Digital Signal Processing for Music

Part 4: Signal Description

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introduction

description of (random) signals

- ergodic signals do not have a functional description

$\Rightarrow$ other ways of describing these signals have to be found

- ergodic signal characteristics are not time variant

$\Rightarrow$ we are looking for **time-independent descriptions**

- these descriptions might also be convenient to use for some deterministic signals

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audio signal description
probability and occurrence

N: number of overall observations
$N(x_i)$: number of occurrences of symbol x_i

- relative number of occurrences:

$$\hat{p}_i = \frac{N(x_i)}{N}$$

- probability:

$$p_i = \lim_{N \to \infty} \frac{N(x_i)}{N}$$

properties

$$\sum_i p_i = 1$$

$$0 \leq p_i \leq 1$$

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audio signal description
probability distribution example

- roll of a die

value	1	2	3	4	5	6
p(x)	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$

audio signal description
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probability distribution for the roll of two dice

audio signal description

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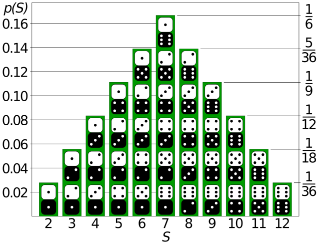
probability distribution for the roll of two dice

value	2	3	4	5	6	7	8	9	10	11	12
p(x)	$1/36$	$1/18$	$1/12$	$1/9$	$5/36$	$1/6$	$5/36$	$1/9$	$1/12$	$1/18$	$1/36$



audio signal description

continuous probability density distribution

$i \rightarrow$ continuous $\Rightarrow$ **PDF**

$$\int_{-\infty}^{\infty} p_X(x)dx = 1$$

$$0 \leq p_X(x)$$

probability of x being a value smaller than or equal x_c

$$\int_{-\infty}^{x_c} p_X(x)dx$$

audio signal description

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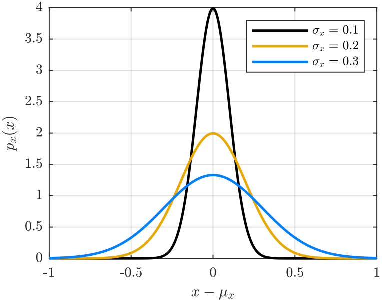
audio signal description

example PDF: Gaussian

$$p_X(x) = \frac{1}{\sqrt{2\pi}\sigma_X} e^{-\frac{(x-\mu_X)^2}{2\sigma_X^2}}$$



matlab source: plotGaussian.m

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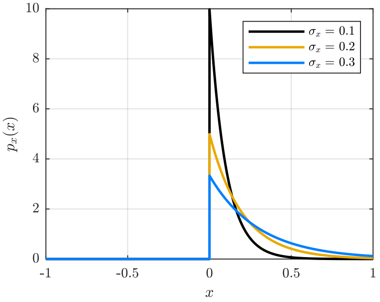
example PDF: Exponential

$$p_X(x) = \begin{cases} \frac{1}{\sigma_X} e^{-\frac{x}{\sigma_X}} & x > 0 \\ 0 & \text{else} \end{cases}$$



matlab source: plotExponential.m

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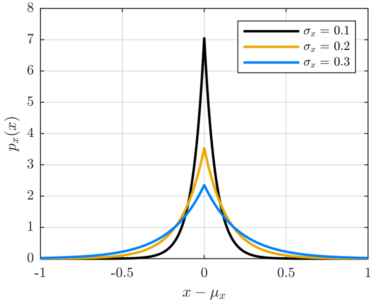
example PDF: Laplace (2-sided exp)

$$p_X(x) = \frac{1}{\sqrt{2}\sigma_X} e^{-\sqrt{2}\frac{|x-\mu_X|}{\sigma_X}}$$



matlab source: plotLaplace.m

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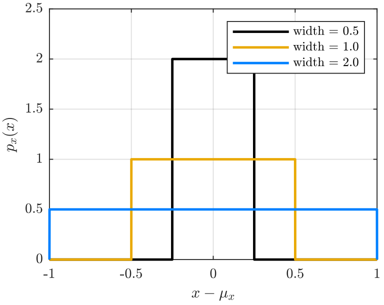
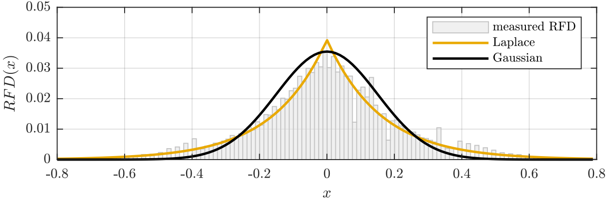
example PDF: Laplace (2-sided exp)

$$p_X(x) = \begin{cases} \frac{1}{width} & |x - mu_x| < width/2 \\ 0 & \text{else} \end{cases}$$



matlab source: plotUniform.m

audio signal description
measured RDF



matlab source: plotRfd.m

audio signal description
PDFs of generated signals 1/2

describe the shape of the following PDFs

audio signal description

PDFs of generated signals 1/2

describe the shape of the following PDFs

- white noise (uniform)
- white noise (Gaussian)
- DC
- square
- sinusoidal
- sawtooth

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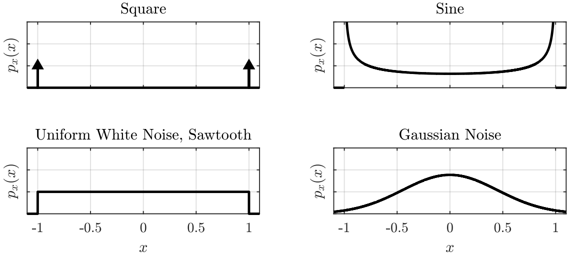
PDFs of generated signals 1/2

describe the shape of the following PDFs



matlab source: plotPdfExamples.m

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expected value 1/3

Example: average grade, five students, grades: 1, 2, 1, 3, 5

$$\hat{\mu}_X = \frac{1+2+1+3+5}{5} = 2.4$$

Grade	# occurrences	relative frequency
1	2	2/5
2	1	1/5
3	1	1/5
4	0	0/5
5	1	1/5

audio signal description

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audio signal description

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$$\mu = \frac{2}{5} \cdot 1 + \frac{1}{5} \cdot 2 + \frac{1}{5} \cdot 3 + \frac{0}{5} \cdot 4 + \frac{1}{5} \cdot 5 = 2.4$$

$$\mu_X = \sum_{\forall x} p(x) \cdot x$$

$$\mu_X = \mathcal{E}\{X\} = \int_{-\infty}^{+\infty} x p_X(x) dx$$

audio signal description

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audio signal description
expected value 3/3

generalization:

$$\mathcal{E}\{f(X)\} = \sum_i f(x)p(x)$$

examples:

- mean: $f(x) = x$
- quad. mean: $f(x) = x^2$

audio signal description

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audio signal description

(central) moments 1/2

- kth moment

$$\mathcal{E}\{X^k\} = \int\limits_{-\infty}^{+\infty} x^k p_X(x)dx$$

- kth central moment

$$\mathcal{E}\{(X-\mu_X)^k\} = \int\limits_{-\infty}^{+\infty} (x-\mu_X)^k p_X(x)dx$$

- example: 2nd order central moment: **Variance**

$$\sigma_X^2 = \mathcal{E}\{(X-\mu_X)^2\} = \int\limits_{-\infty}^{+\infty} (x-\mu_X)^2 p_X(x)dx$$

audio signal description

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audio signal description

(central) moments 2/2

calculation of moments

(central) moments (mean, power, variance, etc.) can be computed from

- the signal
- the signal's PDF

audio signal description

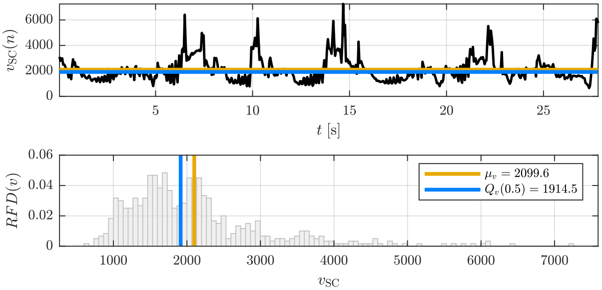
arithmetic mean

- from time series x:

$$\mu_x(n) = \frac{1}{\mathcal{K}} \sum_{i=i_s(n)}^{i_e(n)} x(i)$$

- from distribution p_x:

$$\mu_x(n) = \sum_{x=-\infty}^{\infty} x \cdot p_x(x)$$



matlab source: plotArithmeticMean.m

statistical signal description

variance & standard deviation

measure of *spread* of the signal around its mean

- **variance**
  - from signal block:

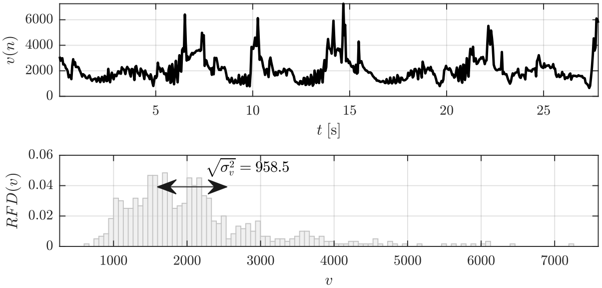
$$\sigma_x^2(n) = \frac{1}{\mathcal{K}} \sum_{i=i_s(n)}^{i_e(n)} \left(x(i) - \mu_x(n)\right)^2$$

  - from distribution:

$$\sigma_x^2(n) = \sum_{x=-\infty}^{\infty} (x - \mu_x)^2 \cdot p_x(x)$$

- **standard deviation**

$$\sigma_x(n) = \sqrt{\sigma_x^2(n)}$$



matlab source: plotStandardDeviation.m

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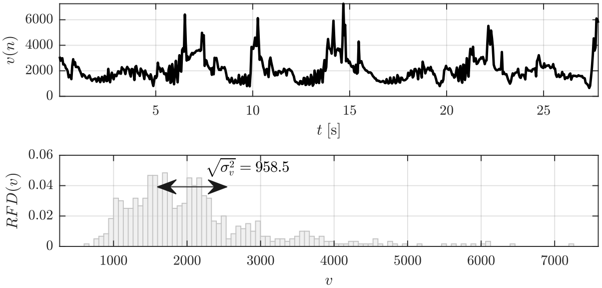

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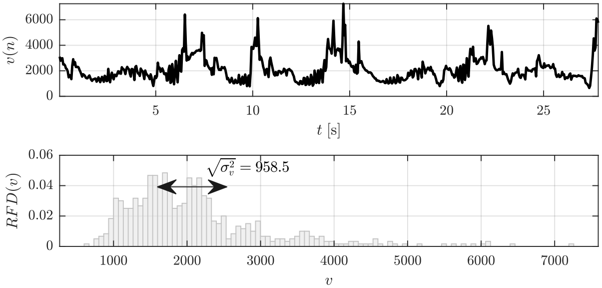

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audio signal description

central moments summary

order	name	obs (cont)	pdf (cont)
1	μ_X	$\frac{1}{T}\int\limits_{-T/2}^{T/2} x(t)dt$	$\int\limits_{-\infty}^{\infty} xp_X(x)dx$
2	σ_X^2	$\frac{1}{T}\int\limits_{-T/2}^{T/2} (x(t)-\mu_X)^2 dt$	$\int\limits_{-\infty}^{\infty} (x-\mu_X)^2 p_X(x)dx$

order	name	obs (disc)	pdf (disc)
1	μ_X	$\frac{1}{N}\sum\limits_{i=0}^{N} x(i)$	$\sum\limits_{\forall x} xp(x)$
2	σ_X^2	$\frac{1}{N}\sum\limits_{i=0}^{N}(x(i)-\mu_X)^2$	$\sum\limits_{\forall x}(x-\mu_X)^2 p(x)$

standard deviation $\sigma_X = \sqrt{\sigma_X^2}$

audio signal description

summary

1. PDF can tell us many important details about a signal
2. statistical measures can be used to describe signal properties, but cannot used to reconstruct the signal
3. statistical measures can be derived from both the time domain signal and its pdf
4. often-used measures are:
   - mean and median
   - variance and standard deviation
   - higher order moments less frequently (skewness, kurtosis)
   - other pdf description possible (quartile-distances etc.)

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