APPENDIX

1 Axioms for the Real Numbers and the Positive Integers

In this book we have assumed an explicit set of axioms for the set of real numbers and for the set of positive integers. In this appendix we will list these axioms and we will illustrate how basic facts, also used without proof in the text, can be derived using them.

Axioms for Real Numbers

The standard axioms for real numbers include both the **field** (or **algebraic) axioms**, used to specify rules for basic arithmetic operations, and the **order axioms**, used to specify properties of the ordering of real numbers.

THE FIELD AXIOMS We begin with the field axioms. As usual, we denote the sum and product of two real numbers x and y by $x + y$ and $x \cdot y$, respectively. (Note that the product of x and y is often denoted by xy without the use of the dot to indicate multiplication. We will not use this abridged notation in this appendix, but will within the text.) Also, by convention, we perform multiplications before additions unless parentheses are used. Although these statements are axioms, they are commonly called *laws* or *rules*. The first two of these axioms tell us that when we add or multiply two real numbers, the result is again a real number; these are the *closure laws*.

- **Closure law for addition** For all real numbers x and y, $x + y$ is a real number.
- **Closure law for multiplication** For all real numbers x and y, $x \cdot y$ is a real number.

The next two axioms tell us that when we add or multiply three real numbers, we get the same result regardless of the order of operations; these are the *associative laws*.

- **Associative law for addition** For all real numbers x, y, and z, $(x + y) + z = x + (y + z)$.
- **Associative law for multiplication** For all real numbers x, y, and z, $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.

Two additional algebraic axioms tell us that the order in which we add or multiply two numbers does not matter; these are the *commutative laws*.

- **Commutative law for addition** For all real numbers x and y, $x + y = y + x$.
- **Commutative law for multiplication** For all real numbers x and y, $x \cdot y = y \cdot x$.

The next two axioms tell us that 0 and 1 are additive and multiplicative identities for the set of real numbers. That is, when we add 0 to a real number or multiply a real number by 1 we do not change this real number. These laws are called *identity laws*.

- **Additive identity law** For every real number x, $x + 0 = 0 + x = x$.
- **Multiplicative identity law** For every real number x, $x \cdot 1 = 1 \cdot x = x$.

Although it seems obvious, we also need the following axiom.

- **Identity elements axiom** The additive identity 0 and the multiplicative identity 1 are distinct, that is $0 \neq 1$.

Two additional axioms tell us that for every real number, there is a real number that can be added to this number to produce 0, and for every nonzero real number, there is a real number by which it can be multiplied to produce 1. These are the *inverse laws*.

- **Inverse law for addition** For every real number x, there exists a real number $-x$ (called the *additive inverse* of x) such that $x + (-x) = (-x) + x = 0$.
- **Inverse law for multiplication** For every nonzero real number x, there exists a real number $1/x$ (called the *multiplicative inverse* of x) such that $x \cdot (1/x) = (1/x) \cdot x = 1$.

The final algebraic axioms for real numbers are the *distributive laws*, which tell us that multiplication distributes over addition; that is, that we obtain the same result when we first add a pair of real numbers and then multiply by a third real number or when we multiply each of these two real numbers by the third real number and then add the two products.

- **Distributive laws** For all real numbers x, y, and z, $x \cdot (y + z) = x \cdot y + x \cdot z$ and $(x + y) \cdot z = x \cdot z + y \cdot z$.

ORDER AXIOMS Next, we will state the *order axioms* for the real numbers, which specify properties of the "greater than" relation, denoted by $>$, on the set of real numbers. We write $x > y$ (and $y < x$) when x is greater than y, and we write $x \geq y$ (and $y \leq x$) when $x > y$ or $x = y$. The first of these axioms tells us that given two real numbers, exactly one of three possibilities occurs: the two numbers are equal, the first is greater than the second, or the second is greater than the first. This rule is called the *trichotomy law*.

- **Trichotomy law** For all real numbers x and y, exactly one of $x = y$, $x > y$, or $y > x$ is true.

Next, we have an axiom, called *the transitivity law*, that tells us that if one number is greater than a second number and this second number is greater than a third, then the first number is greater than the third.

- **Transitivity law** For all real numbers x, y, and z, if $x > y$ and $y > z$, then $x > z$.

We also have two *compatibility laws*, which tell us that when we add a number to both sides in a greater than relationship, the greater than relationship is preserved and when we multiply both sides of a greater than relationship by a *positive real number* (that is, a real number x with $x > 0$), the greater than relationship is preserved.

- **Additive compatibility law** For all real numbers x, y, and z, if $x > y$, then $x + z > y + z$.
- **Multiplicative compatibility law** For all real numbers x, y, and z, if $x > y$ and $z > 0$, then $x \cdot z > y \cdot z$.

We leave it to the reader (see Exercise 15) to prove that for all real numbers x, y, and z, if $x > y$ and $z < 0$, then $x \cdot z < y \cdot z$. That is, multiplication of an inequality by a negative real number reverses the direction of the inequality.

The final axiom for the set of real numbers is the *completeness property*. Before we state this axiom, we need some definitions. First, given a nonempty set A of real numbers, we say that the real number b is an **upper bound** of A if for every real number a in A, $b \geq a$. A real number s is a **least upper bound** of A if s is an upper bound of A and whenever t is an upper bound of A, then we have $s \leq t$.

- **Completeness property** Every nonempty set of real numbers that is bounded above has a least upper bound.

Using Axioms to Prove Basic Facts

The axioms we have listed can be used to prove many properties that are often used without explicit mention. We give several examples of results we can prove using axioms and leave the proof of a variety of other properties as exercises. Although the results we will prove seem quite obvious, proving them using only the axioms we have stated can be challenging.

THEOREM 1 The additive identity element 0 of the real numbers is unique.

Proof: To show that the additive identity element 0 of the real numbers is unique, suppose that $0'$ is also an additive identity for the real numbers. This means that $0' + x = x + 0' = x$ whenever x is a real number. By the additive identity law, it follows that $0 + 0' = 0'$. Because $0'$ is an additive identity, we know that $0 + 0' = 0$. It follows that $0 = 0'$, because both equal $0 + 0'$. This shows that 0 is the unique additive identity for the real numbers. ◁

THEOREM 2 The additive inverse of a real number x is unique.

Proof: Let x be a real number. Suppose that y and z are both additive inverses of x. Then,

$$\begin{aligned} y &= 0 + y && \text{by the additive identity law} \\ &= (z + x) + y && \text{because } z \text{ is an additive inverse of } x \\ &= z + (x + y) && \text{by the associative law for addition} \\ &= z + 0 && \text{because } y \text{ is an additive inverse of } x \\ &= z && \text{by the additive identity law.} \end{aligned}$$

It follows that $y = z$. ◁

Theorems 1 and 2 tell us that the additive identity and additive inverses are unique. Theorems 3 and 4 tell us that the multiplicative identity and multiplicative inverses of nonzero real numbers are also unique. We leave their proofs as exercises.

THEOREM 3 The multiplicative identity element 1 of the real numbers is unique.

THEOREM 4 The multiplicative inverse of a nonzero real number x is unique.

THEOREM 5 For every real number x, $x \cdot 0 = 0$.

Proof: Suppose that x is a real number. By the additive inverse law, there is a real number y that is the additive inverse of $x \cdot 0$, so we have $x \cdot 0 + y = 0$. By the additive identity law, $0 + 0 = 0$. Using the distributive law, we see that $x \cdot 0 = x \cdot (0 + 0) = x \cdot 0 + x \cdot 0$. It follows that

$$0 = x \cdot 0 + y = (x \cdot 0 + x \cdot 0) + y.$$

Next, note that by the associative law for addition and because $x \cdot 0 + y = 0$, it follows that

$$(x \cdot 0 + x \cdot 0) + y = x \cdot 0 + (x \cdot 0 + y) = x \cdot 0 + 0.$$

Finally, by the additive identity law, we know that $x \cdot 0 + 0 = x \cdot 0$. Consequently, $x \cdot 0 = 0$. ◁

THEOREM 6 For all real numbers x and y, if $x \cdot y = 0$, then $x = 0$ or $y = 0$.

Proof: Suppose that x and y are real numbers and $x \cdot y = 0$. If $x \neq 0$, then, by the multiplicative inverse law, x has a multiplicative inverse $1/x$, such that $x \cdot (1/x) = (1/x) \cdot x = 1$. Because $x \cdot y = 0$, we have $(1/x) \cdot (x \cdot y) = (1/x) \cdot 0 = 0$ by Theorem 5. Using the associate law for multiplication, we have $((1/x) \cdot x) \cdot y = 0$. This means that $1 \cdot y = 0$. By the multiplicative identity rule, we see that $1 \cdot y = y$, so $y = 0$. Consequently, either $x = 0$ or $y = 0$. ◁

THEOREM 7 The multiplicative identity element 1 in the set of real numbers is greater than the additive identity element 0.

Proof: By the trichotomy law, either $0 = 1$, $0 > 1$, or $1 > 0$. We know by the identity elements axiom that $0 \neq 1$.

So, assume that $0 > 1$. We will show that this assumption leads to a contradiction. By the additive inverse law, 1 has an additive inverse -1 with $1 + (-1) = 0$. The additive compatibility law tells us that $0 + (-1) > 1 + (-1) = 0$; the additive identity law tells us that $0 + (-1) = -1$. Consequently, $-1 > 0$, and by the multiplicative compatibility law, $(-1) \cdot (-1) > (-1) \cdot 0$. By Theorem 5 the right-hand side of last inequality is 0. By the distributive law, $(-1) \cdot (-1) + (-1) \cdot 1 = (-1) \cdot (-1 + 1) = (-1) \cdot 0 = 0$. Hence, the left-hand side of this last inequality, $(-1) \cdot (-1)$, is the unique additive inverse of -1, so this side of the inequality equals 1. Consequently this last inequality becomes $1 > 0$, contradicting the trichotomy law because we had assumed that $0 > 1$.

Because we know that $0 \neq 1$ and that it is impossible for $0 > 1$, by the trichotomy law, we conclude that $1 > 0$. ◁

ARCHIMEDES (287 B.C.E.–212 B.C.E.) Archimedes was one of the greatest scientists and mathematicians of ancient times. He was born in Syracuse, a Greek city-state in Sicily. His father, Phidias, was an astronomer. Archimedes was educated in Alexandria, Egypt. After completing his studies, he returned to Syracuse, where he spent the rest of his life. Little is known about his personal life; we do not know whether he was ever married or had children. Archimedes was killed in 212 B.C.E. by a Roman soldier when the Romans overran Syracuse.

Archimedes made many important discoveries in geometry. His method for computing the area under a curve was described two thousand years before his ideas were re-invented as part of integral calculus. Archimedes also developed a method for expressing large integers inexpressible by the usual Greek method. He discovered a method for computing the volume of a sphere, as well as of other solids, and he calculated an approximation of π. Archimedes was also an accomplished engineer and inventor; his machine for pumping water, now called *Archimedes' screw*, is still in use today. Perhaps his best known discovery is the *principle of buoyancy*, which tells us that an object submerged in liquid becomes lighter by an amount equal to the weight it displaces. Some histories tell us that Archimedes was an early streaker, running naked through the streets of Syracuse shouting "Eureka" (which means "I have found it") when he made this discovery. He is also known for his clever use of machines that held off Roman forces sieging Syracuse for several years during the Second Punic War.

The next theorem tells us that for every real number there is an integer (where by an *integer*, we mean 0, the sum of any number of 1s, and the additive inverses of these sums) greater than this real number. This result is attributed to the Greek mathematician Archimedes. The result can be found in Book V of Euclid's *Elements*.

THEOREM 8 **ARCHIMEDEAN PROPERTY** For every real number x there exists an integer n such that $n > x$.

Proof: Suppose that x is a real number such that $n \leq x$ for every integer n. Then x is an upper bound of the set of integers. By the completeness property it follows that the set of integers has a least upper bound M. Because $M - 1 < M$ and M is a least upper bound of the set of integers, $M - 1$ is not an upper bound of the set of integers. This means that there is an integer n with $n > M - 1$. This implies that $n + 1 > M$, contradicting the fact that M is an upper bound of the set of integers. ◁

Axioms for the Set of Positive Integers

The axioms we now list specify the set of positive integers as
the subset of the set of integers satisfying four key properties. We assume the truth of these axioms in this textbook.

- **Axiom 1** The number 1 is a positive integer.
- **Axiom 2** If n is a positive integer, then $n + 1$, the *successor* of n, is also a positive integer.
- **Axiom 3** Every positive integer other than 1 is the successor of a positive integer.
- **Axiom 4** **The Well-Ordering Property** Every nonempty subset of the set of positive integers has a least element.

In Sections 5.1 and 5.2 it is shown that the well-ordering principle is equivalent to the principle of mathematical induction.

- **Mathematical induction axiom** If S is a set of positive integers such that $1 \in S$ and for all positive integers n if $n \in S$, then $n + 1 \in S$, then S is the set of positive integers.

Most mathematicians take the real number system as already existing, with the real numbers satisfying the axioms we have listed in this appendix. However, mathematicians in the nineteenth century developed techniques to construct the set of real numbers, starting with more basic sets of numbers. (The process of constructing the real numbers is sometimes studied in advanced undergraduate mathematics classes. A treatment of this can be found in [Mo91], for instance.) The first step in the process is the construction of the set of positive integers using axioms 1–3 and either the well-ordering property or the mathematical induction axiom. Then, the operations of addition and multiplication of positive integers are defined. Once this has been done, the set of integers can be constructed using equivalence classes of pairs of positive integers where $(a, b) \sim (c, d)$ if and only if $a + d = b + c$; addition and multiplication of integers can be defined using these pairs (see Exercise 21). (Equivalence relations and equivalence classes are discussed in Chapter 9.) Next, the set of rational numbers can be constructed using the equivalence classes of pairs of integers where the second integer in the pair is not zero, where $(a, b) \approx (c, d)$ if and only if $a \cdot d = b \cdot c$; addition and multiplication of rational numbers can be defined in terms of these pairs (see Exercise 22). Using infinite sequences, the set of real numbers can then be constructed from the set of rational numbers. The interested reader will find it worthwhile to read through the many details of the steps of this construction.

Exercises

Use only the axioms and theorems in this appendix in the proofs in your answers to these exercises.

1. Prove Theorem 3, which states that the multiplicative identity element of the real numbers is unique.

2. Prove Theorem 4, which states that for every nonzero real number x, the multiplicative inverse of x is unique.

3. Prove that for all real numbers x and y, $(-x) \cdot y = x \cdot (-y) = -(x \cdot y)$.

4. Prove that for all real numbers x and y, $-(x + y) = (-x) + (-y)$.

5. Prove that for all real numbers x and y, $(-x) \cdot (-y) = x \cdot y$.

6. Prove that for all real numbers x, y, and z, if $x + z = y + z$, then $x = y$.

7. Prove that for every real number x, $-(-x) = x$.

Define the **difference** $x - y$ of real numbers x and y by $x - y = x + (-y)$, where $-y$ is the additive inverse of y, and the **quotient** x/y, where $y \neq 0$, by $x/y = x \cdot (1/y)$, where $1/y$ is the multiplicative inverse of y.

8. Prove that for all real numbers x and y, $x = y$ if and only if $x - y = 0$.

9. Prove that for all real numbers x and y, $-x - y = -(x + y)$.

10. Prove that for all nonzero real numbers x and y, $1/(x/y) = y/x$, where $1/(x/y)$ is the multiplicative inverse of x/y.

11. Prove that for all real numbers w, x, y, and z, if $x \neq 0$ and $z \neq 0$, then $(w/x) + (y/z) = (w \cdot z + x \cdot y)/(x \cdot z)$.

12. Prove that for every positive real number x, $1/x$ is also a positive real number.

13. Prove that for all positive real numbers x and y, $x \cdot y$ is also a positive real number.

14. Prove that for all real numbers x and y, if $x > 0$ and $y < 0$, then $x \cdot y < 0$.

15. Prove that for all real numbers x, y, and z, if $x > y$ and $z < 0$, then $x \cdot z < y \cdot z$.

16. Prove that for every real number x, $x \neq 0$ if and only if $x^2 > 0$.

17. Prove that for all real numbers w, x, y, and z, if $w < x$ and $y < z$, then $w + y < x + z$.

18. Prove that for all positive real numbers x and y, if $x < y$, then $1/x > 1/y$.

19. Prove that for every positive real number x, there exists a positive integer n such that $n \cdot x > 1$.

***20.** Prove that between every two distinct real numbers there is a rational number (that is, a number of the form x/y, where x and y are integers with $y \neq 0$).

Exercises 21 and 22 involve the notion of an equivalence relation, discussed in Chapter 9 of the text.

***21.** Define a relation $\sim$ on the set of ordered pairs of positive integers by $(w, x) \sim (y, z)$ if and only if $w + z = x + y$. Show that the operations $[(w, x)]_\sim + [(y, z)]_\sim = [(w + y, x + z)]_\sim$ and $[(w, x)]_\sim \cdot [(y, z)]_\sim = [(w \cdot y + x \cdot z, x \cdot y + w \cdot z)]_\sim$ are well-defined, that is, they do not depend on the representative of the equivalence classes chosen for the computation.

***22.** Define a relation $\approx$ on ordered pairs of integers with second entry nonzero by $(w, x) \approx (y, z)$ if and only if $w \cdot z = x \cdot y$. Show that the operations $[(w, x)]_\approx + [(y, z)]_\approx = [(w \cdot z + x \cdot y, x \cdot z)]_\approx$ and $[(w, x)]_\approx \cdot [(y, z)]_\approx = [(w \cdot y, x \cdot z)]_\approx$ are well-defined, that is, they do not depend on the representative of the equivalence classes chosen for the computation.