CSU44004 Formal Verification: First Order Logic Natural Deduction Solutions

By: Alexander Sepelenco

Note

This document will contain 9 years of first order logic natural deduction solutions ranging from 2022 to 2015.

I will create helper proofs for future questions:
$\neg \rightarrow$ e
$\neg\forall x.A \rightarrow \exists x.\neg A$,
$\neg\exists x.A \rightarrow \forall x.\neg A$,
$\neg\exists x.A \rightarrow \exists x.\neg A$,
$\neg\forall x.A \rightarrow \forall x.\neg A$
$\forall x.\neg A \rightarrow \neg\forall x.A$,
$\exists x.\neg A \rightarrow \neg\exists x.A$,

Prove: $\neg(p \rightarrow q) \vdash p \wedge \neg q$

1.	$\neg(p \rightarrow q)$	premise
2.	$\neg(p \wedge \neg q)$	assumption
3.	p	assumption
4.	$\neg q$	assumption
5.	$p \wedge \neg q$	$\wedge$i, 3, 4
6.	$\perp$	$\neg$e, 5, 2
7.	$\neg\neg q$	$\neg$i, 4–6
8.	q	$\neg\neg$e, 6
9.	$p \rightarrow q$	$\rightarrow$ i, 3–7
10.	$\perp$	$\neg$e, 9, 1
11.	$\neg\neg(p \wedge \neg q)$	$\neg$i, 2–10
12.	$p \wedge \neg q$	$\neg\neg$e, 12

Prove: $\neg\forall x.A \vdash \exists x.\neg A$

1.	$\neg\forall x.A$	premise
2.	$\neg\exists x.\neg A$	assumption
3.	xo	
4.	$\neg A$	assumption
5.	$\exists x.\neg A$	$\exists$i, 4
6.	$\perp$	$\neg$e, 5, 2
7.	A	PBC, 4–6
8.	$\forall x.A$	$\forall$i, 3–7
9.	$\perp$	$\neg$e, 8, 1
10.	$\exists x.\neg A$	PBC, 2–9

Prove: $\neg\exists x.A \vdash \forall x.\neg A$

1.	$\neg\exists x.A$	
2.	$\neg\forall x.\neg A$	assumption
3.	$\exists x.\neg\neg A$	$\neg\forall x.A \to \exists x.\neg A$, 2
4.	xo	
5.	$\neg\neg A$	assumption
6.	A	$\neg\neg$e, 5
7.	$\exists x.A$	$\exists$i, 6
8.	$\bot$	$\neg$e, 7, 1
9.	$\bot$	$\exists$e, 3, 4–8
10.	$\forall x.\neg A$	PBC, 2–4

Prove: $\neg\exists x.A \vdash \exists x.\neg A$

1.	$\neg\exists x.A$	premise
2.	$\neg\exists x.\neg A$	assumption
3.	xo	
4.	$\neg A$	assumption
5.	$\exists x.\neg A$	$\exists$i, 4
6.	$\bot$	$\neg$e, 5, 2
7.	A	PBC, 4–6
8.	$\forall x.A$	$\forall$i, 3–7
9.	A	$\forall$e, 8
10.	$\exists x.A$	$\exists$i, 9
11.	$\bot$	$\neg$e, 10, 1
12.	$\exists x.\neg A$	PBC, 2–11

Prove: $\neg\forall x.A \vdash \forall x.\neg A$

1.	$\neg\forall x.A$	premise
2.	$\neg\forall x.\neg A$	assumption
3.	$\exists x.\neg\neg A$	$\neg\forall x.A \to \exists x.\neg A$, 2
4.	xo	
5.	$\neg\neg A$	assumption
6.	A	$\neg\neg$e, 5
7.	$\forall x.A$	$\forall$i, 4–6
8.	$\bot$	$\neg$e, 7, 1
9.	$\forall x.\neg A$	PBC, 2–8

Prove: $\forall x.\neg A \to \neg\forall x.A$

1.	$\forall x.\neg A$	premise
2.	$\neg\neg\forall x.A$	assumption
3.	$\forall x.A$	$\neg\neg$e, 2
4.	A	$\forall$e, 3
5.	$\neg A$	$\forall$e, 1
6.	$\bot$	$\neg$e, 4, 5
7.	$\neg\forall x.A$	PBC, 2–6

Prove: $\exists x.\neg A \to \neg\exists x.A$

1.	$\exists x.\neg A$	premise
2.	$\neg\neg\exists x.A$	assumption
3.	$\exists x.A$	$\neg\neg$e, 2
4.	xo	
5.	A	assumption
6.	yo	
7.	$\neg A$	assumption
8.	A	copy 5
9.	$\perp$	$\neg$e, 8, 7
10.	$\perp$	$\exists$e, 1, 6–9
11.	$\perp$	$\exists$e, 3, 4–10
12.	$\neg\exists x.A$	PBC, 2–6

2022 Q2

(c) Prove: $\neg\exists x.\forall y.S(x,y) \vdash \forall x.\exists y.\neg S(x,y)$

1.	$\neg\exists x.\forall y.S(x,y)$	premise
2.	$\exists x.\neg\forall y.S(x,y)$	$\neg\exists x.A \to \exists x.\neg A$, 1
3.	xo	
4.	$\neg\forall y.S(xo,y)$	assumption
5.	$\forall y.\neg S(xo,y)$	$\neg\forall x.A \to \forall x.\neg A$, 4
6.	yo	
7.	$\neg S(xo,yo)$	$\forall$e, 5
8.	$\exists y.\neg S(xo,y)$	$\exists$i, 7
9.	$\forall x.\exists y.\neg S(x,y)$	$\forall$i, 6–8
10.	$\forall x.\exists y.\neg S(x,y)$	$\exists$e, 2, 3–9

2021 Q2

(c) This question has been done, view blackboard for solution

2020 Q2

(c) Prove: $\forall x.\exists y.R(x,y) \vdash \neg\forall y.\forall z.(R(a,y) \rightarrow \neg R(y,z))$

1.	$\forall x.\exists y.R(x,y)$	premise
2.	$\neg\neg\forall y.\forall z(R(a,y) \rightarrow \neg R(y,z))$	assumption
3.	$\forall y.\forall z.(R(a,y) \rightarrow \neg R(y,z))$	$\neg\neg$e, 2
4.	xo	
5.	$\exists y.R(xo,y)$	$\forall$e, 1
6.	$\forall z.(R(a,xo) \rightarrow \neg R(xo,z))$	$\forall$e, 3
7.	yo	
8.	$R(xo,yo)$	assumption
9.	$R(a,xo) \rightarrow \neg R(xo,yo)$	$\forall$e, 6
10.	$\neg R(a,xo)$	MT, 8, 9
11.	$\exists y.\neg R(a,y)$	$\exists$i, 10
12.	$\exists y.\neg R(a,y)$	$\exists$e, 5, 7–11
13.	$\neg\exists y.R(a,y)$	$\exists x.\neg A \rightarrow \neg\exists x.A$, 12
14.	$\exists y.R(a,y)[a/xo]$	*copy*, 5
15.	$\perp$	$\neg$e, 14, 13
16.	$\forall x.\perp$	$\forall$i, 4–15
17.	$\perp$	$\neg$e, $\forall$e, 16
18.	$\neg\forall y.\forall z.(R(a,y) \rightarrow \neg R(y,z))$	PBC, 2–17

2019 Q2

(c)

(i) Prove: $\neg\forall x.D(x) \vdash \exists y.\neg D(y)$ Using proof of our rule similar to question.

1.	$\neg\forall x.D(x)$	premise
2.	$\exists x.\neg D(x)$	$\neg\forall x.A \rightarrow \exists x.\neg A$, 1
3.	yo	
4.	$\neg D(yo)$	assumption
5.	$\exists y.\neg D(y)$	$\exists$i, 4
6.	$\exists y.\neg D(y)$	$\exists$e, 2, 3–5

(ii) Prove: $\vdash \exists x.(D(x) \rightarrow \forall y.D(y))$

1.	$\neg\exists x.(D(x) \rightarrow \forall y.D(y))$	assumption
2.	$\forall x.\neg(D(x) \rightarrow \forall y.D(y))$	$\neg\exists x.A \rightarrow \forall x.\neg A$, 1
3.	xo	
4.	$\neg(D(xo) \rightarrow \forall y.D(y))$	$\forall$e, 2
5.	$D(xo) \wedge \neg\forall y.D(y)$	$\neg \rightarrow$ e, 4
6.	$D(xo)$	$\wedge \mathrm{e}_1$, 5
7.	$\exists y.D(y)$	$\exists$i, 6
8.	$\neg\forall y.D(y)$	$\wedge \mathrm{e}_2$, 5
9.	$\exists y.\neg D(y)$	$\neg\forall x.A \rightarrow \exists x.\neg A$, 8
10.	$y0$	
11.	$\neg D(y0)$	assumption
12.	$D(y0)$	copy 6
13.	$\perp$	$\neg$e, 11, 12
14.	$\perp$	$\exists$e, 9, 10–13
15.	$\forall x.\perp$	$\forall$i, 3–14
16.	$\perp$	$\forall$e, 15
17.	$\exists x.(D(x) \rightarrow \forall y.D(y))$	PBC, 1–12