FERTILITY AND EDUCATION PATTERNS ACROSS DIFFERENT PHASES OF DEVELOPMENT

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September 2nd, 2020

Contents

0.1 Theoretical Model

Ohinata & Varvarigos (2020) propose a model where the economy is populated by overlapping generations of households that have a lifespan of two periods: childhood and adulthood. The household's budget constraint is

$$c_t = \omega h_t - n_t(q + x_t) \tag{1}$$

where c_t denotes consumption and h_t is the stock of human capital.Besides, ω is the wage per unit of effective labour, and rearing each child entails a fixed cost of $q > 0$. Parents spend resources towards the education of each of their offspring using x_t amount.

Parents can affect each child's human capital by devoting resources towards their education using x_t units, so the human capital will be determined as :

$$h_{t+1} = \varphi h_t^{\eta} + \psi h_t^{\mu} x_t \tag{2}$$

Where $\phi, \psi > 0$ and $\eta, \mu \in (0,1)$. Note that h_t captures intergenerational externalities that generate dynamics in the formation of human capital. The lifetime utility of the household is given by:

$$u_t = \gamma \ln(c_t) + (1-\gamma)[\beta \ln(n_t) + \theta \ln(n_t h_{t+1})] \tag{3}$$

Households make their choices so as to maximize their lifetime utility in equation (3), subject to the constraints in equations (1) and (2). The first order condition are :

$$\begin{aligned} n_t(q + x_t) &= \frac{(1-\gamma)(\beta+\theta)}{\gamma + (1-\gamma)(\beta+\theta)} \omega h_t \\ x_t &= \frac{(1-\gamma)\theta}{\gamma + (1-\gamma)(\beta+\theta)} \frac{\omega h_t}{n_t} - \frac{\varphi}{\psi} h_t^{\eta-\mu} \end{aligned} \tag{4}$$

The system of equations in (4) can be solved simultaneously to

$$x_t = X(h_t) = \max\{0, \frac{1}{\beta}[\theta q - (\theta+\beta)\frac{\varphi}{\psi} h_t^{\eta-\mu}]\} \tag{5}$$

and

$$n_t = N(h_t) = \begin{cases} \frac{(1-\gamma)(\beta+\theta)}{\gamma+(1-\gamma)(\beta+\theta)} \frac{\omega h_t}{q} & \text{if } x_t = 0 \\ \frac{(1-\gamma)\beta}{\gamma+(1-\gamma)(\beta+\theta)} \frac{\omega \psi h_t^{1+\mu-\eta}}{q \psi h_t^{\mu-\eta} - \varphi} & \text{if } x_t > 0 \end{cases} \tag{6}$$

A closer look at the result in equation (5) reveals that there are circumstances under which parents might find it optimal not to invest any resources towards the education of their offspring.The underlying cause for this possibility lies in the fact that, as long as $\varphi > 0$, each child will still be endowed with units of efficient labour, because of the presence of the intergenerational externality, even though parents might not invest any resources towards their education.

0.1.1 Dynamics

Assumming that $\mu > \eta$, when the stock of human capital is relatively low, the utility cost of foregone consumption outweighs the utility benefit of educating children and increasing their efficiency. Nevertheless, when the stock of human capital is relatively high, its complementary effect becomes strong enough to guarantee that the return to investment in education is sufficiently high to compensate parents for the utility loss due to decreased consumption.

Lemma 1: There exist a threshold which comes when the second element of X functions is equalized to zero :

$$\tilde{h} \equiv \left[\frac{(\theta+\beta)\varphi}{\theta\psi q} \right]^{\frac{1}{\mu-\eta}} \tag{7}$$

such that

$$x_t = X(h_t) = \begin{cases} 0 & \text{if } h_t <= \tilde{h} \\ \frac{1}{\beta}\left[\theta q - (\theta+\beta)\frac{\varphi}{\psi} h_t^{\eta-\mu}\right] & \text{if } h_t > \tilde{h} \end{cases} \tag{8}$$

We can see that $X(\tilde{h}) = 0$ and

$$X'(h_t) = \frac{(\mu-\eta)(\theta+\beta)\varphi}{\beta\psi} h_t^{\eta-\mu-1} > 0 \tag{9}$$

The outcome summarized in (7) allows us to combine equations in order to express human capital accumulation as :

$$h_{t+1} = F(h_t) = \begin{cases} \varphi h_t^{\eta} & \text{if } h_t <= \tilde{h} \\ \frac{\theta(\psi q h_t^{\mu} - \varphi h_t^{\eta})}{\beta} & \text{if } h_t > \tilde{h} \end{cases} \tag{10}$$

Fertility Dynamics: We begin the analysis by using the results in equation (6) in order to examine how fertility varies with the stock of human capital. **Lemma 2** : Consider $n_t = N(h_t)$. It is straightforward to establish that (a) when $x_t = 0$ then $N(h_t) > 0$; (b) when $x_t > 0$ then there exists.

$$\hat{h} \equiv \left[\frac{(1+\mu-\eta)\varphi}{\psi q} \right]^{\frac{1}{\mu-\eta}} \tag{11}$$

such that when $x_t = 0$

$$N'(h_t) = \frac{(1-\gamma)(\beta+\theta)}{\gamma+(1-\gamma)(\beta+\theta)} \frac{\omega}{q} > 0 \tag{12}$$

but when $x_t > 0$

$$\begin{aligned} N'(h_t) = &\frac{(1-\gamma)(\beta)}{\gamma+(1-\gamma)(\beta+\theta)q}\omega\psi \\ X &\left[\frac{(1+\mu+\eta)h_t^{\mu-\eta}(q\psi h_t^{\mu-\eta}-\varphi) - h_t^{1+\mu-\eta}(\mu-\eta)(q\psi h_t^{\mu-\eta-1})}{(q\psi h_t^{\mu-\eta}-\varphi)^2} \right] \end{aligned} \tag{13}$$

The sign of the derivative will depend on the sign of the expression inside the square brackets.

$$N'(h_t) = \begin{cases} <0 & \text{if } h_t <= \hat{h} \\ >0 & \text{if } h_t > \hat{h} \end{cases} \tag{14}$$

Lemma 3: As long as $[(1+\mu-\eta)\theta]/(\theta+\beta) > 1$ then $\hat{h} > \tilde{h}$ Prrof: $\hat{h} > \tilde{h}$ implies

$$\left[\frac{(1+\mu-\eta)\varphi}{\psi q} \right]^{\frac{1}{\mu-\eta}} > \left[\frac{(\theta+\beta)\varphi}{\theta\psi q} \right]^{\frac{1}{\mu-\eta}} \tag{15}$$

And this is only true if $[(1+\mu-\eta)\theta]/(\theta+\beta) > 1$ then $\hat{h} > \tilde{h}$.

Therefore, as the stock of human capital grows, the fertility rate increases for $h_t < \tilde{h}$; it declines for $\tilde{h} < h_t < \hat{h}$; and it increases again for $h_t > \hat{h}$. Formally,

$$N'(h_t) = \begin{cases} >0 & \text{if } h_t < \tilde{h} \\ <0 & \text{if } \tilde{h} < h_t <= \hat{h} \\ >0 & \text{if } h_t > \hat{h} \end{cases} \tag{16}$$

Specifically, the fertility rate increases with h_t at relatively low levels of income; it decreases at intermediate levels of income; and it increases again at relatively high levels of income, as the stock of human capital and therefore fertility converge to their long-run (steady-state) values.

Naturally, our objective is to analyse an economy that goes through all the stages of possible demographic changes, as it converges to the longrun equilibrium that is characterized by h^*.It follows that the subsequent analysis will focus on a scenario where the steady-state equilibrium lies above the two thresholds identified previously.

Lemma 4: Assume that.

$$q\psi > max\left\{\left(\frac{\beta}{\theta}\right)^{(\mu-\eta)/(1-\eta)}\frac{(1+\mu-\eta)\varphi^{(1-\mu)/(1-\eta)}}{(\mu-\eta)^{(\mu-\eta)/(1-\eta)}};\frac{(\theta+\beta)\varphi^{(1-\mu)/(1-\eta)}}{\theta}\right\} \tag{17}$$

holds. Then $h^* > \hat{h}$

0.1.2 The full dynamical system

The full dynamics of this model are characterized by a non-linear three dimensional system of difference equations.

$$\begin{aligned}
h_{t+1} = F(h_t) &= \begin{cases} \varphi h_t^{\eta} & \text{if } h_t <= \tilde{h} \\ \frac{\theta(\psi q h_t^{\mu} - \varphi h_t^{\eta})}{\beta} & \text{if } h_t > \tilde{h} \end{cases} \\
x_t = X(h_t) &= \max\{0, \frac{1}{\beta}[\theta q - (\theta+\beta)\frac{\varphi}{\psi} h_t^{\eta-\mu}]\} \\
n_t = N(h_t) &= \begin{cases} \frac{(1-\gamma)(\beta+\theta)}{\gamma+(1-\gamma)(\beta+\theta)}\frac{\omega h_t}{q} & \text{if } x_t = 0 \\ \frac{(1-\gamma)\beta}{\gamma+(1-\gamma)(\beta+\theta)}\frac{\omega\psi h_t^{1+\mu-\eta}}{q\psi h_t^{\mu-\eta}-\varphi} & \text{if } x_t > 0 \end{cases}
\end{aligned} \tag{18}$$