Digital Signal Processing for Music

Part 14: Improving (Re-)Quantization Quality

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introduction

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we will discuss three ways of “cheating” for better quality

- of quantization:
  - oversampling
  - noise-shaping
- of re-quantization/ word length reduction
  - dither
  - noise shaping

oversampling
introduction

- **oversampling**:
  - ⇒ less steep anti-aliasing filters
  - ⇒ ?

- remember **quantization error properties**
  1. *white* noise: flat spectrum
  2. noise power *sample rate independent*

$$\Rightarrow \quad |Q(j\omega)|^2 \sim \frac{\Delta^2}{12 \cdot \omega_S}$$

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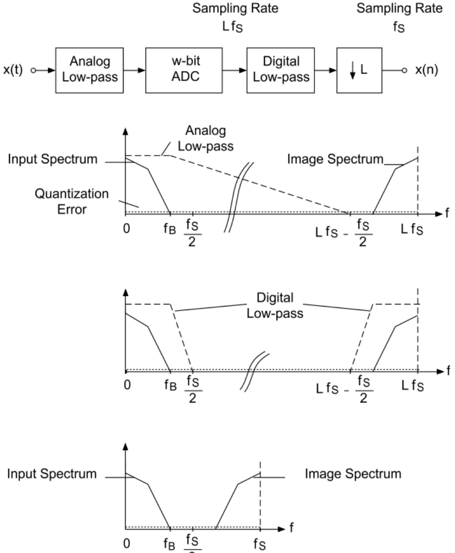
can oversampling take advantage of these properties to reduce quantization error power

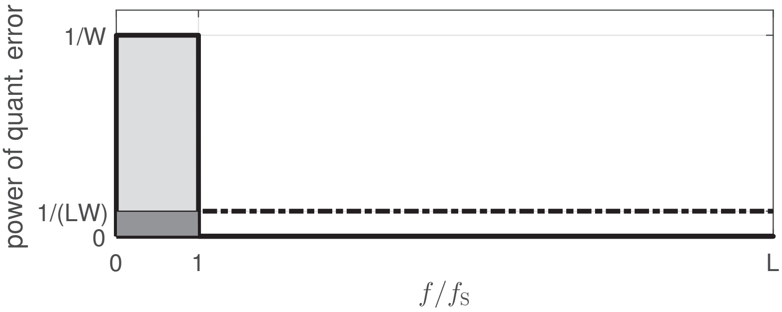
oversampling

visualization



oversampling

visualization



oversampling
SNR gain

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\begin{aligned}
\omega_S^* &= L \cdot \omega_S \\
|Q(\mathrm{j}\omega)|^2 &= \frac{\Delta^2}{12 \cdot \omega_S^*} \\
&= \frac{\Delta^2}{12 \cdot L \cdot \omega_S} \\
W^*_{\mathrm{Q,LP}} &= \frac{\Delta^2}{12 \cdot L} \\
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oversampling

SNR gain

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\Rightarrow & \\
SNR^* &= 6.02 \cdot w + 10\log_{10}(L) + c_S
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oversampling

summary

oversampling

$$SNR = 6.02 \cdot w + c_S + 10 \cdot \log_{10}(L) \quad [dB]$$

- every doubling of f_S adds app. 3 dB SNR

dither

introduction 1/2

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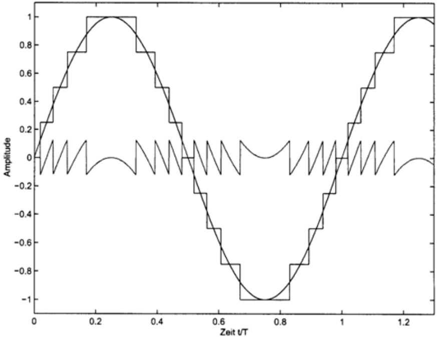

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introduction 1/2

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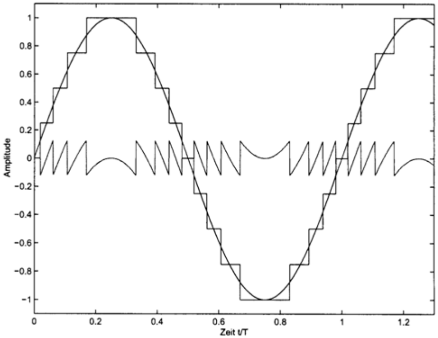
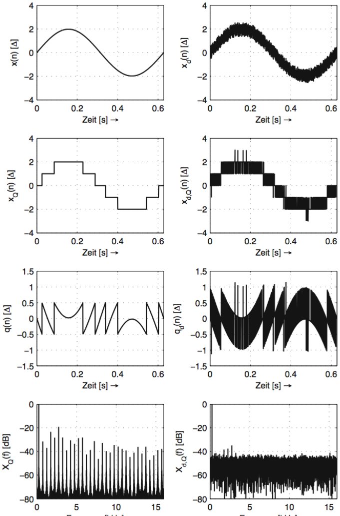
- **solution**:
  - **add noise before quantization: dither**



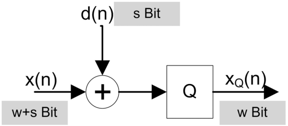
dither

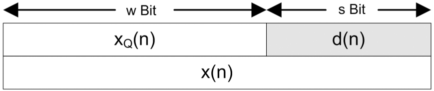
introduction 2/2



dither

process





dither

simple example

input signal: DC at $1.3 \cdot \Delta$

- **w/o dither**:
  - output value: Δ
  - *quantization error constant*: $.3 \cdot \Delta$
- **w/ dither** $(-\Delta/2 \ldots \Delta/2)$:
  - signal is most frequently quantized to Δ ($p = 0.7$), but sometimes to $2 \cdot \Delta$ ($p = 0.3$)
  - *average* output value: $1.3 \cdot \Delta$!
  - *quantization error varying* between $0.3 \cdot \Delta$ and $0.7 \cdot \Delta$

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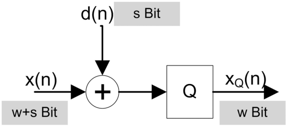
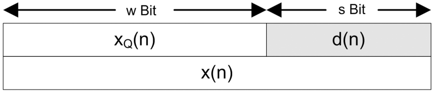
dither

properties





- dither: 2^s possible possible numbers
- in case of *uniform* dist →

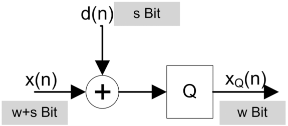
$$p_d(d_n) = \begin{cases} 2^{-s} & -2^{s-1} \leq n \leq 2^{s-1} - 1 \\ 0 & \text{else.} \end{cases}$$

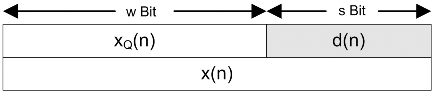
- output (positive X)

$$x_Q(X + d_n) = \Delta \left\lfloor \frac{X + d_n}{\Delta} + 0.5 \right\rfloor$$

dither

properties





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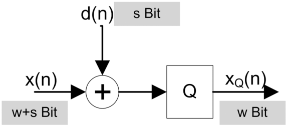

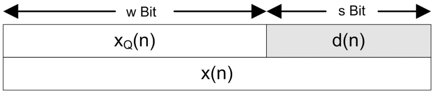

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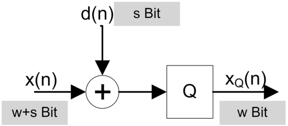

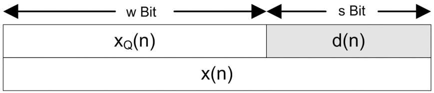

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dither

rectangular dither

dither with rect pdf, $-\Delta/2 \ldots \Delta/2$, not quantized

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x &= 0 \cdot \Delta & &\rightarrow \bar{x_Q} = 0, \\
\sigma_R(x) &= \Delta\sqrt{(-0)^2 \cdot 1.0} & &= 0.0\Delta \\
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\sigma_R(x) &= \Delta\sqrt{(-0.1)^2 \cdot 0.9 + (0.9)^2 \cdot 0.1} & &= 0.3\Delta \\
x &= 0.3 \cdot \Delta & &\rightarrow \bar{x_Q} = 0.3\Delta, \\
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x &= 0.7 \cdot \Delta & &\rightarrow \bar{x_Q} = 0.7\Delta, \\
\sigma_R(x) &= \Delta\sqrt{(-0.7)^2 \cdot 0.3 + (0.3)^2 \cdot 0.7} & &= 0.46\Delta \\
x &= 0.9 \cdot \Delta & &\rightarrow \bar{x_Q} = 0.9\Delta, \\
\sigma_R(x) &= \Delta\sqrt{(-0.9)^2 \cdot 0.1 + (0.1)^2 \cdot 0.9} & &= 0.3\Delta \\
x &= 1 \cdot \Delta & &\rightarrow \bar{x_Q} = 0, \\
\sigma_R(x) &= 0
\end{aligned}
$$

dither

rectangular dither

dither with rect pdf, $-\Delta/2 \ldots \Delta/2$, not quantized

$$
\begin{aligned}
x = 0 \cdot \Delta & & \rightarrow \bar{x}_Q = 0, \\
\sigma_R(x) & = \Delta\sqrt{(-0)^2 \cdot 1.0} & = 0.0\Delta \\
x = 0.1 \cdot \Delta & & \rightarrow \bar{x}_Q = 0.1\Delta, \\
\sigma_R(x) & = \Delta\sqrt{(-0.1)^2 \cdot 0.9 + (0.9)^2 \cdot 0.1} & = 0.3\Delta \\
x = 0.3 \cdot \Delta & & \rightarrow \bar{x}_Q = 0.3\Delta, \\
\sigma_R(x) & = \Delta\sqrt{(-0.3)^2 \cdot 0.7 + (0.7)^2 \cdot 0.3} & = 0.46\Delta \\
x = 0.5 \cdot \Delta & & \rightarrow \bar{x}_Q = 0.5\Delta, \\
\sigma_R(x) & = \Delta\sqrt{(-0.5)^2 \cdot 0.5 + (0.5)^2 \cdot 0.5} & = 0.5\Delta \\
x = 0.7 \cdot \Delta & & \rightarrow \bar{x}_Q = 0.7\Delta, \\
\sigma_R(x) & = \Delta\sqrt{(-0.7)^2 \cdot 0.3 + (0.3)^2 \cdot 0.7} & = 0.46\Delta \\
x = 0.9 \cdot \Delta & & \rightarrow \bar{x}_Q = 0.9\Delta, \\
\sigma_R(x) & = \Delta\sqrt{(-0.9)^2 \cdot 0.1 + (0.1)^2 \cdot 0.9} & = 0.3\Delta \\
x = 1 \cdot \Delta & & \rightarrow \bar{x}_Q = 0, \\
\sigma_R(x) = 0 & &
\end{aligned}
$$

dither

rectangular dither

dither with rect pdf, $-\Delta/2 \ldots \Delta/2$, not quantized

$$x = 0 \cdot \Delta \quad \rightarrow \bar{x_Q} = 0,$$
$$\sigma_R(x) = \Delta\sqrt{(-0)^2 \cdot 1.0} = 0.0\Delta$$
$$x = 0.1 \cdot \Delta \quad \rightarrow \bar{x_Q} = 0.1\Delta,$$
$$\sigma_R(x) = \Delta\sqrt{(-0.1)^2 \cdot 0.9 + (0.9)^2 \cdot 0.1} = 0.3\Delta$$
$$x = 0.3 \cdot \Delta \quad \rightarrow \bar{x_Q} = 0.3\Delta,$$
$$\sigma_R(x) = \Delta\sqrt{(-0.3)^2 \cdot 0.7 + (0.7)^2 \cdot 0.3} = 0.46\Delta$$
$$x = 0.5 \cdot \Delta \quad \rightarrow \bar{x_Q} = 0.5\Delta,$$
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$$x = 0.7 \cdot \Delta \quad \rightarrow \bar{x_Q} = 0.7\Delta,$$
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$$x = 1 \cdot \Delta \quad \rightarrow \bar{x_Q} = 0,$$
$$\sigma_R(x) = 0$$

dither

rectangular dither

dither with rect pdf, $-\Delta/2 \ldots \Delta/2$, not quantized

$$
\begin{aligned}
x &= 0 \cdot \Delta & &\rightarrow \bar{x_Q} = 0,\\
\sigma_R(x) &= \Delta\sqrt{(-0)^2 \cdot 1.0} & &= 0.0\Delta\\
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x &= 0.3 \cdot \Delta & &\rightarrow \bar{x_Q} = 0.3\Delta,\\
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x &= 1 \cdot \Delta & &\rightarrow \bar{x_Q} = 0,\\
\sigma_R(x) &= 0
\end{aligned}
$$

dither

rectangular dither

dither with rect pdf, $-\Delta/2 \dots \Delta/2$, not quantized

$$
\begin{aligned}
x &= 0 \cdot \Delta & &\rightarrow \bar{x_Q} = 0, \\
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\begin{aligned}
x &= 0 \cdot \Delta & & \rightarrow \bar{x_Q} = 0, \\
\sigma_R(x) &= \Delta\sqrt{(-0)^2 \cdot 1.0} & &= 0.0\Delta \\
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x &= 1 \cdot \Delta & & \rightarrow \bar{x_Q} = 0, \\
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\end{aligned}
$$

dither

rectangular dither

Georgia Tech Center for Music Technology College of Design

dither with rect pdf, $-\Delta/2 \ldots \Delta/2$, not quantized

$$
\begin{aligned}
x &= 0\cdot\Delta & &\rightarrow \bar{x_Q} = 0,\\
\sigma_R(x) &= \Delta\sqrt{(-0)^2\cdot 1.0} & &= 0.0\Delta\\
x &= 0.1\cdot\Delta & &\rightarrow \bar{x_Q} = 0.1\Delta,\\
\sigma_R(x) &= \Delta\sqrt{(-0.1)^2\cdot 0.9 + (0.9)^2\cdot 0.1} & &= 0.3\Delta\\
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\sigma_R(x) &= \Delta\sqrt{(-0.5)^2\cdot 0.5 + (0.5)^2\cdot 0.5} & &= 0.5\Delta\\
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\sigma_R(x) &= \Delta\sqrt{(-0.9)^2\cdot 0.1 + (0.1)^2\cdot 0.9} & &= 0.3\Delta\\
x &= 1\cdot\Delta & &\rightarrow \bar{x_Q} = 0,\\
\sigma_R(x) &= 0
\end{aligned}
$$

dither

triangular dither

dither with tri pdf, $-\Delta \ldots \Delta$, not quantized

$$x = 0 \cdot \Delta \quad \rightarrow \bar{x_Q} = 0,$$
$$\sigma_R(x) = 0.5\Delta$$
$$x = 0.1 \cdot \Delta \quad \rightarrow \bar{x_Q} = 0.1\Delta,$$
$$\sigma_R(x) = 0.5\Delta$$
$$x = 0.3 \cdot \Delta \quad \rightarrow \bar{x_Q} = 0.3\Delta,$$
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$$x = 0.9 \cdot \Delta \quad \rightarrow \bar{x_Q} = 0.9\Delta,$$
$$\sigma_R(x) = 0.5\Delta$$
$$x = 1 \cdot \Delta \quad \rightarrow \bar{x_Q} = 0,$$
$$\sigma_R(x) = 0.5\Delta$$

dither

triangular dither

dither with tri pdf, $-\Delta \ldots \Delta$, not quantized

$$
\begin{aligned}
x &= 0 \cdot \Delta & \rightarrow \bar{x_Q} &= 0,\\
\sigma_R(x) &= 0.5\Delta\\
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\sigma_R(x) &= 0.5\Delta\\
x &= 0.3 \cdot \Delta & \rightarrow \bar{x_Q} &= 0.3\Delta,\\
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\sigma_R(x) &= 0.5\Delta\\
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\sigma_R(x) &= 0.5\Delta\\
x &= 1 \cdot \Delta & \rightarrow \bar{x_Q} &= 0,\\
\sigma_R(x) &= 0.5\Delta
\end{aligned}
$$

dither

triangular dither

dither with tri pdf, $-\Delta \ldots \Delta$, not quantized

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\begin{aligned}
x &= 0 \cdot \Delta & &\rightarrow \bar{x_Q} = 0, \\
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\sigma_R(x) &= 0.5\Delta \\
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\sigma_R(x) &= 0.5\Delta \\
x &= 1 \cdot \Delta & &\rightarrow \bar{x_Q} = 0, \\
\sigma_R(x) &= 0.5\Delta
\end{aligned}
$$

dither

triangular dither

dither with tri pdf, $-\Delta \ldots \Delta$, not quantized

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\begin{aligned}
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\sigma_R(x) &= 0.5\Delta \\
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\sigma_R(x) &= 0.5\Delta \\
x &= 0.3 \cdot \Delta & \rightarrow \bar{x_Q} &= 0.3\Delta, \\
\sigma_R(x) &= 0.5\Delta \\
x &= 0.5 \cdot \Delta & \rightarrow \bar{x_Q} &= 0.5\Delta, \\
\sigma_R(x) &= 0.5\Delta \\
x &= 0.7 \cdot \Delta & \rightarrow \bar{x_Q} &= 0.7\Delta, \\
\sigma_R(x) &= 0.5\Delta \\
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\sigma_R(x) &= 0.5\Delta \\
x &= 1 \cdot \Delta & \rightarrow \bar{x_Q} &= 0, \\
\sigma_R(x) &= 0.5\Delta
\end{aligned}
$$

dither

triangular dither

dither with tri pdf, $-\Delta \ldots \Delta$, not quantized

$$
\begin{aligned}
x &= 0 \cdot \Delta && \rightarrow \bar{x_{\mathrm{Q}}} = 0, \\
\sigma_R(x) &= 0.5\Delta \\
x &= 0.1 \cdot \Delta && \rightarrow \bar{x_{\mathrm{Q}}} = 0.1\Delta, \\
\sigma_R(x) &= 0.5\Delta \\
x &= 0.3 \cdot \Delta && \rightarrow \bar{x_{\mathrm{Q}}} = 0.3\Delta, \\
\sigma_R(x) &= 0.5\Delta \\
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\sigma_R(x) &= 0.5\Delta \\
x &= 0.9 \cdot \Delta && \rightarrow \bar{x_{\mathrm{Q}}} = 0.9\Delta, \\
\sigma_R(x) &= 0.5\Delta \\
x &= 1 \cdot \Delta && \rightarrow \bar{x_{\mathrm{Q}}} = 0, \\
\sigma_R(x) &= 0.5\Delta
\end{aligned}
$$

dither

triangular dither

dither with tri pdf, $-\Delta \dots \Delta$, not quantized

$$x = 0 \cdot \Delta \quad \rightarrow \bar{x_Q} = 0,$$
$$\sigma_R(x) = 0.5\Delta$$
$$x = 0.1 \cdot \Delta \quad \rightarrow \bar{x_Q} = 0.1\Delta,$$
$$\sigma_R(x) = 0.5\Delta$$
$$x = 0.3 \cdot \Delta \quad \rightarrow \bar{x_Q} = 0.3\Delta,$$
$$\sigma_R(x) = 0.5\Delta$$
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$$\sigma_R(x) = 0.5\Delta$$
$$x = 1 \cdot \Delta \quad \rightarrow \bar{x_Q} = 0,$$
$$\sigma_R(x) = 0.5\Delta$$

dither

triangular dither

dither with tri pdf, $-\Delta \ldots \Delta$, not quantized

$$\begin{aligned}
x &= 0 \cdot \Delta & \rightarrow \bar{x_Q} &= 0, \\
\sigma_R(x) &= 0.5\Delta \\
x &= 0.1 \cdot \Delta & \rightarrow \bar{x_Q} &= 0.1\Delta, \\
\sigma_R(x) &= 0.5\Delta \\
x &= 0.3 \cdot \Delta & \rightarrow \bar{x_Q} &= 0.3\Delta, \\
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x &= 1 \cdot \Delta & \rightarrow \bar{x_Q} &= 0, \\
\sigma_R(x) &= 0.5\Delta
\end{aligned}$$

dither

triangular dither

dither with tri pdf, $-\Delta \ldots \Delta$, not quantized

$$x = 0 \cdot \Delta \qquad \rightarrow \bar{x_Q} = 0,$$
$$\sigma_R(x) = 0.5\Delta$$
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dither

triangular dither

dither with tri pdf, $-\Delta \ldots \Delta$, not quantized

$$
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x &= 0 \cdot \Delta & &\rightarrow \bar{x_Q} = 0, \\
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dither

triangular dither

dither with tri pdf, $-\Delta \ldots \Delta$, not quantized

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dither

triangular dither

Georgia Tech Center for Music Technology College of Design

dither with tri pdf, $-\Delta \ldots \Delta$, not quantized

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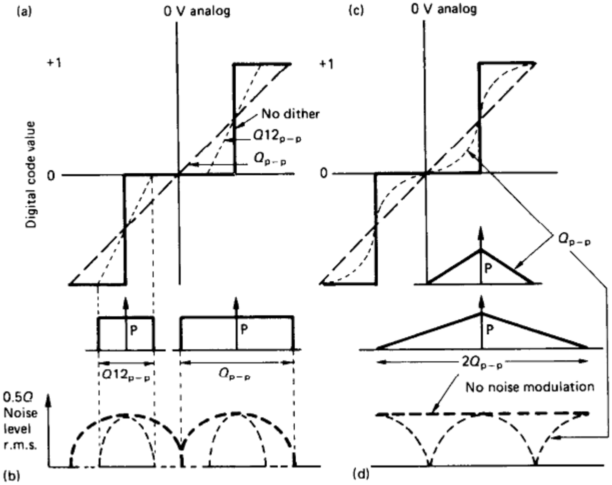
x &= 0.7 \cdot \Delta & \rightarrow \bar{x_Q} &= 0.7\Delta, \\
\sigma_R(x) &= 0.5\Delta \\
x &= 0.9 \cdot \Delta & \rightarrow \bar{x_Q} &= 0.9\Delta, \\
\sigma_R(x) &= 0.5\Delta \\
x &= 1 \cdot \Delta & \rightarrow \bar{x_Q} &= 0, \\
\sigma_R(x) &= 0.5\Delta
\end{aligned}
$$

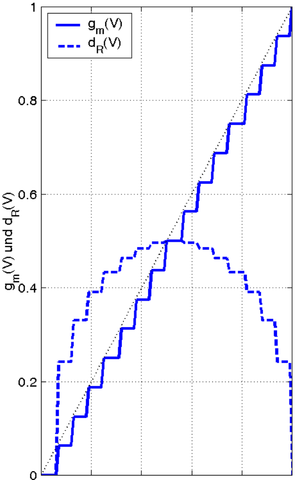
dither

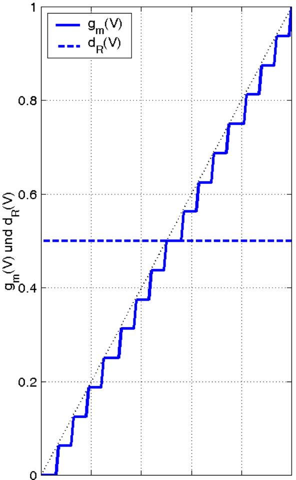
linearization and noise modulation 1/2



dither

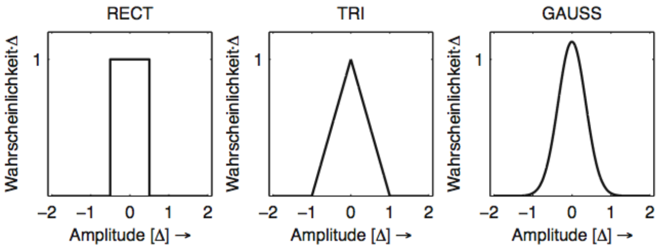
linearization and noise modulation 2/2





dither

noise properties 1/3



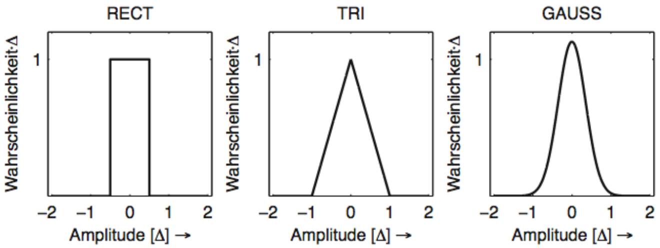
$$
\begin{aligned}
d_{\mathrm{RECT}}(n) &= d(n) \\
d_{\mathrm{TRI}}(n) &= d_{\mathrm{RECT},1}(n) + d_{\mathrm{RECT},2}(n) \\
d_{\mathrm{HP}}(n) &= d(n) - d(n-1)
\end{aligned}
$$

1

[1] note that the gaussian shaped PDF is **not** related to d_{HP}

dither

noise properties 1/3



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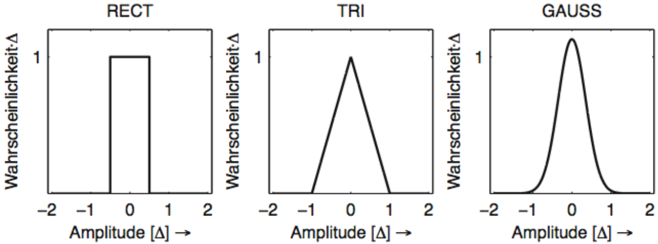
dither

noise properties 1/3



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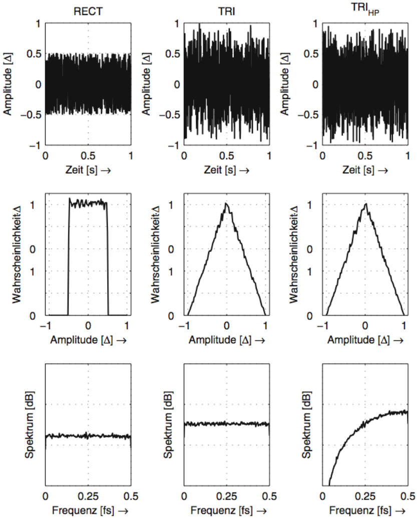
dither

noise properties 2/3



dither

noise properties 3/3

How does the SNR change by adding dither?

dither

noise properties 3/3

How does the SNR change by adding dither?

noise power of d_{RECT} & d_{TRI}

$$
\begin{aligned}
W_{\mathrm{RECT}} &= \frac{\Delta^2}{12} \\
W_{\mathrm{TRI}} &= \frac{\Delta^2}{6}
\end{aligned}
$$

dither
noise properties 3/3

How does the SNR change by adding dither?

noise power of d_{RECT} & d_{TRI}

$$W_{\mathrm{RECT}} = \frac{\Delta^2}{12}$$

$$W_{\mathrm{TRI}} = \frac{\Delta^2}{6}$$

$\Rightarrow$ SNR of dithered full scale signal:

$$SNR_{\mathrm{RECT}} = SNR_{normal} - 3.01 \quad [dB]$$

$$SNR_{\mathrm{TRI}} = SNR_{normal} - 4.77 \quad [dB]$$

dither

audio examples

		sine	speech	music
8 bit	trunc			
	rect			
	tri			
4 bit	trunc			
	rect			
	tri			
2 bit	trunc			
	rect			
	tri			
	trihp			

noise shaping
Z-transform quick and dirty

Z		time
$X(z)$	$\leftrightarrow$	$x(i)$
$X(z) \cdot z^{-k}$	$\leftrightarrow$	$x(n-k)$

transfer function:

$$H(z) = \frac{out}{in} = \frac{Y(z)}{X(z)}$$

spectrum

$$H(\mathrm{j}\Omega) = H(z\big|_{z=e^{\mathrm{j}\Omega}})$$

noise shaping

Z-transform quick and dirty

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noise shaping

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noise shaping

noise shaping introduction

idea:

- filter quantization error, shape its frequency response

$\Rightarrow$ move power to high frequencies

$\Rightarrow$ less recognizable in lower frequencies

noise shaping
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- filter quantization error, shape its frequency response

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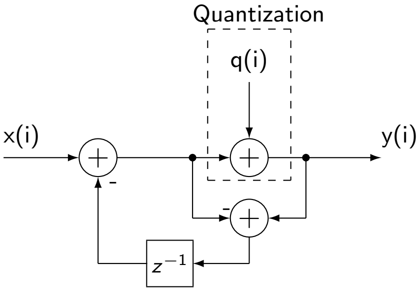
noise shaping

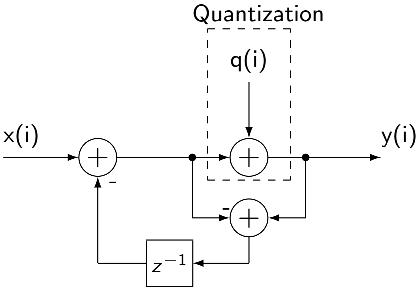
first order noise shaping 1/2



$$
\begin{aligned}
y(i) &= [x(i) - q(i-1)]_Q \\
&= x(i) - q(i-1) + q(i)
\end{aligned}
$$

noise shaping

first order noise shaping 1/2



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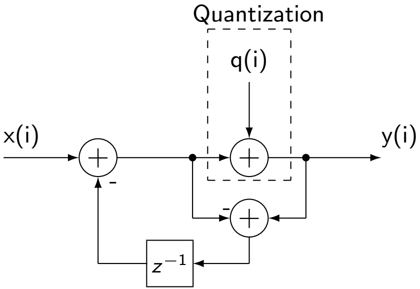
noise shaping

first order noise shaping 1/2



$$\begin{aligned} y(i) &= [x(i) - q(i-1)]_Q \\ &= x(i) - q(i-1) + q(i) \end{aligned}$$

noise shaping

first order noise shaping 2/2

Georgia Tech Center for Music Technology
College of Design

$$
\begin{aligned}
y(i) &= x(i) - q(i-1) + q(i)\\
Y(z) &= X(z) - z^{-1}\cdot Q(z) + Q(z)\\
&= X(z) + \underbrace{\left(1 - z^{-1}\right)}_{H_\mathrm{Q}(z)}\cdot Q(z)\\
\Rightarrow & \\
H_\mathrm{Q}(z) &= 1 - z^{-1}\\
|H_\mathrm{Q}(\mathrm{j}\Omega)| &= |1 - e^{-\mathrm{j}\Omega}|\\
&= 2\cdot\left|\sin\left(\frac{\Omega}{2}\right)\right|
\end{aligned}
$$

noise shaping

first order noise shaping 2/2

Georgia Tech Center for Music Technology College of Design

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$$

$$\Rightarrow$$

$$
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first order noise shaping 2/2

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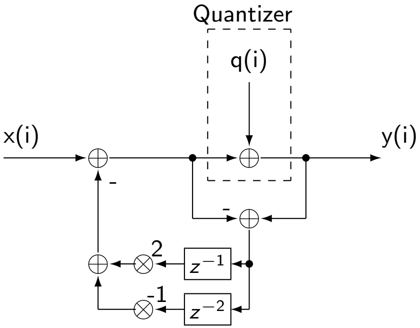
&= 2 \cdot \left|\sin\left(\frac{\Omega}{2}\right)\right|
\end{aligned}
$$

noise shaping

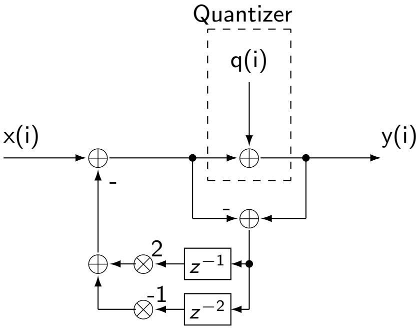
second order noise shaping 1/2



$$
\begin{aligned}
y(i) &= [x(i) - 2 \cdot q(i-1) + q(i-2)]_Q \\
&= x(i) - 2 \cdot q(i-1) + q(i-2) + q(i)
\end{aligned}
$$

noise shaping

second order noise shaping 1/2



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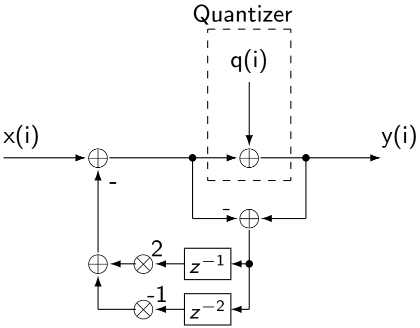
noise shaping

second order noise shaping 1/2



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noise shaping

second order noise shaping 2/2

$$
\begin{aligned}
y(i) &= x(i) - 2\cdot q(i-1) + q(i-2) + q(i)\\
Y(z) &= X(z) - 2\cdot z^{-1}\cdot Q(z) + z^{-2}\cdot Q(z) + Q(z)\\
&= X(z) + (1-z^{-1})^2\cdot Q(z)\\
\Rightarrow &\\
H_\mathrm{Q}(z) &= (1-z^{-1})^2
\end{aligned}
$$

without derivation: nth order noise shaping

$$
\begin{aligned}
Y(z) &= X(z) + (1-z^{-1})^n\cdot Q(z)\\
\Rightarrow &\\
H_\mathrm{Q}(z) &= (1-z^{-1})^n
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$$

noise shaping

second order noise shaping 2/2

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without derivation: nth order noise shaping

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noise shaping

second order noise shaping 2/2

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without derivation: nth order noise shaping

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noise shaping

second order noise shaping 2/2

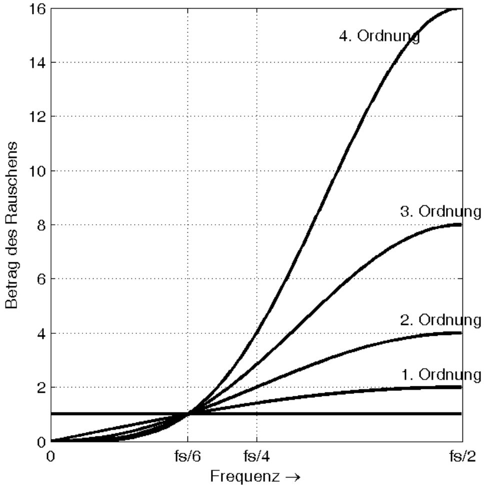
$$\begin{aligned} y(i) &= x(i) - 2\cdot q(i-1) + q(i-2) + q(i) \\ Y(z) &= X(z) - 2\cdot z^{-1}\cdot Q(z) + z^{-2}\cdot Q(z) + Q(z) \\ &= X(z) + (1-z^{-1})^2\cdot Q(z) \\ \Rightarrow & \\ H_\mathrm{Q}(z) &= (1-z^{-1})^2 \end{aligned}$$

without derivation: nth order noise shaping

$$\begin{aligned} Y(z) &= X(z) + (1-z^{-1})^n\cdot Q(z) \\ \Rightarrow & \\ H_\mathrm{Q}(z) &= (1-z^{-1})^n \end{aligned}$$

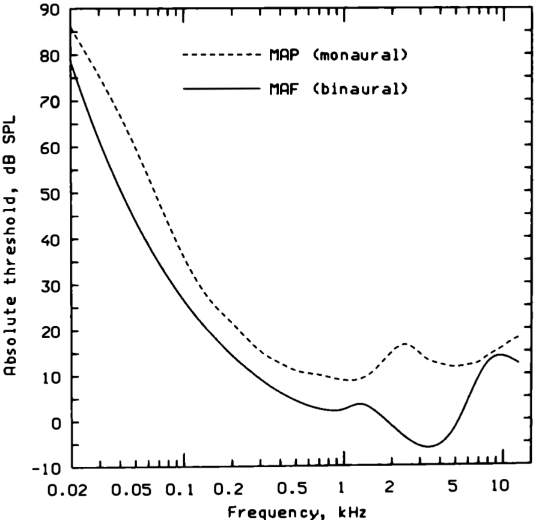
noise shaping

higher order noise shaping



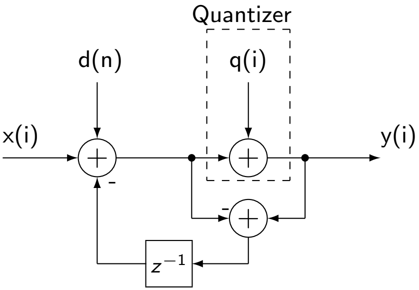
noise shaping

arbitrary noise shaping transfer functions

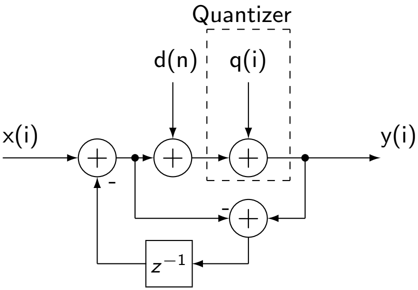


noise shaping

dither & noise shaping 1/3



System A



System B

noise shaping

dither & noise shaping 2/3

System A

$$\begin{aligned} y(i) &= [x(i) + d(n) - q(i-1)]_Q \\ &= x(i) + d(n) - q(i-1) + q(i) \end{aligned}$$

$$\begin{aligned} Y(z) &= X(z) - z^{-1} \cdot Q(z) + Q(z) + D(z) \\ &= X(z) + (1 - z^{-1}) \cdot Q(z) + D(z) \end{aligned}$$

noise shaping

dither & noise shaping 2/3

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noise shaping

dither & noise shaping 3/3

System B

$$\begin{aligned} y(i) &= [x(i) + d(n) - q(i-1) - d(n-1)]_Q \\ &= x(i) - q(i-1) + q(i) - d(n-1) + d(n) \end{aligned}$$

$$\begin{aligned} Y(z) &= X(z) - z^{-1} \cdot Q(z) + Q(z) - z^{-1} \cdot D(z) + D(z) \\ &= X(z) + (1 - z^{-1}) \cdot (Q(z) + D(z)) \end{aligned}$$

noise shaping

dither & noise shaping 3/3

System B

$$\begin{aligned} y(i) &= [x(i) + d(n) - q(i-1) - d(n-1)]_Q \\ &= x(i) - q(i-1) + q(i) - d(n-1) + d(n) \end{aligned}$$

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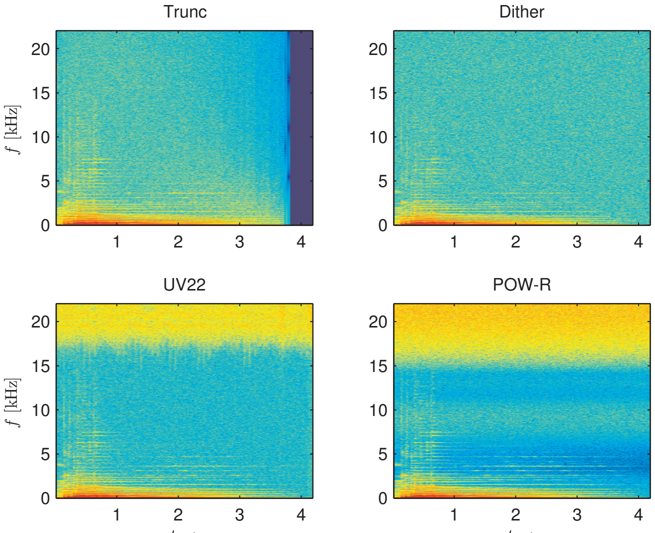
noise shaping
noise shaping audio examples

- 16 bit:
- 8 bit:
- 8 bit dither:
- 8 bit standard noise shaping:
- 8 bit powerful noise shaping:

noise shaping

noise shaping spectrograms



summary

quantization: summary

three ways of improving the (re-)quantization error

- **oversampling**
  - reduces quantization error power in the audible band
  - *process*: oversampling $\rightarrow$ filtering $\rightarrow$ downsampling
- **dither**
  - reduces correlation of error and signal for low amplitude signals
  - increases the power of the quantization error slightly
  - *process*: add triangular shaped low-level noise before word-length reduction
- **noise-shaping**
  - reduces the audibility of the quantization error by shifting it to high frequencies
  - works best at high sample rates
  - *process*: feedback the quantization error

summary

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