# CSU44004 Formal Verification: Natural Deduction Solutions

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## Note

This document will contain 8 years of natural deduction solutions ranging from 2021 to 2016.

## 2021 Q2

I will create a helper proof for this problem to make it easier for myself. Use this proof as a reference as I will not repeat this proof in the solutions.
I will call this rule $\neg \to$ e
Prove: $\neg(p \to q) \vdash p \land \neg q$

| | | |
|---|---|---|
| 1. | $\neg(p \to q)$ | premise |
| 2. | $\neg(p \land \neg q)$ | assumption |
| 3. | $p$ | assumption |
| 4. | $\neg q$ | assumption |
| 5. | $p \land \neg q$ | $\land$i, 3, 4 |
| 6. | $\perp$ | $\neg$e, 5, 2 |
| 7. | $\neg\neg q$ | $\neg$i, 4–6 |
| 8. | $q$ | $\neg\neg$e, 6 |
| 9. | $p \to q$ | $\to$ i, 3–7 |
| 10. | $\perp$ | $\neg$e, 9, 1 |
| 11. | $\neg\neg(p \land \neg q)$ | $\neg$i, 2–10 |
| 12. | $p \land \neg q$ | $\neg\neg$e, 12 |

**(a)** Prove: $\neg q, t \to q, \neg r \to \neg s, p \to u, \neg t \to \neg r, u \to s \vdash p \to x$

| | | |
|---|---|---|
| 1. | $\neg q$ | premise |
| 2. | $t \to q$ | premise |
| 3. | $\neg r \to \neg s$ | premise |
| 4. | $p \to u$ | premise |
| 5. | $\neg t \to \neg r$ | premise |
| 6. | $u \to s$ | premise |
| 7. | $\neg(p \to x)$ | assumption |
| 8. | $p \wedge \neg x$ | $\neg \to$ e, 7 |
| 9. | $\neg t$ | $MT$, 1, 2 |
| 10. | $\neg r$ | $\to$ e, 9, 5 |
| 11. | $\neg s$ | $\to$ e, 10, 3 |
| 12. | $\neg u$ | $MT$, 11, 6 |
| 13. | $\neg p$ | $MT$, 12, 4 |
| 14. | $p$ | $\wedge$e$_1$, 8 |
| 15. | $\bot$ | $\neg$e, 14, 13 |
| 16. | $\neg\neg(p \to x)$ | $\neg$i, 7–15 |
| 17. | $p \to x$ | $\neg\neg$e, 16 |

## 2020 Q2

**(a)** This solution is ever so slightly different to 2021 Q2 (a), so I will do it slightly differently for practice.
Prove: $t \to q, \neg \to \neg s, p \to u, \neg t \to \neg r, u \to s \vdash p \to q$

| | | |
|---|---|---|
| 1. | $t \to q$ | premise |
| 2. | $\neg r \to \neg s$ | premise |
| 3. | $p \to u$ | premise |
| 4. | $\neg t \to \neg r$ | premise |
| 5. | $u \to s$ | premise |
| 6. | $\neg(p \to q)$ | assumption |
| 7. | $p \wedge \neg q$ | $\neg \to$ e, 6 |
| 8. | $\neg q$ | $\wedge e_2$, 7 |
| 9. | $p$ | $\wedge e_1$, 7 |
| 10. | $\neg t$ | $MT$, 8, 1 |
| 11. | $\neg r$ | $\to$ e, 10, 4 |
| 12. | $\neg s$ | $\to$ e, 11, 2 |
| 13. | $\neg u$ | $MT$, 12, 5 |
| 14. | $\neg p$ | $MT$, 13, 3 |
| 15. | $\perp$ | $\neg$e, 14, 9 |
| 16. | $\neg\neg(p \to q)$ | $\neg$i, 6–16 |
| 17. | $p \to q$ | $\neg\neg$e, 16 |

## 2019 Q3

**(a) (i)** Prove: $\neg p \to r \vdash \neg r \to p$

| | | |
|---|---|---|
| 1. | $\neg p \to r$ | premise |
| 2. | $\neg(\neg r \to p)$ | assumption |
| 3. | $\neg r \wedge \neg p$ | $\neg \to$ e, 2 |
| 4. | $\neg p$ | $\wedge e_2$, 3 |
| 5. | $r$ | $\to$ e, 4, 1 |
| 6. | $\neg r$ | $\wedge e_1$, 3 |
| 7. | $\perp$ | $\neg$e, 5, 6 |

**(a) (ii)** Prove: $\neg(p \to r) \to \neg q \vdash q \to p \to r$

| | | |
|---|---|---|
| 1. | $\neg(p \to r) \to \neg q$ | premise |
| 2. | $\neg(q \to p \to r)$ | assumption |
| 3. | $q \wedge \neg(p \to r)$ | $\neg \to$ e, 2 |
| 4. | $\neg(p \to r)$ | $\wedge e_2$, 3 |
| 5. | $\neg q$ | $\to$ e, 4, 1 |
| 6. | $q$ | $\wedge e_1$, 3 |
| 7. | $\perp$ | $\neg$e, 6, 5 |
| 8. | $\neg\neg(q \to p \to r)$ | $\neg$i, 2–7 |
| 9. | $q \to p \to r$ | $\neg\neg$e, 8 |

## 2019 Q1

**(c) (i)** Prove: $p \land q \to \neg r \vdash r \to p \to \neg q$

| | | |
|---|---|---|
| 1. | $p \land q \to \neg r$ | premise |
| 2. | $\neg(r \to p \to \neg q)$ | assumption |
| 3. | $r \land \neg(p \to \neg q)$ | $\neg \to$ e, 2 |
| 4. | $r$ | $\land e_1$, 3 |
| 5. | $\neg(p \to \neg q)$ | $\land e_2$, 3 |
| 6. | $p \land \neg\neg q$ | $\neg \to$ e, 5 |
| 7. | $p$ | $\land e_1$, 6 |
| 8. | $\neg\neg q$ | $\land e_2$, 6 |
| 9. | $q$ | $\neg\neg$e, 8 |
| 10. | $p \land q$ | $\land$i, 7, 8 |
| 11. | $\neg r$ | $\to$ e, 10, 1 |
| 12. | $\bot$ | $\neg$e, 4, 11 |
| 13. | $\neg\neg(r \to p \to \neg q)$ | $\neg$i, 2–12 |
| 14. | $r \to p \to \neg q$ | $\neg\neg$e, 13 |

**(c) (ii)** Sorry, this question is our assignment. I cannot give the answer to this one.

## 2018 Q1

**(c) (i)** Prove: $\vdash \neg((p \to \neg p) \land (\neg p \to p))$

| | | |
|---|---|---|
| 1. | $\neg(\neg((p \to \neg p) \land (\neg p \to p)))$ | assumption |
| 2. | $(p \to \neg p) \land (\neg p \to p)$ | assumption |
| 3. | $p \to \neg p$ | $\land e_1$, 2 |
| 4. | $\neg p \to p$ | $\land e_2$, 2 |
| 5. | $p$ | assumption |
| 6. | $\neg p$ | $\to$ e, 5, 3 |
| 7. | $\bot$ | $\neg$e, 5, 6 |
| 8. | $\neg p$ | $\neg$i, 5–7 |
| 9. | $p$ | $\to$ e, 8, 4 |
| 10. | $\bot$ | $\neg$e, 9, 8 |
| 11. | $\neg((p \to \neg p) \land (\neg p \to p))$ | $\neg$i, 2–11 |
| 12. | $\bot$ | $\neg$e, 11, 1 |
| 13. | $\neg\neg(\neg((p \to \neg p) \land (\neg p \to p)))$ | $\neg$i, 1–12 |
| 14. | $\neg((p \to \neg p) \land (\neg p \to p))$ | $\neg\neg$e, 13 |

**(c) (ii)** Prove: $p \to (q \lor r) \vdash (p \to q) \lor (p \to r)$

| | | |
|---|---|---|
| 1. | $p \to (q \lor r)$ | assumption |
| 2. | $\neg((p \to q) \lor (p \to r))$ | assumption |
| 3. | $p \to q$ | assumption |
| 4. | $(p \to q) \lor (p \to r)$ | assumption |
| 5. | $\bot$ | $\neg$e, 4, 5 |
| 6. | $\neg(p \to q)$ | $\neg$i, 3–5 |
| 7. | $p \land \neg q$ | $\neg \to$ e, 6 |
| 8. | $p$ | $\land$e$_1$, 7 |
| 9. | $(q \lor r)$ | $\to$ e, 8, 1 |
| 10. | $\neg q$ | $\land$e$_2$, 7 |
| 11. | $p \to r$ | assumption |
| 12. | $(p \to q) \lor (p \to r)$ | $\lor$i$_2$, 11 |
| 13. | $\bot$ | $\neg$e, 12, 2 |
| 14. | $\neg(p \to r)$ | $\neg$i, 11-13 |
| 15. | $p \land \neg r$ | $\neg \to$ e, 14 |
| 16. | $\neg r$ | $\land$e$_2$, 15 |
| 17. | $q$ | assumption |
| 18. | $\neg q$ | copy 10 |
| 19. | $\bot$ | $\neg$e, 17, 18 |
| 20. | $r$ | assumption |
| 21. | $\neg r$ | copy 16 |
| 22. | $\bot$ | $\neg$e, 16 |
| 23. | $\bot$ | $\lor$e, 9, 17–9, 20-22 |
| 24. | $\neg\neg((p \to q) \lor (p \to r))$ | $\neg$i, 2–23 |
| 25. | $(p \to q) \lor (p \to r)$ | $\neg\neg$e, 24 |

# 2017 Q1

**(c) (i)** Prove: $p \to r \vdash \neg p \lor r$

| | | |
|---|---|---|
| 1. | $p \to r$ | premise |
| 2. | $\neg(\neg p \lor r)$ | assumption |
| 3. | $p$ | assumption |
| 4. | $r$ | $\to$ e, 3, 1 |
| 5. | $\neg p \lor r$ | $\lor$i$_2$, 4 |
| 6. | $\bot$ | $\neg$e, 5, 2 |
| 7. | $\neg p$ | $\neg$i, 3–6 |
| 8. | $\neg p \lor r$ | $\lor$i$_1$, 7 |
| 9. | $\bot$ | $\neg$e, 8, 2 |
| 10. | $\neg\neg(\neg p \lor r)$ | $\neg$i, 2–9 |
| 11. | $\neg p \lor r$ | $\neg\neg$e, 10 |

**(c) (ii)** Prove: $(\neg p \to r), (r \to p), (p \to s) \vdash p \wedge s$

| | | |
|---|---|---|
| 1. | $\neg p \to r$ | premise |
| 2. | $r \to p$ | premise |
| 3. | $p \to s$ | premise |
| 4. | $\neg(p \wedge s)$ | assumption |
| 5. | $p$ | assumption |
| 6. | $s$ | $\to$ e, 5, 3 |
| 7. | $p \wedge s$ | $\wedge$i, 5, 6 |
| 8. | $\perp$ | $\neg$e, 7, 4 |
| 9. | $\neg p$ | $\neg$i, 5–8 |
| 10. | $r$ | $\to$ e, 9, 1 |
| 11. | $p$ | $\to$ e, 10, 2 |
| 12. | $\perp$ | $\neg$e, 11, 9 |
| 13. | $\neg\neg(p \wedge s)$ | $\neg$i, 4–12 |
| 14. | $p \wedge s$ | $\neg\neg$e, 13 |

## 2016 Q1

I will prove Morgan's Theorem using natural deduction to use as a helper rule for the problem (ii).
I will call this rule $DM_2$
Prove: $\neg(p \vee q) \vdash \neg p \wedge \neg q$

| | | |
|---|---|---|
| 1. | $\neg(p \vee q)$ | premise |
| 2. | $\neg(\neg p \wedge \neg q)$ | assumption |
| 3. | $p$ | assumption |
| 4. | $p \vee q$ | $\vee$i$_1$, 3 |
| 5. | $\perp$ | $\neg$e, 4, 1 |
| 6. | $\neg p$ | $\neg$i, 3–5 |
| 7. | $q$ | assumption |
| 8. | $p \vee q$ | $\vee$i$_2$, 7 |
| 9. | $\perp$ | $\neg$e, 8, 1 |
| 10. | $\neg q$ | $\neg$i, 7–9 |
| 11. | $\neg p \wedge \neg q$ | $\wedge$i, 6, 10 |
| 12. | $\perp$ | $\neg$e, 11, 2 |
| 13. | $\neg\neg(\neg p \wedge \neg q)$ | $\neg$i, 2–12 |
| 14. | $\neg p \wedge \neg q$ | $\neg\neg$e, 13 |

**(d) (i)** Prove: $p \vee (q \wedge r) \vdash (p \vee q) \wedge (p \vee r)$

| | | |
|---|---|---|
| 1. | $p \vee (q \wedge r)$ | premise |
| 2. | $\neg((p \vee q) \wedge (p \vee r))$ | assumption |
| 3. | $p \vee q$ | assumption |
| 4. | $p \vee r$ | assumption |
| 5. | $(p \vee q) \wedge (p \vee r)$ | $\wedge$i, 3, 4 |
| 6. | $\bot$ | $\neg$e, 5, 2 |
| 7. | $\neg(p \vee r)$ | $\neg$i, 4–6 |
| 8. | $\neg p \wedge \neg r$ | $DM_2$, 7 |
| 9. | $\neg p$ | $\wedge e_1$, 8 |
| 10. | $\neg r$ | $\wedge e_2$, 8 |
| 11. | $p$ | assumption |
| 12. | $\neg p$ | copy 9 |
| 13. | $\bot$ | $\neg$e, 11, 12 |
| 14. | $(q \vee r)$ | assumption |
| 15. | $r$ | $\wedge e_2$, 14 |
| 16. | $\bot$ | $\neg$e, 15, 10 |
| 17. | $\bot$ | $\vee$e, 1, 11–13, 14–16 |
| 18. | $\neg(p \vee q)$ | $\neg$i, 2–17 |
| 19. | $\neg p \wedge \neg q$ | $DM_2$, 18 |
| 20. | $\neg p$ | $\wedge e_1$, 19 |
| 21. | $\neg q$ | $\wedge e_2$, 19 |
| 22. | $p$ | assumption |
| 23. | $\bot$ | $\neg$e, 22, 20 |
| 24. | $(q \wedge r)$ | assumption |
| 25. | $q$ | $\wedge e_1$, 24 |
| 26. | $\bot$ | $\neg$e, 24, 21 |
| 27. | $\bot$ | $\vee$e, 1, 22–23, 24–26 |
| 28. | $\neg\neg((p \vee q) \wedge (p \vee r))$ | $\neg$i, 2–27 |
| 29. | $(p \vee q) \wedge (p \vee r)$ | $\neg\neg$e, 28 |

**(d) (ii)** Prove: $\phi \wedge \neg\psi \to \neg\phi \vdash \phi \to \psi$

| | | |
|---|---|---|
| 1. | $\phi \wedge \neg\psi \to \neg\phi$ | premise |
| 2. | $\neg(\phi \to \psi)$ | assumption |
| 3. | $\phi \wedge \neg\psi$ | $\neg \to$ e, 2 |
| 4. | $\neg\phi$ | $\to$ e, 3, 1 |
| 5. | $\phi$ | $\wedge e_1$, 3 |
| 6. | $\bot$ | $\neg$e, 4, 5 |
| 7. | $\neg\neg(\phi \to \psi)$ | $\neg$i, 2–6 |
| 8. | $\phi \to \psi$ | $\neg\neg$e, 7 |

## 2015 Q1

**(e) (i)** Prove: $(\phi_1 \to \psi_1), (\neg\psi_2 \to \neg\phi_2) \vdash (\phi_1 \land \phi_2) \to (\phi_1 \lor \phi_2)$

| | | |
|---|---|---|
| 1. | $(\phi_1 \to \psi_1)$ | premise |
| 2. | $(\neg\psi_2 \to \neg\phi_2)$ | premise |
| 3. | $\neg((\phi_1 \land \phi_2) \to (\psi_1 \lor \psi_2))$ | assumption |
| 4. | $\phi_1 \land \phi_2$ | assumption |
| 5. | $\phi_1$ | $\land e_1$, 4 |
| 6. | $\psi_1$ | $\to$ e, 5, 1 |
| 7. | $\psi_1 \lor \psi_2$ | $\lor$i, 6 |
| 8. | $(\phi_1 \land \phi_2) \to \psi_1 \lor \psi_2$ | $\to$ i, 4–7 |
| 9. | $\perp$ | $\neg$e, 8, 3 |
| 10. | $\neg\neg((\phi_1 \land \phi_2) \to (\psi_1 \lor \psi_2))$ | $\neg$i, 3–9 |
| 11. | $(\phi_1 \land \phi_2) \to (\psi_1 \lor \psi_2)$ | $\neg\neg$e, 10 |

**(e) (ii)** Prove: $\phi \to \neg\psi \vdash \psi \to \neg\phi$

| | | |
|---|---|---|
| 1. | $\phi \to \neg\phi$ | premise |
| 2. | $\neg(\psi \to \neg\phi)$ | assumption |
| 3. | $\psi \land \neg\neg\phi$ | $\neg \to$ e, 2 |
| 4. | $\neg\neg\phi$ | $\land e_2$, 3 |
| 5. | $\phi$ | $\neg\neg$e, 4 |
| 6. | $\neg\phi$ | $\to$ e, 5, 1 |
| 7. | $\perp$ | $\neg$e, 5, 6 |
| 8. | $\neg\neg(\psi \to \neg\psi)$ | $\neg$i, 2–7 |
| 9. | $\psi \to \neg\psi$ | $\neg\neg$e, 8 |