Math 416 Notes

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Definition. The system of equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

where a_{ij} and b_i are scalars in a field F for all i, j and $x_1, ..., x_n$ are variables of values in F, is called a system of m linear equations in n unknowns over the field F. The system can be written succinctly as $Ax = b$, where A is the coefficient matrix, $x = (x_1, ..., x_m)^T$, and $b = (b_1, ..., b_m)^T$.

Definition. An $m \times n$ (real) matrix is an array of real numbers

$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix} \equiv (a_{ij})_{\substack{1 \le i \le m \\ 1 \le j \le n}}$$

Although a real $m \times n$ matrix is an element of $\mathbb{R}^{mn}$, a more useful interpretation is that it is a function

$$\{1, ..., m\} \times \{1, ..., n\} \to \mathbb{R}, (i, j) \mapsto a_{ij}$$

Definition. Suppose $A \in M_{m \times n}$ and $B \in M_{n \times p}$. The augmented matrix, denoted $(A|B)$, is the $m \times (n + p)$ matrix whose first n columns are the columns of A, and the last p columns are the columns of B.

Fact. Every linear system can be represented with a corresponding augmented matrix.

Definition. Suppose $A \in M_{m \times n}$. There are three elementary row operations: (1) Switching any two rows of A. (2) Multiplying any row of A by a nonzero scalar. (3) Adding a scalar multiple of one row of A to another row of A.

Fact. Elementary row operations on a matrix do not change the solution set of the corresponding augmented matrix.

Definition. The leading entry of a row is the leftmost nonzero entry.

Definition. A matrix is in row echelon form (REF) iff: (a) All zero rows are below nonzero rows. (b) The leading entry of each row is to the right of the leading entry of the row above.

Definition. A matrix is in reduced row echelon form (RREF) iff it is in REF and: (c) All leading entries are 1. (d) The leading entires are the only nonzero entries in their columns.

Theorem. Every matrix can be put in RREF by a finite sequence of elementary row operations.

Corollary. Let A and B have the same RREF. Then, A can be transformed into B by a finite sequence of elementary row operations.

Theorem. The RREF of a matrix is unique.

Definition. A linear system is consistent if its solution set is nonempty. Otherwise, it is an inconsistent system.

Fact. Every linear system has one, none, or infinitely many solutions.

Definition. Let k

Thm 2 Every matrix can be put in RREF by a finite sequence of elementary row operations-pf by const =gaussian eliim

Definition. A vector space V over a field F consists of a set closed under addition and scalar multiplication, which satisfy the following conditions: (1) $\forall x, y \in V$, $x + y = y + x$ (2) $\forall x, y, z \in V$, $(x + y) + z = x + (y + z)$ (3) $\exists 0$ such that $\forall x \in V$, $x + 0 = x$. (4) $\forall x \in V$, $\exists y \in V$ such that $x + y = 0$. (5) $\forall x \in V$, $1x = x$. (6) $\forall a, b \in F$ and $x \in V$, $(ab)x = a(bx)$. (7) $\forall a \in F$ and $x, y \in V$, $a(x + y) = ax + ay$. (8) $\forall a, b \in F$ and $x \in V$, $(a + b)x = ax + bx$.

Theorem. In a vector space, the zero vector is unique.

Theorem. Let V be a vector space. For any $v \in V$, $0 \cdot v = 0$.

Theorem. In a vector space, additive inverses are unique. The additive inverse of an element v is denoted as $-v$, and $-v = (-1) \cdot v$.

Example. $\mathbb{R}[x] = \{a_0 + a_1 x + ... + a_n x^n : n \geq 0, a_0, ..., a_n \in \mathbb{R}\}$ is a vector space under the usual definitions of addition and scalar multiplication.

Example. The set of all continuous functions from $[0, 1]$ to $\mathbb{R}$ is a vector space.

Example. The set of all n-tuples with each entry from a field F is denoted by F^n. This set is a vector space over F with component-wise addition and scalar multiplication.

Example. The set of all $m \times n$ matrices with entries from a field F, denoted $M_{m\times n}(F)$, is a vector space.

Example. The set of all functions from a nonempty set S to a field F, denoted $\mathcal{F}(S, F)$, is a vector space.

Example. The set of all polynomials with coefficients from a field F, denoted $P(F)$, is a vector space.

Example. The set of all sequences $\{a_n\}$ in a field F with a finite number of nonzero terms is a vector space.

Definition. Let V and W be vector spaces over a field F. A map $T : V \to W$ between two vector spaces is linear if $forall a, b, \in F$, $x, y, \in V$:

(i) T(ax + by) = aT(x) + bT(y)

Example. Any $m \times n$ matrix A defines a linear map

$$T_A : \mathbb{R}^n \to \mathbb{R}^m, \quad T_A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} := \begin{pmatrix} \sum_i a_{1i} x_i \\ \sum_i a_{mi} x_i \end{pmatrix}$$

Example. Differentiation is linear:

$$\frac{d}{dx} : \mathbb{R}[x] \to \mathbb{R}[x], \quad \frac{d}{dx}(\sum_{i=0}^{n} a_i x^i) = \sum_{i=1}^{n} i a_i x^{i-1}$$

Here, $\mathbb{R}[x]$ represents the set of all polynomials with real coefficients.

Definition. A subset W of a vector space V over a field F is called a subspace of V if W is a vector space over F with the operations of addition and scalar multiplication defined on V.

Fact. In any vector space V, V and $\{0\}$ are subspaces.

Theorem. Let V be a vector space and W a subset of V. Then W is a subspace of V iff the following three conditions hold for the operations defined in V: (a) $0 \in W$.

(b) $\forall x, y \in W$, $x + y \in W$.

(c) $\forall c \in$, $x \in W$, $cx \in W$.

Example. P_k is a subspace of P_{k+1}.
Example. $C^0(\mathbb{R})$ is a subspace of $(\mathbb{R})$.
Example. The set of symmetric matrices, which is a subset of $M_{n\times n}$, is also a subspace.
Definition. A linear combination of the vectors $v_1, ..., v_n$ is the vector $u = a_1v_1+...+a_nv_n$ for some $a_1, ..., a_n \in \mathbb{R}$.
Definition. A set of vectors $\{v_1, ..., v_m\}$ in a vector space V spans V if $\forall u \in V$, $\exists a_1, ..., a_n \in \mathbb{R}$ such that $u = a_1v_1 + ... + a_nv_n$.
Definition. A set of vectors $\{v_1, ..., vn\}$ in a vector space V is linearly dependent if $\exists a_1, ..., a_n \in \mathbb{R}$ not all zero such that $a_1v_1 + ... + a_nv_n = 0$. Otherwise, the vectors are linearly independent.