Math 441 Notes

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First order DEs.

Definition. A first order DE is linear if it can be written in the form

$$\frac{dy}{dt} + p(t)y = q(t)$$

Integrating Factors.

Consider any first order DE, where $p(t)$ and $q(t)$ are continuous. A solution can be found by multiplying both sides by an integrating factor $\mu(t)$, where $\mu(t) = e^{\int p(t)dt}$. Then

$$\mu(t)\frac{dy}{dt} + \mu(t)p(t)y = \mu(t)q(t) \implies (\mu(t)y)' = \mu(t)q(t)$$

Therefore

$$y(t) = \frac{\int e^{\int p(t)dt} q(t)dt}{e^{\int p(t)dt}}$$

Finite Time Blowup. Consider a mass in uniform circular motion. Its velocity can be modeled by

$$\frac{dv}{dt} = kv^2,\ k \in \mathbb{R}$$

Solving for $v(t)$, we have

$$v(t) = \frac{v_0}{1 - ktv_0}$$

The model goes to infinity ("blows up") at $t = \frac{1}{kv_0}$.

Definition. A DE is separable if it can be written in the form

$$\frac{dy}{dx} = \frac{M(x)}{N(y)}$$

Definition. A DE is a Bernoulli equation if it can be written in the form

$$\frac{dy}{dt} + p(t)y = q(t)y^n$$

The equation can be solved using the substitution $y = v^{\frac{1}{1-n}}$.

Another special form. A DE in the form $y' = f(\frac{y}{t})$ can be solved with the substitution $y = tv$.

Definition. Let y be a function of x. A DE is autonomous if it can be written in the form

$$\frac{dy}{dx} = f(y)$$

.

Newton's Law of Cooling. The rate of change of the temperature is proportional to the difference of the temperature and the surrounding (ambient) temperature. In mathematical terms:

$$\frac{dT}{dt} = -k(T - T_s)$$

where t is time, T is a function of t expressing temperature, k is a positive constant, and T_s is the surrounding temperature.

First-order theory.

*Theorem. (Existence and Uniqueness for first-order **linear** DEs)* Consider the first-order linear DE $\frac{dy}{dt}+p(t)y = q(t), y(t_0) = y_0$. If $p(t)$ and $q(t)$ are continuous on the interval (α, β) and $t_0 \in (\alpha, \beta)$, then the IVP has a unique solution on the interval (α, β).

Theorem. (Picard-Lindelof) Consider the first order DE $\frac{dy}{dt} = f(t, y), y(t_0) = y_0$. If $f(t, y)$ and $\frac{df}{dy}$ are continuous on some rectangle $R = (a, b) \times (c, d)$ and $(t_0, y_0) \in R$, then the IVP has a unique solution in some neighborhood of (t_0, y_0).

equivalent? to

Consider the first order DE $\frac{dy}{dt} = f(t, y), y(t_0) = y_0$. If f is Lipschitz continuous in y and continuous in t, then the IVP has a unique solution on the interval $[t_0 - \epsilon, t_0 + \epsilon]$ for some $\epsilon > 0$.

DE models.

- Exponential growth and decay. Model as $\frac{dy}{dt} = ry$, with an initial condition $y(0) = y_0$. The solution is $y(t) = y_0 e^{rt}$. Growth when r is positive. Decay when r is negative. Equilibrium only at $y = 0$.

- Logistic model. Equations of the form $\frac{dy}{dt} = (r - ay)y$, r and k both positive. If we extract r, we have $\frac{dy}{dt} = r(1 - \frac{y}{k})y$, where r is the intrinsic growth rate and k is the environmental carrying capacity or saturation level. Equilibrium at $y = 0$ and $y = k$.

- Critical threshold. Equations of the form $\frac{dy}{dt} = -r(1 - \frac{y}{T})y$,$r$ and T both positive. r is the intrinsic growth rate and T is the critical threshold. Notice this is the same as the logistic model with a negative sign.

Stability. Equilibrium solutions to a DE can be classified by stability - asymptotically stable, semistable, or unstable.

Phase Lines. A diagram called a phase line can be constructed for a DE by marking a line segment with the corresponding critical points (where the derivative is 0), and marking the sign of the DE in every divided segment. Then, the stability of each critical point can be easily determined.

Euler's Method. Given $\frac{dy}{dt} = f(t, y)$, approximate $y(t)$ by $y_{x+1} = y_x + hf(t_x, y_x)$, where h is the step size. As $h \to 0$, the approximation becomes more accurate. (Remark: Euler's method is a first-order method, meaning that the error is proportional to the step size. Other methods exist that are of higher order/lower error.)

Second order DEs.

Special forms of second order DEs. If in the form $y'' = (y, y')$ or $y'' = (t, y')$, let $v = y'$. For the form with no t, use the fact that $\frac{dv}{dt} = \frac{dv}{dy}\frac{dy}{dt}$, and remove all t terms.

Definition. A second order DE is linear if it is in the form $y''+p(t)y'+q(t)y = g(t)$, where y is a function of t.

Operator notation. Define $L[y] = y'' + p(t)y' + q(t)y$. We say that L acts on y.

Definition. A second order DE is of the form $L[y] = g(t)$. If $g(t) = 0$, the DE is homogeneous. Otherwise, it is nonhomogeneous.

Existence and Uniqueness for second order DEs. If $p(t), q(t), and g(t)$ are continuous on $t \in (\alpha, \beta)$ and $t_0 \in (\alpha, \beta)$, then the IVP $L[y] = g(t)$ has a unique solution that satisfies $y(t_0) = y_0$ and $y'(t_0) = y_0'$. This solution exists on all of (α, β).

Superposition Principle. If $L[y_1] = 0$ and $L[y_2] = 0$, then $L[c_1y_1 + c_2y_2] = 0 \quad \forall c_1, c_2 \in \mathbb{R}$.

Definition. Two functions y_1 and y_2 are linearly independent if $y_1 = cy_2$ for some $c \in \mathbb{R}$. Otherwise, they are linearly independent.

Theorem. If $L[y_1] = 0$ and $L[y_2] = 0$ and the two functions are linearly independent, then every solution to $L[y] = 0$ can be written as $c_1y_1c_2y_2$ for some $c_1, c_2 \in \mathbb{R}$.

Fact. $L[y] = 0$ has exactly two linearly independent solutions.

Theorem. If $y(t) = u(t) + iv(t)$ is a solution to $L[y] = 0$, then $u(t)$ and $v(t)$ are both solutions.

Proof.

$$L[y] = (u'' + iv'') + p(t)(u' + iv') + q(t)(u + iv) = 0$$

equivalent to

$$(u'' + p(t)u' + q(t)u) + i(v'' + p(t)v' + q(t)v) = 0$$

Thus, we must have $L[u] = L[v] = 0$.

Characteristic Equation. A second order linear constant coefficient DE has a corresponding polynomial called the characteristic equation. If the characteristic equation has a root r, then e^{rt} solves the DE.

Three cases. Suppose we have a DE $ay'' + by' + cy = 0$, where a, b, c are constants. Then the characteristic equation is $ar^2 + br + c = 0$. Three cases:

- $b^2 > 4ac$. Then, roots are real and distinct ($r_1, r_2 \in \mathbb{R}$, $r_1 \neq r_2$). The solution to the DE is $y(t) = c_1e^{r_1t} + c_2e^{r_2t}$.

- $b^2 < 4ac$. Then, roots are complex ($r_1, r_2 = \alpha \pm i\beta$). The solution is $y(t) = e^{\alpha t}(c_1 cos(\beta t) + c_2 sin(\beta t))$.
- $b^2 = 4ac$. Then, there is one repeated root. The solution is $y(t) = c_1e^{rt} + c_2te^{rt}$.

For all three cases, $c_1, c_2 \in \mathbb{R}$.

Euler Equations. If a second order DE is of the form $at^2y'' + bty' + cy = 0$, where y is a function of t, it can be classified as an Euler Equation, a special form of second order homogeneous non-constant coefficient DE. If there is a root r such that $ar(r-1) + br + c = 0$, then $y(t) = t^r$ solves the DE.

Reduction of Order. Consider the second order linear homogenous DE $y'' + p(t)y' + q(t)y = 0$. Suppose we have a known solution, $y_1(t)$. Assume that the second solution is of the form $y_2(t) = y_1(t)v(t)$, where $v(t)$ is some arbitrary function. After substituting in $y_2(t)$ into the DE and simplifying, we get that

$$y_2(t) = y_1(t) \int \frac{e^{-\int p(t)dt}}{[y_1(t)]^2} dt$$

.

Solving second order linear nonhomogenous equations. Consider $ay'' + by' + cy = g(t)$, where $a, b, c \in \mathbb{R}$. Suppose $y_h(t)$ solves $ay'' + by' + cy = 0$, and $L[y_p(t)] = g(t)$. Then the general solution to the DE is $y(t) = y_h(t) + y_p(t)$, since $L[y_h + y_p] = L[y_p] + L[y_h] = 0 + g(t) = g(t)$. $y_h(t)$ is the homogenous solution, and $y_p(t)$ is the particular solution.

Method of Undetermined Coefficients. This method is used to find the particular solution to some second order nonhomogeneous equations, provided we can solve for the homogenous solution $y_h(t)$. Consider $y'' + p(t)y' + q(t)y = g(t)$. Based on the form of $g(t)$, guess a particular solution form. Adjust the guess if there is duplication with the homogenous solution. Finally, solve for the "undetermined coefficients".

Wronskian. Let $f_1, f_2, ..., f_n$ be a list of n functions. The Wronskian is defined as the determinant of the following matrix:

$$\begin{pmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{n-1} & f_2^{n-2} & \cdots & f_n^{n-1} \end{pmatrix}$$

If the Wronskian is nonzero in some range, then the n functions are linearly independent. If the Wronskian is zero over some range, then the functions are linearly dependent somewhere in that range.

Variation of Parameters. Consider $y'' + p(t)y' + q(t)y = g(t)$. Assume $y_1(t)$ and $y_2(t)$ are linearly independent solutions for the homogenous equation (that is, $y_h(t) = c_1y_1(t) + c_2y_2(t)$). Then a particular solution is

$$y_p(t) = -y_1 \int \frac{y_2 g(t)}{W(y_1, y_2)} + y_2 \int \frac{y_1 g(t)}{W(y_1, y_2)} dt$$

.

Reduction of higher order DEs to a system. Any DE of order n can be written as

$$y^{(n)} = F(t, y, y', y'', ..., y^{(n-1)})$$

Define n new equations $y_i = y^{(i-1)}$ for $i = 2, 3, ..., n$. Then $y^{(n)} = F(t, y_1, y_2, ..., y_{n-1})$. This transforms the original n order DE into a system of n first order DEs:

$$\vec{y}' = F(t, \vec{y}), \vec{y}' = \begin{pmatrix} y_1' \\ y_2' \\ \vdots \\ y_n' \end{pmatrix}, \vec{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

Mechanical Vibrations.

Consider an object that is on a spring, with the spring attached to a flat ceiling and the mass hanging down. Let m be the mass of the object, γ be the damping constant, k be the spring constant, L be the distance the mass pulls down the spring when in the equilibrium position, and $u(t)$ express the displacement of the object with respect to the equilibrium position.

There is a force of gravity $F_g = mg$, a spring force $F_s = -k(L + u(t))$ (varies with time, but $F_s = -kL$ if in constant equilibrium), and possibly external forces, which will be denoted as $F(t)$.

In equilibrium, $F_g = F_s$, so $mg = kL$. This can be used to find unknown values. In all cases, m and k are positive. γ may be 0 or positive.

Free, undamped vibrations. In this case, $F(t) = 0$ for all t, and $\gamma = 0$. Then the DE $mu'' + ku = 0$ models this case. The solution is of the form $u(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t)$, where $\omega_0 = \sqrt{\frac{k}{m}}$ is called the natural frequency. The period of oscillation is denoted as $T = \frac{2\pi}{\omega_0}$. The amplitude of motion is denoted as $R = \sqrt{A^2 + B^2}$. The phase shift is denoted as $\delta, \cos\delta = \frac{A}{R}, \sin\delta = \frac{B}{R}$, $\arctan\frac{B}{A} = \delta$. The solution for $u(t)$ can be written in a more convenient form, $u(t) = R\cos(\omega_0 t - \delta)$. This case exhibits simple harmonic motion.

Free, damped vibrations. In this case, $F(t) = 0$ for all t, and $\gamma > 0$. There are three cases:

- $\gamma > \sqrt{4mk}$. This leads to overdamping. The solution is of the form $u(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$.
- $\gamma = \sqrt{4mk}$. This leads to critical damping, which is the fastest approach towards $u(t) = 0$. The solution is of the form $u(t) = e^{\frac{-\gamma t}{2m}}(c_1 + c_2 t)$.
- $\gamma < \sqrt{4mk}$. This leads to underdamping. The solution is of the form $u(t) = e^{\frac{-\gamma t}{2m}}(c_1\cos(\mu t)c_2\sin(\mu t)) = Re^{\frac{-\gamma t}{2m}}\cos(\mu t - \delta)$, $c_1, c_2 \in \mathbb{R}$, where μ is the quasi-frequency, and the quasi-period is defined as $\frac{2\pi}{\omega_0}$.

Forced Oscillations. We have $mu''+\gamma u'+ku = F(t)$, with $F(t) = F_0\cos(\omega_0 t)$ for some F_0 and ω. Resonance occurs when $\omega = \sqrt{\frac{k}{m}} = \omega_0$. Assume $\omega \neq \omega_0$. Then the particular solution $u_p(t) = A\cos(\omega t) + B\sin(\omega t)$, also called the steady state solution. This can be rewritten as $u_p(t) = R\cos(\omega t - \delta)$, where $R = \frac{F_0}{\sqrt{m^2(\omega_0^2-\omega^2)^2+\omega^2\gamma^2}}$. If the forcing frequency ω is 0, then $R = \frac{F_0}{k}$. As $\omega \to \infty$, $R \to 0$. Note that $\omega \in [0,\infty)$.

Forced Oscillations, $\gamma = 0$. If there is no damping ($\gamma = 0$), and $\omega \neq \omega_0$, then we have simple harmonic motion. If $\omega = \omega_0$, then there is duplication between the homogeneous and particular solutions. Thus, the particular must be in the form $y_p(t) = At\cos(\omega t) + Bt\sin(\omega t)$. This graph has waves of increasing amplitude bounded by $y = t$ and $y = -t$. However, in any realistic situation, zero damping is impossible.

This case exhibits simple harmonic motion for all ω except for $\omega = \omega_0$, which is resonance. The amplitude varies as well: it is largest when $\omega \approx \omega_0$.

Forced Oscillations, $\gamma \neq 0$. Though pure resonance is only possible if the damping constant is zero, we have practical resonance with a nonzero damping constant. With practical resonance, ω is tuned so that the amplitude of the steady state solution is maximized. In fact, $\omega_{max} = \omega_0\sqrt{1-\frac{\gamma^2}{2km}}$. The max amplitude is $R(\omega_{max}) = \frac{F_0}{\gamma\omega_0\sqrt{1-\frac{\gamma^2}{4km}}}$. For very small values of γ, $R \approx \frac{F_0}{\gamma\omega_0}$.

Power Series. Some ODEs may be difficult to solve. One technique to find approximate solutions is to use power series. Consider $y'' + p(t)y' + q(t)y = 0$. We will assume $y(t) = \sum_{n=0}^{\infty} a_n(x-x_0)^n$, with $a_i \in \mathbb{R}\forall i$. Since differentiation is a linear operator, we have $y'(t) = \sum_{n=1}^{\infty} na_n(x-x_0)^{n-1}$, and $y''(t) = \sum_{n=2}^{\infty} n(n-1)a_n(x-x_0)^{n-2}$.

Systems of ODEs.

Phase Plane. Consider a system of two ODEs, $\vec{x}' = A\vec{x}$, where $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. The system can be graphed with one trajectory in the phase plane, where the horizontal axis represents $x_1(t)$ and the vertical axis represents $x_2(t)$. The trajectory must have arrows to indicate direction as t increases.

Equilibrium Solutions. We often look for equilibrium solutions of a system of ODEs. In general, we seek values of $x_1, x_2, ..., x_n$ such that $\vec{x}' = \vec{0}$, where $\vec{0}$ is the zero vector.

SIR Model.

Lotka-Volterra System.

Theory of systems.

Linear systems.

Superposition principle for linear systems.

Existence and Uniqueness for linear systems.

Definition. A set of vectors $\vec{v_1}, \vec{v_2}, ..., \vec{v_n} \in \mathbb{R}^n$ are linearly dependent if $\vec{v_i} = c\vec{v_j}, i \neq j$ for some $c \in \mathbb{R}$. Otherwise, they are linearly independent.

Spanning/Basis Property.

General Solution Theorem.

Solutions and portraits of $\vec{x}' = A\vec{x}$, where A is a 2×2 matrix constant coefficient matrix.

Finding solutions to $\vec{x}' = A\vec{x}$. Expanded, this system can also be written as

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

where $a, b, c, d \in \mathbb{R}$. We will guess a solution in the form $\vec{x}(t) = e^{\lambda t}\vec{v}$, where $\vec{v}$ is some constant vector. Substituting this form, we have

$$\lambda e^{\lambda t}\vec{v} = Ae^{\lambda t}\vec{v}$$

so

$$(A\vec{v} - \lambda\vec{v})e^{\lambda t} = \vec{0}$$

where $\vec{0}$ is the zero vector. Then

$$(A - \lambda I)\vec{v} = \vec{0}$$

As an intuitive explanation, we must find vectors $\vec{v}$ such that when acted upon by a linear transformation $(A - \lambda I)$, it results in the zero vector. That is, which vectors in $\mathbb{R}^2$ are in the null space of $A - \lambda I$? For any linear transformation (i.e. matrix) that is invertible, the null space only consists of the zero vector. Thus, we must have $A - \lambda I$ noninvertible; that is, $det(A - \lambda I) = 0$. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $A - \lambda I = \begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix}$. So we must have $det(A - \lambda I) = (a - \lambda)(d - \lambda) - bc = 0$. From this, we can derive the characteristic equation: $\lambda^2 - T\lambda + D = 0$, where $T = a + d$, $D = ad - bc$, and solutions for λ are the eigenvalues of A.

Theorem. If $\vec{v_1}, \vec{v_2}$ are both eigenvectors of a 2 by 2 matrix A with different eigenvalues (that is, $A\vec{v_1} = \lambda_1 v_1, A\vec{v_2} = \lambda_2 v_2, \lambda_1 \neq \lambda_2$), then the two eigenvectors are linearly independent.

Eigenvalue cases. From the characteristic equation $\lambda^2 - T\lambda + D = 0$, there are three cases:

- $T^2 > 4D$. Then $\lambda_1, \lambda_2 \in \mathbb{R}$, $\lambda_1 \neq \lambda_2$; real and distinct eigenvalues. Four half-line solutions that split $\mathbb{R}^2$ into four regions.
- $T^2 < 4D$. Then $\lambda_1, \lambda_2 \in \mathbb{C}$; complex eigenvalues with real part zero or nonzero.
- $T^2 = 4D$. Then there is one repeated eigenvalue.

The matrix exponential.

Definition.

Let $\mathbf{A}$ be an $n \times n$ matrix. Then the matrix exponential is a matrix function from $\mathbb{R}^{n\times n}$ to $\mathbb{R}^{n\times n}$ defined by

$$e^{\mathbf{A}} = \sum_{k=0}^{\infty} \frac{1}{k!}\mathbf{A}^k = I + X + \frac{1}{2!}\mathbf{A}^2 + \frac{1}{3!}\mathbf{A}^3 + ...$$

We bring the $\frac{1}{k!}$ out front to emphasize the scalar-vector multiplication. We also define $\mathbf{A}^0 = I$, where I is the n by n identity matrix. Some notes:

- If $n = 1$, then $e^{\mathbf{A}} = e^x$, where x is the lone entry of $\mathbf{A}$.
- If all entries of $\mathbf{A}$ are 0, then $e^{\mathbf{A}} = I$.
- If $\mathbf{A}$ is a diagonal matrix (that is, $\forall i \neq j$, $\mathbf{A}_{ij} = 0$), then

$$\mathbf{A} = \begin{pmatrix} x_1 & 0 & \dots & 0 \\ 0 & x_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & x_n \end{pmatrix} \quad \implies \quad e^{\mathbf{A}} = \begin{pmatrix} e^{x_1} & 0 & \dots & 0 \\ 0 & e^{x_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e^{x_n} \end{pmatrix}$$

Fact. For any square matrix $\mathbf{A}$, $e^{\mathbf{A}}$ converges.

Proof.

Law of exponents for commuting matrices. If $\mathbf{A}$ and $\mathbf{B}$ are square matrices of the same size and they commute (i.e. $\mathbf{AB} = \mathbf{BA}$), then

$$e^{\mathbf{A}}e^{\mathbf{B}} = e^{\mathbf{A}+\mathbf{B}}$$

Since matrix addition is commutative, $e^{\mathbf{A}+\mathbf{B}} = e^{\mathbf{B}+\mathbf{A}} = e^{\mathbf{B}}e^{\mathbf{A}}$.

Proof.

Lemma. $(e^{\mathbf{A}})^{-1} = e^{-\mathbf{A}}$.

Proof. It can be easily shown that $(\mathbf{A})(-\mathbf{A}) = (-\mathbf{A})(\mathbf{A})$. Then, by the law of exponents for commuting matrices

$$e^{\mathbf{A}+(-\mathbf{A})} = e^{\mathbf{A}}e^{-\mathbf{A}} \implies \mathbf{I} = e^{\mathbf{A}}e^{-\mathbf{A}}$$

Hence, the inverse of the matrix $e^{\mathbf{A}}$ is $e^{-\mathbf{A}}$.

Fact. If $\mathbf{A}$ is a square matrix, then $\frac{d}{dt}(e^{\mathbf{A}t}) = \mathbf{A}e^{\mathbf{A}t}$. (Note: $\mathbf{A}e^{\mathbf{A}t} = e^{\mathbf{A}t}\mathbf{A}$)
Proof.

$$\begin{aligned}
\frac{d}{dt}(e^{\mathbf{A}t}) = \frac{d}{dt}(\mathbf{I} + \mathbf{A}t + \frac{1}{2!}\mathbf{A}^2t^2 + \frac{1}{3!}\mathbf{A}^3t^3...) &= \frac{d}{dt}\left(\sum_{k=0}^{\infty}\frac{\mathbf{A}^k t^k}{k!}\right) \\
= \sum_{k=0}^{\infty}\frac{d}{dt}\frac{\mathbf{A}^k t^k}{k!} = \sum_{k=1}^{\infty}\frac{k\mathbf{A}^k t^{k-1}}{k!} &= \sum_{k=1}^{\infty}\frac{\mathbf{A}^k t^{k-1}}{(k-1)!} \\
= \mathbf{A}\sum_{k=1}^{\infty}\frac{\mathbf{A}^{k-1} t^{k-1}}{(k-1)!} = \mathbf{A}\sum_{k=0}^{\infty}\frac{\mathbf{A}^k t^k}{k!} &= \mathbf{A}(e^{\mathrm{X}t})
\end{aligned}$$

Definition. Let $\mathbf{A}$ be an $n \times n$ matrix. $\mathbf{A}$ is diagonalizable iff there exists matrices $\mathbf{V}$ and $\mathbf{\Lambda}$ such that $\mathbf{A} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1}$. (Note: another equivalent condition is that $\mathbf{A}$ has n linearly independent eigenvectors)

Motivation for diagonalization. Calculating the matrix exponential for a matrix $\mathbf{A}$ requires computing $\mathbf{A}^k$, which is usually the most tedious step. If $\mathbf{A}$ is not diagonal, then computing $\mathbf{A}^k$ becomes involved. Therefore, diagonalizing the matrix makes calculating the matrix exponential much easier.

Process. Let $\mathbf{A}$ be an $n \times n$ matrix with n linearly independent eigenvectors. By the definition above, $\mathbf{A}$ is diagonalizable. We know that $\mathbf{A}\vec{\mathbf{v}}_i = \lambda_i\vec{\mathbf{v}}_i$ for $i \in \{1, 2, ..., n\}$. Let $\mathbf{V}$ represent the matrix containing all n linearly independent eigenvectors of $\mathbf{A}$. Then

$$\mathbf{AV} = \mathbf{A}\begin{bmatrix}\vec{\mathbf{v}}_1 ... \vec{\mathbf{v}}_n\end{bmatrix} = \begin{bmatrix}\lambda_1\vec{\mathbf{v}}_1 ... \lambda_n\vec{\mathbf{v}}_n\end{bmatrix} = \begin{bmatrix}\vec{\mathbf{v}}_1 ... \vec{\mathbf{v}}_n\end{bmatrix}\begin{bmatrix}\lambda_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_n\end{bmatrix}$$

If we let $\mathbf{\Lambda}$ represent the diagonal matrix containing all n (not necessarily unique) eigenvalues, then we have

$$\mathbf{AV} = \mathbf{V\Lambda} \implies \mathbf{A} = \mathbf{V\Lambda V}^{-1}$$

Furthermore, we can use this to easily find the matrix exponential of a non-diagonal matrix:

$$e^{\mathbf{A}t} = \sum_{k=0}^{\infty}\frac{\mathbf{A}^k t^k}{k!} = \sum_{k=0}^{\infty}\frac{\mathbf{V\Lambda}^k\mathbf{V}^{-1}t^k}{k!} = \mathbf{V}\left(\sum_{k=0}^{\infty}\frac{\mathbf{\Lambda}^k t^k}{k!}\right)\mathbf{V}^{-1} = \mathbf{V}e^{\mathbf{\Lambda}t}\mathbf{V}^{-1}$$

Solving a non-homogeneous linear system with constant coefficients.
Recall that

$$y' = ay - b \implies y = \frac{b}{a} + Ce^{at}, \quad \mu = e^{-at}$$

where μ is the integrating factor. Now, consider $\vec{x}' = \mathbf{A}\vec{x} - \vec{b}$, a non-homogeneous linear system with constant coefficients. Let $\mu = e^{-\mathbf{A}t}$. Then

$$(e^{-\mathbf{A}t}\vec{x})' = -e^{\mathbf{A}\vec{x}} + e^{-\mathbf{A}t}\vec{x}' = e^{-\mathbf{A}t}(\vec{x}' - \mathbf{A}\vec{x}) = -e^{-\mathbf{A}t}\vec{b}$$

Integrating gives

$$e^{-\mathbf{A}t}\vec{x} = e^{-\mathbf{A}t}\mathbf{A}^{-1}\vec{b} + \vec{c}$$

Solving for $\vec{x}$, we have

$$\vec{x} = \mathbf{A}^{-1}\vec{b} + e^{\mathbf{A}t}\vec{c}$$

Note the similarity between this result and the solution to the single non-homogeneous linear equation.

Linearization.

Motivation. With a single differential equation, we drew the phase line to represent the behavior of solutions in certain intervals of y. With systems of differential equations, we use a phase portrait for the same purpose. However, phase portraits are difficult to draw when the system isn't a homogeneous, constant coefficient system. Thus, we use linearization to approximate the actual system at its equilibrium points so we can estimate its actual phase portrait. (Note: we can also use linearization for single, nonlinear differential equations)

Process. Recall the Taylor series of a function $f(x)$ around the point $x = x_0$:

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + ... = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

Now, if $f(x_0) = 0$, the terms $(x - x_0)^k$ become increasingly smaller around the point $x = x_0$ as $k \to \infty$. Thus, if we let $x' = f(x)$, we can approximate x' as

$$x' = f(x) \approx f'(x_0)(x - x_0)$$

Letting $u = x - x_0$, we have

$$u' = f'(x_0)u$$

Now we have the linearized version of the original equation with equilibrium at 0.

Example. Let $x' = 4\sin(x)$. We recognize that one equilibrium point is $x_0 = 0$. Then, the linearization is

$$x' = f(x) \approx f'(x_0)(x - x_0) = f'(0)(x - 0) = 4\cos(0)x = 4x$$

Letting $u = x - x_0$, we get

$$u' = 4u$$

An alternative method is to use the Taylor expansion for $\sin(x)$:

$$x' = f(x) = 4\sin(x) = 4(x - \frac{x^3}{3!} + \frac{x^5}{5!}...)$$

Ignoring all $O(x^2)$ terms, we get

$$x' = f(x) \approx 4x$$

With the same change of variables, we get the same result as above.

Linearizing a system. Previously, we had one equation $x' = f(x)$. Now, consider a system of two equations:

$$x' = f(x, y)$$
$$y' = g(x, y)$$

Previously, we had an equilibrium solution $x*$ if $x' = f(x*) = 0$. In a system, an equilibrium point is a point $(x*, y*)$ such that $x' = f(x*, y*) = 0$ and $y' = g(x*, y*) = 0$. We want to linearize a system around each of its equilibrium points to estimate the behavior of the solutions (phase portrait) near that equilibrium point.

Example. Consider the system

$$x' = 2x + x^3 + 3\sin(y)$$
$$y' = -x + e^y - 1$$

We will linearize around the trivial equilibrium point $(0, 0)$. Use use Taylor expansions to convert the trig functions into polynomials (this will allow us to omit $O(x^2)$ terms easily):

$$x' = 2x + x^3 + 3\sin(y) = 2x + x^3 + 3(x - O(y^3)) = 2x + 3y$$
$$y' = -x + e^y - 1 = -x + (1 + y + O(y^2)) - 1 = -x + y$$

Our linearized system is

$$\vec{x}' = \begin{pmatrix} 2 & 3 \\ -1 & 1 \end{pmatrix} \vec{x}$$

The Jacobian.

What if the equilibrium point is not at the origin? Recall that with a single equation $x' = f(x)$, we used a change of variables $u' = f'(x_0)u$. With a system, we have $\vec{x}' = f(x, y)$ with the following change of variables:

$$\vec{u}' = \mathbf{J}\vec{u}$$

where $\mathbf{J}$ is a constant matrix called the Jacobian, defined by

$$\mathbf{J} = \begin{pmatrix} f_x(x, y) & f_y(x, y) \\ g_x(x, y) & g_y(x, y) \end{pmatrix}$$

In the case of one equation, we used a tangent line approximation. The Jacobian offers a tangent-plane approximation. We have the two-variable Taylor expansion about the point (x_0, y_0):

$$f(x, y) \approx f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + ...$$

Letting $u = x - x_0$ and $v = y - y_0$, we have

$$u' = f_x(x_0, y_0)u + f_y(x_0, y_0)v$$
$$v' = g_x(x_0, y_0)u + g_y(x_0, y_0)v$$

Example. Consider the system

$$x' = 4(1 - x - y)x$$
$$y' = 4(4 - 7x - 3y)y$$

Besides the trivial equilibrium point $(0, 0)$, we have another nontrivial equilibrium point at $(\frac{1}{4}, \frac{3}{4})$. Next, we compute the partial derivatives for the Jacobian matrix, and evaluate them at the equilibrium:

$$f_x = 4 - 8x - 4y \to f_x(\frac{1}{4}, \frac{3}{4}) = -1$$
$$f_y = -4x \to f_y(\frac{1}{4}, \frac{3}{4}) = -1$$
$$g_x = -28y \to g_x(\frac{1}{4}, \frac{3}{4}) = -21$$
$$g_y = 16 - 28x - 24y \to g_y(\frac{1}{4}, \frac{3}{4}) = -9$$

Hence, the linearized system is

$$\vec{u}' = \begin{pmatrix} -1 & -1 \\ -21 & -9 \end{pmatrix} \vec{u}$$

We can classify this as an unstable saddle point because the determinant of the matrix is negative.

Definition. The following phase portraits are generic: saddle, nodal sink, nodal source, spiral sink, spiral source. Thus, the following phase portraits are non-generic: center, proper node, improper node.

Linearization Theorem. If the linearized system has a generic phase portrait, then the nonlinear system has a qualitatively similar phase portrait near the equilibrium point.

Example. Consider the following system:

$$x' = -y + ax(x^2 + y^2)$$
$$y' = x + ay(x^2 + y^2)$$

Linearizing around the equilibrium point at $(0, 0)$, we have

$$x' = -y$$
$$y' = x$$

Hence

$$\vec{u}' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \vec{u}$$

Since the trace is 0 and the determinant is positive, we conclude that the linearized system behaves like a center near the point $(0, 0)$. However, by the linearization theorem, we cannot be sure that the linearized behavior is the same as the actual behavior, since a center is a non-generic phase portrait. Furthermore, we rewrite the system using polar coordinates:

$$r^2 = x^2 + y^2, \quad x = r\cos(\theta), \quad y = r\sin(\theta), \quad \theta = \arctan(\frac{y}{x})$$

Taking derivatives, we obtain

$$2rr' = 2xx' + 2yy' \implies r' = \frac{xx' + yy'}{r}$$

$$\theta' = \frac{1}{1 + (\frac{y}{x})^2} \frac{y}{x}' = \frac{xy' - yx'}{x^2 + y^2} \implies \theta' = \frac{xy' - yx'}{r^2}$$

Then

$$r' = \frac{x(-y + ax(x^2 + y^2)) + y(x + ay(x^2 + y^2))}{r} = \frac{(-xy + ax^2r^2) + (yx + ay^2r^2)}{r} = ar^3$$

$$\theta' = \frac{x(x + ay(x^2 + y^2)) - y(-y + ax(x^2 + y^2))}{r^2} = \frac{(x^2 + ayxr^2) - (-y^2 + axyr^2)}{r^2} = 1$$

If $a = 0$, the actual solution would have a center near the equilibrium point. If $a < 0$, it would be a spiral sink. If $a > 0$, it would be a spiral source.

Example. Consider the system

$$x' = -y + x(1 - x^2 - y^2)$$
$$y' = x + y(1 - x^2 - y^2)$$

Using the previously derived formulas for r' and θ', we have

$$r' = \frac{xx' + yy'}{r} = r(1 - r^2)$$
$$\theta' = \frac{xy' - yx'}{r^2} = 1$$

Hence, we have equilibrium when $r = 0$. At $r = 1$, we have a closed loop, called a limit cycle. All trajectories inside and outside this loop spiral counter-clockwise and approach the limit cycle as $t \to \infty$.

Lorentz System. Consider the following system of ODEs:

$$\frac{dx}{dt} = a(y - x)$$

$$\frac{dy}{dt} = x(r - z) - y$$
$$\frac{dz}{dt} = -bz + xy$$

This system has chaotic solutions for certain choices of the parameters a, b, r.

Undamped Spring. Let $x(t)$ express the position of a spring with no damping. Let $v(t)$ be the velocity of the spring, and $x'(t) = v(t)$. Then we can reduce into a system of equations:

$$x' = v$$
$$v' = \frac{-k}{m}x$$

There is equilibrium at (0,0). To find the energy of the system, we look at the quotient $\frac{v'(t)}{x'(t)}$:

$$\frac{v'(t)}{x'(t)} = \frac{\frac{dv}{dt}}{\frac{dx}{dt}} = \frac{dv}{dx} = \frac{-kx}{mv}$$

Integrating, we have

$$\int mvdv = -\int kxdx \implies \frac{mv^2}{2} + \frac{kx^2}{2} = E$$

This expresses the energy of the system. In fact, the kinetic energy is $\frac{mv^2}{2}$ and the spring's potential energy is $\frac{kx^2}{2}$.

$$\frac{dE}{dt} = \frac{1}{2}m(2v\frac{dv}{dt}) + \frac{1}{2}k(2x\frac{dv}{dt}) = mv(\frac{-k}{m}x) + kx(v) = 0$$

Thus, in the undamped case, energy is conserved. With the damped case, the system is

$$x' = v$$
$$v' = \frac{-k}{m}x - \frac{\gamma}{m}v$$

In this case, the expression for the energy E is the same. However, $\frac{dE}{dt} = -\gamma v^2 \leq 0$, since $\gamma > 0$ and $v^2 > 0$. This means energy is dissipating from the system.

In the case of underdamping, trajectories in the x-v phase plane will be spiral sinks. In the case of overdamping, the trajectories will be nodal sinks. In either case, trajectories will cross level curves that represent constant energy (undamped systems).