

Lecture 3

Jakob Sverre Alexandersen
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1 Moments

Central moment:

$$\begin{aligned}\mathbb{E}(x - \mathbb{E}x)^k & \quad (k^{\text{th}} \text{ central moment}) \\ z &= x - \mathbb{E}x \\ \mathbb{E}z &= \mathbb{E}x - \mathbb{E}(\mathbb{E}x) = 0 \\ \mathbb{E}(x - \mathbb{E}x)^2 &= \text{Var}(x)\end{aligned}$$

First moment is the mean, second is the variance

Properties of variance:

If a and b are constants, then

$$\text{Var}(ax + b) = a^2 \text{Var}(x)$$

Indicator functions ($\mathbb{I}_A(x)$)

$$\mathbb{I}_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

$$\mathbb{I}_{\{x \in A\}}$$

A, B are r.v. $x :$

$$i) \quad \mathbb{I}_A \mathbb{I}_B = \mathbb{I}_{AB}$$

$$ii) \quad P(x \in A) = \mathbb{E}[\mathbb{I}_A(x)] = \mathbb{E}[\mathbb{I}_{\{x \in A\}}]$$

$$iii) \quad P(x \in A)(1 - P(x \in A)) = \text{Var}(\mathbb{I}_A(x))$$

x = value g a fair die

$$\begin{aligned}\mathbb{E}x &= \sum_{i=1}^6 P(x=i) = \frac{1}{6} \sum_{i=1}^6 i = \frac{7}{2} \\ \text{Var}(x) &= \sum_{i=1}^6 \left(i - \frac{7}{2}\right)^2 P(x=i) \\ &= \mathbb{E}(x^2) - (\mathbb{E}(x))^2 \\ &= \frac{1}{6} \sum_{i=1}^6 i^2 - (\mathbb{E}(x))^2 = \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \frac{35}{12}\end{aligned}$$

$$X \sim N(\mu, \sigma^2)$$

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

$$\mathbb{E}x = \mu$$

$$\text{Var}(x) = \sigma^2$$

$$z = \frac{(x-\mu)}{\sigma} \sim N(0, 1)$$

$$z \sim N(0, 1)$$

$$\mathbb{E}z = 0$$

$$\text{Var}(z) = 1$$

$$x = \sigma z + \mu$$

$$P(Z \leq z) \quad (\text{show that this is a normal dist(??)})$$

$$P(Z \leq z) = P\left(\frac{(x-\mu)}{\sigma} \leq z\right)$$

$$P(x \leq \sigma^2 + \mu)$$

$$= \int_{-\infty}^{\sigma^2 + \mu} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx$$

$$t = \frac{x-\mu}{\sigma} \quad \text{integration by substitution}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^2 \exp(-t^2/2) dt$$

$$\rightarrow z \sim N(0, 1)$$

$$\begin{aligned} \mathbb{E}Z &= \int z f(z) dz = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z \exp\left(-\frac{z^2}{2}\right) dz \\ &= 0 \end{aligned}$$

For the second moment, the same method is applied, only with x^2 (i.e. the variance):

$$\begin{aligned}\mathbb{E}z^2 &= \int z^2 f(z) dz \\ &= 1 \quad \text{due to it being a pdf (the prob across the entire range must be 1)} \\ &\quad \text{OR} \\ \mathbb{E}X^2 &= \text{Var}(x) + (\mathbb{E}(x))^2\end{aligned}$$

2 Conditional probability

BAYES WOHOOOO LETS GO CHAT

$$P(A | B) = \frac{P(A \cap B)}{P(B)} \quad P(B) > 0$$

$$\begin{aligned}P(A \cap B) &= P(A|B)P(B) = P(B|A)P(A) \\ P(A|B) &= P(B|A) \frac{P(A)}{P(B)} \quad (\text{bayes' thm})\end{aligned}$$

A and B are independent iff.:

$$P(A \cap B) = P(A)P(B)$$

A_1, \dots, A_n for a subcollection A_1, \dots, A_{k_k}

$$P\left(\prod_{j=1}^k A_{k_j}\right) = \prod_{j=1}^k P(A_{k_j})$$

Random vectors:

$$\begin{aligned}X &\in \mathbb{R}^k \\ X &= \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_k \end{pmatrix} \\ X : S &\rightarrow \mathbb{R}^k \\ F_X(x) &= P(X_1 \leq x_1, \dots, x_k \leq X_k), \quad x \in \mathbb{R}^k = P(\{X_1 \in (-\infty, x_1]\} \cap \dots \cap \{X_k \in (-\infty, x_k]\}) \\ &= F_{X_1}(x_1) \times \dots \times F_{X_K}(x_K)\end{aligned}$$

Marginal g_{X_k} : distribution of X_k is a random variable

X_1, \dots, X_k are independent if:

$$P\left(\prod_{i=1}^K \{x_i \in A_i\}\right) = \prod_{i=1}^K P(X_i \in A_i)$$

$$X = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_k \end{pmatrix}$$

pmf: $f : \mathbb{R}^k \rightarrow \mathbb{R}$

$$f(x) = P(X_1 = x_1, \dots, X_K = x_K)$$

$$P(X \in A) = \sum_{x \in A} f(x) = \sum_{x \in A} P(X_1 = x_1, \dots, X_K = x_K)$$

Marginal pmf:

$$f_{X_1}(x_1) = P(X_1 = x_1)$$

Conditional pmf:

$$\begin{aligned} & f_{x_{k+1}, \dots, x_K | x_1, \dots, x_k} \\ &= P(X_{k+1} = x_{k+1}, \dots, X_K = x_K | X_1 = x_1, \dots, X_k = x_k) \\ &= \frac{f_X(x)}{f_{X_1: X_k}(x_1, \dots, x_k)} \\ &= \frac{P(X_1 = x_1, \dots, X_K = x_K)}{P(X_1 = x_1, \dots, X_k = x_k)} \end{aligned}$$

Marginal pdfs can be recovered from joint pdfs:

$$f_{\vec{X}}(\vec{x})$$

$$f_{X_1, \dots, X_k}(x_1, \dots, x_k) = \sum_{x_{k+1}, \dots, x_K \in \mathbb{R}^{K-k}} f_X(x_1, \dots, x_k)$$

If x is a continuous random vector (i.e. its dcf is continuous) it has a joint pdf if there is a (non-negative) function $f : \mathbb{R}^K \rightarrow \mathbb{R}$ such that for any event $A \subset \mathbb{R}^K$:

$$P(X \in A) = \int_A f(x) dx$$

, therefore:

$$\begin{aligned} P((X_1, \dots, X_k) \in A) &= P(X \in B) \\ &= \int_A f_{X_1, \dots, X_k}(x_1, \dots, x_k) d(x_1, \dots, x_k) \end{aligned}$$

Suppose that $X = (X_1, \dots, X_K)$ is continuous. Then the marginal pdf of X_1, \dots, X_k is given by:

$$f_{X_1, \dots, X_k}(x_1, \dots, x_k) = \int_{\mathbb{R}^{K-k}} f_X(x_1, \dots, x_K) d(x_{k+1}, \dots, x_K)$$