Digital Signal Processing for Music

Part 11: Discretization 2—Quantization

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sampling and quantization
introduction

digital signals can only be represented with a limited number of values

$\Rightarrow$

- time discretization:
  sampling

- amplitude discretization:
  quantization

sampling and quantization
introduction

digital signals can only be represented with a limited number of values

⇒

- time discretization:
  sampling

- amplitude discretization:
  quantization

sampling and quantization
quantization introduction

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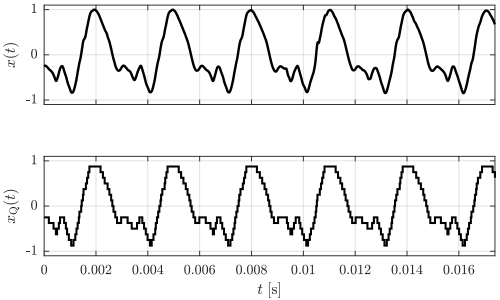
quantizer:
continuous $\mapsto$ discrete (pre-defined set of allowed values)



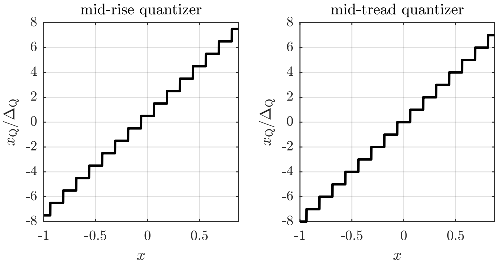
matlab source: plotQuantizedAudio.m

sampling and quantization

quantization: characteristic curve

quantizer **characteristic curve**:

- plots output amplitude over input amplitude
- given even number of quantization steps, characteristic curve can be either
  - symmetric, or
  - include 0



matlab source: plotQuantizationCharacteristicCurve.m

sampling and quantization

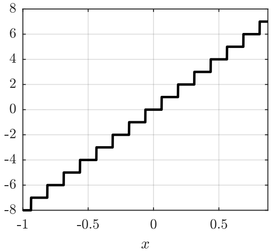
basic quantization properties

quantization fundamentals

- quantization is **non-linear**:
  signal is distorted
- quantization is **irreversible**:
  signal cannot be perfectly restored

sampling and quantization

quantization: word length & number of steps



Given a number of quantization steps $\mathcal{M} = 16$ what is the required word length (bits)

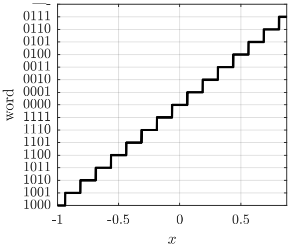
sampling and quantization

quantization: word length & number of steps

Given a number of quantization steps $\mathcal{M} = 16$ what is the required word length (bits)

$$\Rightarrow w = \log_2(16) = 4\,\mathrm{bit}$$

$$\begin{aligned} w &= \log_2(\mathcal{M}) \\ \mathcal{M} &= 2^w \end{aligned}$$



sampling and quantization
quantization: word length examples

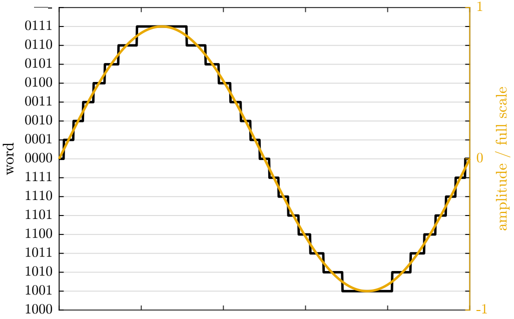
w	$\mathcal{M} = 2^w$
1	2
2	4
4	16
8	256
12	4096
16	65536
20	1048576
24	16777216

sampling and quantization

scaling, ranges, and words

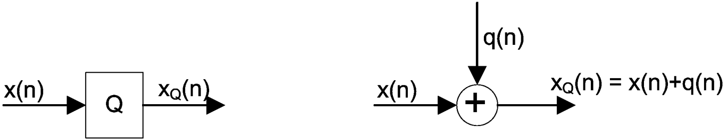
values are encoded and interpretation is up to the user:

- word can be translated to integer $\Rightarrow [-8 \ldots 7]$
- word can be scaled to range of $[-1 \ldots 1 - 1/2^{\mathcal{M}-1}]$
  - standard for floating point systems
  - -1/1 means full scale
  - internal representation independent of quantization word length



matlab source: plotQuantNumbers.m

sampling and quantization
quantization error: definition



$$q(i) = x_\mathrm{Q}(i) - x(i)$$

sampling and quantization

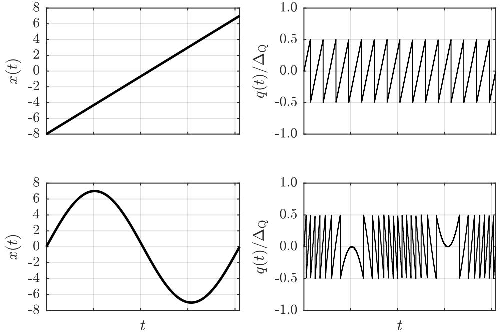
quantization error: max. amplitude

What is the maximum amplitude of the quantization error

matlab source: plotQuantizationError.m

sampling and quantization
quantization error: max. amplitude

What is the maximum amplitude of the quantization error



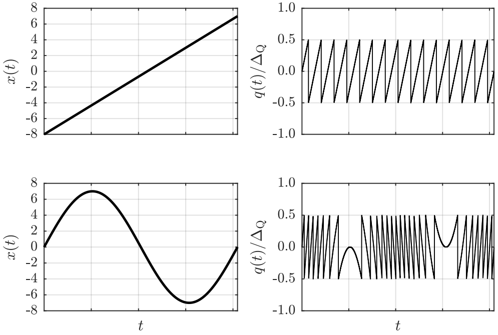
matlab source: plotQuantizationError.m

sampling and quantization

quantization error: max. amplitude

What is the maximum amplitude of the quantization error

$$|q(i)| \leq \frac{\Delta}{2}$$



matlab source: plotQuantizationError.m

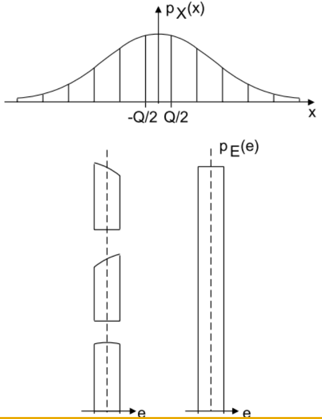
sampling and quantization
quantization error: pdf 1/2

What is the pdf of the quantization error

sampling and quantization
quantization error: pdf 1/2

What is the pdf of the quantization error

assuming $\Delta \ll max(|x(i)|)$



sampling and quantization

quantization error: pdf 1/2

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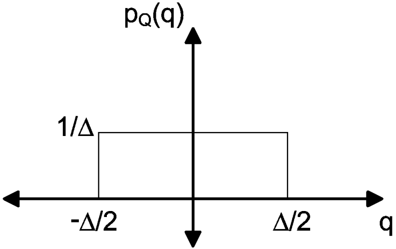

sampling and quantization

quantization error: pdf 2/2

it can be shown that the pdf of the quantization error depends (without derivation)

- on the **variance of the input** signal in relation to the step size
- on the **pdf of the input** signal

$\rightarrow$ will be **uniform (and white) for large values of** $\frac{\sigma_x}{\Delta}$

sampling and quantization

quantization error: pdf 2/2

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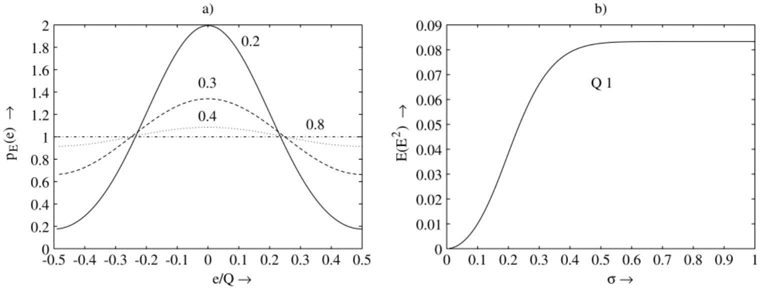

Figure 2.16 (a) PDF of quantization error for different standard deviations of a Gaussian PDF input. (b) Variance of quantization error for different standard deviations of a Gaussian PDF input.

sampling and quantization

quantization error: power

how to compute the power W_Q of the quantization error

sampling and quantization

quantization error: power

how to compute the power W_Q of the quantization error

from PDF:

$$W_\mathrm{Q} = \int\limits_{-\Delta/2}^{\Delta/2} q^2 \cdot \underbrace{p_\mathrm{Q}(q)}_{1/\Delta} dq$$

sampling and quantization

quantization error: power

how to compute the power W_Q of the quantization error

from PDF:

$$W_Q = \int\limits_{-\Delta/2}^{\Delta/2} q^2 \cdot \underbrace{p_Q(q)}_{1/\Delta} \, dq$$

$$= \frac{1}{\Delta} \int\limits_{-\Delta/2}^{\Delta/2} q^2 \, dq$$

sampling and quantization
quantization error: power

how to compute the power W_Q of the quantization error

from PDF:

$$
\begin{aligned}
W_\mathrm{Q} &= \int\limits_{-\Delta/2}^{\Delta/2} q^2 \cdot \underbrace{p_\mathrm{Q}(q)}_{1/\Delta}\, dq \\
&= \frac{1}{\Delta} \int\limits_{-\Delta/2}^{\Delta/2} q^2 \, dq \\
&= \frac{1}{\Delta} \left[\frac{1}{3} q^3\right]_{-\Delta/2}^{\Delta/2}
\end{aligned}
$$

sampling and quantization

quantization error: power

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&= \frac{1}{\Delta} \left[\frac{1}{3} q^3\right]_{-\Delta/2}^{\Delta/2} \\
&= \frac{1}{3\Delta} \left(\frac{\Delta^3}{8} + \frac{\Delta^3}{8}\right)
\end{aligned}$$

sampling and quantization

quantization error: power

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W_\mathrm{Q} &= \int\limits_{-\Delta/2}^{\Delta/2} q^2 \cdot \underbrace{p_\mathrm{Q}(q)}_{1/\Delta}\, dq \\
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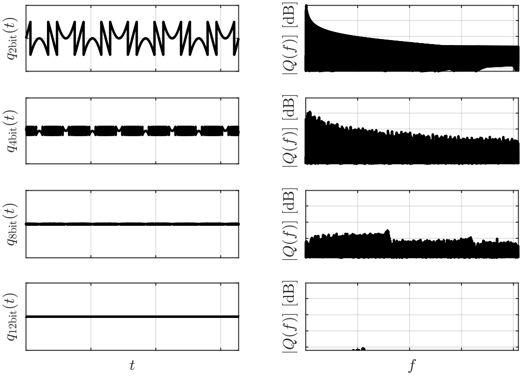
&= \frac{1}{3\Delta} \left(\frac{\Delta^3}{8} + \frac{\Delta^3}{8}\right) \\
&= \frac{\Delta^2}{12}
\end{aligned}$$

sampling and quantization

quantization error: spectrum

quantization error of a full-scale sinusoidal (2,4,8,12 bits)



matlab source: plotQuantizationErrorSpectrum.m

sampling and quantization

quantization: audio examples

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w	$x_{\mathrm{Q,sine}}(i)$	$q_{\mathrm{sine}}(i)$	$x_{\mathrm{Q,speech}}(i)$	$q_{\mathrm{speech}}(i)$	$x_{\mathrm{Q,music}}(i)$	$q_{\mathrm{music}}(i)$
16						
12						
8						
6						
4						
2						

sampling and quantization

quality assessment of a quantizer: SNR

Signal-to-Noise Ratio (SNR):

- power of the signal in relation to power of the (quantization) noise

$$SNR' = \frac{\text{signal energy}}{\text{noise energy}} = \frac{W_\mathrm{S}}{W_\mathrm{Q}}$$

- often in decibel

$$SNR = 10 \cdot \log_{10}\left(\frac{W_\mathrm{S}}{W_\mathrm{Q}}\right) \; [dB]$$

- SNR grows by
  - reducing the noise power
  - increasing the signal power

sampling and quantization

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sampling and quantization

quantization: SNR 1/3

What is the SNR of a quantized full-scale sinusoidal

sampling and quantization

quantization: SNR 1/3

What is the SNR of a quantized full-scale sinusoidal

$$SNR = 10 \cdot \log_{10}\left(\frac{W_\mathrm{S}}{W_\mathrm{Q}}\right) \; [dB]$$

use: $sin^2(t) = \frac{1-cos(2t)}{2}$

sampling and quantization
quantization: SNR 1/3

What is the SNR of a quantized full-scale sinusoidal

$$SNR = 10 \cdot \log_{10}\left(\frac{W_\mathrm{S}}{W_\mathrm{Q}}\right)\ [dB]$$

use: $sin^2(t) = \frac{1-cos(2t)}{2}$

$$W_\mathrm{S} = \frac{A^2}{2} \underset{\text{full-scale}}{\rightarrow} W_\mathrm{S} = \frac{(\Delta \cdot 2^{w-1})^2}{2}$$

$$W_\mathrm{Q} = \frac{\Delta^2}{12}$$

sampling and quantization
quantization: SNR 1/3

What is the SNR of a quantized full-scale sinusoidal

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W_\mathrm{Q} &= \frac{\Delta^2}{12} \\
\frac{W_\mathrm{S}}{W_\mathrm{Q}} &= \frac{3}{2} \cdot 2^{2w}
\end{aligned}$$

sampling and quantization

quantization: SNR 1/3

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$$SNR = w \cdot 20 \log_{10}(2) + 10 \cdot \log_{10}\left(\frac{3}{2}\right)\ [dB]$$

sampling and quantization

quantization: SNR 2/3

derive the SNR for a full-scale square wave

sampling and quantization

quantization: SNR 2/3

derive the SNR for a full-scale square wave

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sampling and quantization

quantization: SNR 2/3

derive the SNR for a full-scale square wave

$$SNR = 10 \cdot \log_{10}\left(\frac{W_\mathrm{S}}{W_\mathrm{Q}}\right) \; [dB]$$

$$\begin{aligned}
W_\mathrm{S} &= A^2 \xrightarrow{\text{full-scale}} W_\mathrm{S} = (\Delta \cdot 2^{w-1})^2 \\
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\end{aligned}$$

sampling and quantization

quantization: SNR 2/3

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\end{aligned}$$

$$SNR = w \cdot 20 \log_{10}(2) + 10 \cdot \log_{10}(3) \;[dB]$$

sampling and quantization

quantization: SNR 3/3

Signal-to-Noise Ratio

$$SNR = 6.02 \cdot w + c_{\mathrm{S}} \quad [dB]$$

- every additional bit adds app. 6 dB SNR
- constant c_{S} depends on signal (scaling and PDF shape)

SNR for different input signal examples

- square wave (full scale): $c_{\mathrm{S}} = 4.77\,\mathrm{dB}$
- sinusoidal wave (full scale): $c_{\mathrm{S}} = 1.76\,\mathrm{dB}$
- rectangular PDF (full scale): $c_{\mathrm{S}} = 0\,\mathrm{dB}$
- Gaussian PDF (full scale $= 4\sigma_g$): $c_{\mathrm{S}} = -7.27\,\mathrm{dB}$

sampling and quantization

quantization: SNR 3/3

Signal-to-Noise Ratio

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sampling and quantization

quantization: word length and SNR

w	**Δ**	**Max. Amp**	**theo. SNR**
8 (Int)	± 1	$0 \dots 255$	$\approx 48\,\text{dB}$
16 (Int)	± 1	$-32768 \dots 32767$	$\approx 96\,\text{dB}$
20 (Int)	± 1	$-524288 \dots 524287$	$\approx 120\,\text{dB}$
24 (Int)	± 1	$-16777216 \dots 16777215$	$\approx 144\,\text{dB}$
32 (Float)	$\pm 1.175 \cdot 10^{-38}$	$\pm 3.403 \cdot 10^{1038}$	$1529\,\text{dB}$
64 (Float)	$\pm 2.225 \cdot 10^{-308}$	$\pm 1.798 \cdot 10^{10308}$	$12318\,\text{dB}$

sampling and quantization

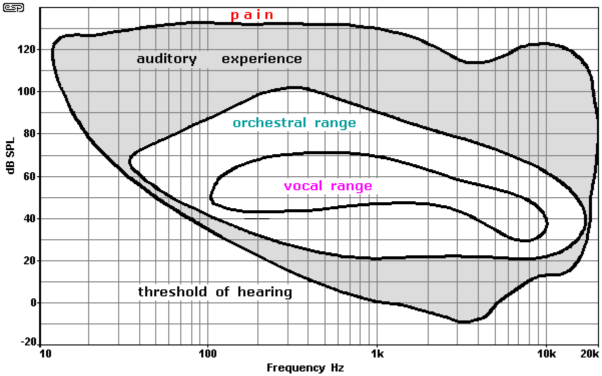
quantization: SNR and auditory sensation area

so how many bits do we need

sampling and quantization

quantization: SNR and auditory sensation area

so how many bits do we need



sampling and quantization
quantization: SNR and auditory sensation area

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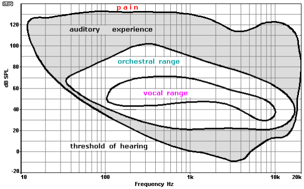
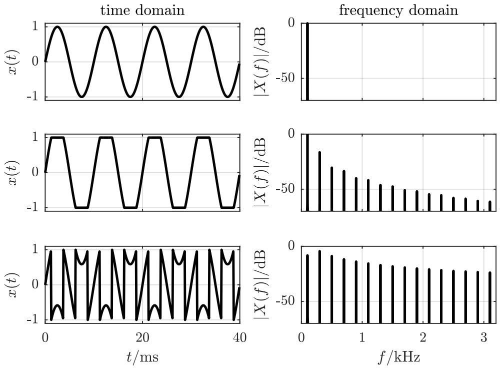


- to cover the whole range of hearing: 20–24 bit
- practically, a lower range is sufficient as the dynamic range of recordings has to be much lower
- in production with many processing and possible requantization steps, high resolution (if possible floating point) is recommended

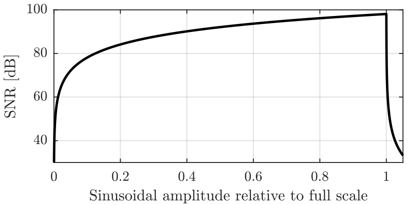
sampling and quantization
quantization: clipping



matlab source: plotClipping.m

sampling and quantization

quantization: clipping and SNR



full scale:

- absolute maximum before clipping
- usually 1 (in floating point systems)
- marks 0 dbFS

matlab source: plotSnr.m

sampling and quantization

quantization: summary

- quantization is **non-linear** & **irreversible**
  - information is lost
  - error is introduced
- quantization **error**
  - power is determined by number of bits (wordlength)
  - is approximately white noise (flat spectrum and uncorrelated to signal) when the signal power is much higher than the quantization step size
  - special severe case: clipping
- **SNR** is used to assess quantizer quality
  - depends on both signal power and quant error power (ratio)
  - each additional bit gains 6 dB SNR
  - different signals with identical maximum amplitude yield different SNRs
- **typical word lengths** include
  - 8 bit: phone
  - 16 bit: consumer audio
  - 24 bit and higher: production audio

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