STAT 482 Project.

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This project aims to investigate the properties of signal power. The impact of the variables that define signal power are assessed in several situations. The Gamma distribution is tested as an approximation for signal power in each situation.

The variable being tested is signal power, $Y = \Sigma a_i X_i$. X_i is exponential with $\lambda = 1$. A values are positive constants. The sum is for N antennae, with 4 situations being tested: 2, 6, 30 and 1000 antennae. A values are constrained to sum to 1, so their impact can be compared across all situations. The a values are from 4 situations: equal values, low decay (0.95), high decay (0.65) and extreme decay (0.2). Decay defines the relation of a value to its predecessor, with the first value being 1. All situations are normalised to sum to 1.

$Y = \Sigma a_i X_i$ Thus: And:

$$
\begin{aligned}
E[Y] &= E[\Sigma a_i X_i] \\
&= \Sigma E[a_i] E[X_i] \\
&= \Sigma a_i . 1 \\
&= \Sigma a_i = 1
\end{aligned}
\qquad
\begin{aligned}
Var[Y] &= Var[\Sigma a_i X_i] \\
&= \Sigma Var[a_i X_i] \\
&= \Sigma a_i^2 Var[X_i] \\
&= \Sigma a_i^2 . 1 \\
&= \Sigma a_i^2
\end{aligned}
$$

The case of a single antenna was not tested because the value of Y in that case is directly proportionate to the value of a, which would have to equal 1. In that case Y is exponential.

The exponential variables were generated using the random package in python. I simulated 10000 runs, changing the seed each time.

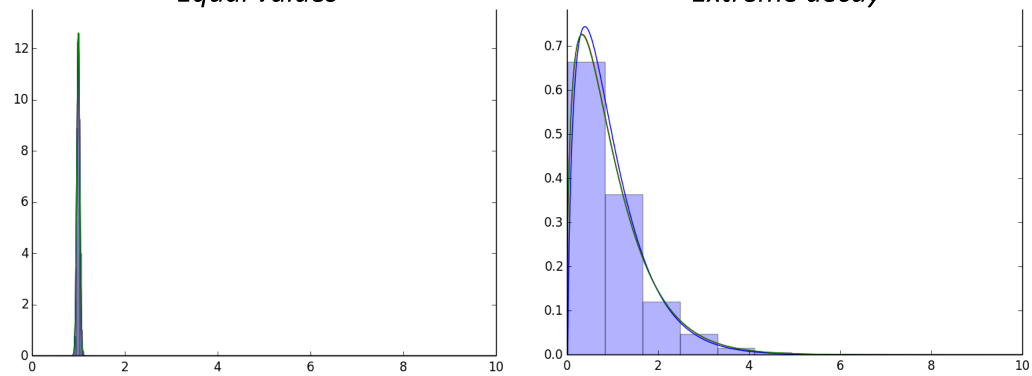
In the simulation of Y, the sample means and variances conformed to mathematical expectations.

In extreme cases, a large a value dominates signal power and its variance. As the rate of decay increases the variance does too, moving towards the variance of the exponential variable. In situations with one massively dominant a value signal power will be approximated by an exponential variable, like in the one antenna case.

Full numerical results are shown in the table of results in the appendix.

Graphs showing simulation results for 1000 antennae.

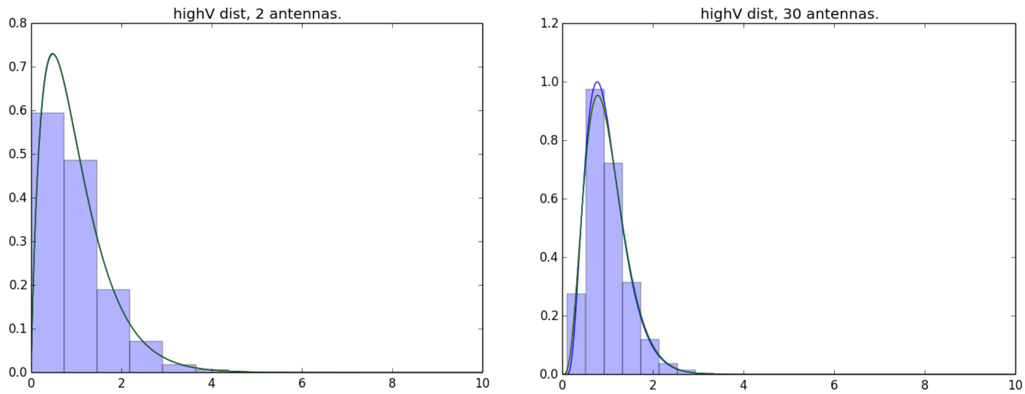
Equal values *Extreme decay*



As I have shown mathematically, the variance of Y is relative to the relationship between the a values. Thus, the scale of the differences between a values in a set has a significant impact on signal power. In the case of equal values Y is an average of N exponential variables. With many antennae, the graph clusters tightly around the expected mean. The extreme decay case shows greater propensity for smaller and larger values because one random variable will dominate the sum.

Increasing the number of antennae reduces the variance. This effect can be seen in all the graphs. As the sum increases in size the ability of one value to dominate signal power is reduced in the milder distributions.

The High decay cases showing moderate clustering around the mean as the number of antennae increases.



Having generated samples from different signal power distributions, I tested the Gamma distribution as an approximation. The moments are as follows:

$$\mu = \alpha/\beta \qquad \sigma^2 = \alpha/\beta^2 \qquad \text{and thus:}$$

$$\alpha = \mu\beta \qquad = \mu\beta/\beta^2 = \mu/\beta \qquad \alpha = \mu\beta = \mu.(\mu/\sigma^2)$$

$$\beta = \mu/\sigma^2 \qquad \alpha = \mu^2/\sigma^2$$

Signal power and Gamma are continuous. They both have a positive range. Signal power is concentrated around a mean, with the variance reflected in the spread of the graph. The distribution rises from 0 to the mean and tails off to the right. The Gamma distribution should be a good fit.

I used all 3 methods of fitting the Gamma distribution:

- The method of moments using mathematical derivation of the expectation and variance,
- The method of moments using sample parameters as estimates, and
- Maximum likelihood estimation using the scipy package in python.

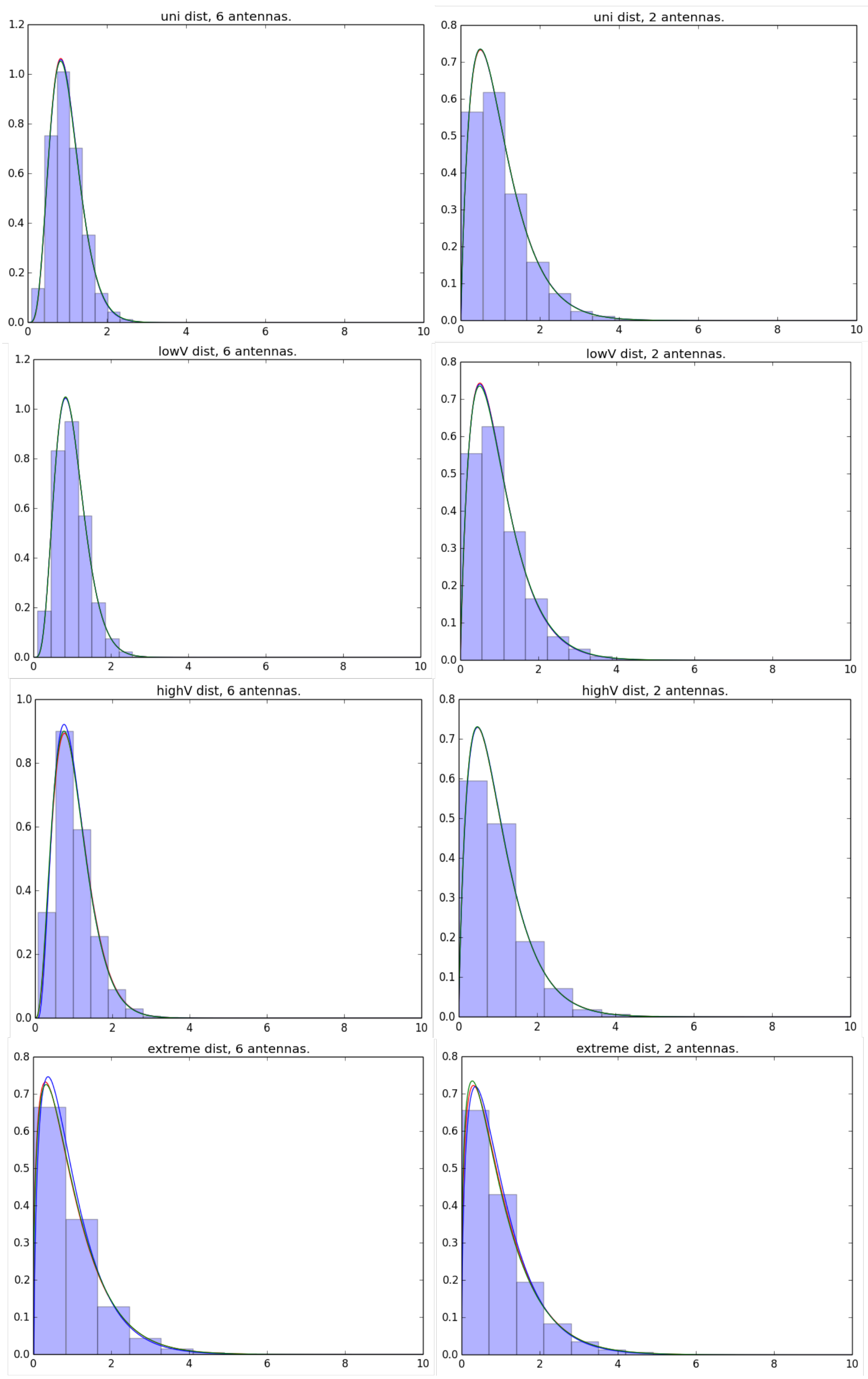
Visually, the Gamma distributions fits very well. All three methods are plotted in each graph, mostly overlapping one another.

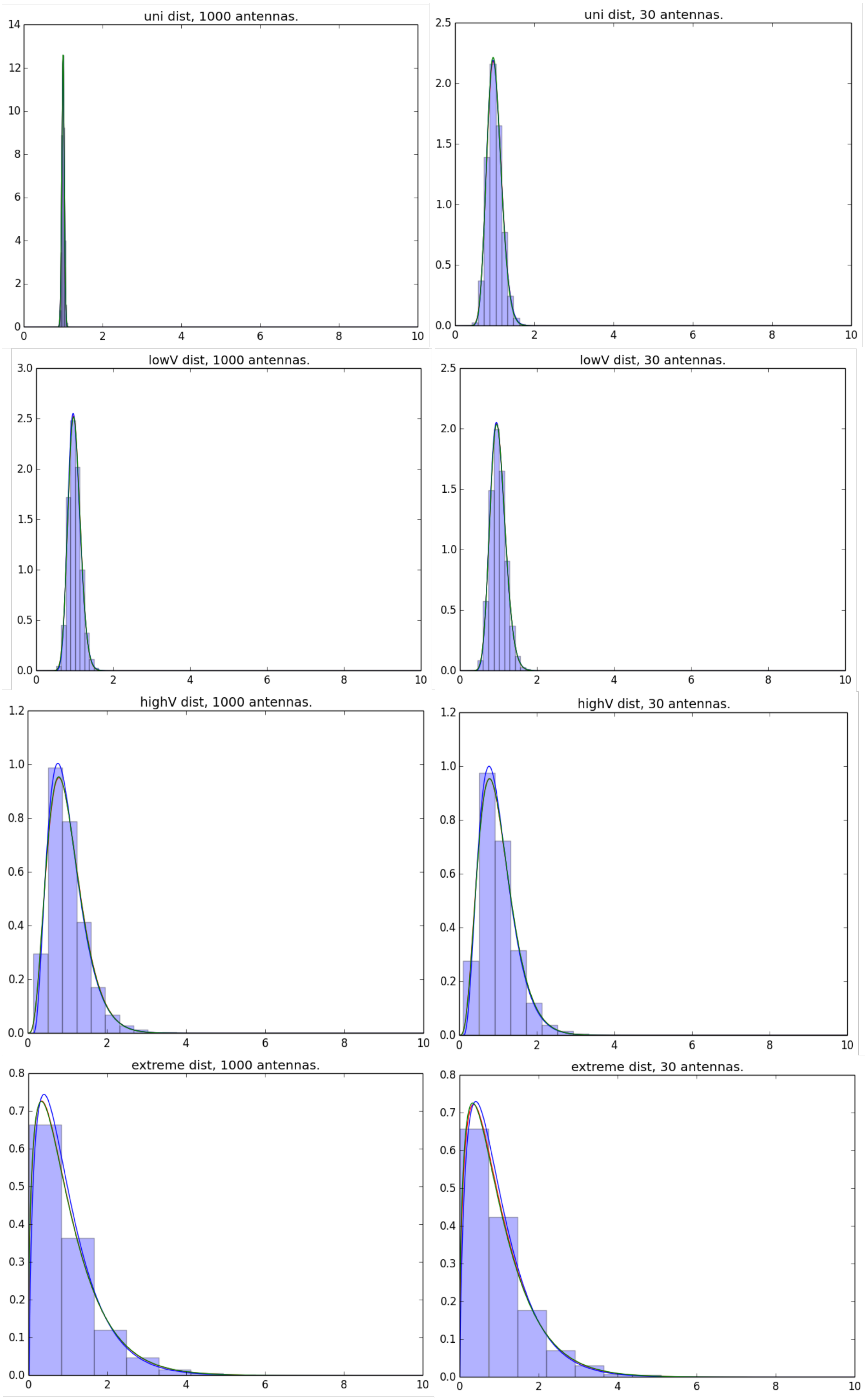
Each fit in each situation was tested using the Kolmogrov Smirnoff test. The statistical results are in the appendix.

- The equal case was an average of a sum of exponential variables. This is approximated extremely well by the Gamma distribution for every number of antennae.
- The low decay case was only marginally different to the equal case and also approximated by a Gamma distribution well.
- The high decay case was fit well by the Gamma but would underestimate the probability of low values.
- The extreme case visually fit the sample well but suffers from a more acute version of the problem the high decay case had- underestimating the probability of low values.

Overall the Gamma is an excellent fit for the samples I created. The case of most interest that I didn’t test was one where the decay had no uniformity. In that case there may be more than one dominant value, thus reducing the approximation of the Gamma by distorting the relationship between the exponential and signal power.

	Antenna	Sample mean	sum(A^2)	Sample variance	Kolmogrov-Smirnoff tests: Theoretical parameters Value	Theoretical parameters %	MLE Parameters Value	MLE Parameters %	Sample moments Value	Sample moments %
Equal	*2*	1.0034	0.5	0.5038	0.0064	0.8049	0.0049	0.969	0.0052	0.9502
	6	0.9987	0.1667	0.1627	0.0057	0.9037	0.0043	0.9929	0.0048	0.9747
	30	1.0031	0.0333	0.0344	0.0098	0.2884	0.0079	0.5669	0.0082	0.5068
	1000	1.0001	0.001	0.001	0.0084	0.481	0.0077	0.5997	0.0078	0.5703
Low decay	*2*	0.9921	0.5003	0.4839	0.011	0.1811	0.0097	0.3007	0.0101	0.2559
	6	1.0003	0.1679	0.169	0.007	0.7046	0.0061	0.8483	0.0067	0.7532
	30	0.9996	0.0397	0.0394	0.0069	0.728	0.0062	0.8367	0.0058	0.8861
	1000	0.9994	0.0256	0.0254	0.0135	0.051	0.0056	0.9167	0.012	0.1127
High decay	*2*	1.0019	0.5225	0.5244	0.0057	0.9054	0.0065	0.7991	0.0066	0.7774
	6	1.0114	0.2467	0.2498	0.019	0.0015	0.0123	0.0987	0.0163	0.0098
	30	0.9966	0.2121	0.2121	0.0238	0	0.0147	0.0267	0.021	0.0003
	1000	1.0044	0.2121	0.2132	0.0237	0	0.0121	0.105	0.022	0.0001
Extreme decay	*2*	1.0098	0.7222	0.71	0.029	0	0.0217	0.0002	0.0206	0.0004
	6	0.993	0.6668	0.6711	0.0327	0	0.0201	0.0006	0.0365	0
	30	1.0022	0.6667	0.6466	0.0333	0	0.0255	0	0.0275	0
	1000	0.9975	0.6667	0.6566	0.0361	0	0.0238	0	0.0349	0


uni dist, 6 antennas.
uni dist, 2 antennas.
lowV dist, 6 antennas.
lowV dist, 2 antennas.
highV dist, 6 antennas.
highV dist, 2 antennas.
extreme dist, 6 antennas.
extreme dist, 2 antennas.


uni dist, 1000 antennas.
uni dist, 30 antennas.
lowV dist, 1000 antennas.
lowV dist, 30 antennas.
highV dist, 1000 antennas.
highV dist, 30 antennas.
extreme dist, 1000 antennas.
extreme dist, 30 antennas.