Promblems 1

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Contents

1 Probabilities, random variables

1. A fair die is thrown until a 6 appears. What is the probability that it must be thrown at least k times?

$$\begin{aligned} P(\text{at least } k \text{ throws}) &= 1 - P(\text{fewer than } k \text{ throws}) \\ P(\text{fewer than } k \text{ throws}) &= P(\text{success in first } k-1 \text{ throws}) \\ P(\text{success in first } k-1 \text{ throws}) &= 1 - \left(\frac{5}{6}\right)^{k-1} \end{aligned}$$

2. For $x \in \mathbb{R}^K$, let $f_\theta(x) := h(x)\exp(\eta(\theta)'T(x) - A(\theta))$ for functions h, η, T, A. When is this a pdf? i.e. when is $f(x) \geq 0$ and such that its integral is equal to one?

Non-negativity: $f_\theta(x) \geq 0$

$$\begin{aligned} h(x) &\geq 0 \quad \forall x \text{ in the support} \\ \exp(\eta(\theta)'T(x) - A(\theta)) &\geq 0 \end{aligned}$$

Integrating to 1:

This is where the log-partition function $A(\theta)$ plays a crucial role:

$$\begin{aligned} \int f_\theta(x)dx = \int h(x)\exp(\eta(\theta)'T(x) - A(\theta))dx &= 1 \\ = \exp(-A(\theta)) \int h(x)\exp(\eta(\theta)'T(x))dx &= 1 \\ \therefore A(\theta) = \log \int h(x)\exp(\eta(\theta)'T(x))dx & \end{aligned}$$

3. For each $i = 1, \ldots, K$, let f_i be pdfs (resp.) and define $f(x) := \sum_{i=1}^{K} w_i f_i(x)$ where $w_i \geq 0$ and $\sum_{i=1}^{K} w_i = 1$. Show that f is a probability density (resp. probability mass) function. i.e. show that $f(x) \geq 0$ and its integral / sum is equal to one in the density / mass case respectively

Here we need to verify two conditions:

- **Non-negativity:** Show that $f(x) \geq 0 \quad \forall x$
- **Normalization:** Show that the integral (or sum) equals 1

Non-negativity:

Since each $f_i(x)$ is a valid pdf/pmf, we have $f_i(x) \geq 0 \quad \forall x \wedge (i = 1, \ldots, K)$

Additionally, we have given that $w_i \geq 0 \quad \forall i = 1, \ldots, K$

$$\therefore f(x) = \sum_{i=1}^{K} w_i f_i(x) \geq 0$$

Since we are summing non-negative terms $(w_i \geq 0 \wedge f_i \geq 0)$

Normalization:

Case 1: pdf

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x)dx &= \int_{-\infty}^{\infty} \sum_{i=1}^{K} w_i f_i(x)dx \\
&= \sum_{i=1}^{K} w_i \int_{-\infty}^{\infty} f_i(x)dx \quad \text{(linearity of integration)} \\
&= \sum_{i=1}^{K} w_i \times 1 \quad \text{(since each } f_i \text{ is a valid pdf)} \\
&= \sum_{i=1}^{K} w_i \\
&= 1 \quad \text{(given the constraint } \sum_{i=1}^{K} w_i = 1\text{)}
\end{aligned}$$

Case 2: pmf

$$\begin{aligned}
\sum_x f(x) &= \sum_x \sum_{i=1}^{K} w_i f_i(x) \\
&= \sum_{i=1}^{K} w_i \sum_x f_i(x) \quad \text{(linearity of summation)} \\
&= \sum_{i=1}^{K} w_i \times 1 \quad \text{(since each } f_i \text{ is a valid pmf)} \\
&= \sum_{i=1}^{K} w_i \\
&= 1 \quad \text{(given constraint)}
\end{aligned}$$

4. Let X be a Poisson r.v. with mass function $f(x) = \lambda^x \exp(-\lambda)/x!, \quad x = 0, 1, \ldots$ for $\lambda > 0$. Find the probability that X is odd

$$\exp(\lambda) = \sum_{x=0}^{\infty} \frac{\lambda^x}{x!} = 1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \ldots + \frac{\lambda^n}{n!}$$

$$\exp(-\lambda) = \sum_{x=0}^{\infty} \frac{-\lambda^x}{x!} = 1 + \frac{-\lambda}{1!} + \frac{-\lambda^2}{2!} + \ldots + \frac{-\lambda^n}{n!}$$

$$P(X \text{ is odd}) = \sum_{x \text{ odd}} \frac{\lambda^x e^{-\lambda}}{x!} = e^{-\lambda} \sum_{x \text{ odd}} \frac{\lambda^x}{x!}$$

$$e^{\lambda} = \sum_{x=0}^{\infty} \frac{\lambda^x}{x!} \quad \text{(sum of all terms)}$$

$$e^{-\lambda} = \sum_{x=0}^{\infty} \frac{(-1)^x \lambda^x}{x!} \quad \text{(alternating signs)}$$

$$e^{\lambda} + e^{-\lambda} = 2 \sum_{x \text{ even}} \frac{\lambda^x}{x!} \quad \text{(even terms don't cancel)}$$

$$e^{\lambda} - e^{-\lambda} = 2 \sum_{x \text{ odd}} \frac{\lambda^x}{x!} \quad \text{(odd terms don't cancel)}$$

$$\rightarrow \sum_{x \text{ odd}} \frac{\lambda^x}{x!} = \frac{e^{\lambda} - e^{-\lambda}}{2}$$

$$P(X \text{ is odd}) = e^{-\lambda} \cdot \frac{e^{\lambda} - e^{-\lambda}}{2} = \frac{1 - e^{-2\lambda}}{2}$$

5. Prove that $F(x) := (1 + \exp(-x))^{-1} \quad x \in \mathbb{R}$ is a CDF

Recall that a CDF must satisfy these requirements:

- **Monotonicity:** F is non-decreasing (i.e. $x_1 \leq x_2 \to F(x_1) \leq F(x_2)$)
- **Right-continuity:** F is right-continuous at every point
- **Limit conditions:**
  - $\lim_{x\to-\infty} F(x) = 0$
  - $\lim_{x\to\infty} F(x) = 1$

Limit conditions: It is trivial that the function satisfies these two conditions.

Monotonicity: We prove that $F'(x) \geq 0$:

$$F'(x) = \frac{\exp(-x)}{\big(1 + \exp(-x)\big)^2} \quad \forall x \in \mathbb{R}$$

Right-continuity:

Since $F(x)$ is continuous everywhere (as a composition of continuous functions), it is automatically right-continuous

$\therefore F(x)$ is a CDF $\square$

6. Show that any CDF F, i.e. $F(x) := P(X \leq x)$, can have at most a countable number of discontinuities

The key here is to use the monotonicity of CDFs combined with the fact that rational numbers are countable.

For any CDF F, discontinuities can only by "jump" discontinuities due to monotonicity. At each discontinuity point x_0 we have:

- Left limit: $F(x_0^-) = \lim_{x \to x_0^-} F(x)$ exists
- Right limit: $F(x_0^+) = \lim_{x \to x_0^+} F(x) = F(x_0) \quad$ (right-continuity)
- Jump size: $F(x_0) - F(x_0^-) > 0$

Associate each discontinuity with a rational numebr:

- Let D be the set of discontinuity points
- For each $x \in D$, the jump size is $F(x) - F(x^-) > 0$
- Between any two consecutive jumps $F(x^-)$ and $F(x)$, there exists a rational number
- Since F is monotonic, these rational intervals are disjoint

Rationals are countable:

- Each discontinuity corresponds to a unique rational number in $(F(x^-), F(x)]$
- Since $\mathbb{Q} \cap [0, 1]$ is countable, and all these rationals are distinct
- Therefore D is at most countable