Lecture 1

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GRA4153 Advanced Statistics

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Contents

1 Introduction

Finna learn which analyses to undertake and how to interpret them

Divided into 4 parts:

1. Basic probability theory
2. Statistical inference
3. Linear (regression) models
4. Time series analysis

Lecture notes are the primary course material

Exercises are heavily useful → do these every week

1.1 Asessment

- Part 1 – 40% – midterm (in groups of up to 3)
- Part 2 – 60% – final take-home exams

1.2 Contact

- Office Hours: by apt
- Contact: Adam Lee
- for god's sake use email

2 Probability

$$S : \text{Sample space} \to \{x_1, \ldots, x_n\} \to \text{cannot have duplicate numbers}$$
$$\mathcal{S} : \sigma\text{-algebra set of sets}$$
$$S = \{1, 2, 3\}$$
$$A \subset S$$
$$A = \{1, 2\} \quad A = \{1\} \quad A = \{\} \to \text{This is the empty set } \emptyset$$
$$P : \mathcal{S} \to \mathbb{R} \text{ is a probability IF}$$
$$i) P(A) \geq 0 \quad \forall A \in \mathcal{S}$$
$$ii) P(\mathcal{S}) = 1$$
$$iii) A_1, \ldots, A_n \text{ s.t. } i \neq j =) A_i \cap A_j = \emptyset \to P(u_{1,2}^{\infty}) = \sum_{i=1}^{\infty} P(A_i)$$

2.1 2.1.1 Events and probabilities

$$\text{Prop: if } P : \mathcal{S} \to \mathbb{R} \text{ is a probability s.t. } A \in \mathcal{S}$$
$$\text{then:}$$
$$i) P(\emptyset) = 0$$
$$\mathcal{S} \cap \emptyset = \emptyset$$
$$ii) P(A) \leq 1$$
$$A^C = \{x \in \mathcal{S} : x \notin A\}$$
$$S = A \cap A^C$$
$$A \cup A^C = \emptyset$$
$$1 = P(\mathcal{S}) = P(A) + P(A^C)$$
$$iii) P(A^C) = 1 - P(A)$$

$$A_n \uparrow A = U_{n=1}^{\infty} A \to P(A_n) \uparrow P(A)$$
$$A_n \subset A_{n+1}$$
$$B_1 = A_1$$
$$B_k = A_k \setminus A_{k-1}$$

$$U_{n=1}^{\infty} A_i = \{x \in \mathcal{S} : \exists i : x \in A_i\}$$

$$P : \mathcal{S} \to [0,1]$$
$$A, B \in \mathcal{S}$$

then:

$i)$

$$P(A \cap A^C) = P(B) - P(A \cap B)$$

$B = (B \cap A) \cup (B \cap A^C)$ B and A complement have no commen elements

$$P(B) = P(B \cap A) + P(B \cap A^C)$$

$ii)$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$A = \{1,2,3\} \quad B = \{3,4,5\}$$

$$A \cup B = A \cup (B \cap A^C)$$
$$P(A \cup B) = P(A) + P(B \cap A^C) \quad = P(A) + P(B) - P(A \cap B)$$

$iii)$

$$A \subset B \to P(A) \leq P(B)$$
$$P(B) = P(A) + P(B \cap A^C)$$

$iv)$

$$(A_n)$$
$$A \subset U_{n=1}^{\infty} A_n$$
$$P(A) \leq \sum_{i=1}^{\infty} P(A_n)$$
$$A'_n = A_n \cap A$$
$$B_n = A'_n \setminus U_{m=1}^{n-1} A'_m$$
$$U_{n=1}^{\infty} B_n = A \to P(A) = \sum_{n=1}^{\infty} P(B_n) \leq \sum_{n=1}^{\infty} P(A_n)$$

$$X : S \to \mathbb{R} \quad \text{(R is the sample space)}$$
$$\mathcal{X} \subset \mathbb{R}$$
$$\mathcal{B} \quad \text{(collection of subsets)}$$
$$B \in \mathcal{B}$$
$$P_x(B) = P(\{s \in S : X(s) \in B\}) \to P(X^{-1}(B)) \to P(X \in B) = P(\{x \in B\})$$

$$S = \{(1,1),(1,2),\dots,(6,5),(6,6)\} \quad \text{(throwing two dice)}$$
$$\mathcal{S} = \text{all subsets of } S$$
$$P(\{(i,j)\}) = \frac{1}{36}$$

$$A = \{(1,1),\dots,(6,6)\} \to A_1 = \{1,1\},\dots,A_6 = \{6,6\}$$
$$U_{i=1}^{6} A_i = A$$
$$P(A) = P(U_{i=1}^{6} A_i) = \sum_{i=1}^{6} P(A_i) = \sum_{i=1}^{6} \frac{i}{36} = \frac{6}{36} = \frac{1}{6}$$

$$X : S \to \mathbb{R}$$
$$X\Big((i,j)\Big) = i + j$$
$$\mathcal{X} = \{\text{all the possible sums you can run through}\} \to \mathcal{X} = \{(2,\ 3,\ 4,\ \dots,\ 12)\}$$
$$\mathcal{B} = \text{subset of } \mathcal{X}$$
$$P_x(\{3\}) = P\Big(X^{-1}(\{3\})\Big) = P\Big((i,j) \in \mathcal{S} : X((i,j)) \in \{3\}\Big)$$

$$\mathcal{X} \supset A = \{2,4,6,8,10,12\}$$
$$P_x(A) =$$
$$P(\text{both even}) = P(\text{first even}) \cdot P(\text{second even}) = \frac{3}{6} \cdot \frac{3}{6} = \frac{9}{36}$$
$$P(\text{both odd}) = P(\text{first odd}) \cdot P(\text{second odd}) = \frac{3}{6} \cdot \frac{3}{6} = \frac{9}{36}$$
$$P_x(A) = \frac{9}{36} + \frac{9}{36} = \frac{18}{36} = \frac{1}{2}$$

3 CDF

Cumulative Distribution Function

$$
\begin{aligned}
P_x \\
F : \mathbb{R} \to [0,1] \quad \text{cdf of } X_1 \\
F(x) = P(X \leq x) = P(s \in S : X(s) \leq x) = P_x\big((-\infty, x]\big) \\
\big(P_x(A) : A \in B\big)
\end{aligned}
$$

Properties of CDF: if F is the cdf of a random variable X, then:

$$i) \lim_{x \to -\infty} F(x) = 0 \text{ and } \lim_{x \to \infty} F(x) = 1$$

$$ii) F(x) \leq F(y) \text{ whenever } y \geq x$$

$$iii) \lim_{x \downarrow z} F(x) = F(z)$$

$$
\begin{aligned}
x_n \to -\infty \\
F(x_n) = P_x\big((-\infty, x_n]\big)\big) \downarrow P_x(\emptyset) \quad \text{(converges downwards)} \\
(-\infty, x_n] \downarrow \emptyset
\end{aligned}
$$

$$(-\infty, x_n] \uparrow (-\infty, \infty) = \mathbb{R}$$

$$
\begin{aligned}
ii) (-\infty, x] \subset (-\infty, y] \quad \text{if } y \geq x \\
F(x) = P_x\big((-\infty, x]\big) \leq P_x\big((-\infty, y]\big) = F(y)
\end{aligned}
$$

$$z_n \downarrow x \to (-\infty, z_n] \downarrow (-\infty, x] \to F(z_n) \to F(x)$$

$$z_n \uparrow x \to (-\infty, z_n] \uparrow (-\infty, x)$$

Identically distributed:

$$
\begin{aligned}
\text{let } x \text{ and } y \text{ be a random variable} \\
\forall B \in \mathcal{B} \\
P(x \in B) = P(y \in B) \\
P_x(B) = P_y(B)
\end{aligned}
$$

This is like flipping two coins, coin one is x and coin two is y (they are not the same coin, hence the same probability)

random variables can also be Identically distributed:

$$\text{if } F_x = F_y \quad \forall x \in \mathbb{R}$$

$$
\begin{aligned}
F_x(x) = P\big(x \in (-\infty, x]\big) &= P_x\big((-\infty, x]\big) \\
&= P_y\big((-\infty, x]\big) \\
&= P\big(y \in (-\infty, x]\big) \\
&= F_y(x)
\end{aligned}
$$

3.1 Continuous and discrete distributions:

x is discrete:

x has a pmf: $f(x) = P(X = x), \quad x \in \mathbb{R}$

$$
\begin{aligned}
B &\in \mathcal{B} \\
P_x(B) = P(x \in B) &= \sum_{t \in B} f(t) \\
P(\text{even}) = f(2) + f(4) + f(6) \quad &\text{(like the example about the dice)}
\end{aligned}
$$

$$
\begin{aligned}
F(x) &= P_x\big((-\infty, x]\big) \\
&= \sum_{t \le x} f(t) \quad (\textbf{countable} \text{ points that are more than zero})
\end{aligned}
$$

4 Three common distributions:

1. **Bernoulli:**

$$X \sim \text{Bernoulli}(p), \quad p \in [0,1]$$

$$X = \begin{cases} 1 & \text{if prob. } p \\ 0 & \text{if prob. } 1-p \end{cases}$$

$$f(x) \begin{cases} p & x = 1 \\ p & x01 \\ p & x \notin \{0,1\} \end{cases}$$

Ex: we have two different websites and we want to measure website performance of how many people bought stuff on each of the websites → **proportions**

2. **Binomial:**

$$X \sim \text{Binom}(n,p)$$
$$f(x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0, 1, \ldots, n$$

3. **Poisson:**

$$X \sim \text{Poisson}(\lambda)$$
$$f(x) = \frac{\exp(-\lambda)\lambda^x}{x!}, \quad x = 0, 1, 2, \ldots$$

Example: ordered packages, how likely is it that 6 of my packages are going to arrive before lunch? lambda is package arrival intensity

4.1 Continuous random variables

X is continuous

$$f.$$

$$f(x) \neq P(X = x)$$

$$P(X = x) = 0 \quad \forall x \in \mathbb{R} \quad \text{(The probability of hitting any specific exact value is essentially zero)}$$

$$P(X \leq x) = F(x) = \int_{-\infty}^{x} f(t)dt$$

$$P(x \in B) = \int_{-B} f(t)dt = \int_{-\infty}^{\infty} \not\Vdash_B(t) f(t)dt$$

Uniform distribution

$X \sim \text{uniform}(a, b)$

$$f(x) = \begin{cases} \frac{1}{b-a} & x \in [a, b] \\ 0 & \text{otherwise} \end{cases}$$

Normal distribution:

$X \sim \text{Normal}(\mu, \sigma^2)$

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-(x-\mu^2)}{2\sigma^2}\right)$$

Student's distribution

$X \sim t(v)$

$$f(x) = \frac{\Gamma\big((v+1)/2\big)}{\Gamma(v/2)\sqrt{v\pi}} \left(1 + \frac{x^2}{v}\right)^{-(v\pi)/2}$$

Chi-squared - related to the N dist

$X \sim \chi^2(p)$

Ugly formula – google it

f is a pdf/pmf of r.v. x

$$i) f(x) \geq 0$$

$$ii) \int_{-\infty}^{\infty} f(x)dx = 1; \quad \sum_{x \in \mathcal{X}} f(x) = 0$$

i) continuous

$$\text{as } F(x) \uparrow \quad F(x) = \int_{-\infty}^{x} f(t)dt$$

ii) discrete

$$\begin{aligned} \sum_{x \in \mathcal{X}} f(x) &= \lim_{t \to \infty} \sum_{x \leq t, x \in \mathcal{X}} f(x) \\ &= \lim_{t \to \infty} F(t) \\ &= 1 \end{aligned}$$

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{t \to \infty} \int_{-\infty}^{t} f(x)dx = \lim_{t \to \infty} F(t) = 1$$

5 Creating r.vs from an r.v.

$X : S \to \mathbb{R}$

$g : \mathbb{R} \to \mathbb{R}$

$Y = g(X) : S \to \mathbb{R}$

$P(y \in A) = P(g(x) \in A) = P\big(\{x \in \mathbb{R} : g(x) \in A\}\big) = P\big(x \in g^{-1}(A)\big)$

If this transformation is well ´-behaved, we can compute the pdf of g(x)

i) If x is discrete, then $Y = g(x)$ is also discrete: $f_y(y) = \sum_{x \in g^{-1}(\{y\})} f_x(x)$

ii) If x is cont., then the output will be discrete:

$F_Y(y) = \int_{\{x : g(x) \leq y\}} f_X(x) dx = P(g(x) \leq y) = P\Big(x \in g^{-1}\big(\{y\}\big)\Big) = P(Y \leq y)$

iii) if g is strictly monotone increasing, then $h = g^{-1}$ is differentiable, THEN

$$F_Y(y) = F_X\big(h(y)\big), \quad f_Y(y) = f_X\big(h(x)\big)h'(y)$$

$$F_X(y) = \int_{-\infty}^{g^{-1}(y)} f_X(x) dx = F_X\big(h(y)\big)$$

6 Mean variance of an arbitrary distribution

$$\begin{aligned}\mathbb{E}g(x) &= \int_{-\infty}^{\infty} g(x)f(x)dx \\ &= \sum_{x\in\mathcal{X}} g(x)f(x) \quad \text{(in the discrete case)}\end{aligned}$$

$$\mathbb{E}X = \int_{-\infty}^{\infty} x(fx)dx \quad \text{this technically does not exist, hope this helps}$$

we can always write

$$g(x) = g^{+}(x) - g^{-}(x) \quad \text{(both funcs are positive)}$$

expectation exists if:

$$\mathbb{E}|g(x)| < \infty$$

in the case that

$$\mathbb{E}|g(x)| \geq \infty$$

we can say that it *may* not exist

$$\begin{aligned}i)\mathbb{E}[\alpha g_1(x)] &= \alpha\mathbb{E}[g(x)] \\ \mathbb{E}[g_1(x) + g_2(x)] &= \mathbb{E}[g_1(x)] + \mathbb{E}[g_2(x)]\end{aligned}$$

$$ii)g_1(x) \geq 0 \quad \forall f(x) > 0 \to \mathbb{E}g_1(x) \geq 0 \quad \text{(proof below)}$$

$\pm$ inf are implied

$$\begin{aligned}\mathbb{E}\alpha g_1(x) &= \int \alpha g_1(x)f(x)dx = \alpha \int g(x)f(x)dx = \alpha\mathbb{E}g_1(x) \\ \mathbb{E}\big(g_1(x) + g_2(x)\big) &= \int \big(g_1(x) + g_2(x)\big)f(x)dx = \int g_1(x)f(x) + g_2(x)f(x)dx \\ &= \int g_1(x)f(x)dx + \int g_2(x)f(x)dx\end{aligned}$$

remember to look at ii) from the notes