repetition allowed from S, formed by subtracting $k-1$ from the kth element.]

c) Conclude that there are $C(n+r-1, r)$ r-combinations with repetition allowed from a set with n elements.

50. How many ways are there to distribute five distinguishable objects into three indistinguishable boxes?

51. How many ways are there to distribute six distinguishable objects into four indistinguishable boxes so that each of the boxes contains at least one object?

52. How many ways are there to put five temporary employees into four identical offices?

53. How many ways are there to put six temporary employees into four identical offices so that there is at least one temporary employee in each of these four offices?

54. How many ways are there to distribute five indistinguishable objects into three indistinguishable boxes?

55. How many ways are there to distribute six indistinguishable objects into four indistinguishable boxes so that each of the boxes contains at least one object?

56. How many ways are there to pack eight identical DVDs into five indistinguishable boxes so that each box contains at least one DVD?

57. How many ways are there to pack nine identical DVDs into three indistinguishable boxes so that each box contains at least two DVDs?

58. How many ways are there to distribute five balls into seven boxes if each box must have at most one ball in it if

a) both the balls and boxes are labeled?

b) the balls are labeled, but the boxes are unlabeled?

c) the balls are unlabeled, but the boxes are labeled?

d) both the balls and boxes are unlabeled?

59. How many ways are there to distribute five balls into three boxes if each box must have at least one ball in it if

a) both the balls and boxes are labeled?

b) the balls are labeled, but the boxes are unlabeled?

c) the balls are unlabeled, but the boxes are labeled?

d) both the balls and boxes are unlabeled?

60. Suppose that a basketball league has 32 teams, split into two conferences of 16 teams each. Each conference is split into three divisions. Suppose that the North Central Division has five teams. Each of the teams in the North Central Division plays four games against each of the other teams in this division, three games against each of the 11 remaining teams in the conference, and two games against each of the 16 teams in the other conference. In how many different orders can the games of one of the teams in the North Central Division be scheduled?

***61.** Suppose that a weapons inspector must inspect each of five different sites twice, visiting one site per day. The inspector is free to select the order in which to visit these sites, but cannot visit site X, the most suspicious site, on two consecutive days. In how many different orders can the inspector visit these sites?

62. How many different terms are there in the expansion of $(x_1 + x_2 + \cdots + x_m)^n$ after all terms with identical sets of exponents are added?

***63.** Prove the **Multinomial Theorem:** If n is a positive integer, then

$$(x_1 + x_2 + \cdots + x_m)^n = \sum_{n_1+n_2+\cdots+n_m=n} C(n; n_1, n_2, \ldots, n_m) x_1^{n_1} x_2^{n_2} \cdots x_m^{n_m},$$

where

$$C(n; n_1, n_2, \ldots, n_m) = \frac{n!}{n_1!\, n_2! \cdots n_m!}$$

is a **multinomial coefficient**.

64. Find the expansion of $(x+y+z)^4$.

65. Find the coefficient of $x^3y^2z^5$ in $(x+y+z)^{10}$.

66. How many terms are there in the expansion of

$$(x+y+z)^{100}?$$

6.6 Generating Permutations and Combinations

Introduction

Methods for counting various types of permutations and combinations were described in the previous sections of this chapter, but sometimes permutations or combinations need to be generated, not just counted. Consider the following three problems. First, suppose that a salesperson must visit six different cities. In which order should these cities be visited to minimize total travel time? One way to determine the best order is to determine the travel time for each of the $6! = 720$ different orders in which the cities can be visited and choose the one with the smallest travel time. Second, suppose we are given a set of six positive integers and wish to find a subset of them that has 100 as their sum, if such a subset exists. One way to find these numbers is to generate all $2^6 = 64$ subsets and check the sum of their elements. Third, suppose a laboratory has 95 employees. A group of 12 of these employees with a particular set of 25 skills is needed for a project. (Each employee can have one or more of these skills.) One way to find such a set of employees is to generate all sets of 12 of these employees and check whether they have the desired skills. These examples show that it is often necessary to generate permutations and combinations to solve problems.

Generating Permutations

Links

Any set with n elements can be placed in one-to-one correspondence with the set $\{1, 2, 3, \ldots, n\}$. We can list the permutations of any set of n elements by generating the permutations of the n smallest positive integers and then replacing these integers with the corresponding elements. Many different algorithms have been developed to generate the $n!$ permutations of this set. We will describe one of these that is based on the **lexicographic** (or **dictionary) ordering** of the set of permutations of $\{1, 2, 3, \ldots, n\}$. In this ordering, the permutation $a_1a_2 \cdots a_n$ precedes the permutation of $b_1b_2 \cdots b_n$, if for some k, with $1 \leq k \leq n$, $a_1 = b_1, a_2 = b_2, \ldots, a_{k-1} = b_{k-1}$, and $a_k < b_k$. In other words, a permutation of the set of the n smallest positive integers precedes (in lexicographic order) a second permutation if the number in this permutation in the first position where the two permutations disagree is smaller than the number in that position in the second permutation.

EXAMPLE 1 The permutation 23415 of the set $\{1, 2, 3, 4, 5\}$ precedes the permutation 23514, because these permutations agree in the first two positions, but the number in the third position in the first permutation, 4, is smaller than the number in the third position in the second permutation, 5. Similarly, the permutation 41532 precedes 52143. ◀

An algorithm for generating the permutations of $\{1, 2, \ldots, n\}$ can be based on a procedure that constructs the next permutation in lexicographic order following a given permutation $a_1a_2 \cdots a_n$. We will show how this can be done. First, suppose that $a_{n-1} < a_n$. Interchange a_{n-1} and a_n to obtain a larger permutation. No other permutation is both larger than the original permutation and smaller than the permutation obtained by interchanging a_{n-1} and a_n. For instance, the next larger permutation after 234156 is 234165. On the other hand, if $a_{n-1} > a_n$, then a larger permutation cannot be obtained by interchanging these last two terms in the permutation. Look at the last three integers in the permutation. If $a_{n-2} < a_{n-1}$, then the last three integers in the permutation can be rearranged to obtain the next largest permutation. Put the smaller of the two integers a_{n-1} and a_n that is greater than a_{n-2} in position $n - 2$. Then, place the remaining integer and a_{n-2} into the last two positions in increasing order. For instance, the next larger permutation after 234165 is 234516.

On the other hand, if $a_{n-2} > a_{n-1}$ (and $a_{n-1} > a_n$), then a larger permutation cannot be obtained by permuting the last three terms in the permutation. Based on these observations, a general method can be described for producing the next larger permutation in increasing order following a given permutation $a_1a_2 \cdots a_n$. First, find the integers a_j and a_{j+1} with $a_j < a_{j+1}$ and

$$a_{j+1} > a_{j+2} > \cdots > a_n,$$

that is, the last pair of adjacent integers in the permutation where the first integer in the pair is smaller than the second. Then, the next larger permutation in lexicographic order is obtained by putting in the jth position the least integer among $a_{j+1}, a_{j+2}, \ldots,$ and a_n that is greater than a_j and listing in increasing order the rest of the integers $a_j, a_{j+1}, \ldots, a_n$ in positions $j + 1$ to n. It is easy to see that there is no other permutation larger than the permutation $a_1a_2 \cdots a_n$ but smaller than the new permutation produced. (The verification of this fact is left as an exercise for the reader.)

EXAMPLE 2 What is the next permutation in lexicographic order after 362541?

Extra Examples

Solution: The last pair of integers a_j and a_{j+1} where $a_j < a_{j+1}$ is $a_3 = 2$ and $a_4 = 5$. The least integer to the right of 2 that is greater than 2 in the permutation is $a_5 = 4$. Hence, 4 is placed in the third position. Then the integers 2, 5, and 1 are placed in order in the last three positions, giving 125 as the last three positions of the permutation. Hence, the next permutation is 364125. ◀

To produce the $n!$ permutations of the integers $1, 2, 3, \ldots, n$, begin with the smallest permutation in lexicographic order, namely, $123 \cdots n$, and successively apply the procedure described for producing the next larger permutation of $n! - 1$ times. This yields all the permutations of the n smallest integers in lexicographic order.

EXAMPLE 3 Generate the permutations of the integers 1, 2, 3 in lexicographic order.

Solution: Begin with 123. The next permutation is obtained by interchanging 3 and 2 to obtain 132. Next, because $3 > 2$ and $1 < 3$, permute the three integers in 132. Put the smaller of 3 and 2 in the first position, and then put 1 and 3 in increasing order in positions 2 and 3 to obtain 213. This is followed by 231, obtained by interchanging 1 and 3, because $1 < 3$. The next larger permutation has 3 in the first position, followed by 1 and 2 in increasing order, namely, 312. Finally, interchange 1 and 2 to obtain the last permutation, 321. We have generated the permutations of 1, 2, 3 in lexicographic order. They are 123, 132, 213, 231, 312, and 321. ◀

Algorithm 1 displays the procedure for finding the next permutation in lexicographic order after a permutation that is not $n \;\; n-1 \;\; n-2 \;\; \ldots \;\; 2 \;\; 1$, which is the largest permutation.

ALGORITHM 1 Generating the Next Permutation in Lexicographic Order.

procedure *next permutation*($a_1a_2 \ldots a_n$: permutation of
        $\{1, 2, \ldots, n\}$ not equal to $n \;\; n-1 \;\; \ldots \;\; 2 \;\; 1$)
$j := n - 1$
while $a_j > a_{j+1}$
    $j := j - 1$
$\{j$ is the largest subscript with $a_j < a_{j+1}\}$
$k := n$
while $a_j > a_k$
    $k := k - 1$
$\{a_k$ is the smallest integer greater than a_j to the right of $a_j\}$
interchange a_j and a_k
$r := n$
$s := j + 1$
while $r > s$
    interchange a_r and a_s
    $r := r - 1$
    $s := s + 1$
{this puts the tail end of the permutation after the jth position in increasing order}
$\{a_1a_2 \ldots a_n$ is now the next permutation$\}$

Generating Combinations

How can we generate all the combinations of the elements of a finite set? Because a combination is just a subset, we can use the correspondence between subsets of $\{a_1, a_2, \ldots, a_n\}$ and bit strings of length n.

Links

Recall that the bit string corresponding to a subset has a 1 in position k if a_k is in the subset, and has a 0 in this position if a_k is not in the subset. If all the bit strings of length n can be listed, then by the correspondence between subsets and bit strings, a list of all the subsets is obtained.

Recall that a bit string of length n is also the binary expansion of an integer between 0 and $2^n - 1$. The 2^n bit strings can be listed in order of their increasing size as integers in their binary expansions. To produce all binary expansions of length n, start with the bit string $000\ldots00$, with n zeros. Then, successively find the next expansion until the bit string $111\ldots11$ is obtained. At each stage the next binary expansion is found by locating the first position from the right that is not a 1, then changing all the 1s to the right of this position to 0s and making this first 0 (from the right) a 1.

EXAMPLE 4 Find the next bit string after 10 0010 0111.

Solution: The first bit from the right that is not a 1 is the fourth bit from the right. Change this bit to a 1 and change all the following bits to 0s. This produces the next larger bit string, 10 0010 1000. ◀

The procedure for producing the next larger bit string after $b_{n-1}b_{n-2}\ldots b_1b_0$ is given as Algorithm 2.

ALGORITHM 2 Generating the Next Larger Bit String.

procedure *next bit string*($b_{n-1}\,b_{n-2}\ldots b_1b_0$: bit string not equal to $11\ldots11$)
$i := 0$
while $b_i = 1$
  $b_i := 0$
  $i := i + 1$
$b_i := 1$
$\{b_{n-1}\,b_{n-2}\ldots b_1b_0$ is now the next bit string$\}$

Next, an algorithm for generating the r-combinations of the set $\{1, 2, 3, \ldots, n\}$ will be given. An r-combination can be represented by a sequence containing the elements in the subset in increasing order. The r-combinations can be listed using lexicographic order on these sequences. In this lexicographic ordering, the first r-combination is $\{1, 2, \ldots, r-1, r\}$ and the last r-combination is $\{n-r+1, n-r+2, \ldots, n-1, n\}$. The next r-combination after $a_1a_2\cdots a_r$ can be obtained in the following way: First, locate the last element a_i in the sequence such that $a_i \neq n - r + i$. Then, replace a_i with $a_i + 1$ and a_j with $a_i + j - i + 1$, for $j = i + 1, i + 2, \ldots, r$. It is left for the reader to show that this produces the next larger r-combination in lexicographic order. This procedure is illustrated with Example 5.

EXAMPLE 5 Find the next larger 4-combination of the set $\{1, 2, 3, 4, 5, 6\}$ after $\{1, 2, 5, 6\}$.

Solution: The last term among the terms a_i with $a_1 = 1$, $a_2 = 2$, $a_3 = 5$, and $a_4 = 6$ such that $a_i \neq 6 - 4 + i$ is $a_2 = 2$. To obtain the next larger 4-combination, increment a_2 by 1 to obtain $a_2 = 3$. Then set $a_3 = 3 + 1 = 4$ and $a_4 = 3 + 2 = 5$. Hence the next larger 4-combination is $\{1, 3, 4, 5\}$. ◀

Algorithm 3 displays pseudocode for this procedure.

ALGORITHM 3 Generating the Next r-Combination in Lexicographic Order.

procedure *next r-combination*($\{a_1, a_2, \ldots, a_r\}$: proper subset of
$\{1, 2, \ldots, n\}$ not equal to $\{n - r + 1, \ldots, n\}$ with
$a_1 < a_2 < \cdots < a_r$)
$i := r$
while $a_i = n - r + i$
  $i := i - 1$
$a_i := a_i + 1$
for $j := i + 1$ **to** r
  $a_j := a_i + j - i$
$\{\{a_1, a_2, \ldots, a_r\}$ is now the next combination$\}$

Exercises

1. Place these permutations of $\{1, 2, 3, 4, 5\}$ in lexicographic order: 43521, 15432, 45321, 23451, 23514, 14532, 21345, 45213, 31452, 31542.

2. Place these permutations of $\{1,2,3,4,5,6\}$ in lexicographic order: 234561, 231456, 165432, 156423, 543216, 541236, 231465, 314562, 432561, 654321, 654312, 435612.

3. The name of a file in a computer directory consists of three uppercase letters followed by a digit, where each letter is either A, B, or C, and each digit is either 1 or 2. List the name of these files in lexicographic order, where we order letters using the usual alphabetic order of letters.

4. Suppose that the name of a file in a computer directory consists of three digits followed by two lowercase letters and each digit is 0, 1, or 2, and each letter is either a or b. List the name of these files in lexicographic order, where we order letters using the usual alphabetic order of letters.

5. Find the next larger permutation in lexicographic order after each of these permutations.

a) 1432 **b)** 54123 **c)** 12453
d) 45231 **e)** 6714235 **f)** 31528764

6. Find the next larger permutation in lexicographic order after each of these permutations.

a) 1342 **b)** 45321 **c)** 13245
d) 612345 **e)** 1623547 **f)** 23587416

7. Use Algorithm 1 to generate the 24 permutations of the first four positive integers in lexicographic order.

8. Use Algorithm 2 to list all the subsets of the set $\{1, 2, 3, 4\}$.

9. Use Algorithm 3 to list all the 3-combinations of $\{1, 2, 3, 4, 5\}$.

10. Show that Algorithm 1 produces the next larger permutation in lexicographic order.

11. Show that Algorithm 3 produces the next larger r-combination in lexicographic order after a given r-combination.

12. Develop an algorithm for generating the r-permutations of a set of n elements.

13. List all 3-permutations of $\{1, 2, 3, 4, 5\}$.

The remaining exercises in this section develop another algorithm for generating the permutations of $\{1, 2, 3, \ldots, n\}$. This algorithm is based on Cantor expansions of integers. Every nonnegative integer less than $n!$ has a unique Cantor expansion

$$a_1 1! + a_2 2! + \cdots + a_{n-1}(n-1)!$$

where a_i is a nonnegative integer not exceeding i, for $i = 1, 2, \ldots, n - 1$. The integers $a_1, a_2, \ldots, a_{n-1}$ are called the **Cantor digits** of this integer.

Given a permutation of $\{1, 2, \ldots, n\}$, let $a_{k-1}, k = 2, 3, \ldots, n$, be the number of integers less than k that follow k in the permutation. For instance, in the permutation 43215, a_1 is the number of integers less than 2 that follow 2, so $a_1 = 1$. Similarly, for this example $a_2 = 2$, $a_3 = 3$, and $a_4 = 0$. Consider the function from the set of permutations of $\{1, 2, 3, \ldots, n\}$ to the set of nonnegative integers less than $n!$ that sends a permutation to the integer that has $a_1, a_2, \ldots, a_{n-1}$, defined in this way, as its Cantor digits.

14. Find the Cantor digits $a_1, a_2, \ldots, a_{n-1}$ that correspond to these permutations.

a) 246531 **b)** 12345 **c)** 654321

***15.** Show that the correspondence described in the preamble is a bijection between the set of permutations of $\{1, 2, 3, \ldots, n\}$ and the nonnegative integers less than $n!$.

16. Find the permutations of $\{1, 2, 3, 4, 5\}$ that correspond to these integers with respect to the correspondence between Cantor expansions and permutations as described in the preamble to Exercise 14.

a) 3 **b)** 89 **c)** 111

17. Develop an algorithm for producing all permutations of a set of n elements based on the correspondence described in the preamble to Exercise 14.

Key Terms and Results

TERMS

combinatorics: the study of arrangements of objects
enumeration: the counting of arrangements of objects
tree diagram: a diagram made up of a root, branches leaving the root, and other branches leaving some of the endpoints of branches
permutation: an ordered arrangement of the elements of a set
***r*-permutation:** an ordered arrangement of r elements of a set
***P*(*n*,*r*):** the number of r-permutations of a set with n elements
***r*-combination:** an unordered selection of r elements of a set
***C*(*n*,*r*):** the number of r-combinations of a set with n elements
binomial coefficient $\binom{n}{r}$: also the number of r-combinations of a set with n elements
combinatorial proof: a proof that uses counting arguments rather than algebraic manipulation to prove a result
Pascal's triangle: a representation of the binomial coefficients where the ith row of the triangle contains $\binom{i}{j}$ for $j = 0, 1, 2, \ldots, i$
***S*(*n*,*j*):** the Stirling number of the second kind denoting the number of ways to distribute n distinguishable objects into j indistinguishable boxes so that no box is empty

RESULTS

product rule for counting: The number of ways to do a procedure that consists of two tasks is the product of the number of ways to do the first task and the number of ways to do the second task after the first task has been done.
product rule for sets: The number of elements in the Cartesian product of finite sets is the product of the number of elements in each set.
sum rule for counting: The number of ways to do a task in one of two ways is the sum of the number of ways to do these tasks if they cannot be done simultaneously.
sum rule for sets: The number of elements in the union of pairwise disjoint finite sets is the sum of the numbers of elements in these sets.
subtraction rule for counting or **inclusion–exclusion for sets:** If a task can be done in either n_1 ways or n_2 ways, then the number of ways to do the task is $n_1 + n_2$ minus the number of ways to do the task that are common to the two different ways.
subtraction rule or **inclusion–exclusion for sets:** The number of elements in the union of two sets is the sum of the number of elements in these sets minus the number of elements in their intersection.
division rule for counting: There are n/d ways to do a task if it can be done using a procedure that can be carried out in n ways, and for every way w, exactly d of the n ways correspond to way w.
division rule for sets: Suppose that a finite set A is the union of n disjoint subsets each with d elements. Then $n = |A|/d$.
the pigeonhole principle: When more than k objects are placed in k boxes, there must be a box containing more than one object.
the generalized pigeonhole principle: When N objects are placed in k boxes, there must be a box containing at least $\lceil N/k \rceil$ objects.

$$P(n, r) = \frac{n!}{(n-r)!}$$

$$C(n, r) = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Pascal's identity: $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$
the binomial theorem: $(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$
There are n^r r-permutations of a set with n elements when repetition is allowed.
There are $C(n + r - 1, r)$ r-combinations of a set with n elements when repetition is allowed.
There are $n!/(n_1! n_2! \cdots n_k!)$ permutations of n objects of k types where there are n_i indistinguishable objects of type i for $i = 1, 2, 3, \ldots, k$.
the algorithm for generating the permutations of the set $\{1, 2, \ldots, n\}$

Review Questions

1. Explain how the sum and product rules can be used to find the number of bit strings with a length not exceeding 10.

2. Explain how to find the number of bit strings of length not exceeding 10 that have at least one 0 bit.

3. a) How can the product rule be used to find the number of functions from a set with m elements to a set with n elements?

b) How many functions are there from a set with five elements to a set with 10 elements?