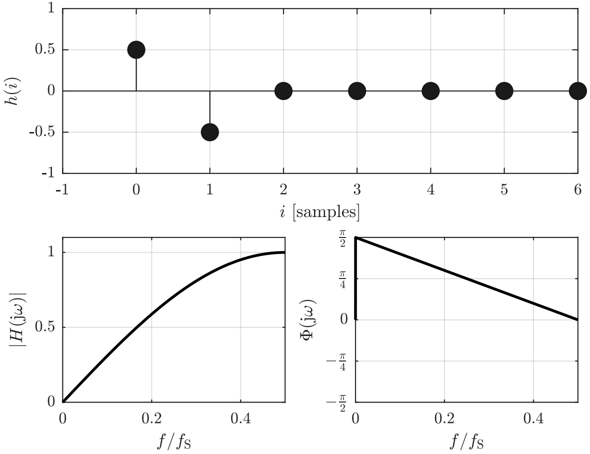
matlab source: plotSimpleFilterZplane.m

filters

example 2 revisited



$$y(i) = 0.5 \cdot x(i) - 0.5 \cdot x(i-1)$$



matlab source: plotSimpleFilter.m

filters

example 2 revisited: zplane

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filters

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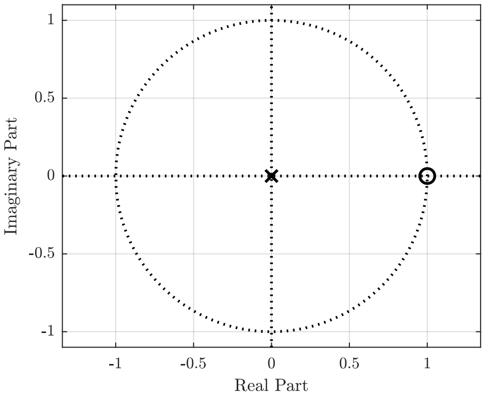
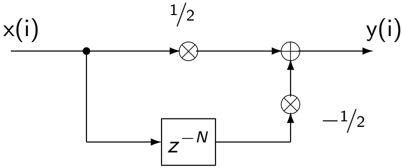


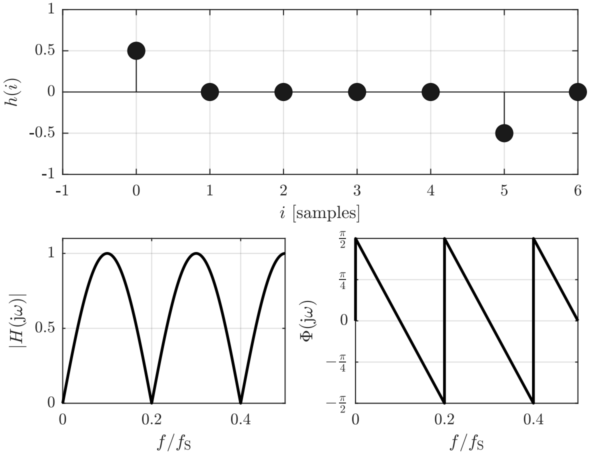
matlab source: plotSimpleFilterZplane.m

filters

example 3 revisited



$$y(i) = 0.5 \cdot x(i) - 0.5 \cdot x(i - N)$$



matlab source: plotSimpleFilter.m

filters

example 3 revisited: zplane

$$H(z) = 0.5 - 0.5 \cdot z^{-N}$$

$$\Rightarrow$$

$$H(0, \omega = \frac{\pi}{5}, \omega = \frac{2\pi}{5}) = 0$$

$$H(0) = \infty$$

filters

example 3 revisited: zplane

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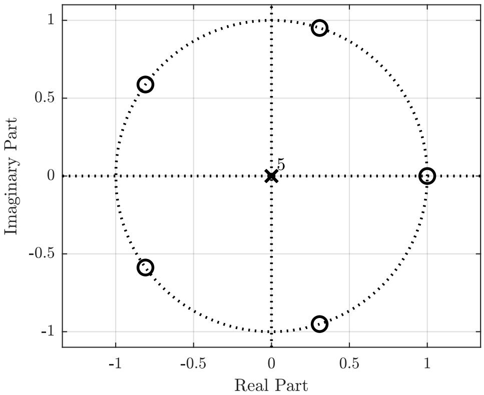
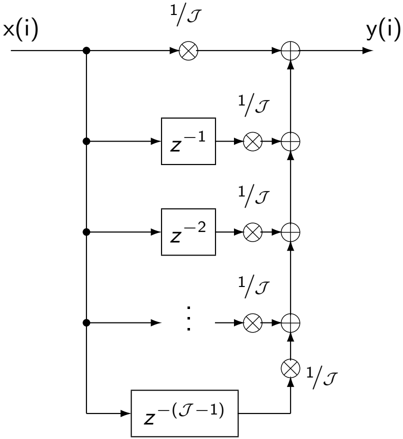
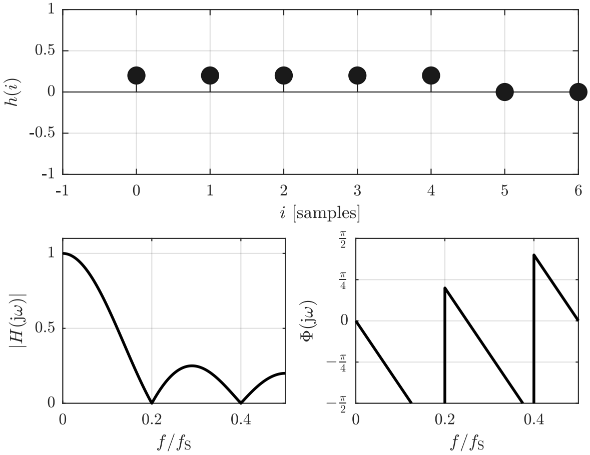


matlab source: plotSimpleFilterZplane.m

filters

example 4 revisited





matlab source: plotSimpleFilter.m

filters

example 4 revisited: zplane

$$H(z) = \frac{1}{\mathcal{J}}\sum_{j=0}^{\mathcal{J}-1} z^{-j}$$

$$\Rightarrow$$

$$H(z) = 0 \text{ for } \omega = \frac{\pi}{5}, \omega = \frac{2\pi}{5}, |z| = 1$$

$$H(0) = \infty$$

filters

example 4 revisited: zplane

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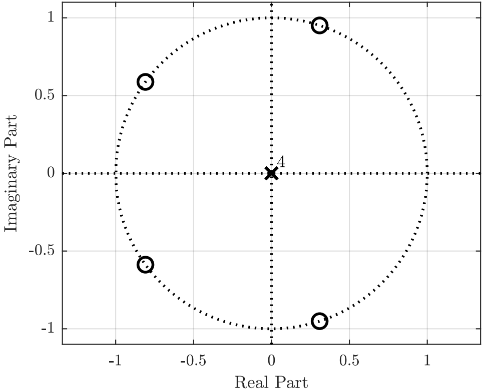
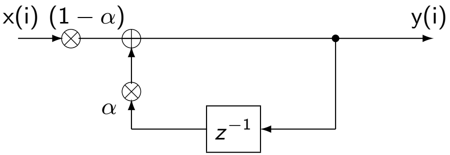


matlab source: plotSimpleFilterZplane.m

filters

example 5 revisited



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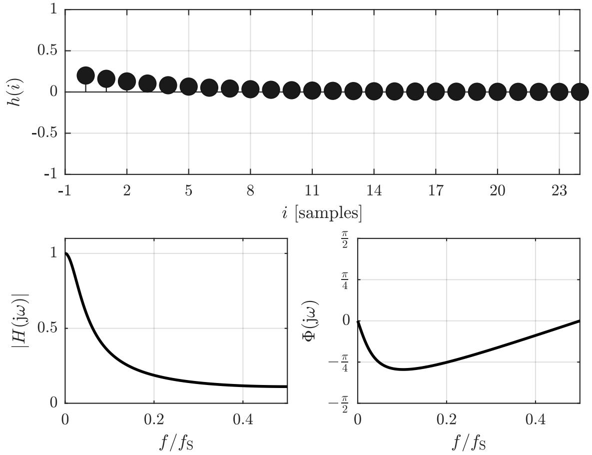

matlab source: plotSimpleFilter.m

filters

example 5 revisited: zplane

$$H(z) = \frac{1-\alpha}{1-\alpha x^{-1}}$$

$$\Rightarrow$$

$$H(0) = 0$$

$$H(\alpha) = \infty$$

filters

example 5 revisited: zplane

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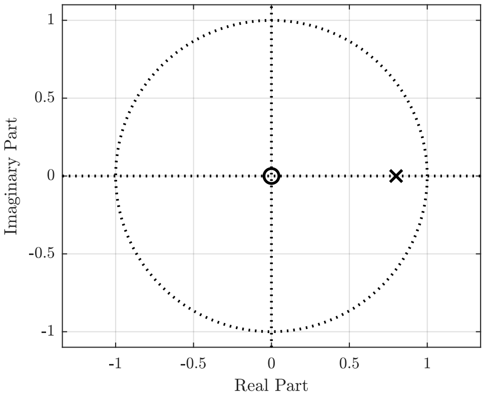
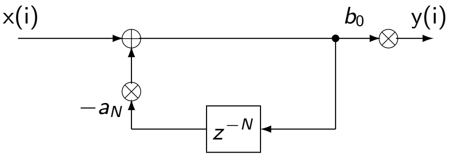


matlab source: plotSimpleFilterZplane.m

filters

example 6 revisited



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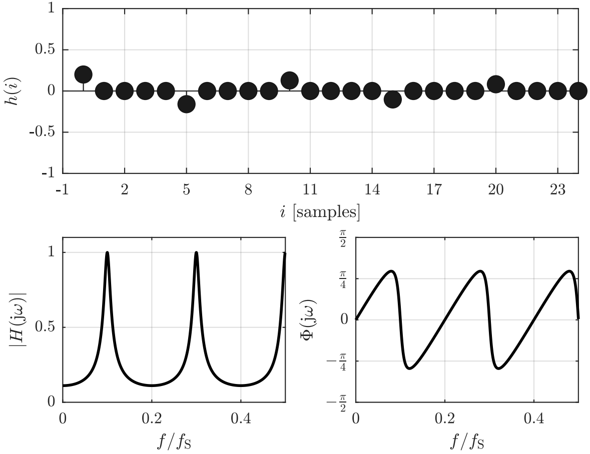

matlab source: plotSimpleFilter.m

filters

example 6 revisited: zplane

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H(z) &= \frac{b_0}{1 - a_N x^{-N}} \\
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\end{aligned}
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filters

example 6 revisited: zplane

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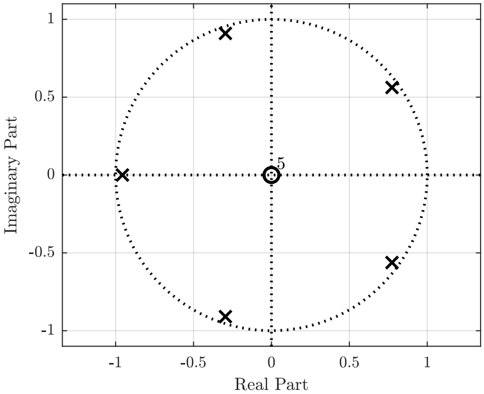

matlab source: plotSimpleFilterZplane.m

filters

z-plane characteristics

- **stability**:
  poles within unit circle
- **zero points and poles**
  are either real or complex conjugate
- **minimal phase systems**:
  no zero points outside of unit circle
- **all pass system**:
  poles and zeros symmetric wrt unit circle
- **linear phase**:
  zero points within and outside unit circle symmetric wrt unit circle

filters

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- **impulse invariance**: sample impulse response
  - if continuous system is band-limited, frequency response will be approximately equal (below $f_S/2$)
  - special case: no filter definition available $\rightarrow$ FIR coefficients
- **bi-linear transform**
  - map filter from (analogue) Laplace-plane to (digital) z-plane
  - introduces frequency warping (increasing towards Nyquist frequency)
- **frequency transformation**
  - transform a (low-pass) prototype filter
  - usually via all-pass mapping filter
- **iterative approximation** of the magnitude response
- **intuitive methods**
  - manually move zeros and poles in z-plane
  - draw magnitude response in frequency domain

filters

filter design

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$$s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}$$

$$z = \frac{1+sT_S/2}{1-sT_S/2}$$

  - introduces frequency warping (increasing towards Nyquist frequency)
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filters

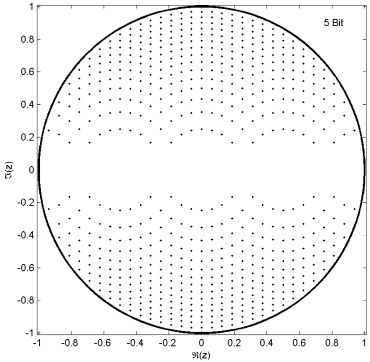
effects of word length

- quantization of filter coefficients can lead to problems
- effects depend on filter type and structure:
  - changes of transfer function
  - instability
  - quantization noise $\rightarrow$ SNR

filters

effects of word length

- quantization of filter coefficients can lead to problems
- effects depend on filter type and structure:
  - changes of transfer function
  - instability
  - quantization noise $\rightarrow$ SNR



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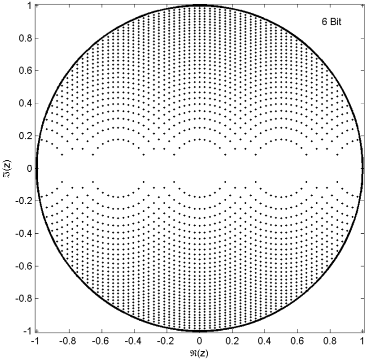

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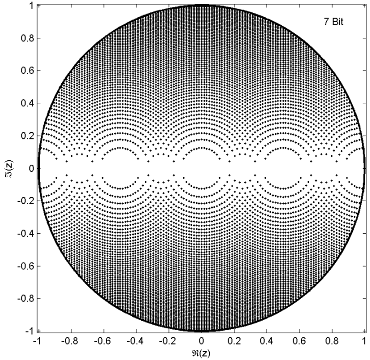

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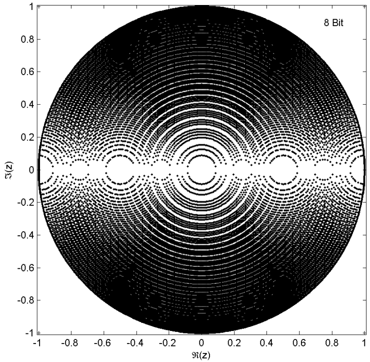

filter summary

FIR & IIR

	FIR	**IIR**
IR length	finite	infinite
structure	non-recursive	recursive
phase linearity	possible	impossible
ratio steepness/workload	low	high
stability	guaranteed	possibly unstable

- every LTI system is **completely described** either by
  - its complex transfer function,
  - its impulse response, or
  - its pole and zero positions in the z-plane