Lecture 1

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Contents

1 Moments

Central moment:

$$\begin{aligned}
\mathbb{E}(x-\mathbb{E}x)^k \quad (k^{\text{th}} \text{ central moment})\\
z = x - \mathbb{E}x\\
\mathbb{E}z = \mathbb{E}x - \mathbb{E}(\mathbb{E}x) = 0\\
\mathbb{E}(x-\mathbb{E}x)^2 = \mathrm{Var}(x)
\end{aligned}$$

First moment is the mean, second is the variance

Properties of variance:

If a and b are constants, then

$$\mathrm{Var}(ax+b) = a^2\mathrm{Var}(x)$$

Indicator functions ($\nVdash(x)$)

$$\nVdash_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

$$\begin{aligned}
\nVdash\{x \in A\}\\
\text{A, B are r.v. } x:\\
i) \quad \nVdash_A \nVdash_B = \nVdash_{AB}\\
ii) \quad P(x \in A) = \mathbb{E}[\nVdash_A(x)] = \mathbb{E}[\nVdash\{x \in A\}]\\
iii) \quad P(x \in A)(1 - P(x \in A)) = \mathrm{Var}(\nVdash_A(x))
\end{aligned}$$

$x =$ value g a fair die

$$\begin{aligned}
\mathbb{E}x = \sum_{i=1}^{6} P(x=i) = \frac{1}{6}\sum_{i=1}^{6} i = \frac{7}{2}\\
\mathrm{Var}(x) = \sum_{i=1}^{6}\left(i - \frac{7}{2}\right)^2 P(x=i)\\
= \mathbb{E}(x^2) - \big(\mathbb{E}(x)\big)^2\\
= \frac{1}{6}\sum_{i=1}^{6} i^2 - \big(\mathbb{E}(x)\big)^2 = \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \frac{35}{12}
\end{aligned}$$

$$X \sim N(\mu, \sigma^2)$$

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\Big(-\frac{(x-\mu)^2}{2\sigma^2}\Big)$$

$$\mathbb{E}x = \mu$$
$$\mathrm{Var}(x) = \sigma^2$$
$$z = \frac{(x-\mu)}{\sigma} \sim N(0,1)$$

$$z \sim N(0,1)$$
$$\mathbb{E}z = 0$$
$$\mathrm{Var}(z) = 1$$
$$x = \sigma z + \mu$$

$$P(Z \le z) \quad \text{(show that this is a normal dist(??))}$$
$$P(Z \le z) = P\Big(\frac{(x-\mu)}{\sigma} \le z\Big)$$
$$P(x \le \sigma^2 + \mu)$$
$$= \int_{-\infty}^{\sigma^2+\mu} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\Big(-\frac{(x-\mu)^2}{2\sigma^2}\Big)dx$$
$$t = \frac{x-\mu}{\sigma} \quad \text{integration by substitution}$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{2} \exp(-t^2/z)dx$$
$$\to z \sim N(0,1)$$

$$\mathbb{E}Z = \int zf(z)dz = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} 2\exp\Big(-\frac{2^2}{2}\Big)dz$$
$$= 0$$

For the second moment, the same method is applied, only with x^2 (i.e. the variance):

$$\begin{aligned}\mathbb{E}z^2 &= \int z^2 f(z)dz \\ &= 1 \quad \text{due to it being a pdf (the prob across the entire range must be 1)} \\ &\text{OR} \\ \mathbb{E}X^2 &= \text{Var}(x) + \big(\mathbb{E}(x)\big)^2\end{aligned}$$

2 Conditional probability

BAYES WOHOOOO LETS GO CHAT

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} \quad P(B) < 0$$

$$\begin{aligned}P(A \cap B) &= P(A|B)P(B) = P(B|A)P(A) \\ P(A|B) &= P(B|A)\frac{P(A)}{P(B)} \quad \text{(bayes' thm)}\end{aligned}$$

A and B are independent iff.:

$$P(A \cap B) = P(A)P(B)$$

$$\begin{aligned}A_1, \ldots, A_n \text{ for a subcolecction } A_k, \ldots, A_{k_k} \\ P(\prod_{j=1}^{k})A_{k_j} = \prod_{j=1}^{k} P(A_{k_j})\end{aligned}$$

Random vectors:

$$\begin{aligned}X &\in \mathbb{R}^k \\ X &= \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_k \end{pmatrix} \\ X &: S \to \mathbb{R}^k \\ F_X(x) = P(X_1 \leq x_1, \ldots, x_k \leq X_k), \quad x &\in \mathbb{R}^k = P\big(\{X_1 \in (-\infty, x_1]\}\cap, \ldots, \cap X_k \in (-\infty, x_k]\}\big) \\ &F_{X_1}(x_1)\times, \ldots, \times F_{X_K}(x_K)\end{aligned}$$

Marginal $g \quad X_k$: distribution of X_k is a random variable

$X_1, \ldots, X_k$ are independent if:

$$P(\prod_{i=1}^{K}\{x_i \in A_i\}) = \prod_{i=1}^{K} P(X_i \in A_i)$$

$$X = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_k \end{pmatrix}$$
$$\text{pmf: } f : \mathbb{R}^k \to \mathbb{R}$$
$$f(x) = P(X_1 = x_1, \ldots, X_K = x_K)$$
$$P(X \in A) = \sum_{x \in A} f(x) = \sum_{x \in A} P(X_1 = x_1, \ldots, X_K = x_K)$$

Marginal pmf:

$$f_{X_1}(x_1) = P(X_1 = x_1)$$

Conditional pmf:

$$f_{x_{k+1},...,x_K|x_1,...,x_k}$$
$$= P(X_{k+1} = x_{k+1}, \ldots, X_K = x_K | X_1 = x_1, \ldots, X_k = x_k)$$
$$= \frac{f_X(x)}{f_{X_1:X_k}(x_1, \ldots, x_k)}$$

$$= \frac{P(X_1 = x_1, \ldots, X_K = x_K)}{P(X_1 = x_1, \ldots, X_k = x_k)}$$

Marginal pdfs can be recovered from joint pdfs:

$$f_{\vec{X}}(\vec{x})$$

$$f_{X_1,...,X_k}(x_1, \ldots, x_k) = \sum_{x_{k+1},...,x_K \in \mathbb{R}^{K-k}} f_X(x_1, \ldots, x_k)$$

If x is a continuous random vector (i.e. its dcf is continuous) it has a joint pdf if there is a (non-negative) function $f : \mathbb{R}^K \to \mathbb{R}$ such that for any event $A \subset \mathbb{R}^K$:

$$P(X \in A) = \int_A f(x)dx$$

, therefore:

$$P((X_1, \ldots, X_k) \in A) = P(X \in B)$$
$$= \int_A f_{X_1, \ldots, x_k}(x_1, \ldots, x_k)d(x_1, \ldots, x_k)$$

Suppose that $X = (X_1, \ldots, X_K)$ is continuous. Then the marginal pdf of $X_1, \ldots, X_k$ is given by:

$$f_{X_1, \ldots, x_k}(x_1, \ldots, x_k) = \int_{\mathbb{R}^{K-k}} f_X(x_1, \ldots, x_K)d(x_{k+1}, \ldots, x_K)$$