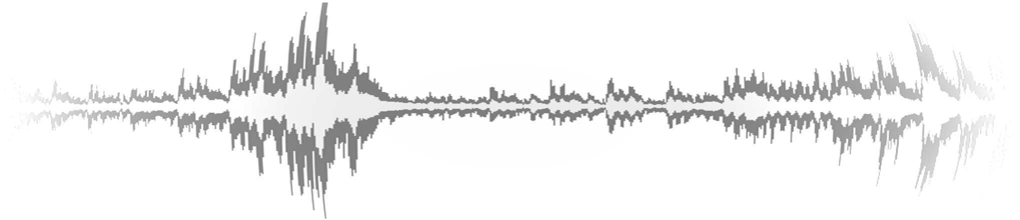
Digital Signal Processing for Music

Part 6: LTI Systems & Convolution

alexander lerch



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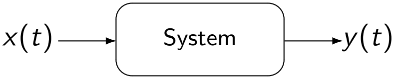
systems
introduction

a system:

- any process producing an output signal in response to an input signal



name examples for systems in signal processing

systems
introduction

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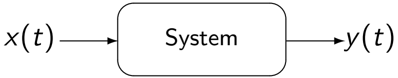

name examples for systems in signal processing

- filters, effects
- vocal tract
- room
- (audio) cable
- ...

systems

linearity and non-linearity

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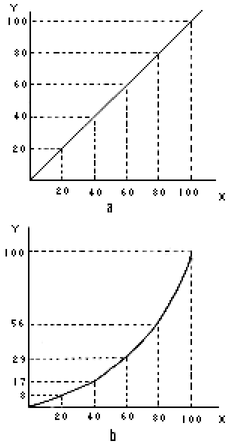
- examples for mostly linear systems:
  - room
  - eq
- examples for non-linear systems:
  - diode
  - vacuum tube



http://www.sfu.ca/sonic-studio/handbook/Linear.html

linearity is defined by two properties

1. **homogeneity**:

$$f(ax) = af(x)$$

2. **superposition** (additivity):

$$f(x+y) = f(x) + f(y)$$

time invariance

- does not change with time:

$$f\left(x(t-\tau)\right) = f(x)(t-\tau)$$

LTI: Linear Time-invariant Systems

are a great simplification for many real-world systems we would like to model — circuits, spring-mass-damper systems, etc.

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LTI system example

velocity of mass an a table

1. hammer gives *impulse*
2. system *responds* with velocity

linearity:
double force, double velocity, multiple strikes add up

time invariance:
system reacts the same whether I do it now or tomorrow

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other system characteristics

- **causality**:
  output depends only on past and present input
- **BIBO stability**:
  output is bounded for bounded input

convolution
introduction

we know how a system reacts to an impulse, but what of a more complex input signal

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- assume that the signal is constructed from many densely packed impulses

$\Rightarrow$ output is superposition of all individual responses

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convolution

$$y(t) = (x * h)(t) := \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

convolution

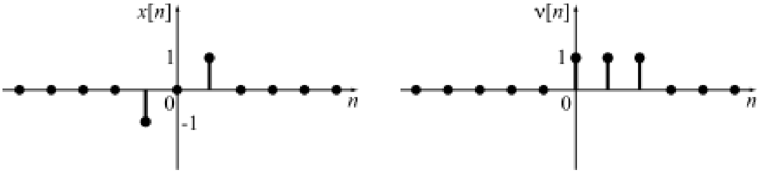
animation

matlab source: [matlab/animateConvolution.m]

convolution

exercise — convolution by hand

compute the convolution of the following two signals



steps:

1. flip one signal
2. multiply the two signals
3. integrate the result
4. shift
5. go to 2.

convolution

identity and impulse response

$$
\begin{aligned}
x(t) &= \delta(t) * x(t) \\
h(t) &= \delta(t) * h(t)
\end{aligned}
$$

- describes the response of a system to an impulse as a function of time
- as an impulse includes all frequencies, the resulting IR defines the response for all frequencies
- the convolution of $\delta(t)$ with a signal/impulse response results in that impulse response

convolution

identity and impulse response

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convolution

properties

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} h(\tau) \cdot x(t-\tau) d\tau$$

- **commutativity**

$$h(t) * x(t) = x(t) * h(t)$$

- **associativity**

$$\left(g(t) * h(t)\right) * x(t) = g(t) * \left(h(t) * x(t)\right)$$

- **distributivity**

$$g(t) * \left(h(t) + x(t)\right) = \left(g(t) * h(t)\right) + \left(g(t) * x(t)\right)$$

convolution
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derivation: commutativity

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substituting $\tau' = t - \tau$:

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systems
summary

- many real-world systems can be approximated by an **LTI system**
- properties of an LTI system:
  - linearity 1: homogeneity (scaling)
  - linearity 2: superposition (additivity)
  - time invariance (system doesn't change)
- additional properties:
  - causality (no future input)
  - BIBO — bounded input bounded output
- impulse response is a complete description of an LTI system
- convolution:
  - describes the process of generating the output of an LTI system from the input
  - is commutative
  - is associative
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