Problems 1

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Contents

1 Probabilities, random variables

1. A fair die is thrown until a 6 appears. What is the probability that it must be thrown at least k times?

$$P(\text{at least } k \text{ throws}) = 1 - P(\text{fewer than } k \text{ throws})$$
$$P(\text{fewer than } k \text{ throws}) = P(\text{success in first } k-1 \text{ throws})$$
$$P(\text{success in first } k-1 \text{ throws}) = 1 - \left(\frac{5}{6}\right)^{k-1}$$

2. For $x \in \mathbb{R}^K$, let $f_\theta(x) := h(x)\exp(\eta(\theta)'T(x) - A(\theta))$ for functions h, η, T, A. When is this a pdf? i.e. when is $f(x) \geq 0$ and such that its integral is equal to one?

Non-negativity: $f_\theta(x) \geq 0$

$$h(x) \geq 0 \quad \forall x \text{ in the support}$$
$$\exp(\eta(\theta)'T(x) - A(\theta)) \geq 0$$

Integrating to 1:

This is where the log-partition function $A(\theta)$ plays a crucial role:

$$\int f_\theta(x)dx = \int h(x)\exp(\eta(\theta)'T(x) - A(\theta))dx = 1$$
$$= \exp(-A(\theta)) \int h(x)\exp(\eta(\theta)'T(x))dx = 1$$
$$\therefore A(\theta) = \log \int h(x)\exp(\eta(\theta)'T(x))dx$$

3. For each $i = 1, \ldots, K$, let f_i be pdfs (resp.) and define $f(x) := \sum_{i=1}^{K} w_i f_i(x)$ where $w_i \geq 0$ and $\sum_{i=1}^{K} w_i = 1$. Show that f is a probability density (resp. probability mass) function. i.e. show that $f(x) \geq 0$ and its integral / sum is equal to one in the density / mass case respectively

Here we need to verify two conditions:

- **Non-negativity:** Show that $f(x) \geq 0 \quad \forall x$
- **Normalization:** Show that the integral (or sum) equals 1

Non-negativity:

Since each $f_i(x)$ is a valid pdf/pmf, we have $f_i(x) \geq 0 \quad \forall x \wedge (i = 1, \ldots, K)$

Additionally, we have given that $w_i \geq 0 \quad \forall i = 1, \ldots, K$

$$\therefore f(x) = \sum_{i=1}^{K} w_i f_i(x) \geq 0$$

Since we are summing non-negative terms $(w_i \geq 0 \wedge f_i \geq 0)$

Normalization:

Case 1: pdf

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} \sum_{i=1}^{K} w_i f_i(x)dx$$

$$= \sum_{i=1}^{K} w_i \int_{-\infty}^{\infty} f_i(x)dx \quad \text{(linearity of integration)}$$

$$= \sum_{i=1}^{K} w_i \times 1 \quad \text{(since each } f_i \text{ is a valid pdf)}$$

$$= \sum_{i=1}^{K} w_i$$

$$= 1 \quad \text{(given the constraint } \sum_{i=1}^{K} w_i = 1\text{)}$$

Case 2: pmf

$$\sum_{x} f(x) = \sum_{x} \sum_{i=1}^{K} w_i f_i(x)$$

$$= \sum_{i=1}^{K} w_i \sum_{x} f_i(x) \quad \text{(linearity of summation)}$$

$$= \sum_{i=1}^{K} w_i \times 1 \quad \text{(since each } f_i \text{ is a valid pmf)}$$

$$= \sum_{i=1}^{K} w_i$$

$$= 1 \quad \text{(given constraint)}$$

4. Let X be a Poisson r.v. with mass function $f(x) = \lambda^x \exp(-\lambda)/x!, \quad x = 0, 1, \ldots$ for $\lambda > 0$. Find the probability that X is odd

$$\exp(\lambda) = \sum_{x=0}^{\infty} \frac{\lambda^x}{x!} = 1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \ldots + \frac{\lambda^n}{n!}$$

$$\exp(-\lambda) = \sum_{x=0}^{\infty} \frac{-\lambda^x}{x!} = 1 + \frac{-\lambda}{1!} + \frac{-\lambda^2}{2!} + \ldots + \frac{-\lambda^n}{n!}$$

$$P(X \text{ is odd}) = \sum_{x \text{ odd}} \frac{\lambda^x e^{-\lambda}}{x!} = e^{-\lambda} \sum_{x \text{ odd}} \frac{\lambda^x}{x!}$$

$$e^{\lambda} = \sum_{x=0}^{\infty} \frac{\lambda^x}{x!} \quad \text{(sum of all terms)}$$

$$e^{-\lambda} = \sum_{x=0}^{\infty} \frac{(-1)^x \lambda^x}{x!} \quad \text{(alternating signs)}$$

$$e^{\lambda} + e^{-\lambda} = 2 \sum_{x \text{ even}} \frac{\lambda^x}{x!} \quad \text{(even terms don't cancel)}$$

$$e^{\lambda} - e^{-\lambda} = 2 \sum_{x \text{ odd}} \frac{\lambda^x}{x!} \quad \text{(odd terms don't cancel)}$$

$$\rightarrow \sum_{x \text{ odd}} \frac{\lambda^x}{x!} = \frac{e^{\lambda} - e^{-\lambda}}{2}$$

$$P(X \text{ is odd}) = e^{-\lambda} \cdot \frac{e^{\lambda} - e^{-\lambda}}{2} = \frac{1 - e^{-2\lambda}}{2}$$

5. Prove that $F(x) := (1+\exp(-x))^{-1} \quad x \in \mathbb{R}$ is a CDF

Recall that a CDF must satisfy these requirements:

- **Monotonicity:** F is non-decreasing (i.e. $x_1 \leq x_2 \to F(x_1) \leq F(x_2)$)
- **Right-continuity:** F is right-continuous at every point
- **Limit conditions:**
  - $\lim_{x\to-\infty} F(x) = 0$
  - $\lim_{x\to\infty} F(x) = 1$

Limit conditions: It is trivial that the function satisfies these two conditions.

Monotonicity: We prove that $F'(x) \geq 0$:

$$F'(x) = \frac{\exp(-x)}{\big(1+\exp(-x)\big)^2} \quad \forall x \in \mathbb{R}$$

Right-continuity:

Since $F(x)$ is continuous everywhere (as a composition of continuous functions), it is automatically right-continuous

$\therefore F(x)$ is a CDF $\square$

6. Show that any CDF F, i.e. $F(x) := P(X \leq x)$, can have at most a countable number of discontinuities

The key here is to use the monotonicity of CDFs combined with the fact that rational numbers are countable.

For any CDF F, discontinuities can only by "jump" discontinuities due to monotonicity. At each discontinuity point x_0 we have:

- Left limit: $F(x_0^-) = \lim_{x \to x_0^-} F(x)$ exists
- Right limit: $F(x_0^+) = \lim_{x \to x_0^+} F(x) = F(x_0) \quad$ (right-continuity)
- Jump size: $F(x_0) - F(x_0^-) > 0$

Associate each discontinuity with a rational numebr:

- Let D be the set of discontinuity points
- For each $x \in D$, the jump size is $F(x) - F(x^-) > 0$
- Between any two consecutive jumps $F(x^-)$ and $F(x)$, there exists a rational number
- Since F is monotonic, these rational intervals are disjoint

Rationals are countable:

- Each discontinuity corresponds to a unique rational number in $(F(x^-), F(x)]$
- Since $\mathbb{Q} \cap [0, 1]$ is countable, and all these rationals are distinct
- Therefore D is at most countable

2 Expectations

1. Show that $\mathbb{E}[\alpha] = \alpha$ for any non-random α

By the definition of the expected value (and the fact that all pdfs integrate to one):

$$\mathbb{E}[c] = \int_{-\infty}^{\infty} cf(x)\ dx = c\int_{-\infty}^{\infty} f(x)\ dx = c \cdot 1 = c$$

2. Let X be the sum of two rolls of a fair die. What is the mean and variance of X?

X itself is defined as $X = X_1 + X_2$, where $X_1, X_2 \sim \text{Uniform}\{1, \ldots, 6\}$, independent.

This means that:

$$\mathbb{E}[X] = \mathbb{E}[X_1 + X_2] = \mathbb{E}[X_1] + \mathbb{E}[X_2]$$

Since the die are identical, $\mathbb{E}[X_1] = \mathbb{E}[X_2]$, and thus:

$$\mathbb{E}[X_1] = \sum_{i=1}^{6} P(x = i) = \frac{1}{6}\sum_{i=1}^{6} i = \frac{7}{2}$$

$$\mathbb{E}[X] = \mathbb{E}[X_1 + X_2] = \mathbb{E}[X_1] + \mathbb{E}[X_2] = \frac{7}{2} + \frac{7}{2} = 7$$

Since the die are independent, we only need to compute the one variance:

$$\text{Var}(X) = \mathbb{E}[X^2] - \big(\mathbb{E}[X]\big)^2$$
$$\mathbb{E}[X_1^2] = \frac{1}{6}\sum_{i=1}^{6} i^2 = \frac{91}{6}$$
$$\text{Variance of one die: } \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \frac{35}{12}$$

$$\text{Var}(X) = \text{Var}(X_1) + \text{Var}(X_2) = 2 \cdot \frac{35}{12} = \frac{35}{6} = 5.8\bar{3}$$

3. X is uniformly distributed on $[a, b]$ if its density is $f(x) = \frac{1}{b-a}$. Compute the mean and variance of X.

Recall that $\mathbb{E}[x] = \int_{-\infty}^{\infty} xf(x)\, dx$

$$\mathbb{E}[X] = \int_a^b x \cdot \frac{1}{b-a} dx = \frac{1}{b-a} \cdot \int_a^b x dx$$

$$= \frac{1}{b-a} \cdot \frac{x^2}{2} = \frac{x^2}{2(b-a)}\bigg|_a^b = \frac{b^2 - a^2}{2(b-a)}$$

$$= \frac{b+a}{2}$$

Nice. Also recall that $\text{Var}(X) = \mathbb{E}[X^2] - \big(\mathbb{E}[X]\big)^2$:

$$\mathbb{E}[X^2] = \int_a^b x^2 \cdot \frac{1}{b-a} dx = \frac{1}{b-a} \cdot \int_a^b x^2 dx$$

$$= \frac{1}{b-a} \cdot \frac{x^3}{3} = \frac{x^3}{3(b-a)}\bigg|_a^b = \frac{b^3 - a^3}{3(b-a)} = \frac{b^2 + ab + a^2}{3}$$

Thus:

$$\text{Var}[X] = \frac{b^2 + ab + a^2}{3} - \Big(\frac{b+a}{2}\Big)^2 = \frac{b^2 + ab + a^2}{3} - \frac{b^2 + 2ab + a^2}{4}$$

$$= \frac{(b-a)^2}{12}$$

4. Calculate the mean of $X \sim t(v)$. Are restrictions on v required for the mean to exist?

First we should define the student's t-distribution:

$$f(t) = \frac{\Gamma\left(\frac{v+1}{2}\right)}{\sqrt{\pi v}\Gamma\left(\frac{v}{2}\right)} \left(1 + \frac{t^2}{v}\right)^{\frac{-(v+1)}{2}}$$

By symmetry the integrand $xf(x)$ is an odd function, so if the expectation integral converges, then

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} xf(x)\ dx = 0$$

Existence: for large $|x|$, the density behaves like a constant times $|x|^{-(v+1)}$. Hence:

$$\mathbb{E}[|X|] \asymp \int_{1}^{\infty} x \cdot x^{-(v+1)}\ dx = \int_{1}^{\infty} x^{-v}\ dx$$

Which converges iff $v > 1$

5. Prove Proposition 2.6 for the discrete case.

Prop. 6:

Let X be a random variable, α a constant and g_1 and g_2 such that $\mathbb{E}g_1(X)$ and $\mathbb{E}g_2(X)$ exist. Then

$$i) \quad \mathbb{E}[\alpha g_1(X)] = \alpha \mathbb{E}g_1(X) \textit{ and } \mathbb{E}[g_1(X) + g_2(X)] = \mathbb{E}g_1(X) + \mathbb{E}g_2(X);$$
$$ii) \quad \textit{If } g_1(x) > 0 \quad \forall x \textit{ with } f(x) > 0, \textit{ then } \mathbb{E}g_1(X) \geq 0$$

Let X be a discrete r.v. with pmf $P(X = x)$, where x ranges over the support of X

Part i): linearity of expectation

For scalar multiplication:

$$\mathbb{E}[\alpha g_1(X)] = \sum_{x \in \mathcal{X}} \alpha g_1(x) P(X = x) = \alpha \sum_{x \in \mathcal{X}} g_1 P(X = x) = \alpha \mathbb{E}[g_1(X)]$$

For addition:

$$\mathbb{E}[g_1(X) + g_2(X)] = \sum_{x \in \mathcal{X}} [g_1(x) + g_2(x)] P(X = x)$$

$$= \sum_{x \in \mathcal{X}} g_1 P(X = x) + \sum_{x \in \mathcal{X}} g_2 P(X = x)$$

$$= \mathbb{E}[g_1(X)] + \mathbb{E}[g_2(X)]$$

Part ii): non-negative property:

If $g_1(x) \geq 0 \quad \forall x$ with $P(X = x)$, then:

$$\mathbb{E}[g_q(X)] = \sum_{x \in \mathcal{X}} g_1 P(X = x)$$

Since each term $g_1(x) P(X = x) \geq 0$ (because $g_1(x) \geq 0$ and $P(X = x) \geq 0$), we have:

$$\mathbb{E}[g_1(X)] = \sum_{x \in \mathcal{X}} g_1(x) P(X = x) \geq 0$$

6. Prove Lemma 2.1:

It states that *If* $\mathrm{Var}(X)$ *exists, then for any constants* a, b

$$\mathrm{Var}(aX+b) = a^2\mathrm{Var}(X)$$

Recall that $\mathrm{Var}(X) = \mathbb{E}[X^2] - \big(\mathbb{E}[X]\big)^2$. Therefore,

$$\begin{aligned}
\mathbb{E}[(aX+b)^2] - \big(\mathbb{E}[(aX+b)]\big)^2 &= \mathbb{E}\big[(aX+b) - \mathbb{E}[(aX+b)]^2\big] \\
&= \mathbb{E}\big[(aX - a\mathbb{E}[X])^2\big] = a^2\mathbb{E}\big[(X-\mathbb{E}[X])^2\big] \\
&= a^2\mathrm{Var}(X)
\end{aligned}$$

$\square$

7. Prove Lemma 2.2:

It states that [Properties of indicators]: *If* A, B *are events and* X *an r.v.:*

$$\begin{aligned}
(i) &\quad \nVdash_A \nVdash_B = \nVdash_{A\cap B} \\
(ii) &\quad P(X\in A) = \mathbb{E}[\nVdash_A(X)] \\
(iii) &\quad P(X\in A)[1-P(X\in A)] = \mathrm{Var}(\nVdash_A(X))
\end{aligned}$$

For an event E, its indicator $\nVdash_E$ takes value 1 on E and 0 on E^c. If X is an r.v. and A is a measurable set in the state space of X, then $\nVdash_A(X)$ means the indicator of the event $\{\omega : X(\omega)\in A\}$, i.e. $\nVdash_{X\in A}$

$$(i) \quad \nVdash_A \nVdash_B = \nVdash_{A\cap B}$$

Pointwise check: for any ω, the left side is 1 iff. $\omega\in A \ \wedge \ \omega \in B$, otherwise it is 0. That is exactly the indicator of $A\cap B$ $\square$

$$(ii) \quad P(X\in A) = \mathbb{E}[\nVdash_A(X)]$$

Let $E = \{\omega : X(\omega)\in A\} = X^{-1}(A)$. Then $\nVdash_A(X) = \nVdash_E$

For any event E, $\mathbb{E}[\nVdash_E] = \int \nVdash_E \, dP = P(E)$. Hence $\mathbb{E}[\nVdash_A(X)] = P(X\in A)$ $\square$

$$(iii) \quad P(X\in A)[1-P(X\in A)] = \mathrm{Var}(\nVdash_A(X))$$

Set $Y = \nVdash_A(X)$. Then $Y\in\{0,1\}$ and $\mathbb{E}[Y] = P(X\in A) =: p$

Since $Y^2 = Y,\ \mathrm{Var}(Y) = \mathbb{E}[Y^2] - (\mathbb{E}[Y])^2 = p - p^2 = p(1-p)$

$\therefore \mathrm{Var}\big(\nVdash_A(X)\big) = P(X\in A)\big[1-P(X\in A)\big]$ $\square$

8. Let X and Y be r.v.s with $\mathbb{E}|X| < \infty$, $\mathbb{E}|Y| < \infty$ and let $X \wedge Y := \min\{X, Y\}$ and $X \vee Y := \max\{X, Y\}$. Show that $\mathbb{E}[X \vee Y] = \mathbb{E}[X] + \mathbb{E}[Y] - \mathbb{E}[X \wedge Y]$ [Hint: What is $(X \vee Y) + (X \wedge Y)$?]

Consider what happens when we add the max and min of two numbers: $\min\{a, b\} + \max\{a, b\} = a + b$

$\therefore (X \vee Y) + (X \wedge Y) = X + Y$

Since $(X \vee Y) + (X \wedge Y) = X + Y$, we can carry expectation operations:

$$\mathbb{E}[(X \vee Y) + (X \wedge Y)] = \mathbb{E}[X + Y]$$

We can use linearity of expectation since $\mathbb{E}|X|, \mathbb{E}|Y| < \infty$:

$$\mathbb{E}[X \vee Y] + \mathbb{E}[X \wedge Y] = \mathbb{E}[X] + \mathbb{E}[Y]$$

$$= \mathbb{E}[X \vee Y] = \mathbb{E}[X] + \mathbb{E}[Y] - \mathbb{E}[X \wedge Y]$$

$\square$

3 Conditioning & independence

1. If P is a probability and B an event with $P(B) > 0$, show that $P(\cdot|B)$ is also a probability

$P(\cdot|B)$ must satisfy the three axioms of probability.

Definition: For an event A, we have $P(A|B) = \frac{P(A\cap B)}{P(B)}$ where $P(B) > 0$

Axiom 1: Non-negativity: For any event A, we need $P(A|B) \geq 0$.

Since P is a probability measure:

$$P(A \cap B) \geq 0 \quad \text{(by non-negativity of } P\text{)}$$
$$P(B) > 0 \quad \text{(given condition)}$$
$$\therefore P(A|B) = \frac{P(A \cap B)}{P(B)} \geq 0$$

Axiom 2: Normalization: we need $P(\Omega|B) = 1$, where Ω is the sample space.

$$P(\Omega|B) = \frac{P(\Omega \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1 \quad \text{(since } \Omega \cap B = B\text{)}$$

Axiom 3: Countable additivity: For countably many disjoints events $A_1, \ldots, A_n$, we need:

$$P\left(\bigcup_{i=1}^{\infty} A_i \middle| B\right) = \sum_{i=1}^{\infty} P(A_i|B)$$

$$P\left(\bigcup_{i=1}^{\infty} A_i \middle| B\right) = \frac{P\big((\bigcup_{i=1}^{\infty} A_i) \cap B\big)}{P(B)}$$

$$= \frac{P\big(\bigcup_{i=1}^{\infty} (A_i \cap B)\big)}{P(B)} \quad \text{(using the distributive property of intersection over union)}$$

$$= \sum_{i=1}^{\infty} \frac{P(A_i \cap B)}{P(B)} = \sum_{i=1}^{\infty} P(A_i|B)$$

Since the A_i are disjoint, the events $A_i \cap B$ are also disjoint. Therefore, by cointable additivity of P:

Since $P(\cdot|B)$ satisfies all three axioms of probability, it is indeed a probability measure $\square$

2. If $P(B \cap C) > 0$ show that $P(A \cap B \cap C) = P(A|B \cap C)P(B|C)P(C)$

We know that $P(A \cap B) = P(A|B)P(B)$

Let $B_c = B \cap C \quad \therefore P(A \cap B_c) = P(A|B \cap C)P(B \cap C)$

By the definition: $P(B \cap C) = P(B|C)P(C)$

Subsitute back in: $P(A \cap B \cap C) = P(A|B \cap C)P(B|C)P(C)$ $\square$

3. Prove Lemma 2.3

It states that *If* $X = (X_1, \ldots, X_K)$ *is a random vector and* $\mathrm{Var}(X)$ *exists then for any (constant) vector* $\vec{b} \in \mathbb{R}^K$ *and any (constant) matrix* $A \in \mathbb{R}^{m \times K}$,

$$\mathrm{Var}(AX + b) = A\mathrm{Var}(X)A'$$

Fix dimensions: $X \in \mathbb{R}^K$ and $A \in \mathbb{R}^{m \times K} \to AX \in \mathbb{R}^m$

Let $Y = AX + b$, where $X \in \mathbb{R}^K, A \in \mathbb{R}^{m \times K}$, and $b \in \mathbb{R}^m$ (note that the dimensions of b must match AX)

Thus,

$$\begin{aligned}
\mathbb{E}[YY'] &= \mathbb{E}[(AX + b)(AX + b)'] \\
&= \mathbb{E}[AXX'A' + AX'b + bX'A' + bb'] \\
&= A\mathbb{E}[XX']A' + A\mathbb{E}[X]b' + b\mathbb{E}[X]'A' + bb'
\end{aligned}$$

Also,

$$\begin{aligned}
\mathbb{E}[Y]\mathbb{E}[Y]' &= (A\mathbb{E}[X] + b)(A\mathbb{E}[X] + b)' \\
&= A\mathbb{E}[X]\mathbb{E}[X]'A' + A\mathbb{E}[X]b' + b\mathbb{E}[X]'A' + bb'
\end{aligned}$$

$$\begin{aligned}
\therefore \mathrm{Var}(AX + b) &= \mathbb{E}[Y]\mathbb{E}[Y]' \\
&= A(\mathbb{E}[XX'] - \mathbb{E}[X]\mathbb{E}[X]')A' \\
&= A\mathrm{Var}(X)A'
\end{aligned}$$

4. Prove Lemma 2.4

Lemma 2.4: *If X, Z and Y are random vectors in* $\mathbb{R}^K, \mathbb{R}^K$ *and* $\mathbb{R}^L$*, respectively,* $a \in \mathbb{R}^K, b \in \mathbb{R}^L$ *are constant vectors and A and B are constant matrices with K and L columns respectively,*

$$\text{Cov}(X + Z, Y) = \text{Cov}(X, Y) + \text{Cov}(Z, Y)$$
$$\textit{and}$$
$$\text{Cov}(AX + a, BY + b) = A\text{Cov}(X, Y)B'$$

Recall that the matrix covariance of random vectors $X \in \mathbb{R}^K$ and $Y \in \mathbb{R}^L$ is

$\text{Cov}(X, Y) = \mathbb{E}[(X - \mathbb{E}X)(Y - \mathbb{E}Y)']$

Additivity in the first argument:

Let $\mu_X = \mathbb{E}X, \quad \mu_Z = \mathbb{E}Z, \quad \mu_Y = \mathbb{E}Y$, then:

$$\begin{gathered} \text{Cov}(X + Z, Y) \\ = \mathbb{E}[(X + Z - \mu_X - \mu_Z)(Y - \mu_Y)'] \\ = \mathbb{E}[(X - \mu_X)(Y - \mu_Y)' + (Z - \mu_Z)(Y - \mu_Y)'] \\ = \text{Cov}(X, Y) + \text{Cov}(Z, Y) \end{gathered}$$

$\square$

Affine equivariance:

Let $U = AX + a$ and $V = BY + b$. Then $\mathbb{E}U = A\mathbb{E}X + a$ and $\mathbb{E}V = B\mathbb{E}Y + b$, so

$$\begin{gathered} \text{Cov}(U, V) \\ = \mathbb{E}[(U - \mathbb{E}U)(V - \mathbb{E}V)'] \\ = \mathbb{E}[(AX + a - (A\mathbb{E}X + a))(BY + b - (B\mathbb{E}Y + b))'] \\ = \mathbb{E}[(A(X - \mathbb{E}X))(B(Y - \mathbb{E}Y))'] \\ = \mathbb{E}[A(X - \mathbb{E}X)(Y - \mathbb{E}Y)'B'] \\ = A\mathbb{E}[(X - \mathbb{E}X)(Y - \mathbb{E}Y)']B' \\ = A\text{Cov}(X, Y)B' \end{gathered}$$

$\square$

5. Prove Lemma 2.5

If X, Z and Y are r.vs. then

$$\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\mathrm{Cov}(X,Y)$$

If X and Y are random vectors of the same dimension then

$$\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) + \mathrm{Cov}(X,Y) + \mathrm{Cov}(X,Y)'$$

Key identity: $\mathrm{Var}(Z) = \mathrm{Cov}(Z,Z)$, and for any r.v./vectors U, V, W,

$$\begin{aligned}\mathrm{Cov}(U+V,W) &= \mathrm{Cov}(U,W) + \mathrm{Cov}(V,W)\\ \mathrm{Cov}(U,V+W) &= \mathrm{Cov}(U,V) + \mathrm{Cov}(U,W)\end{aligned}$$

Scalar case (real-valued X, Y):

$$\begin{gathered}\mathrm{Var}(X+Y) = \mathrm{Cov}(X+Y, X+Y)\\ = \mathrm{Cov}(X,X) + \mathrm{Cov}(X,Y) + \mathrm{Cov}(Y,X) + \mathrm{Cov}(Y,Y)\\ = \mathrm{Var}(X) + \mathrm{Cov}(X,Y) + \mathrm{Cov}(X,Y) + \mathrm{Var}(Y)\\ = \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\mathrm{Cov}(X,Y)\end{gathered}$$

Vector case ($X, Y \in \mathbb{R}^d$):

Use $\mathrm{Cov}(X,Y) = \mathbb{E}[(X-\mathbb{E}X)(Y-\mathbb{E}Y)']$ (a $d \times d$ matrix) so $\mathrm{Cov}(Y,X) = \mathrm{Cov}(X,Y)'$:

$$\begin{gathered}\mathrm{Var}(X+Y) = \mathrm{Cov}(X+Y, X+Y)\\ = \mathrm{Cov}(X,X) + \mathrm{Cov}(X,Y) + \mathrm{Cov}(Y,X) + \mathrm{Cov}(Y,Y)\\ = \mathrm{Var}(X) + \mathrm{Var} + \mathrm{Cov}(X,Y) + \mathrm{Cov}(X,Y)'\end{gathered}$$

$\square$

6. Prove Corollary 2.1

If X and Y are independent, then $\mathrm{Cov} = 0$

$$\mathrm{Cov}(X,Y) = \mathbb{E}\big[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])\big] = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$$

Using conditional expectation,

$$\mathbb{E}[XY] = \mathbb{E}\big[\mathbb{E}[XY|X]\big] = \mathbb{E}\big[X\mathbb{E}[Y|X]\big]$$

If X and Y are independent, then $\mathbb{E}[Y|X] = \mathbb{E}[Y]$. Hence $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$, and therefore $\mathrm{Cov}(X,Y) = 0$

□

7. Prove Proposition 2.10

Let X and Y be random vectors, α a constant and g_1 and g_2 such that $\mathbb{E}g_1(X)$ and $\mathbb{E}g_2(X)$ exist. Then

$$(i) \quad \mathbb{E}[\alpha g_1(X)|Y] = \alpha\mathbb{E}[g_1(X)|Y] \text{ and } \mathbb{E}[g_1(X) + g_2(X)|Y] = \mathbb{E}[g_1(X)|Y] + \mathbb{E}[g_2(X)|Y];$$
$$(ii) \quad \text{If } g_1 \geq 0 \quad \forall x \text{ with } f(x|y) > 0, \text{ then } \mathbb{E}[g_1(X)|Y] \geq 0$$

Let $Z_1 := g_1(X)$ and $Z_2 := g_2(X)$, and let $\mathcal{G} := \sigma(Y)$. Recall the defining property of conditional expectation: For any integrable Z, a version of $\mathbb{E}[Z|\mathcal{G}]$ is the $\mathcal{G}$-measurable W s.t. for every $A \in \mathcal{G}$, $\mathbb{E}[Z\mathbf{1}_A] = \mathbb{E}[W\mathbf{1}_A]$

Such W is unique a.s.

(i) Linearity:

Scalar multiple: for any $A \in \mathcal{G}$,

$$\mathbb{E}[(\alpha Z_1)\mathbf{1}_A] = \alpha\mathbb{E}[Z_1\mathbf{1}_A] = \alpha\mathbb{E}\big[\mathbb{E}[Z_1|\mathcal{G}]\mathbf{1}_A\big] = \mathbb{E}\big[\alpha\mathbb{E}[Z_1|\mathcal{G}]\mathbf{1}_A\big]$$

Both $\mathbb{E}[\alpha Z_1|\mathcal{G}]$ and $\alpha\mathbb{E}[Z_1]\mathcal{G}$ are $\mathcal{G}$-measurable and satisfy the same defining identity, hence

$$\mathbb{E}[\alpha Z_1|\mathcal{G}] = \alpha\mathbb{E}[Z_1|\mathcal{G}] \quad \text{a.s.}$$

Replacing Z_1 by $g_1(X)$ and $\mathcal{G}$ by $\sigma(Y)$ gives $\mathbb{E}[\alpha g_1(X)|Y] = \alpha\mathbb{E}[g_1(X)|Y]$ a.s.

Sum: For any $A \in \mathcal{G}$,

$$\mathbb{E}[(Z_1 + Z_2)\mathbf{1}_A] = \mathbb{E}[Z_1\mathbf{1}_A] + \mathbb{E}[Z_2\mathbf{1}_A] = \mathbb{E}\big[\mathbb{E}[Z_1|\mathcal{G}]\mathbf{1}_A\big] + \mathbb{E}\big[\mathbb{E}[Z_2|\mathcal{G}]\mathbf{1}_A\big] = \mathbb{E}\big[(\mathbb{E}[Z_1|\mathcal{G}] + \mathbb{E}[Z_2|\mathcal{G}])\mathbf{1}_A\big]$$

By the same uniqueness argument,

$$\mathbb{E}[Z_1 + Z_2|\mathcal{G}] = \mathbb{E}[Z_1|\mathcal{G}] + \mathbb{E}[Z_2|\mathcal{G}] \quad \text{a.s.}$$

Substituting $Z_i = g_i(X)$ and $\mathcal{G} = \sigma(Y)$ gives the desired result □

(ii) Positivity:

Assume $g_1(X) \geq 0$ a.s. Let $W := \mathbb{E}[g_1(X)|\mathcal{G}]$. For any $A \in \mathcal{G}$,

$$\mathbb{E}[W\mathbf{1}_A] = \mathbb{E}[g_1(X)\mathbf{1}_A] \geq 0$$

If $\mathbb{P}(W < 0) > 0$, take $A = \{W < 0\} \in \mathcal{G}$; then $\mathbb{E}[W\mathbf{1}_A] < 0$, a contradiction. Hence $\mathbb{P}(W \geq 0) = 1$, i.e.

$$\mathbb{E}[g_1(X)|Y] \geq 0 \quad \text{a.s.}$$

$\square$

7. Prove the "law of total variance": if $\mathrm{Var}(X) < \infty$ then $\mathrm{Var}(X) = \mathbb{E}[\mathrm{Var}(X|Y)] + \mathrm{Var}(\mathbb{E}[X|Y])$ [Hint: $\mathrm{Var}(X|Y) = \mathbb{E}\big[(X - \mathbb{E}[Y|Y])^2\big]$].

Recall the definition of variance in terms of $\mathbb{E}$:

$$\begin{gathered}
\mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 \\
\to \mathbb{E}[X^2] = \mathrm{Var}(X) + (\mathbb{E}[X])^2 \\
\to \mathbb{E}[X^2] = \mathbb{E}\big[\mathrm{Var}(X|Y) + \mathbb{E}[X|Y]^2\big] \quad \text{(applying the law of total expectation)} \\
\to \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \mathbb{E}\big[\mathrm{Var}(X|Y) + \mathbb{E}[X|Y]^2\big] - (\mathbb{E}[X])^2 \\
\to \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \mathbb{E}\big[\mathrm{Var}(X|Y) + \mathbb{E}[X|Y]^2\big] - \mathbb{E}\big[\mathbb{E}[X|Y]\big]^2 \quad \text{(applying the law of total expectation)} \\
\to \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \mathbb{E}[\mathrm{Var}(X|Y)] + \Big(\mathbb{E}\big[\mathbb{E}[X|Y]^2\big] - \mathbb{E}\big[\mathbb{E}[X|Y]\big]^2\Big) \\
\mathrm{Var}(X) = \mathbb{E}[\mathrm{Var}(X|Y)] + \mathrm{Var}(\mathbb{E}[X|Y])
\end{gathered}$$

$\square$

4 Key results

1. Prove Corollary 2.2:

If $t > 0$ then

$$P(||X|| > t) \leq t^{-p}\mathbb{E}||X||^p$$

This is Markov's inequality applied to the nonnegative r.v. $||X||^p$

Let $Z := ||X||^p \geq 0$ and assume $\mathbb{E}||X||^p < \infty$

For any $a > 0$:

$$\mathbb{P}(Z > a) \leq \frac{\mathbb{E}Z}{a}$$

Taking $a = t^p$ with $t > 0$,

$$\mathbb{P}(||X|| > t) = \mathbb{P}(||X||^p > t^p) = \mathbb{P}(Z > t^p) \leq \frac{\mathbb{E}Z}{t^p} = t^{-p}\mathbb{E}||X||^p$$

□

2. Let X be a random vector. Prove that its characteristic function exists:

Proposition: Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $X : \Omega \to \mathbb{R}^d$ a random vector. For each $t \in \mathbb{R}^d$

$$\varphi_X(t) := \mathbb{E}\big[e^{i\langle t,X\rangle}\big]$$

exists.

Fix $t \in \mathbb{R}^d$. The map $g_t : \mathbb{R}^d \to \mathbb{C}, g_t(x) = e^{i\langle t,X\rangle}$, is continuous, hence Borel measurable, and bounded with $|g_t(x)| = 1 \quad \forall x$.

Since X is Borel measurable, the composition $c_t \circ X : \Omega \to \mathbb{C}$ is $\mathcal{F}$-measurable. Moreover,

$$\Big|g_t\big(X(\omega)\big)\Big| = 1 \quad \forall \omega \in \Omega$$

so $g_t \circ X$ is bounded and therefore integrable. Consequently, the expectation

$$\varphi_X(t) = \mathbb{E}[g_t(X)] = \int_\Omega e^{i\langle t,X(\omega)\rangle} d\mathbb{P}(\omega)$$

is well-defined. Equivalently, if $\mu_X = \mathbb{P} \circ X^{-1}$ denotes the law of X, then

$$\varphi_X(t) = \int_{\mathbb{R}^d} e^{i\langle t,X\rangle}\mu_X(dx),$$

which exists because the integrand is bounded by 1 and μ_X is a probability measure □

3. Complete the proof of Proposition 2.13

It states that *For any random vector X its characteristic function ψ has the following properties:*

$$
\begin{gathered}
(i) \quad \psi(0) = 1; \\
(ii) \quad \psi(-t) = \overline{\psi(t)}; \\
(iii) \quad |\psi(t)| \leq \mathbb{E}|\exp(it'X)| = 1; \\
(iv) \quad |\psi(t+h) - \psi(t)| \leq \mathbb{E}|\exp(ih'X) - 1|; \\
(v) \quad \mathbb{E}\exp(it'[AX+b]) = \exp(it'b)\psi(A't).
\end{gathered}
$$

(i) Let $\psi_X(t) = \mathbb{E}\big[e^{i\langle t,X\rangle}\big]$. By definition the **dot product** of $t = 0$ and any vector $\vec{X}$ is 0.

Thus, $\mathbb{E}[e^0] = \mathbb{E}[1] = 1$ $\square$

(ii) $\phi(-t) = \mathbb{E}\big[e^{-i\langle t,X\rangle}\big] = \mathbb{E}\big[\overline{e^{i\langle t,X\rangle}}\big] = \overline{\psi(t)}$. (conjugation passes through the expectation) $\square$

(iii) Let $Z = \exp(i\langle t, X\rangle)$. Then $\psi(t) = \mathbb{E}[Z]$.

Thus, $|\psi(t)| = |\mathbb{E}[Z]| \leq \mathbb{E}[|Z|]$

But $|Z| = |\exp(i\theta)| = 1 \quad \forall\theta$, hence $\mathbb{E}[|Z|] = 1, \quad \therefore |\psi(t)| \leq 1$ $\square$

(iv) Let X be a random vector in $\mathbb{R}^d$ and let $\psi_X(t) = \mathbb{E}\big[e^{i\langle t,X\rangle}\big]$ for $t, h \in \mathbb{R}^d$

Using linearity, $|\mathbb{E}Z| \leq \mathbb{E}|Z|$ and $|e^{i\theta}| = 1$

$$
\begin{aligned}
|\psi_X(t+h) - \psi_X(t)| &= \Big|\mathbb{E}\big[e^{1\langle t+h,X\rangle} - e^{i\langle t,X\rangle}\big]\Big| \\
&= \Big|\mathbb{E}\big[e^{1\langle t+h,X\rangle}\big(e^{i\langle h,X\rangle} - 1\big)\big]\Big| \\
&\leq \mathbb{E}\big[|e^{i\langle t,X\rangle}||e^{i\langle h,X\rangle} - 1|\big] \\
&= \mathbb{E}\big[|e^{i\langle h,X\rangle} - 1|\big]
\end{aligned}
$$

$\square$

(v) With $A : d \times d$ matrix and $b \in \mathbb{R}^d$, let $Y = AX + b$. Then, using $e^{i(a+b)} = e^{ia}e^{ib}$ and $\langle t, AX\rangle = \langle A't, X\rangle$:

$$
\begin{aligned}
\mathbb{E}[e^{i\langle t,Y\rangle}] &= \mathbb{E}[e^{i\langle t,AX+b\rangle}] = \mathbb{E}[e^{i\langle t,b\rangle}e^{i\langle t,AX\rangle}] \\
&= e^{i\langle t,b\rangle}\mathbb{E}[e^{i\langle A't,X\rangle}] = e^{i\langle t,b\rangle}\psi_X(A't)
\end{aligned}
$$

Thus,

$$\mathbb{E}\exp(it'[AX+b]) = \exp(it'b)\psi_X(A't)$$

4. Let ψ be a characteristic function. Prove that ψ is uniformly continuous on $\mathbb{R}$. [Hint: use prop 2.13]

Let $\psi(t) = \mathbb{E}\big[e^{i\langle t,X\rangle}\big], \quad t \in \mathbb{R}^d$

By property *(iv)* in prop 2.13, $\forall t, h \in \mathbb{R}^d$,

$$|\psi(t+h) - \psi(t)| \leq \mathbb{E}|\exp(ih'X) - 1|$$

RHS depends only on h, not on t. We claim that

$$\mathbb{E}|e^{i\langle,h,X\rangle} - 1| \xrightarrow[h\to 0]{} 0$$

For each outcome ω,

$$|e^{i\langle,h,X(\omega)\rangle} - 1| \xrightarrow[h\to 0]{} 0$$

by continuity of $z \mapsto e^{iz}$, and

$$0 \leq |e^{i\langle h,X\rangle} - 1| \leq 2 \quad \text{a.s.}$$

Hence, by the dominated convergence thm.,

$$\lim_{h\to 0} \mathbb{E}|e^{i\langle h,X\rangle} - 1| = 0$$

Therefore, for every $\epsilon > 0$ there exists $\delta > 0$ s.t. $||h|| < \delta$ implies

$$\mathbb{E}|e^{i\langle h,X\rangle} - 1| < \epsilon$$

Combining with property *(iv)*, we get $\forall t \in \mathbb{R}^d$ and all $||h|| < \delta$,

$$|\psi(t+h) - \psi(t)| \leq \mathbb{E}|e^{i\langle h,X\rangle} - 1| < \epsilon.$$

This shows that ψ is uniformly continuous on $\mathbb{R}^d$ $\square$

5. Given that (i) if $Z \sim \mathcal{N}(\mu, \Sigma)$, $Z = AX + \mu$ for $X \sim \mathcal{N}(0, I)$ and $\Sigma = AA'$ and (ii) the characteristic function of a standard normal r.v. is $\exp(-t^2/2)$, compute the characteristic function of $Z \sim \mathcal{N}(\mu, \Sigma)$

Let $Z \sim \mathcal{N}(\mu, \Sigma)$ in $\mathbb{R}^d$. By (i), $\exists A$ with $\Sigma = AA'$ and

$$Z = \mu + AX, \quad X \sim \mathcal{N}(0, I_d)$$

For any $t \in \mathbb{R}^d$, the characteristic function of Z is

$$\varphi_Z(t) = \mathbb{E}[e^{it'Z}] = \mathbb{E}[e^{it'(\mu+AX)}] = e^{it'\mu}\mathbb{E}[e^{it'AX}] = e^{it'\mu}\mathbb{E}[e^{i(A't)'X}]$$

Set $s := A't$. Since $X \sim \mathcal{N}(0, I_d)$, the scalar $s'X$ is univariate normal with mean 0 and variance

$$\mathrm{Var}(s'X) = s'\mathrm{Cov}(X)s = s'Is = ||s||^2$$
$$(\text{see } ||s||^2 = s's)$$

By (ii), the characteristic function of a univariate $\mathcal{N}(0, ||s||^2)$ variable is $\exp(-\frac{1}{2}t'AA't) = \exp(-\frac{1}{2}t'\Sigma t)$. Hence,

$$\mathbb{E}[e^{is'X}] = \exp(-\frac{1}{2}||s||^2) = \exp(-\frac{1}{2}t'AA't) = \exp(-\frac{1}{2}t'\Sigma t)$$
$$\therefore \varphi_Z(t) = \exp(it'\mu - \frac{1}{2}t'\Sigma t), \quad t \in \mathbb{R}^d$$

5 Stochastic convergence

1. Prove Lemma 2.7

Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of random vectors and X a random vector. For $\epsilon > 0$, let $E_{n,\epsilon} := \{||X_n - X|| > \epsilon\}$. Then $X_n \xrightarrow{a.s.} X$ iff. for each $\epsilon > 0$

$$P(\limsup_{n\to\infty} E_{n,\epsilon}) = 0 \quad \text{(limit superior – limiting upper bounds of sequence)}$$

For $\epsilon > 0$ set $E_{n,\epsilon} := \{||X_n - X|| > \epsilon\}$. Then $X_n \xrightarrow{a.s.} X \iff \forall\epsilon > 0, \mathbb{P}\Big(\limsup_{n\to\infty} E_{n,\epsilon}\Big) = 0$

$$\text{where} \limsup_{n\to\infty} E_{n,\epsilon} := \bigcap_{m=1}^{\infty} \bigcup_{n\geq m} E_{n,\epsilon} = \{\omega : \omega \in E_{n,\epsilon}\text{i.o.}\}$$

($\Rightarrow$) Assume $X_n \to X$ a.s. Then $\exists N_\epsilon(\omega)$ s.t. $\forall n \geq N_\epsilon(\omega), ||X_n(\omega) - X(\omega)||$

Hence $E_{n,\epsilon}(\omega)$ happens only finitely often, i.e., $\omega \notin \limsup_n E_{n,\epsilon}$. $\therefore \mathbb{P}\limsup_n E_{n,\epsilon} = 0$

$$(\Leftarrow)\text{Assume } \forall\epsilon > 0, \mathbb{P}\limsup_n E_{n,\epsilon} = 0. \text{ Set } A := \bigcup_{k=1}^{\infty} \limsup_n E_{n,1/k},$$
$$\text{so } \exists N_k(\omega)\text{s.t. } n \geq N_k(\omega) \implies ||X_n(\omega) - X(\omega)|| < \delta,$$
$$\text{hence } ||X_n(\omega) - X(\omega)|| \to 0. \text{ Since } \mathbb{P}(A^c) = 1, \text{ we have } X_n \to X \text{ a.s.} \quad \square$$

2. Fill in the missing details in the proof of thm. 2.8:

Thm: *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of random vectors and X a random vector. The following are equivalent:*

(i) $X_n \xrightarrow{D} X$,

(ii) $\lim_{n\to\infty} \mathbb{E}f(X_n) = \mathbb{E}f(X)$,

(iii) $\limsup_{n\to\infty} P(X_n \in F) \leq P(X \in F)$ for all closed sets F,

(iv) $\liminf_{n\to\infty} P(X_n \in G) \geq P(X \in G)$ for all open sets G,

(v) $\lim_{n\to\infty} P(X - n \in A) = P(X \in A)$ for all X-continuity sets A,

(vi) *if F_n is the CDF of X_n and F that of X, $F_n(X) \to F(x)$ for all x at which F is continuous*

Proof:

(i) $\Longrightarrow (ii)$: obvious since every Lipschitz-continuous function is continuous.

(ii) $\Longrightarrow (iii)$: let F be a closed set, $F^\epsilon := \{x : \rho(x, F) < \epsilon\}$ and define $f(x) := \max\{1 - \rho(x, F)/\epsilon, 0\}$. We note that $\not\Vdash_F(x) \leq \not\Vdash_{F^\epsilon}(x)$ and f is (Lipschitz) continuous [Exercise].

Then by (i) or (ii)

$$P(X_n \in F) = \mathbb{E}\not\Vdash_F(X_n) \leq \mathbb{E}f(X_n) \to \mathbb{E}f(X) \leq \mathbb{E}\not\Vdash_{F^\epsilon}(X) = P(X \in F^\epsilon)$$

As F is closed, taking the limit as $\epsilon \downarrow 0$ yields the required inequality [Exercise].

(iii) $\Longrightarrow (iv)$: Take complements [Exercise]

(iv) & $(iv) \Longrightarrow (v)$: Combining (iii) and (iv):

$$\begin{aligned} P(X \in \text{cl } A) &\geq \limsup_{n\to\infty} P(X_n \in \text{cl } A) \geq \limsup_{n\to\infty} P(X_n \in A) \\ &\geq \liminf P(X_n \in A) \geq \liminf_{n\to\infty} P(X_n \in \text{int } A) \geq P(X \in \text{int } A) \end{aligned}$$

If $P(X \in \delta A) = 0$, i.e. A is an X-continuity set the left and right hand side terms coincide (both are equal to $P(X \in A)$), which implies (v)

(v) $\Longrightarrow (vi)$: Let x be a continuity point of F and set $A = (-\infty, x]$. Then A is an X-continuity set as $\delta A = \{x\}$ and $P(X = x) = 0$ [Exercise]. Then by (v) $F_n(x) = P(X_n \in A) \to P(X \in A) = F(x)$

$(v) \Longrightarrow (i)$: It is enough to consider the case that $0 \leq f \leq 1$ [Exercise]. Then

$$\mathbb{E}f(X) = \int_0^1 P\big(f(X) > t\big)\ dt, \quad \mathbb{E}f(X_n) = \int_0^1 P\big(f(X_n) > t\big)\ dt,$$

by e.g. Lemma 2.2.13 in [5] Since f is continuous, $\delta\{x : f(x) > t\} \subset \{x : f(x) = t\}$ and so $\{x : f(x) > t\}$ is an X-continuity set except for at countably many t [Exercise]. By (v) and the bounded convergence thm

$$\mathbb{E}f(X_n) = \int_0^1 P\big(f(X_n) > t\big)\ dt \to \int_0^1 P\big(f(X) > t\big)\ dt = \mathbb{E}f(X)$$

(vi) $\implies$ *(iv)* : Define $D^i := \{x : P(X \in H_c^i) > 0\}$ for $H_c^i := \{x : x_i = c\}$. Note that D^i is at most countable. Let A be a rectangle $A = \Pi_{i=1}^K (a_i, b_i]$ with $a_i, b_i \notin D^i$ for each i. By $F_n(x) \to F(x)$, for any such $A, P(X_n \in A) \to P(X \in A)$ and therefore for any finite collection of disjoint such rectangles $A_1, \ldots, A_k$, their union B_k also satisfies $P(X_n \in B_k) \to P(X \in B_k)$. As any open set $G \in \mathbb{R}^K$ can be written as the increasing limit of such a disjoint union of rectangles with $a_i, b_i \notin D^i$,

$$\liminf_{n\to\infty} P(X_n \in G) \geq \liminf_{n\to\infty} P(X_n \in B_k) = P(X \in B_k)$$

Taking $B_k \uparrow G$ completes the proof. $\square$

Answers:

(i) $\implies$ *(ii)* : This is immediate: convergence in distribution means $\mathbb{E}g(X_n) \to \mathbb{E}(X)$ for every bounded continuous g. Every Lipschitz function is bounded and continuous on $\mathbb{R}^k$ $\square$

(ii) $\implies$ *(iii)* : Let $F \subset \mathbb{R}^k$ be closed. For $\epsilon > 0$ define the open ϵ-neighborhood $F^\epsilon = \{x : \rho(x, F) < \epsilon\}$ and the function

$$f_\epsilon(x) = \max\{1 - \rho(x, F)/\epsilon, 0\}$$

Properties of f_ϵ:

- f_ϵ is continuous (indeed Lipschitz with constant $1/\epsilon$)
- $f_\epsilon(x) = 1 \quad \forall x \in F$
- $f_\epsilon(x) = 0 \quad \forall x \notin F^\epsilon$
  Hence $\nVdash_F \leq f_\epsilon \leq 1_{F^\epsilon}$

Now apply (ii):

$$P(X_n \in F) = \mathbb{E}\nVdash_F(X_n) \leq \mathbb{E}f_\epsilon(X_n) \xrightarrow[n\to\infty]{} \mathbb{E}f_\epsilon(X)$$

But $\mathbb{E}f_\epsilon(X) \leq \mathbb{E}\nVdash_{F^\epsilon}(X) = P(X \in F^\epsilon)$. So

$$\limsup_{n\to\infty} P(X_n \in F) \leq P(X \in F^\epsilon)$$

Finally let $\epsilon \downarrow 0$. Because F is closed the sets F^ϵ decrease to F, so by continuity from above of probability measures,

$$\lim_{\epsilon\downarrow 0} P(X \in F^\epsilon) = P\Big(\bigcap_{\epsilon>0} F^\epsilon\Big) = P(F)$$

Thus, $\limsup_n P(X_n \in F) \leq P(X \in F)$, proving (iii) $\square$

(iii) $\implies$ *(iv)* : Take complements. if G is open then G^c is closed, so by (iii):

$$\limsup_n P(X_n \in G^c) \leq P(X \in G^c)$$

But $P(X_n \in G^c) = 1 - P(X_n \in G)$ and similarly for X. Rearranging gives

$$\liminf_n P(X_n \in G) \geq P(X \in G)$$

which is (iv) □

(v) $\implies$ *(vi)* : Take a real x at which the (univariate) distribution function F of X is continuous. Set $A = (-\infty, x]$. Then $\delta A = \{x\}$, and since F is continuous at x we have $P(X = x) = 0$. Hence A is an X-continuity set. By (v),

$$F_n(x) = P(X_n \leq x) = P(X_n \in A) \xrightarrow[n\to\infty]{} P(X \in A) = F(x),$$

so $F_n(x) \to F(x)$ at every continuity point x of F. This is (vi) □

(vi) $\implies$ *(iv)* : We now show that convergence of CDFs at continuity points (componentwise) implies (iv) for multivariate open sets.

The countability fact: For each coordinate i, define

$$D^i = \{c \in \mathbb{R} : P(X_i) = c > 0\}$$

Each D^i is at most countable because $\sum_{c \in D^i} P(X_i = c) \leq 1$ and only countably many positive numbers can sum to ≤ 1

Rectanges: Let $A = \Pi_{i=1}^k (a_i, b_i]$ be a rectangle with $a_i, b_i \notin D^i$ for all i. Then each enpoint is a continuity point of the univariate marginal CDFs, so by (vi) every joint CDF evaluation at the 2^k corner points of the rectangle converges to the corresponding value for X. The probability of the rectangle can be written by the inclusion-exclusion formula as a finite alternating sum of the joint CDF values at the corners. Since each corner CDF converges, the finite sum converges:

$$P(X_n \in A) \xrightarrow[n\to\infty]{} P(X \in A)$$

Finite unions: If $A_1, \ldots, A_m$ are disjoint such rectangles then the same is true for the union $B = \bigsqcup_{j=1}^m A_j$: one adds the limits, so $P(X_n \in B) \to P(X \in B)$

Approximation of open sets: Any open $G \subset \mathbb{R}^k$ can be written as an increasing union of countably many disjoint rectangles with rational endpoints (standard fact using a grid with rational coordinates). Because the sets D^i are countable we can choose rational endpoints that avoid D^i (rationals are dense and uncountable choices exist). Hence one can construct an increasing sequence of finite unions $B_1 \subset B_2 \subset \ldots \subset B_k$ of disjoint rectangles (each rectangle having endpoints not in D^i) with $B_m \uparrow G$. For each m,

$$P(X_n \in B_m) \xrightarrow[n\to\infty]{} P(X \in B_m)$$

Thus

$$\liminf_{n\to\infty} P(X_n \in G) \geq \liminf_{n\to\infty} P(X_n \in B_m) = P(X \in B_m)$$

Let $m \to \infty$ and use monotone convergence of Probabilities ($B_m \uparrow G$) to get

$$\liminf_{n\to\infty} P(X_n \in G) \geq P(X \in G)$$

This is exactly (iv) □

(v) $\implies (i)$: **Detail of the layer-cake argument:** We now sketch the standard argument that (v) implies convergence in distribution, i.e. $\mathbb{E}f(X_n) \to \mathbb{E}f(X)$ for every bounded continuous f. It suffices to check this for $0 \leq f \leq 1$ (any bounded f is an affine transform of such a functino and linearity handles the rest)

Use the identity (layer-cake representation)

$$\mathbb{E}f(X) = \int_0^1 P\big(f(X) > t\big)\ dt,$$

and similarly for X_n. For each fixed t the level set A_t :)$\{x : f(x) > t\}$ is open. Its boundary δA_t is contained in $\{x : f(x) = 1\}$. The set of t for which $P\big(f(X) = t\big) > 0$ is at most countable because $\sum_{t \in \mathbb{R}} P\big(f(X) = t\big) \leq 1$. Hence for all t except a countable set, A_t is an X-continuity set. for such t, by (v),

$$P\big(f(X_n) > t\big) = P(X_n \in A) \xrightarrow[n\to\infty]{} P(X \in A_t) = P\big(f(X) > t\big)$$

Dominated convergence (the integrands are bounded by 1) gives

$$\mathbb{E}f(X_n) = \int_0^1 P\big(f(X_n) > t\big)\ dt \xrightarrow[n\to\infty]{} \int_0^1 P\big(f(X) > t\big)\ dt = \mathbb{E}f(X)$$

Thus we obtain convergence of expectations for every bounded continuous f, i.e. convergence in distribution (i) $\square$

also please explain X-continuity set