Chapter 1
Graph Spectrum

This chapter presents some simple results on graph spectra. We assume the reader is familiar with elementary linear algebra and graph theory. Throughout, J will denote the all-1 matrix, and $\mathbf{1}$ is the all-1 vector.

1.1 Matrices associated to a graph

Let Γ be a graph without multiple edges. The *adjacency matrix* of Γ is the 0-1 matrix A indexed by the vertex set $V\Gamma$ of Γ, where $A_{xy} = 1$ when there is an edge from x to y in Γ and $A_{xy} = 0$ otherwise. Occasionally we consider multigraphs (possibly with loops), in which case A_{xy} equals the number of edges from x to y.

Let Γ be an undirected graph without loops. The (vertex-edge) *incidence matrix* of Γ is the 0-1 matrix M, with rows indexed by the vertices and columns indexed by the edges, where $M_{xe} = 1$ when vertex x is an endpoint of edge e.

Let Γ be a directed graph without loops. The *directed incidence matrix* of Γ is the matrix N, with rows indexed by the vertices and columns by the edges, where $N_{xe} = -1, 1, 0$ when x is the head of e, the tail of e, or not on e, respectively.

Let Γ be an undirected graph without loops. The *Laplace matrix* of Γ is the matrix L indexed by the vertex set of Γ, with zero row sums, where $L_{xy} = -A_{xy}$ for $x \neq y$. If D is the diagonal matrix, indexed by the vertex set of Γ such that D_{xx} is the degree (valency) of x, then $L = D - A$. The matrix $Q = D + A$ is called the *signless Laplace matrix* of Γ.

An important property of the Laplace matrix L and the signless Laplace matrix Q is that they are positive semidefinite. Indeed, one has $Q = MM^\top$ and $L = NN^\top$ if M is the incidence matrix of Γ and N the directed incidence matrix of the directed graph obtained by orienting the edges of Γ in an arbitrary way. It follows that for any vector u one has $u^\top Lu = \sum_{xy}(u_x - u_y)^2$ and $u^\top Qu = \sum_{xy}(u_x + u_y)^2$, where the sum is over the edges of Γ.

A.E. Brouwer and W.H. Haemers, *Spectra of Graphs*, Universitext,
DOI 10.1007/978-1-4614-1939-6_1,

1.2 The spectrum of a graph

The (ordinary) *spectrum* of a finite graph Γ is by definition the spectrum of the adjacency matrix A, that is, its set of eigenvalues together with their multiplicities. The *Laplace spectrum* of a finite undirected graph without loops is the spectrum of the Laplace matrix L.

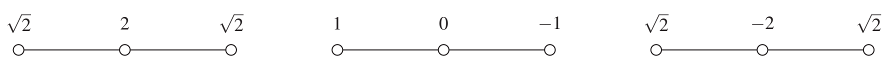
The rows and columns of a matrix of order n are numbered from 1 to n, while A is indexed by the vertices of Γ, so that writing down A requires one to assign some numbering to the vertices. However, the spectrum of the matrix obtained does not depend on the numbering chosen. It is the spectrum of the linear transformation A on the vector space K^X of maps from X into K, where X is the vertex set and K is some field such as $\mathbb{R}$ or $\mathbb{C}$.

The *characteristic polynomial* of Γ is that of A, that is, the polynomial p_A defined by $p_A(\theta) = \det(\theta I - A)$.

Example Let Γ be the path P_3 with three vertices and two edges. Assigning some arbitrary order to the three vertices of Γ, we find that the adjacency matrix A becomes one of

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

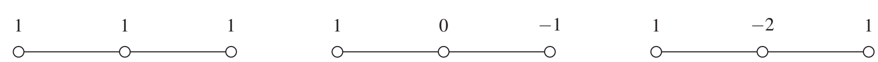
The characteristic polynomial is $p_A(\theta) = \theta^3 - 2\theta$. The spectrum is $\sqrt{2}$, 0, $-\sqrt{2}$. The eigenvectors are:



Here, for an eigenvector u, we write u_x as a label at the vertex x. One has $Au = \theta u$ if and only if $\sum_{y \leftarrow x} u_y = \theta u_x$ for all x. The Laplace matrix L of this graph is one of

$$\begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$

Its eigenvalues are 0, 1 and 3. The Laplace eigenvectors are:



One has $Lu = \theta u$ if and only if $\sum_{y \sim x} u_y = (d_x - \theta) u_x$ for all x, where d_x is the degree of the vertex x.

Example Let Γ be the directed triangle with adjacency matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Then A has characteristic polynomial $p_A(\theta) = \theta^3 - 1$ and spectrum 1, ω, ω^2, where ω is a primitive cube root of unity.

Example Let Γ be the directed graph with two vertices and a single directed edge. Then $A = \begin{bmatrix} 0 1 \\ 0 0 \end{bmatrix}$ with $p_A(\theta) = \theta^2$, so A has the eigenvalue 0 with geometric multiplicity (that is, the dimension of the corresponding eigenspace) equal to 1 and algebraic multiplicity (that is, its multiplicity as a root of the polynomial p_A) equal to 2.

1.2.1 Characteristic polynomial

Let Γ be a directed graph on n vertices. For any directed subgraph C of Γ that is a union of directed cycles, let $c(C)$ be its number of cycles. Then the characteristic polynomial $p_A(t) = \det(tI - A)$ of Γ can be expanded as $\sum c_i t^{n-i}$, where $c_i = \sum_C (-1)^{c(C)}$, with C running over all regular directed subgraphs with in- and outdegree 1 on i vertices.

(Indeed, this is just a reformulation of the definition of the determinant as $\det M = \sum_\sigma \operatorname{sgn}(\sigma) M_{1\sigma(1)} \cdots M_{n\sigma(n)}$. Note that when the permutation σ with $n - i$ fixed points is written as a product of nonidentity cycles, its sign is $(-1)^e$, where e is the number of even cycles in this product. Since the number of odd nonidentity cycles is congruent to i (mod 2), we have $\operatorname{sgn}(\sigma) = (-1)^{i+c(\sigma)}$.)

For example, the directed triangle has $c_0 = 1$, $c_3 = -1$. Directed edges that do not occur in directed cycles do not influence the (ordinary) spectrum.

The same description of $p_A(t)$ holds for undirected graphs (with each edge viewed as a pair of opposite directed edges).

Since $\frac{d}{dt} \det(tI - A) = \sum_x \det(tI - A_x)$ where A_x is the submatrix of A obtained by deleting row and column x, it follows that $p'_A(t)$ is the sum of the characteristic polynomials of all single-vertex-deleted subgraphs of Γ.

1.3 The spectrum of an undirected graph

Suppose Γ is undirected and simple with n vertices. Since A is real and symmetric, all its eigenvalues are real. Also, for each eigenvalue θ, its algebraic multiplicity coincides with its geometric multiplicity, so that we may omit the adjective and just speak about “multiplicity”. Conjugate algebraic integers have the same multiplicity. Since A has zero diagonal, its trace tr$A$, and hence the sum of the eigenvalues, is zero.

Similarly, L is real and symmetric, so that the Laplace spectrum is real. Moreover, L is positive semidefinite and singular, so we may denote the eigenvalues by

$\mu_1, \ldots, \mu_n$, where $0 = \mu_1 \le \mu_2 \le \ldots \le \mu_n$. The sum of these eigenvalues is $\mathrm{tr}\, L$, which is twice the number of edges of Γ.

Finally, also Q has real spectrum and nonnegative eigenvalues (but is not necessarily singular). We have $\mathrm{tr}\, Q = \mathrm{tr}\, L$.

1.3.1 Regular graphs

A graph Γ is called *regular* of degree (or valency) k when every vertex has precisely k neighbors. So, Γ is regular of degree k precisely when its adjacency matrix A has row sums k, i.e., when $A\mathbf{1} = k\mathbf{1}$ (or $AJ = kJ$).

If Γ is regular of degree k, then for every eigenvalue θ we have $|\theta| \le k$. (One way to see this is by observing that if $|t| > k$ then the matrix $tI - A$ is strictly diagonally dominant, and hence nonsingular, so that t is not an eigenvalue of A.)

If Γ is regular of degree k, then $L = kI - A$. It follows that if Γ has ordinary eigenvalues $k = \theta_1 \ge \ldots \ge \theta_n$ and Laplace eigenvalues $0 = \mu_1 \le \mu_2 \le \ldots \le \mu_n$, then $\theta_i = k - \mu_i$ for $i = 1, \ldots, n$. The eigenvalues of $Q = kI + A$ are $2k, k + \theta_2, \ldots, k + \theta_n$.

1.3.2 Complements

The *complement* $\overline{\Gamma}$ of Γ is the graph with the same vertex set as Γ, where two distinct vertices are adjacent whenever they are nonadjacent in Γ. So, if Γ has adjacency matrix A, then $\overline{\Gamma}$ has adjacency matrix $\overline{A} = J - I - A$ and Laplace matrix $\overline{L} = nI - J - L$.

Because eigenvectors of L are also eigenvectors of J, the eigenvalues of $\overline{L}$ are $0, n - \mu_n, \ldots, n - \mu_2$. (In particular, $\mu_n \le n$.)

If Γ is regular we have a similar result for the ordinary eigenvalues: if Γ is k-regular with eigenvalues $\theta_1 \ge \ldots \ge \theta_n$, then the eigenvalues of the complement are $n - k - 1$, $-1 - \theta_n$, $\ldots$, $-1 - \theta_2$.

1.3.3 Walks

From the spectrum one can read off the number of closed walks of a given length.

Proposition 1.3.1 *Let h be a nonnegative integer. Then $(A^h)_{xy}$ is the number of walks of length h from x to y. In particular, $(A^2)_{xx}$ is the degree of the vertex x, and* $\mathrm{tr}\, A^2$ *equals twice the number of edges of Γ; similarly,* $\mathrm{tr}\, A^3$ *is six times the number of triangles in Γ.*

1.3.4 Diameter

We saw that all eigenvalues of a single directed edge are zero. For undirected graphs this does not happen.

Proposition 1.3.2 *Let Γ be an undirected graph. All its eigenvalues are zero if and only if Γ has no edges. The same holds for the Laplace eigenvalues and the signless Laplace eigenvalues.*

More generally, we find a lower bound for the diameter:

Proposition 1.3.3 *Let Γ be a connected graph with diameter d. Then Γ has at least $d+1$ distinct eigenvalues, at least $d+1$ distinct Laplace eigenvalues, and at least $d+1$ distinct signless Laplace eigenvalues.*

Proof Let M be any nonnegative symmetric matrix with rows and columns indexed by $V\Gamma$ and such that for distinct vertices x,y we have $M_{xy} > 0$ if and only if $x \sim y$. Let the distinct eigenvalues of M be $\theta_1,\dots,\theta_t$. Then $(M-\theta_1 I)\cdots(M-\theta_t I) = 0$, so that M^t is a linear combination of $I, M, \dots, M^{t-1}$. But if $d(x,y) = t$ for two vertices x,y of Γ, then $(M^i)_{xy} = 0$ for $0 \le i \le t-1$ and $(M^t)_{xy} > 0$, a contradiction. Hence $t > d$. This applies to $M = A$, to $M = nI - L$, and to $M = Q$, where A is the adjacency matrix, L is the Laplace matrix, and Q is the signless Laplace matrix of Γ. $\square$

Distance-regular graphs, discussed in Chapter 12, have equality here. For an upper bound on the diameter, see §4.7.

1.3.5 Spanning trees

From the Laplace spectrum of a graph one can determine the number of spanning trees (which will be nonzero only if the graph is connected).

Proposition 1.3.4 *Let Γ be an undirected (multi)graph with at least one vertex, and Laplace matrix L with eigenvalues $0 = \mu_1 \le \mu_2 \le \ldots \le \mu_n$. Let ℓ_{xy} be the (x,y)-cofactor of L. Then the number N of spanning trees of Γ equals*

$$N = \ell_{xy} = \det(L + \frac{1}{n^2}J) = \frac{1}{n}\mu_2\cdots\mu_n \ \textit{ for any } \ x,y \in V\Gamma.$$

(The (i,j)-*cofactor* of a matrix M is by definition $(-1)^{i+j}\det M(i,j)$, where $M(i,j)$ is the matrix obtained from M by deleting row i and column j. Note that ℓ_{xy} does not depend on an ordering of the vertices of Γ.)

Proof Let L^S, for $S \subseteq V\Gamma$, denote the matrix obtained from L by deleting the rows and columns indexed by S, so that $\ell_{xx} = \det L^{\{x\}}$. The equality $N = \ell_{xx}$ follows by induction on n, and for fixed $n > 1$ on the number of edges incident with x. Indeed, if $n = 1$ then $\ell_{xx} = 1$. Otherwise, if x has degree 0, then $\ell_{xx} = 0$ since $L^{\{x\}}$ has zero row sums. Finally, if xy is an edge, then deleting this edge from Γ diminishes ℓ_{xx} by $\det L^{\{x,y\}}$, which by induction is the number of spanning trees of Γ with edge xy contracted, which is the number of spanning trees containing the edge xy. This shows $N = \ell_{xx}$.

Now $\det(tI - L) = t\prod_{i=2}^{n}(t - \mu_i)$ and $(-1)^{n-1}\mu_2 \cdots \mu_n$ is the coefficient of t, that is, is $\frac{d}{dt}\det(tI - L)|_{t=0}$. But $\frac{d}{dt}\det(tI - L) = \sum_x \det(tI - L^{\{x\}})$, so $\mu_2 \cdots \mu_n = \sum_x \ell_{xx} = nN$.

Since the sum of the columns of L is zero, so that one column is minus the sum of the other columns, we have $\ell_{xx} = \ell_{xy}$ for any x, y. Finally, the eigenvalues of $L + \frac{1}{n^2}J$ are $\frac{1}{n}$ and $\mu_2, \ldots, \mu_n$, so $\det(L + \frac{1}{n^2}J) = \frac{1}{n}\mu_2 \cdots \mu_n$. □

For example, the multigraph of valency k on two vertices has Laplace matrix $L = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}$ so $\mu_1 = 0$, $\mu_2 = 2k$, and $N = \frac{1}{2}.2k = k$.

If we consider the complete graph K_n, then $\mu_2 = \ldots = \mu_n = n$, and therefore K_n has $N = n^{n-2}$ spanning trees. This formula is due to CAYLEY [85]. Proposition 1.3.4 is implicit in KIRCHHOFF [242] and known as the *matrix-tree theorem*. There is a "1-line proof" of the above result using the *Cauchy-Binet formula*.

Proposition 1.3.5 (Cauchy-Binet) *Let A and B be $m \times n$ matrices. Then*

$$\det AB^\top = \sum_S \det A_S \det B_S,$$

where the sum is over the $\binom{n}{m}$ m-subsets S of the set of columns, and A_S (B_S) is the square submatrix of order m of A (resp. B) with columns indexed by S.

Second proof of Proposition 1.3.4 (sketch) Let N_x be the directed incidence matrix of Γ with row x deleted. Then $l_{xx} = \det N_x N_x^\top$. Apply the Cauchy-Binet formula to get l_{xx} as a sum of squares of determinants of size $n-1$. These determinants vanish unless the set S of columns is the set of edges of a spanning tree, in which case the determinant is ± 1. □

1.3.6 Bipartite graphs

A graph Γ is called *bipartite* when its vertex set can be partitioned into two disjoint parts X_1, X_2 such that all edges of Γ meet both X_1 and X_2. The adjacency matrix of a bipartite graph has the form $A = \begin{bmatrix} 0 & B \\ B^\top & 0 \end{bmatrix}$. It follows that the spectrum of a bipartite graph is symmetric w.r.t. 0: if $\begin{bmatrix} u \\ v \end{bmatrix}$ is an eigenvector with eigenvalue θ, then $\begin{bmatrix} u \\ -v \end{bmatrix}$ is an eigenvector with eigenvalue $-\theta$. (The converse also holds, see Proposition 3.4.1.)

For the ranks one has $\mathrm{rk}A = 2\,\mathrm{rk}B$. If $n_i = |X_i|$ $(i = 1, 2)$ and $n_1 \geq n_2$, then $\mathrm{rk}A \leq 2n_2$, so Γ has eigenvalue 0 with multiplicity at least $n_1 - n_2$.

One cannot, in general, recognize bipartiteness from the Laplace or signless Laplace spectrum. For example, $K_{1,3}$ and $K_1 + K_3$ have the same signless Laplace spectrum and only the former is bipartite. And Figure 14.4 gives an example of a bipartite and a nonbipartite graph with the same Laplace spectrum. However, by Proposition 1.3.10 below, a graph is bipartite precisely when its Laplace spectrum and signless Laplace spectrum coincide.

1.3.7 Connectedness

The spectrum of a disconnected graph is easily found from the spectra of its connected components:

Proposition 1.3.6 *Let Γ be a graph with connected components Γ_i $(1 \le i \le s)$. Then the spectrum of Γ is the union of the spectra of Γ_i (and multiplicities are added). The same holds for the Laplace spectrum and the signless Laplace spectrum.* □

Proposition 1.3.7 *The multiplicity of 0 as a Laplace eigenvalue of an undirected graph Γ equals the number of connected components of Γ.*

Proof We have to show that a connected graph has Laplace eigenvalue 0 with multiplicity 1. As we saw earlier, $L = NN^\top$, where N is the incidence matrix of an orientation of Γ. Now $Lu = 0$ is equivalent to $N^\top u = 0$ (since $0 = u^\top Lu = ||N^\top u||^2$), that is, for every edge the vector u takes the same value on both endpoints. Since Γ is connected, that means that u is constant. □

Proposition 1.3.8 *Let the undirected graph Γ be regular of valency k. Then k is the largest eigenvalue of Γ, and its multiplicity equals the number of connected components of Γ.*

Proof We have $L = kI - A$. □

One cannot see from the spectrum alone whether a (nonregular) graph is connected: both $K_{1,4}$ and $K_1 + C_4$ have spectrum 2^1, 0^3, $(-2)^1$ (we write multiplicities as exponents). And both $\hat{E}_6$ and $K_1 + C_6$ have spectrum 2^1, 1^2, 0, $(-1)^2$, $(-2)^1$.

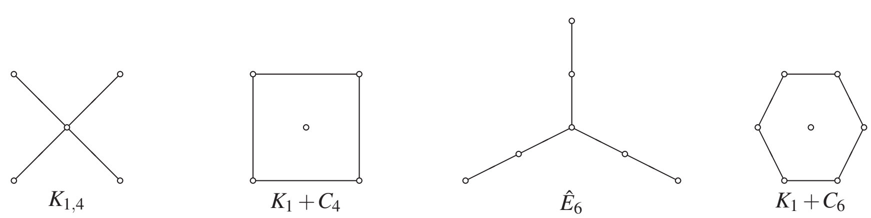


Fig. 1.1 Two pairs of cospectral graphs

Proposition 1.3.9 *The multiplicity of 0 as a signless Laplace eigenvalue of an undirected graph Γ equals the number of bipartite connected components of Γ.*

Proof Let M be the vertex-edge incidence matrix of Γ, so that $Q = MM^\top$. If $MM^\top u = 0$, then $M^\top u = 0$, so $u_x = -u_y$ for all edges xy, and the support of u is the union of a number of bipartite components of Γ. □

Proposition 1.3.10 *A graph Γ is bipartite if and only if the Laplace spectrum and the signless Laplace spectrum of Γ are equal.*

Proof If Γ is bipartite, the Laplace matrix L and the signless Laplace matrix Q are similar by a diagonal matrix D with diagonal entries ± 1 (that is, $Q = DLD^{-1}$). Therefore Q and L have the same spectrum. Conversely, if both spectra are the same, then by Propositions 1.3.7 and 1.3.9 the number of connected components equals the number of bipartite components. Hence Γ is bipartite. □

1.4 Spectrum of some graphs

In this section we discuss some special graphs and their spectra. All graphs in this section are finite, undirected, and simple. Observe that the all-1 matrix J of order n has rank 1, and that the all-1 vector $\mathbf{1}$ is an eigenvector with eigenvalue n, so the spectrum of J is $n^1,\ 0^{n-1}$. (Here and throughout, we write multiplicities as exponents where convenient and no confusion seems likely.)

1.4.1 The complete graph

Let Γ be the complete graph K_n on n vertices. Its adjacency matrix is $A = J - I$, and the spectrum is $(n-1)^1,\ (-1)^{n-1}$. The Laplace matrix is $nI - J$, which has spectrum $0^1,\ n^{n-1}$.

1.4.2 The complete bipartite graph

The spectrum of the complete bipartite graph $K_{m,n}$ is $\pm\sqrt{mn},\ 0^{m+n-2}$. The Laplace spectrum is $0^1,\ m^{n-1},\ n^{m-1},\ (m+n)^1$.

1.4.3 The cycle

Let Γ be the directed n-cycle D_n. Eigenvectors are $(1, \zeta, \zeta^2, \ldots, \zeta^{n-1})^\top$, where $\zeta^n = 1$, and the corresponding eigenvalue is ζ. Thus, the spectrum consists precisely of the complex n-th roots of unity $e^{2\pi ij/n}$ $(j = 0, \ldots, n-1)$.

Now consider the undirected n-cycle C_n. If B is the adjacency matrix of D_n, then $A = B + B^\top$ is the adjacency matrix of C_n. We find the same eigenvectors as before, with eigenvalues $\zeta + \zeta^{-1}$, so that the spectrum consists of the numbers $2\cos(2\pi j/n)$ $(j = 0,\ldots,n-1)$.

This graph is regular of valency 2, so the Laplace spectrum consists of the numbers $2 - 2\cos(2\pi j/n)$ $(j = 0,\ldots,n-1)$.

1.4.4 The path

Let Γ be the undirected path P_n with n vertices. The ordinary spectrum consists of the numbers $2\cos(\pi j/(n+1))$ $(j = 1,\ldots,n)$. The Laplace spectrum is $2 - 2\cos(\pi j/n)$ $(j = 0,\ldots,n-1)$.

The ordinary spectrum follows by looking at C_{2n+2}. If $u(\zeta) = (1,\zeta,\zeta^2,\ldots,\zeta^{2n+1})^\top$ is an eigenvector of C_{2n+2}, where $\zeta^{2n+2} = 1$, then $u(\zeta)$ and $u(\zeta^{-1})$ have the same eigenvalue, $2\cos(\pi j/(n+1))$, and hence so has $u(\zeta) - u(\zeta^{-1})$. This latter vector has two zero coordinates distance $n+1$ apart and (for $\zeta \neq \pm 1$) induces an eigenvector on the two paths obtained by removing the two points where it is zero.

Eigenvectors of L with eigenvalue $2 - \zeta - \zeta^{-1}$ are $(1+\zeta^{2n-1},\ldots,\zeta^j + \zeta^{2n-1-j},\ldots,\zeta^{n-1}+\zeta^n)$, where $\zeta^{2n} = 1$. One can check this directly, or view P_n as the result of folding C_{2n}, where the folding has no fixed vertices. An eigenvector of C_{2n} that is constant on the preimages of the folding yields an eigenvector of P_n with the same eigenvalue.

1.4.5 Line graphs

The *line graph* $L(\Gamma)$ of Γ is the graph with the edge set of Γ as vertex set, where two vertices are adjacent if the corresponding edges of Γ have an endpoint in common. If N is the incidence matrix of Γ, then $N^\top N - 2I$ is the adjacency matrix of $L(\Gamma)$. Since $N^\top N$ is positive semidefinite, the eigenvalues of a line graph are not smaller than -2. We have an explicit formula for the eigenvalues of $L(\Gamma)$ in terms of the signless Laplace eigenvalues of Γ.

Proposition 1.4.1 *Suppose Γ has m edges, and let $\rho_1 \geq \ldots \geq \rho_r$ be the positive signless Laplace eigenvalues of Γ. Then the eigenvalues of $L(\Gamma)$ are $\theta_i = \rho_i - 2$ for $i = 1,\ldots,r$, and $\theta_i = -2$ if $r < i \leq m$.*

Proof The signless Laplace matrix Q of Γ and the adjacency matrix B of $L(\Gamma)$ satisfy $Q = NN^\top$ and $B + 2I = N^\top N$. Because $NN^\top$ and $N^\top N$ have the same nonzero eigenvalues (multiplicities included), the result follows. $\square$

Example Since the path P_n has line graph P_{n-1} and is bipartite, the Laplace and the signless Laplace eigenvalues of P_n are $2 + 2\cos\frac{\pi i}{n}$, $i = 1,\ldots,n$.

Corollary 1.4.2 *If Γ is a k-regular graph ($k \geq 2$) with n vertices, $e = kn/2$ edges, and eigenvalues θ_i ($i = 1, \ldots, n$), then $L(\Gamma)$ is $(2k-2)$-regular with eigenvalues $\theta_i + k - 2$ ($i = 1, \ldots, n$) and $e - n$ times -2.* □

The line graph of the complete graph K_n ($n \geq 2$) is known as the *triangular graph* $T(n)$. It has spectrum $2(n-2)^1$, $(n-4)^{n-1}$, $(-2)^{n(n-3)/2}$. The line graph of the regular complete bipartite graph $K_{m,m}$ ($m \geq 2$) is known as the *lattice graph* $L_2(m)$. It has spectrum $2(m-1)^1$, $(m-2)^{2m-2}$, $(-2)^{(m-1)^2}$. These two families of graphs, and their complements, are examples of strongly regular graphs, which will be the subject of Chapter 9. The complement of $T(5)$ is the famous *Petersen graph*. It has spectrum $3^1\ 1^5\ (-2)^4$.

1.4.6 Cartesian products

Given graphs Γ and Δ with vertex sets V and W, respectively, their *Cartesian product* $\Gamma \square \Delta$ is the graph with vertex set $V \times W$, where $(v,w) \sim (v',w')$ when either $v = v'$ and $w \sim w'$ or $w = w'$ and $v \sim v'$. For the adjacency matrices we have $A_{\Gamma \square \Delta} = A_\Gamma \otimes I + I \otimes A_\Delta$.

If u and v are eigenvectors for Γ and Δ with ordinary or Laplace eigenvalues θ and η, respectively, then the vector w defined by $w_{(x,y)} = u_x v_y$ is an eigenvector of $\Gamma \square \Delta$ with ordinary or Laplace eigenvalue $\theta + \eta$.

For example, $L_2(m) = K_m \square K_m$.

For example, the *hypercube* 2^n, also called Q_n, is the Cartesian product of n factors K_2. The spectrum of K_2 is $1, -1$, and hence the spectrum of 2^n consists of the numbers $n - 2i$ with multiplicity $\binom{n}{i}$ ($i = 0, 1, \ldots, n$).

1.4.7 Kronecker products and bipartite double

Given graphs Γ and Δ with vertex sets V and W, respectively, their *Kronecker product* (or *direct product*, or *conjunction*) $\Gamma \otimes \Delta$ is the graph with vertex set $V \times W$, where $(v,w) \sim (v',w')$ when $v \sim v'$ and $w \sim w'$. The adjacency matrix of $\Gamma \otimes \Delta$ is the Kronecker product of the adjacency matrices of Γ and Δ.

If u and v are eigenvectors for Γ and Δ with eigenvalues θ and η, respectively, then the vector $w = u \otimes v$ (with $w_{(x,y)} = u_x v_y$) is an eigenvector of $\Gamma \otimes \Delta$ with eigenvalue $\theta\eta$. Thus, the spectrum of $\Gamma \otimes \Delta$ consists of the products of the eigenvalues of Γ and Δ.

Given a graph Γ, its *bipartite double* is the graph $\Gamma \otimes K_2$ (with for each vertex x of Γ two vertices x' and x'', and for each edge xy of Γ two edges $x'y''$ and $x''y'$). If Γ is bipartite, its double is just the union of two disjoint copies. If Γ is connected and not bipartite, then its double is connected and bipartite. If Γ has spectrum Φ, then $\Gamma \otimes K_2$ has spectrum $\Phi \cup -\Phi$.

The notation $\Gamma \times \Delta$ is used in the literature both for the Cartesian product and for the Kronecker product of two graphs. We avoid it here.

1.4.8 Strong products

Given graphs Γ and Δ with vertex sets V and W, respectively, their *strong product* $\Gamma \boxtimes \Delta$ is the graph with vertex set $V \times W$, where two distinct vertices (v,w) and (v',w') are adjacent whenever v and v' are equal or adjacent in Γ, and w and w' are equal or adjacent in Δ. If A_Γ and A_Δ are the adjacency matrices of Γ and Δ, then $((A_\Gamma + I) \otimes (A_\Delta + I)) - I$ is the adjacency matrix of $\Gamma \boxtimes \Delta$. It follows that the eigenvalues of $\Gamma \boxtimes \Delta$ are the numbers $(\theta+1)(\eta+1)-1$, where θ and η run through the eigenvalues of Γ and Δ, respectively.

Note that the edge set of the strong product of Γ and Δ is the union of the edge sets of the Cartesian product and the Kronecker product of Γ and Δ.

For example, $K_{m+n} = K_m \boxtimes K_n$.

1.4.9 Cayley graphs

Let G be an Abelian group and $S \subseteq G$. The *Cayley graph* on G with difference set S is the (directed) graph Γ with vertex set G and edge set $E = \{(x,y) \mid y - x \in S\}$. Now Γ is regular with in- and outvalency $|S|$. The graph Γ will be undirected when $S = -S$.

It is easy to compute the spectrum of finite Cayley graphs (on an Abelian group). Let χ be a character of G, that is, a map $\chi : G \to \mathbb{C}^*$ such that $\chi(x+y) = \chi(x)\chi(y)$. Then $\sum_{y \sim x} \chi(y) = (\sum_{s \in S} \chi(s))\chi(x)$, so the vector $(\chi(x))_{x \in G}$ is a right eigenvector of the adjacency matrix A of Γ with eigenvalue $\chi(S) := \sum_{s \in S} \chi(s)$. The $n = |G|$ distinct characters give independent eigenvectors, so one obtains the entire spectrum in this way.

For example, the directed pentagon (with in- and outvalency 1) is a Cayley graph for $G = \mathbb{Z}_5$ and $S = \{1\}$. The characters of G are the maps $i \mapsto \zeta^i$ for some fixed fifth root of unity ζ. Hence the directed pentagon has spectrum $\{\zeta \mid \zeta^5 = 1\}$.

The undirected pentagon (with valency 2) is the Cayley graph for $G = \mathbb{Z}_5$ and $S = \{-1, 1\}$. The spectrum of the pentagon becomes $\{\zeta + \zeta^{-1} \mid \zeta^5 = 1\}$, that is, consists of 2 and $\frac{1}{2}(-1 \pm \sqrt{5})$ (both with multiplicity 2).

1.5 Decompositions

Here we present two nontrivial applications of linear algebra to graph decompositions.

1.5.1 Decomposing K_{10} into Petersen graphs

An amusing application ([35, 310]) is the following. Can the edges of the complete graph K_{10} be colored with three colors such that each color induces a graph isomorphic to the Petersen graph? K_{10} has 45 edges, 9 on each vertex, and the Petersen graph has 15 edges, 3 on each vertex, so at first sight this might seem possible. Let the adjacency matrices of the three color classes be P_1, P_2 and P_3, so that $P_1+P_2+P_3 = J-I$. If P_1 and P_2 are Petersen graphs, they both have a 5-dimensional eigenspace for eigenvalue 1, contained in the 9-space $\mathbf{1}^{\perp}$. Therefore, there is a common 1-eigenvector u and $P_3u = (J-I)u - P_1u - P_2u = -3u$ so that u is an eigenvector for P_3 with eigenvalue -3. But the Petersen graph does not have eigenvalue -3, so the result of removing two edge-disjoint Petersen graphs from K_{10} is not a Petersen graph. (In fact, it follows that P_3 is connected and bipartite.)

1.5.2 Decomposing K_n into complete bipartite graphs

A famous result is the fact that for any edge decomposition of K_n into complete bipartite graphs one needs to use at least $n-1$ summands. Since K_n has eigenvalue -1 with multiplicity $n-1$, this follows directly from the following:

Proposition 1.5.1 (H. S. Witsenhausen; GRAHAM & POLLAK [181]) *Suppose a graph Γ with adjacency matrix A has an edge decomposition into r complete bipartite graphs. Then $r \ge n_+(A)$ and $r \ge n_-(A)$, where $n_+(A)$ and $n_-(A)$ are the numbers of positive and negative eigenvalues of A, respectively.*

Proof Let u_i and v_i be the characteristic vectors of both sides of a bipartition of the i-th complete bipartite graph. Then that graph has adjacency matrix $D_i = u_i v_i^\top + v_i u_i^\top$, and $A = \sum D_i$. Let w be a vector orthogonal to all u_i. Then $w^\top A w = 0$ and it follows that w cannot be chosen in the span of eigenvectors of A with positive (negative) eigenvalue. $\square$

1.6 Automorphisms

An *automorphism* of a graph Γ is a permutation π of its point set X such that $x \sim y$ if and only if $\pi(x) \sim \pi(y)$. Given π, we have a linear transformation P_π on V defined by $(P_\pi(u))_x = u_{\pi(x)}$ for $u \in V$, $x \in X$. That π is an automorphism is expressed by $AP_\pi = P_\pi A$. It follows that P_π preserves the eigenspace V_θ for each eigenvalue θ of A.

More generally, if G is a group of automorphisms of Γ, then we find a linear representation of degree $m(\theta) = \dim V_\theta$ of G.

We denote the group of all automorphisms of Γ by Aut Γ. One would expect that when Aut Γ is large then $m(\theta)$ tends to be large, so that Γ has only few distinct eigenvalues. And indeed, the arguments below will show that a transitive group of automorphisms does not go together very well with simple eigenvalues.

Suppose $\dim V_\theta = 1$, say $V_\theta = \langle u \rangle$. Since P_π preserves V_θ, we must have $P_\pi u = \pm u$. So either u is constant on the orbits of π or π has even order, $P_\pi(u) = -u$, and u is constant on the orbits of π^2. For the Perron-Frobenius eigenvector (cf. §2.2) we always have the former case.

Corollary 1.6.1 *If all eigenvalues are simple, then* Aut Γ *is an elementary Abelian* 2-*group.*

Proof If π has order larger than 2, then there are two distinct vertices x, y in an orbit of π^2, and all eigenvectors have identical x- and y-coordinates, a contradiction. □

Corollary 1.6.2 *Let* Aut Γ *be transitive on* X. *(Then* Γ *is regular of degree* k, *say.)*

(i) If $m(\theta) = 1$ *for some eigenvalue* $\theta \neq k$, *then* $v = |X|$ *is even and* $\theta \equiv k$ (mod 2). *If* Aut Γ *is, moreover, edge-transitive then* Γ *is bipartite and* $\theta = -k$.
(ii) If $m(\theta) = 1$ *for two distinct eigenvalues* $\theta \neq k$, *then* $v \equiv 0$ (mod 4).
(iii) If $m(\theta) = 1$ *for all eigenvalues* θ, *then* Γ *has at most two vertices.*

Proof (i) Suppose $V_\theta = \langle u \rangle$. Then u induces a partition of X into two equal parts: $X = X_+ \cup X_-$, where $u_x = a$ for $x \in X_+$ and $u_x = -a$ for $x \in X_-$. Now $\theta = k - 2|\Gamma(x) \cap X_-|$ for $x \in X_+$.

(ii) If $m(k) = m(\theta) = m(\theta') = 1$, then we find three pairwise orthogonal (± 1)-vectors, and a partition of X into four equal parts.

(iii) There are not enough integers $\theta \equiv k$ (mod 2) between $-k$ and k. □

For more details, see CVETKOVIĆ, DOOB & SACHS [115], Ch. 5.

1.7 Algebraic connectivity

Let Γ be a graph with at least two vertices. The second-smallest Laplace eigenvalue $\mu_2(\Gamma)$ is called the *algebraic connectivity* of the graph Γ. This concept was introduced by FIEDLER [156]. Now, by Proposition 1.3.7, $\mu_2(\Gamma) \geq 0$, with equality if and only if Γ is disconnected.

The algebraic connectivity is monotone: it does not decrease when edges are added to the graph:

Proposition 1.7.1 *Let* Γ *and* Δ *be two edge-disjoint graphs on the same vertex set, and* $\Gamma \cup \Delta$ *their union. We have* $\mu_2(\Gamma \cup \Delta) \geq \mu_2(\Gamma) + \mu_2(\Delta) \geq \mu_2(\Gamma)$.

Proof Use $\mu_2(\Gamma) = \min_u \{u^\perp L u \mid (u,u) = 1,\ (u, \mathbf{1}) = 0\}$. □

The algebraic connectivity is a lower bound for the vertex connectivity:

Proposition 1.7.2 *Let Γ be a graph with vertex set X. Suppose $D \subset X$ is a set of vertices such that the subgraph induced by Γ on $X \setminus D$ is disconnected. Then $|D| \geq \mu_2(\Gamma)$.*

Proof By monotonicity we may assume that Γ contains all edges between D and $X \setminus D$. Now a nonzero vector u that is 0 on D and constant on each component of $X \setminus D$ and satisfies $(u, \mathbf{1}) = 0$, is a Laplace eigenvector with Laplace eigenvalue $|D|$. □

1.8 Cospectral graphs

As noted above (in §1.3.7), there exist pairs of nonisomorphic graphs with the same spectrum. Graphs with the same (adjacency) spectrum are called *cospectral* (or *isospectral*). The two graphs of Figure 1.2 below are nonisomorphic and cospectral. Both graphs are regular, which means that they are also cospectral for the Laplace matrix and any other linear combination of A, I, and J, including the Seidel matrix (see §1.8.2) and the adjacency matrix of the complement.

Fig. 1.2 Two cospectral regular graphs
(Spectrum: 4, 1, $(-1)^4$, $\pm\sqrt{5}$, $\frac{1}{2}(1 \pm \sqrt{17})$)

Let us give some more examples and families of examples. A more extensive discussion is found in Chapter 14.

1.8.1 The 4-cube

The hypercube 2^n is determined by its spectrum for $n < 4$, but not for $n \geq 4$. Indeed, there are precisely two graphs with spectrum 4^1, 2^4, 0^6, $(-2)^4$, $(-4)^1$ (Hoffman [218]). Consider the two binary codes of word length 4 and dimension 3 given by $C_1 = \mathbf{1}^\perp$ and $C_2 = (0111)^\perp$. Construct a bipartite graph, where one class of the bipartition consists of the pairs $(i, x) \in \{1,2,3,4\} \times \{0,1\}$ of coordinate position and value, and the other class of the bipartition consists of the code words, and code word u is adjacent to the pairs (i, u_i) for $i \in \{1,2,3,4\}$. For the code C_1 this yields the 4-cube (tesseract), and for C_2 we get its unique cospectral mate.

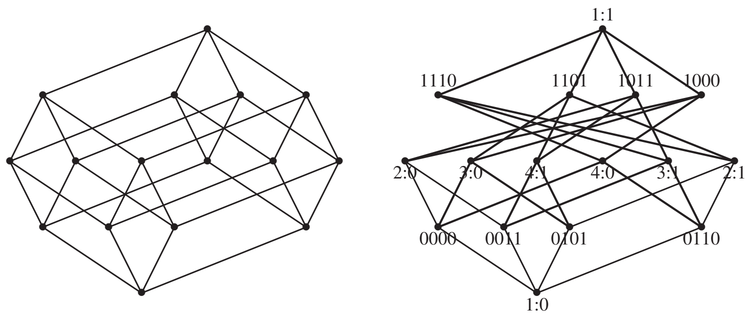


Fig. 1.3 Tesseract and cospectral switched version

1.8.2 Seidel switching

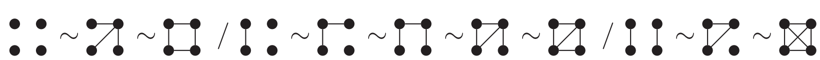
The *Seidel adjacency matrix* of a graph Γ with adjacency matrix A is the matrix S defined by

$$S_{uv} = \begin{cases} 0 \text{ if } u = v \\ -1 \text{ if } u \sim v \\ 1 \text{ if } u \not\sim v \end{cases}$$

so that $S = J - I - 2A$. The *Seidel spectrum* of a graph is the spectrum of its Seidel adjacency matrix. For a regular graph on n vertices with valency k and other eigenvalues θ, the Seidel spectrum consists of $n-1-2k$ and the values $-1-2\theta$.

Let Γ have vertex set X, and let $Y \subset X$. Let D be the diagonal matrix indexed by X with $D_{xx} = -1$ for $x \in Y$, and $D_{xx} = 1$ otherwise. Then DSD has the same spectrum as S. It is the Seidel adjacency matrix of the graph obtained from Γ by leaving adjacency and nonadjacency inside Y and $X \setminus Y$ as it was, and interchanging adjacency and nonadjacency between Y and $X \setminus Y$. This new graph, Seidel-cospectral with Γ, is said to be obtained by *Seidel switching* with respect to the set of vertices Y.

Being related by Seidel switching is an equivalence relation, and the equivalence classes are called *switching classes*. Here are the three switching classes of graphs with four vertices.



The Seidel matrix of the complementary graph $\overline{\Gamma}$ is $-S$, so a graph and its complement have opposite Seidel eigenvalues.

If two regular graphs of the same valency are Seidel-cospectral, then they are also cospectral.

Figure 1.2 shows an example of two cospectral graphs related by Seidel switching (with respect to the four corners). These graphs are nonisomorphic: they have different local structure.

The Seidel adjacency matrix plays a role in the description of regular two-graphs (see §§10.1–10.3) and equiangular lines (see §10.6).

1.8.3 Godsil-McKay switching

Let Γ be a graph with vertex set X, and let $\{C_1,\dots,C_t,D\}$ be a partition of X such that $\{C_1,\dots,C_t\}$ is an equitable partition of $X\setminus D$ (that is, any two vertices in C_i have the same number of neighbors in C_j for all i,j), and for every $x\in D$ and every $i\in\{1,\dots,t\}$ the vertex x has either 0, $\frac{1}{2}|C_i|$ or $|C_i|$ neighbors in C_i. Construct a new graph Γ' by interchanging adjacency and nonadjacency between $x\in D$ and the vertices in C_i whenever x has $\frac{1}{2}|C_i|$ neighbors in C_i. Then Γ and Γ' are cospectral ([176]).

Indeed, let Q_m be the matrix $\frac{2}{m}J-I$ of order m, so that $Q_m^2=I$. Let $n_i=|C_i|$. Then the adjacency matrix A' of Γ' is found to be QAQ where Q is the block diagonal matrix with blocks Q_{n_i} $(1\le i\le t)$ and I (of order $|D|$).

The same argument also applies to the complementary graphs, so that also the complements of Γ and Γ' are cospectral. Thus, for example, the second pair of graphs in Figure 1.1 is related by GM switching, and hence has cospectral complements. The first pair does not have cospectral complements and hence does not arise by GM switching.

The 4-cube and its cospectral mate (Figure 1.3) can be obtained from each other by GM switching with respect to the neighborhood of a vertex. Figure 1.2 is also an example of GM switching. Indeed, when two regular graphs of the same degree are related by Seidel switching, the switch is also a case of GM switching.

1.8.4 Reconstruction

The famous Kelly-Ulam conjecture (1941) asks whether a graph Γ can be reconstructed when the (isomorphism types of) the n vertex-deleted graphs $\Gamma\setminus x$ are given. The conjecture is still open (see Bondy [34] for a discussion), but Tutte [339] showed that one can reconstruct the characteristic polynomial of Γ, so any counterexample to the reconstruction conjecture must be a pair of cospectral graphs.

1.9 Very small graphs

Table 1.1 gives various spectra for the graphs on at most four vertices. The columns with heading A, L, Q, S give the spectrum for the adjacency matrix, the Laplace matrix $L=D-A$ (where D is the diagonal matrix of degrees), the signless Laplace matrix $Q=D+A$, and the Seidel matrix $S=J-I-2A$, respectively.

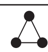
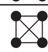
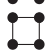
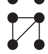
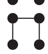
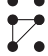
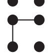
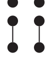
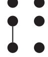
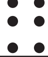
Label	Picture	A	L	Q	S
0.1					
1.1		0	0	0	0
2.1		$1,-1$	$0,2$	$2,0$	$-1,1$
2.2		$0,0$	$0,0$	$0,0$	$-1,1$
3.1		$2,-1,-1$	$0,3,3$	$4,1,1$	$-2,1,1$
3.2		$\sqrt{2},0,-\sqrt{2}$	$0,1,3$	$3,1,0$	$-1,-1,2$
3.3		$1,0,-1$	$0,0,2$	$2,0,0$	$-2,1,1$
3.4		$0,0,0$	$0,0,0$	$0,0,0$	$-1,-1,2$
4.1		$3,-1,-1,-1$	$0,4,4,4$	$6,2,2,2$	$-3,1,1,1$
4.2		$\rho,0,-1,1-\rho$	$0,2,4,4$	$2+2\tau,2,2,4-2\tau$	$-\sqrt{5},-1,1,\sqrt{5}$
4.3		$2,0,0,-2$	$0,2,2,4$	$4,2,2,0$	$-1,-1,-1,3$
4.4		$\theta_1,\theta_2,-1,\theta_3$	$0,1,3,4$	$2+\rho,2,1,3-\rho$	$-\sqrt{5},-1,1,\sqrt{5}$
4.5		$\sqrt{3},0,0,-\sqrt{3}$	$0,1,1,4$	$4,1,1,0$	$-1,-1,-1,3$
4.6		$\tau,\tau-1,1-\tau,-\tau$	$0,4-\alpha,2,\alpha$	$\alpha,2,4-\alpha,0$	$-\sqrt{5},-1,1,\sqrt{5}$
4.7		$2,0,-1,-1$	$0,0,3,3$	$4,1,1,0$	$-3,1,1,1$
4.8		$\sqrt{2},0,0,-\sqrt{2}$	$0,0,1,3$	$3,1,0,0$	$-\sqrt{5},-1,1,\sqrt{5}$
4.9		$1,1,-1,-1$	$0,0,2,2$	$2,2,0,0$	$-3,1,1,1$
4.10		$1,0,0,-1$	$0,0,0,2$	$2,0,0,0$	$-\sqrt{5},-1,1,\sqrt{5}$
4.11		$0,0,0,0$	$0,0,0,0$	$0,0,0,0$	$-1,-1,-1,3$

Table 1.1 Spectra of very small graphs

Here $\alpha = 2+\sqrt{2}$, $\tau = (1+\sqrt{5})/2$, and $\rho = (1+\sqrt{17})/2$, and $\theta_1 \approx 2.17009$, $\theta_2 \approx 0.31111$, $\theta_3 \approx -1.48119$ are the three roots of $\theta^3-\theta^2-3\theta+1=0$.

1.10 Exercises

Exercise 1.1 Show that no graph has eigenvalue $-1/2$. Show that no undirected graph has eigenvalue $\sqrt{2+\sqrt{5}}$. (Hint: Consider the algebraic conjugates of this number.)

Exercise 1.2 Let Γ be an undirected graph with eigenvalues $\theta_1,\dots,\theta_n$. Show that for any two vertices a and b of Γ there are constants $c_1,\dots,c_n$ such that the number of walks of length h from a to b equals $\sum c_i\theta_i^h$ for all h.

Exercise 1.3 Let Γ be a directed graph with constant outdegree $k>0$ and without directed 2-cycles. Show that Γ has a nonreal eigenvalue.

Exercise 1.4 A *perfect e-error-correcting code* in an undirected graph Γ is a set of vertices C such that each vertex of Γ has distance at most e to precisely one vertex in C. For $e = 1$, this is also known as a *perfect dominating set*. Show that if Γ is regular of degree $k > 0$, and has a perfect dominating set, it has an eigenvalue -1.

Exercise 1.5 (i) Let Γ be a directed graph on n vertices such that there is an h with the property that for any two vertices a and b (distinct or not) there is a unique directed path of length h from a to b. Prove that Γ has constant in-degree and out-degree k, where $n = k^h$, and has spectrum $k^1\ 0^{n-1}$.

(ii) The *de Bruijn graph* of order m is the directed graph with as vertices the 2^m binary sequences of length m, where there is an arrow from $a_1 \ldots a_m$ to $b_1 \ldots b_m$ when the tail $a_2 \ldots a_m$ of the first equals the head $b_1 \ldots b_{m-1}$ of the second. (For $m = 0$ we take a single vertex with two loops.) Determine the spectrum of the de Bruijn graph.

(iii) A *de Bruijn cycle* of order $m \geq 1$ ([70, 71, 160]) is a circular arrangement of 2^m zeros and ones such that each binary sequence of length m occurs once in this cycle. (In other words, it is a Hamiltonian cycle in the de Bruijn graph of order m, and a Eulerian cycle in the de Bruijn graph of order $m-1$.) Show that there are precisely $2^{2^{m-1}-m}$ de Bruijn cycles of order m.

Exercise 1.6 ([43, 306]) Let Γ be a *tournament*, that is, a directed graph in which there is precisely one edge between any two distinct vertices, or, in other words, of which the adjacency matrix A satisfies $A^\top + A = J - I$.

(i) Show that all eigenvalues have real part not less than $-1/2$.
(ii) The tournament Γ is called *transitive* if (x,z) is an edge whenever both (x,y) and (y,z) are edges. Show that all eigenvalues of a transitive tournament are zero.
(iii) The tournament Γ is called *regular* when each vertex has the same number of out-arrows. Clearly, when there are n vertices, this number of out-arrows is $(n-1)/2$. Show that all eigenvalues θ have real part at most $(n-1)/2$ and that $\mathrm{Re}(\theta) = (n-1)/2$ occurs if and only if Γ is regular (and then $\theta = (n-1)/2$).
(iv) Show that A either has full rank n or has rank $n-1$, and that A has full rank when Γ is regular and $n > 1$.

(Hint: For a vector u, consider the expression $\bar{u}^\top (A^\top + A)u$.)

Exercise 1.7 Let Γ be bipartite and consider its line graph $L(\Gamma)$.

(i) Show that Γ admits a directed incidence matrix N such that $N^\top N - 2I$ is the adjacency matrix of $L(\Gamma)$.
(ii) Give a relation between the Laplace eigenvalues of Γ and the ordinary eigenvalues of $L(\Gamma)$.
(iii) Verify this relation in case Γ is the path P_n.

Exercise 1.8 ([102]) Verify (see §1.2.1) that both graphs pictured here have characteristic polynomial $t^4(t^4 - 7t^2 + 9)$, so that these two trees are cospectral.

Note how the coefficients of the characteristic polynomial of a tree count partial matchings (sets of pairwise disjoint edges) in the tree.

Exercise 1.9 ([17]) Verify that both graphs pictured here have characteristic polynomial $(t-1)(t+1)^2(t^3-t^2-5t+1)$ by computing eigenvectors and eigenvalues. Use the observation (§1.6) that the image of an eigenvector under an automorphism is again an eigenvector. In particular, when two vertices x,y are interchanged by an involution (automorphism of order 2), then the eigenspace has a basis consisting of vectors where the x- and y-coordinates are either equal or opposite.

Exercise 1.10 Show that the disjoint union $\Gamma+\Delta$ of two graphs Γ and Δ has characteristic polynomial $p(x)=p_\Gamma(x)p_\Delta(x)$.

Exercise 1.11 If Γ is regular of valency k on n vertices, then show that its complement $\overline{\Gamma}$ has characteristic polynomial

$$p(x)=(-1)^n\frac{x-n+k+1}{x+k+1}p_\Gamma(-x-1).$$

Exercise 1.12 Let the *cone* over a graph Γ be the graph obtained by adding a new vertex and joining that to all vertices of Γ. If Γ is regular of valency k on n vertices, then show that the cone over Γ has characteristic polynomial

$$p(x)=(x^2-kx-n)p_\Gamma(x)/(x-k).$$

Exercise 1.13 Let the *join* of two graphs Γ and Δ be $\overline{\overline{\Gamma}+\overline{\Delta}}$, the result of joining each vertex of Γ to each vertex of (a disjoint copy of) Δ. If Γ and Δ are regular of valencies k and ℓ, and have m and n vertices, respectively, then the join of Γ and Δ has characteristic polynomial

$$p(x)=((x-k)(x-\ell)-mn)\frac{p_\Gamma(x)p_\Delta(x)}{(x-k)(x-\ell)}.$$

Exercise 1.14 Let $\Gamma=(V,E)$ be a graph with n vertices and m edges. Construct a new graph Δ with vertex set $V\cup E$ (of size $n+m$), where Γ is the induced subgraph on V and E is a coclique, and each edge $e=xy$ in E is adjacent to its two endpoints x,y in V. Show that if Γ is k-regular, with $k>1$, then the spectrum of Δ consists of two eigenvalues $\frac{1}{2}(\theta\pm\sqrt{\theta^2+4\theta+4k})$ for each eigenvalue θ of A, together with 0 of multiplicity $m-n$.

Exercise 1.15 Show that the Seidel adjacency matrix S of a graph on n vertices has rank $n-1$ or n. (Hint: $\det S \equiv n-1 \pmod 2$.)

Exercise 1.16 Let Γ be a graph with at least one vertex such that any two distinct vertices have an odd number of common neighbors. Show that Γ has an odd number of vertices. (Hint: Consider $A\mathbf{1}$ and $A^2\mathbf{1} \pmod 2$.)

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