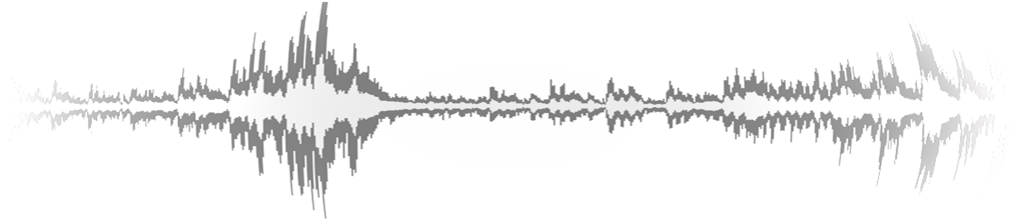
Digital Signal Processing for Music

Part 8: Fourier Transform

alexander lerch



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Fourier transform

overview

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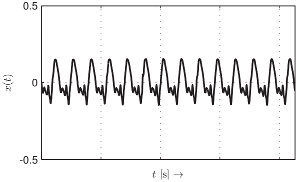
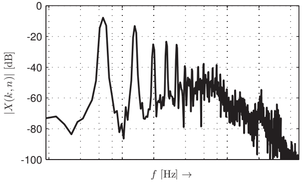
1. Fourier series to Fourier transform
2. properties of the Fourier transform
3. windowed Fourier transform (STFT)
4. transform of sampled time signals
5. Discrete Fourier Transform (DFT)

Fourier transform

introduction





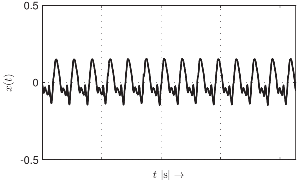
Fourier series is cool, but:

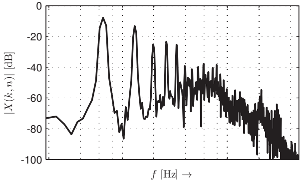
- works only for periodic signals
- difficult to use for real-world analysis as it requires knowledge of fundamental frequency

⇒ Fourier transform

Fourier transform

introduction





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- works only for periodic signals
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⇒ **Fourier transform**

Fourier transform

Fourier series revisited

$$c_k = \frac{1}{T_0}\int\limits_{-T_0/2}^{T_0/2} x(t)e^{-\mathrm{j}\omega_0 kt}\, dt$$

- Fourier series coefficients can be interpreted as **correlation coefficient** between signal and sinusoidals of different frequencies
- only frequencies $k\omega_0$ are used (ω_0 *has to be known*)

$\Rightarrow$ Fourier series produces a **'line spectrum'**

- distance between frequency components decreases as T_0 increases

$\Rightarrow$ aperiodic functions could be analyzed by increasing $T_0 \rightarrow \infty$

Fourier transform

Fourier series revisited

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Fourier transform
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Fourier transform

Fourier series revisited

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$$\begin{aligned} & T_0 \rightarrow \infty \\ \Rightarrow \quad & k\omega_0 \rightarrow \omega \\ \Rightarrow \quad & \frac{1}{T_0} \rightarrow 0 \end{aligned}$$

to avoid Zero result, multiply with T_0

Fourier transform

definition (continuous)

$$X(\mathrm{j}\omega) = \mathfrak{F}[x(t)] = \int\limits_{-\infty}^{\infty} x(t)e^{-\mathrm{j}\omega t}\,dt$$

Fourier transform

example 1: rect window

$$w_{\mathrm{R}}(t) = \begin{cases} 1, & -\frac{1}{2} \leq t \leq \frac{1}{2} \\ 0, & \text{otherwise} \end{cases}.$$

$$\begin{aligned}
W_{\mathrm{R}}(\mathrm{j}\omega) &= \int_{-\infty}^{\infty} w_{\mathrm{R}}(t) e^{-\mathrm{j}\omega t}\, dt \\
&= \int_{-1/2}^{1/2} e^{-\mathrm{j}\omega t}\, dt \\
&= \frac{1}{\mathrm{j}\omega} \underbrace{\left(e^{-\mathrm{j}\omega/2} - e^{\mathrm{j}\omega/2}\right)}_{=-2j\sin(\omega/2)} \\
&= \frac{-\sin\left(\omega/2\right)}{\omega/2} = -\operatorname{sinc}\left(\frac{\omega}{2}\right).
\end{aligned}$$

How will this change for different widths of w_R?

Fourier transform

example 1: rect window

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Fourier transform

example 2: dirac

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1,$$

$$\delta(t) = 0 \text{ for all } t \neq 0.$$

$$\Rightarrow \Delta(\mathrm{j}\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-\mathrm{j}\omega t} dt = e^{-\mathrm{j}\omega \cdot 0} = 1$$

shifted dirac: $\delta(t - \tau_0)$

$$\Rightarrow \Delta(\mathrm{j}\omega) = \int_{-\infty}^{\infty} \delta(t - \tau_0) e^{-\mathrm{j}\omega t} dt = e^{-\mathrm{j}\omega \tau_0}$$

Fourier transform
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Fourier transform

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Fourier transform

property 1: invertibility

$$
\begin{aligned}
x(t) &= \mathfrak{F}^{-1}[X(\mathrm{j}\omega)] \\
&= \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} X(\mathrm{j}\omega)e^{\mathrm{j}\omega t}\,d\omega
\end{aligned}
$$

reminder: signal reconstruction with Fourier series coefficients

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{\mathrm{j}\omega_0 kt}$$

- **comments**:
  - invertibility: no information is lost during this process!
  - FT and IFT are very similar, largely equivalent

Fourier transform

property 1: invertibility

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- **comments**:
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Fourier transform

property 2: superposition

$$\begin{aligned} y(t) &= c_1 \cdot x_1(t) + c_2 \cdot x_2(t) \\ \mapsto & \\ Y(\mathrm{j}\omega) &= c_1 \cdot X_1(\mathrm{j}\omega) + c_2 \cdot X_2(\mathrm{j}\omega) \end{aligned}$$

derivation

$$\begin{aligned} Y(\mathrm{j}\omega) &= \int\limits_{-\infty}^{\infty} (c_1 \cdot x_1(t) + c_2 \cdot x_2(t)) \cdot e^{-\mathrm{j}\omega t}\, dt \\ &= c_1 \cdot \int\limits_{-\infty}^{\infty} x_1(t) e^{-\mathrm{j}\omega t}\, dt + c_2 \cdot \int\limits_{-\infty}^{\infty} x_2(t) e^{-\mathrm{j}\omega t}\, dt \\ &= c_1 \cdot X_1(\mathrm{j}\omega) + c_2 \cdot X_2(\mathrm{j}\omega) \end{aligned}$$

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Fourier transform

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Fourier transform

property 3: convolution and multiplication

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$$y(t) = \int_{-\infty}^{\infty} h(\tau)\cdot x(t-\tau)\,d\tau \mapsto Y(\mathrm{j}\omega) = H(\mathrm{j}\omega)\cdot X(\mathrm{j}\omega)$$

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Fourier transform

property 3: convolution and multiplication

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Fourier transform

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Fourier transform

property 4: Parseval's theorem

$$\int_{-\infty}^{\infty} x^2(t)\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|X(\mathrm{j}\omega)\right|^2\, d\omega$$

derivation

$$\int_{-\infty}^{\infty} h(\tau)\cdot x(t-\tau)\, d\tau = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\mathrm{j}\omega)\cdot X(\mathrm{j}\omega)e^{\mathrm{j}\omega t} d\omega$$

$$H(\mathrm{j}\omega) \longrightarrow X^*(\mathrm{j}\omega) / h(\tau) \longrightarrow x(-\tau),\ t=0$$

$$\begin{aligned}\int_{-\infty}^{\infty} x(-\tau)\cdot x(-\tau)\, d\tau &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(\mathrm{j}\omega)\cdot X(\mathrm{j}\omega)\, d\omega \\ \int_{-\infty}^{\infty} x^2(t)\, dt &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|X(\mathrm{j}\omega)\right|^2\, d\omega\end{aligned}$$

Fourier transform

property 4: Parseval's theorem

$$\int_{-\infty}^{\infty} x^2(t)\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\mathrm{j}\omega)|^2\ d\omega$$

derivation

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Fourier transform

property 4: Parseval's theorem

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$$\int_{-\infty}^{\infty} h(\tau)\cdot x(t-\tau)\,d\tau = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\mathrm{j}\omega)\cdot X(\mathrm{j}\omega)e^{\mathrm{j}\omega t}d\omega$$

$$H(\mathrm{j}\omega) \longrightarrow X^*(\mathrm{j}\omega)/h(\tau) \longrightarrow x(-\tau),\ t=0$$

$$\begin{aligned}\int_{-\infty}^{\infty} x(-\tau)\cdot x(-\tau)\,d\tau &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(\mathrm{j}\omega)\cdot X(\mathrm{j}\omega)\,d\omega\\ \int_{-\infty}^{\infty} x^2(t)\,dt &= \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\mathrm{j}\omega)|^2\ d\omega\end{aligned}$$

Fourier transform

property 5: time & frequency shift

$$y(t) = x(t - t_0) \rightarrow Y(\mathrm{j}\omega) = X(\mathrm{j}\omega)e^{-\mathrm{j}\omega t_0}$$

derivation

$$\begin{aligned}\int_{-\infty}^{\infty} x(t - t_0)e^{-\mathrm{j}\omega t}\, dt &= \int_{-\infty}^{\infty} x(\tau)e^{-\mathrm{j}\omega(\tau + t_0)}\, d\tau \\ &= e^{-\mathrm{j}\omega t_0}\int_{-\infty}^{\infty} x(\tau)e^{-\mathrm{j}\omega\tau}\, d\tau \\ &= e^{-\mathrm{j}\omega t_0}\cdot X(\mathrm{j}\omega)\end{aligned}$$

frequency shift:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\mathrm{j}(\omega - \omega_0))e^{\mathrm{j}\omega t}\, d\omega = e^{\mathrm{j}\omega_0 t}\cdot x(t)$$

Fourier transform

property 5: time & frequency shift

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$$y(t) = x(t - t_0) \rightarrow Y(\mathrm{j}\omega) = X(\mathrm{j}\omega)e^{-\mathrm{j}\omega t_0}$$

derivation

$$\int\limits_{-\infty}^{\infty} x(t - t_0)e^{-\mathrm{j}\omega t}\, dt = \int\limits_{-\infty}^{\infty} x(\tau)e^{-\mathrm{j}\omega(\tau + t_0)}\, d\tau$$

$$= e^{-\mathrm{j}\omega t_0} \int\limits_{-\infty}^{\infty} x(\tau)e^{-\mathrm{j}\omega\tau}\, d\tau$$

$$= e^{-\mathrm{j}\omega t_0} \cdot X(\mathrm{j}\omega)$$

frequency shift:

$$\frac{1}{2\pi}\int\limits_{-\infty}^{\infty} X(\mathrm{j}(\omega - \omega_0))e^{\mathrm{j}\omega t}\, d\omega = e^{\mathrm{j}\omega_0 t} \cdot x(t)$$

Fourier transform
property 5: time & frequency shift

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Fourier transform

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\end{aligned}$$

frequency shift:

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Fourier transform

property 6: symmetry 1/2

$$
\begin{aligned}
|X(\mathrm{j}\omega)| &= |X(-\mathrm{j}\omega)| \\
\Phi_{\mathrm{X}}(\omega) &= -\Phi_{\mathrm{X}}(-\omega)
\end{aligned}
$$

derivation

time signal sum of even and odd component $x_e(t), x_o(t)$:

$$x(t) = \underbrace{\frac{1}{2}(x(t) + x(-t))}_{x_e(t)} + \underbrace{\frac{1}{2}(x(t) - x(-t))}_{x_o(t)}$$

$$X_e(\mathrm{j}\omega) = \int_{-\infty}^{\infty} x_e(t)\cos(\omega t)\,dt - \mathrm{j}\underbrace{\int_{-\infty}^{\infty} x_e(t)\sin(\omega t)\,dt}_{=0}$$

- $X_e(\mathrm{j}\omega)$ is real
- $X_e(\mathrm{j}\omega) = X_e(-\mathrm{j}\omega)$ (substitute $x(t)$ with $x(-t)$)

Fourier transform

property 6: symmetry 1/2

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Fourier transform

property 6: symmetry 1/2

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Fourier transform

property 6: symmetry 2/2

$$\begin{aligned}|X(\mathrm{j}\omega)| &= |X(-\mathrm{j}\omega)|\\ \Phi_{\mathrm{X}}(\omega) &= -\Phi_{\mathrm{X}}(-\omega)\end{aligned}$$

derivation

time signal sum of even and odd component $x_e(t), x_o(t)$:

$$x(t) = \underbrace{\frac{1}{2}(x(t) + x(-t))}_{x_e(t)} + \underbrace{\frac{1}{2}(x(t) - x(-t))}_{x_o(t)}$$

$$X_o(\mathrm{j}\omega) = \underbrace{\int_{-\infty}^{\infty} x_o(t)\cos(\omega t)\,dt}_{=0} - \mathrm{j}\int_{-\infty}^{\infty} x_o(t)\sin(\omega t)\,dt$$

- $\Rightarrow$ $X_o(\mathrm{j}\omega)$ is imaginary
- $\Rightarrow$ $X_o(\mathrm{j}\omega) = -X_o(-\mathrm{j}\omega)$ (substitute $x(t)$ with $-x(-t)$)

Fourier transform
property 6: symmetry 2/2

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\begin{aligned}
|X(\mathrm{j}\omega)| &= |X(-\mathrm{j}\omega)| \\
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derivation

time signal sum of even and odd component $x_e(t), x_o(t)$:

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Fourier transform

property 7: time & frequency scaling

$$y(t) = x(c\cdot t) \mapsto Y(\mathrm{j}\omega) = \frac{1}{|c|}X\left(j\frac{\omega}{c}\right)$$

derivation (positive c)

$$\begin{aligned} Y(\mathrm{j}\omega) &= \int\limits_{-\infty}^{\infty} x(c\cdot t)e^{-\mathrm{j}\omega t}\,dt \\ &= \int\limits_{-\infty}^{\infty} x(\tau)e^{-\mathrm{j}\omega\frac{\tau}{c}}\,d\frac{\tau}{c} \\ &= \frac{1}{c}\int\limits_{-\infty}^{\infty} x(\tau)e^{-\mathrm{j}\frac{\omega}{c}\tau}\,d\tau \\ &= \frac{1}{c}X\left(\mathrm{j}\frac{\omega}{c}\right) \end{aligned}$$

Fourier transform

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Fourier transform

property 7: time & frequency scaling

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Fourier transform

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&= \frac{1}{c} X\left(\mathrm{j}\frac{\omega}{c}\right)
\end{aligned}$$

Fourier transform

examples

What is the FT of

Fourier transform

examples

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What is the FT of

- delta function
- constant
- cosine
- rectangular window
- delta pulse

Fourier transform
STFT introduction

short time Fourier transform (STFT):
compute Fourier transform only over a segment

reasons:

- **signal properties**: choose quasi-periodic segment
- **perception**: ear analyzes short segments of signal
- **hardware**: Fourier transform is inefficient and memory consuming for very long input segments

$\Rightarrow$ multiply a **window** with the signal

Fourier transform

STFT introduction

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Fourier transform
STFT introduction

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$\Rightarrow$ multiply a **window** with the signal

Fourier transform
STFT: windowing

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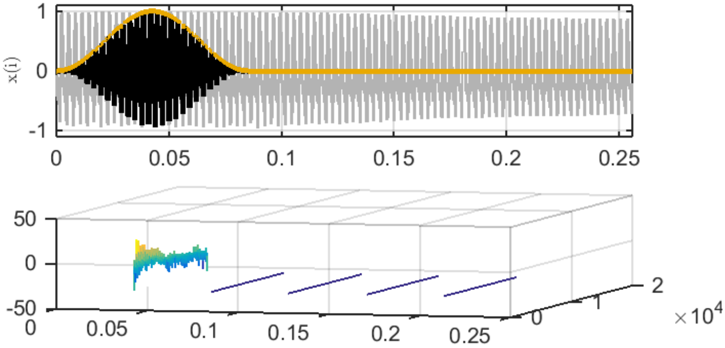
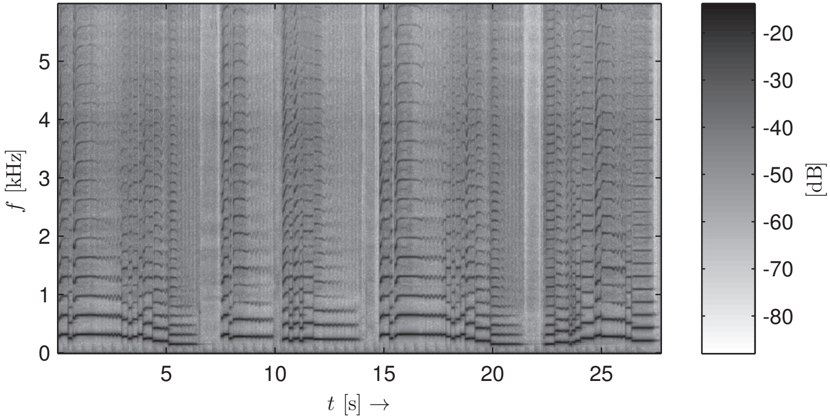


Fourier transform

STFT: spectrogram



Fourier transform

reminder: FT of rectangular window

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$$w_{\mathrm{R}}(t) = \begin{cases} 1, & -L \leq t \leq L \\ 0, & \text{otherwise} \end{cases}.$$

$$\begin{aligned} W_{\mathrm{R}}(\mathrm{j}\omega) &= \int_{-\infty}^{\infty} w_{\mathrm{R}}(t) e^{-\mathrm{j}\omega t}\, dt \\ &= \int_{-L}^{L} e^{-\mathrm{j}\omega t}\, dt \\ &= \frac{1}{-\mathrm{j}\omega} \underbrace{\left(e^{-\mathrm{j}\omega L} - e^{\mathrm{j}\omega L}\right)}_{=-2j\sin(L\omega)} \\ &= \frac{2\sin(L\omega)}{\omega}. \end{aligned}$$

Fourier transform

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Fourier transform

question: FT of triangular window

Given your knowledge of the frequency transform of the rect window and both the FT & LTI system properties, what is the FT of a triangular window

Fourier transform

question: FT of triangular window

Given your knowledge of the frequency transform of the rect window and both the FT & LTI system properties, what is the FT of a triangular window

1. triangular window is output convolution of two rectangular windows
2. convolution in time domain is multiplication in the frequency domain

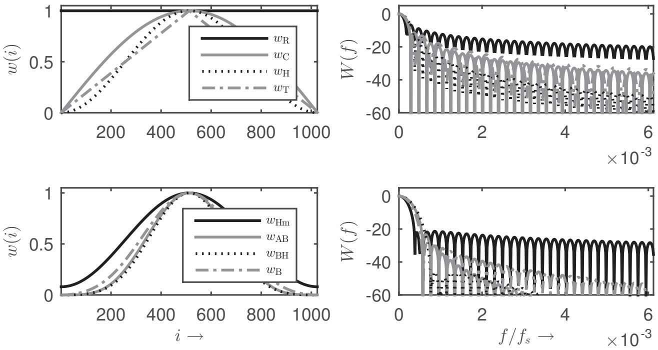
Fourier transform

STFT: window functions

multiplication in time domain $\rightarrow$ convolution in frequency domain

$$x_{\mathrm{W}}(t) = x(t) \cdot w(t) \rightarrow X_{\mathrm{W}}(\mathrm{j}\omega) = X(\mathrm{j}\omega) * W(\mathrm{j}\omega)$$

$\Rightarrow$**spectral leakage**



Fourier transform

STFT: window functions

multiplication in time domain $\rightarrow$ convolution in frequency domain

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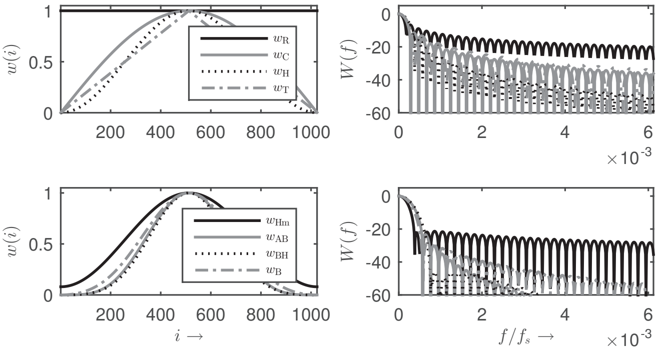

Fourier transform

STFT: window function properties

- **main lobe width**
  - how much does the main lobe “smear” a peak
- **side lobe height**
  - how dominant is the (highest) side lobe
- **side lobe attenuation/fall-off**
  - how much influence have distant sidelobes
- **process and scalloping loss** (DFT)
  - how accurate is the amplitude (best case and worst case)

Fourier transform

STFT: window function properties

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Fourier transform

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Fourier transform

STFT: typical window functions

- (rectangular window) $w_\mathrm{R}(t)$
- **von-Hann window**: $w_\mathrm{H}(t) = w_\mathrm{R}(t) \cdot {}^1\!/_2 \left(1 + \cos\left(\frac{\pi}{2} t\right)\right)$
- **Hamming window**: $w_\mathrm{Hm}(t) = w_\mathrm{R}(t) \cdot {}^{25}\!/_{46} + {}^{42}\!/_{46} \cos\left(\frac{\pi}{2} t\right)$
- **Cosine window**: $w_\mathrm{C}(t) = w_\mathrm{R}(t) \cdot \cos\left(\frac{\pi}{2} t\right)$
- **Blackman-Harris window**:

$$w_\mathrm{BH}(t) = w_\mathrm{R}(t) \sum_{m=0}^{3} b_m \cos\left(\frac{\pi}{2} m t\right).$$

with $b_0 = 0.35875$, $b_1 = 0.48829$, $b_2 = 0.14128$, $b_3 = 0.01168$

Fourier transform

sampled time signals 1/2

$$\begin{aligned}
\mathfrak{F}[x(i)] &= \mathfrak{F}[x(t)\cdot\delta_{\mathrm{T}}(t)]\\
&= \mathfrak{F}[x(t)] * \mathfrak{F}[\delta_{\mathrm{T}}(t)]\\
&= X(\mathrm{j}\omega) * \Delta_{\mathrm{T}}(\mathrm{j}\omega)
\end{aligned}$$

Fourier transform

sampled time signals 1/2

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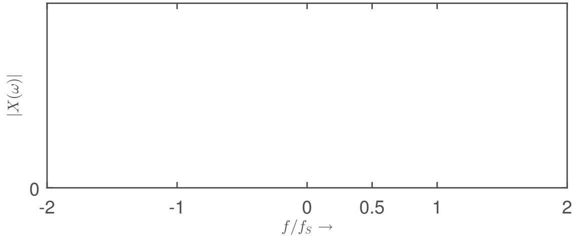

Fourier transform

sampled time signals 1/2

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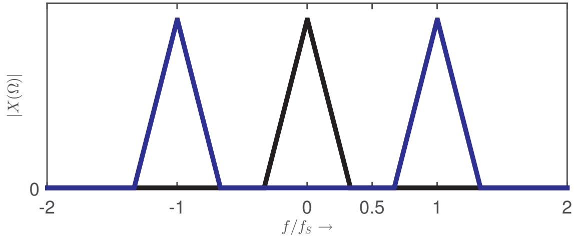
transformed signal is

- still **continuous**
- **periodic**



Fourier transform

sampled time signals 1/2

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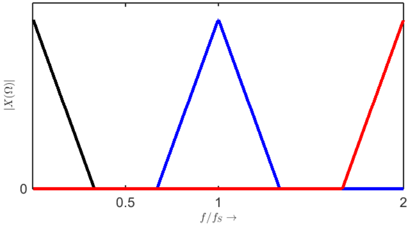

Fourier transform

sampled time signals 2/2

$$
\begin{aligned}
X(\mathrm{j}\Omega) &= \sum_{i=-\infty}^{\infty} x(i) e^{-\mathrm{j}\Omega i} \\
x(i) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\mathrm{j}\Omega) e^{\mathrm{j}\Omega i} d\Omega \\
\Omega &= 2\pi \frac{\omega}{\omega_T}
\end{aligned}
$$

Fourier transform

DFT

digital domain: requires discrete frequency values:

$\Rightarrow$ **Discrete Fourier transform (DFT)**

$$X(k) = \sum_{i=0}^{\mathcal{K}-1} x(i) e^{-\mathrm{j}ki\frac{2\pi}{\mathcal{K}}}$$

alternative notation

$$X(k\Omega_{\mathcal{K}}) = \sum_{i=0}^{\mathcal{K}-1} x(i) e^{-\mathrm{j}ki\Omega_{\mathcal{K}}}$$

2 interpretations:

- sampled continuous Fourier transform
- continuous Fourier transform of periodically extended time domain segment

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DFT

digital domain: requires discrete frequency values:
$\Rightarrow$ **Discrete Fourier transform (DFT)**

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DFT frequency resolution

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  - block length $\mathcal{K}$
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  - decreasing time resolution
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Fourier transform
DFT vs FFT

- FFT is an algorithm to efficiently calculate the DFT
- result is **identical**
- efficiency:
  - DFT: $\mathcal{K}^2$ complex multiplications
  - FFT: $\mathcal{K}/2\,\log_2(\mathcal{K})$ complex multiplications

$\mathcal{K}$	DFT mult	FFT mult	efficiency
256	2^{16}	1024	64 : 1
512	2^{18}	2304	114 : 1
1024	2^{20}	5120	205 : 1
2048	2^{22}	11264	372 : 1
4096	2^{24}	24576	683 : 1

Fourier transform

summary 1/2

FT properties

1. invertibility
2. linearity
3. convolution — multiplication
4. Parseval's theorem
5. time shift — phase shift
6. symmetry
7. time scaling — frequency scaling

Fourier transform

summary 2/2

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2. continuous FT transforms any continuous function $\rightarrow$ continuous spectrum
3. STFT transforms a segment of the signal $\rightarrow$ convolution with window spectrum
4. FT of sampled signals $\rightarrow$ periodic
5. DFT $\rightarrow$ sampled FT of periodic continuation

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Fourier transform

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