CSU44004 Formal Verification: Natural Deduction Solutions

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Note

This document will contain 9 years of propositional natural deduction solutions ranging from 2022 to 2015.

2022 Q2

(a) Prove: $\vdash (A \to B) \lor (B \to A)$

1.	$\neg((A \to B) \lor (B \to A))$	assumption
2.	$A \to B$	assumption
3.	$(A \to B) \lor (B \to A)$	$\lor i_1$, 2
4.	$\bot$	$\neg$e, 3, 1
5.	$\neg(A \to B)$	$\neg$i, 2–4
6.	B	assumption
7.	A	assumption
8.	B	copy 6
9.	$A \to B$	$\to$ i, 7–8
10.	$\bot$	$\neg$e, 9, 5
11.	$\neg B$	$\neg$i, 6–10
12.	$\neg A$	assumption
13.	$\neg B$	copy 11
14.	$\neg A \to \neg B$	$\to$ i, 12–13
15.	B	assumption
16.	$\neg\neg A$	MT, 15, 14
17.	A	$\neg\neg$e, 16
18.	$B \to A$	$\to$ i, 15–17
19.	$(A \to B) \lor (B \to A)$	$\lor i_2$, 18
20.	$\bot$	$\neg$e, 19, 1
21.	$\neg\neg((A \to B) \lor (B \to A))$	$\neg$i, 1–20
22.	$(A \to B) \lor (B \to A)$	$\neg\neg$e, 21

2021 Q2

I will create a helper proof for this problem to make it easier for myself. Use this proof as a reference as I will not repeat this proof in the solutions.
I will call this rule $\neg \rightarrow$ e
Prove: $\neg(p \rightarrow q) \vdash p \land \neg q$

1.	$\neg(p \rightarrow q)$	premise
2.	$\neg(p \land \neg q)$	assumption
3.	p	assumption
4.	$\neg q$	assumption
5.	$p \land \neg q$	$\land$i, 3, 4
6.	$\bot$	$\neg$e, 5, 2
7.	$\neg\neg q$	$\neg$i, 4–6
8.	q	$\neg\neg$e, 6
9.	$p \rightarrow q$	$\rightarrow$ i, 3–7
10.	$\bot$	$\neg$e, 9, 1
11.	$\neg\neg(p \land \neg q)$	$\neg$i, 2–10
12.	$p \land \neg q$	$\neg\neg$e, 12

(a) Prove: $\neg q, t \rightarrow q, \neg r \rightarrow \neg s, p \rightarrow u, \neg t \rightarrow \neg r, u \rightarrow s \vdash p \rightarrow x$

1.	$\neg q$	premise
2.	$t \rightarrow q$	premise
3.	$\neg r \rightarrow \neg s$	premise
4.	$p \rightarrow u$	premise
5.	$\neg t \rightarrow \neg r$	premise
6.	$u \rightarrow s$	premise
7.	$\neg(p \rightarrow x)$	assumption
8.	$p \land \neg x$	$\neg \rightarrow$ e, 7
9.	$\neg t$	MT, 1, 2
10.	$\neg r$	$\rightarrow$ e, 9, 5
11.	$\neg s$	$\rightarrow$ e, 10, 3
12.	$\neg u$	MT, 11, 6
13.	$\neg p$	MT, 12, 4
14.	p	$\land e_1$, 8
15.	$\bot$	$\neg$e, 14, 13
16.	$\neg\neg(p \rightarrow x)$	$\neg$i, 7–15
17.	$p \rightarrow x$	$\neg\neg$e, 16

2020 Q2

(a) This solution is ever so slightly different to 2021 Q2 (a), so I will do it slightly differently for practice.
Prove: $t \to q, \neg r \to \neg s, p \to u, \neg t \to \neg r, u \to s \vdash p \to q$

1.	$t \to q$	premise
2.	$\neg r \to \neg s$	premise
3.	$p \to u$	premise
4.	$\neg t \to \neg r$	premise
5.	$u \to s$	premise
6.	$\neg(p \to q)$	assumption
7.	$p \wedge \neg q$	$\neg \to$ e, 6
8.	$\neg q$	$\wedge$e$_2$, 7
9.	p	$\wedge$e$_1$, 7
10.	$\neg t$	MT, 8, 1
11.	$\neg r$	$\to$ e, 10, 4
12.	$\neg s$	$\to$ e, 11, 2
13.	$\neg u$	MT, 12, 5
14.	$\neg p$	MT, 13, 3
15.	$\perp$	$\neg$e, 14, 9
16.	$\neg\neg(p \to q)$	$\neg$i, 6–16
17.	$p \to q$	$\neg\neg$e, 16

2019 Q3

(a) (i) Prove: $\neg p \to r \vdash \neg r \to p$

1.	$\neg p \to r$	premise
2.	$\neg(\neg r \to p)$	assumption
3.	$\neg r \wedge \neg p$	$\neg \to$ e, 2
4.	$\neg p$	$\wedge$e$_2$, 3
5.	r	$\to$ e, 4, 1
6.	$\neg r$	$\wedge$e$_1$, 3
7.	$\perp$	$\neg$e, 5, 6

(a) (ii) Prove: $\neg(p \to r) \to \neg q \vdash q \to p \to r$

1.	$\neg(p \to r) \to \neg q$	premise
2.	$\neg(q \to p \to r)$	assumption
3.	$q \wedge \neg(p \to r)$	$\neg \to$ e, 2
4.	$\neg(p \to r)$	$\wedge$e$_2$, 3
5.	$\neg q$	$\to$ e, 4, 1
6.	q	$\wedge$e$_1$, 3
7.	$\perp$	$\neg$e, 6, 5
8.	$\neg\neg(q \to p \to r)$	$\neg$i, 2–7
9.	$q \to p \to r$	$\neg\neg$e, 8

2019 Q1

(c) (i) Prove: $p \wedge q \rightarrow \neg r \vdash r \rightarrow p \rightarrow \neg q$

1.	$p \wedge q \rightarrow \neg r$	premise
2.	$\neg(r \rightarrow p \rightarrow \neg q)$	assumption
3.	$r \wedge \neg(p \rightarrow \neg q)$	$\neg \rightarrow$ e, 2
4.	r	$\wedge e_1$, 3
5.	$\neg(p \rightarrow \neg q)$	$\wedge e_2$, 3
6.	$p \wedge \neg\neg q$	$\neg \rightarrow$ e, 5
7.	p	$\wedge e_1$, 6
8.	$\neg\neg q$	$\wedge e_2$, 6
9.	q	$\neg\neg$e, 8
10.	$p \wedge q$	$\wedge$i, 7, 8
11.	$\neg r$	$\rightarrow$ e, 10, 1
12.	$\perp$	$\neg$e, 4, 11
13.	$\neg\neg(r \rightarrow p \rightarrow \neg q)$	$\neg$i, 2–12
14.	$r \rightarrow p \rightarrow \neg q$	$\neg\neg$e, 13

(c) (ii) Sorry, this question is our assignment. I cannot give the answer to this one.

2018 Q1

(c) (i) Prove: $\vdash \neg((p \rightarrow \neg p) \wedge (\neg p \rightarrow p))$

1.	$\neg(\neg((p \rightarrow \neg p) \wedge (\neg p \rightarrow p)))$	assumption
2.	$(p \rightarrow \neg p) \wedge (\neg p \rightarrow p)$	assumption
3.	$p \rightarrow \neg p$	$\wedge e_1$, 2
4.	$\neg p \rightarrow p$	$\wedge e_2$, 2
5.	p	assumption
6.	$\neg p$	$\rightarrow$ e, 5, 3
7.	$\perp$	$\neg$e, 5, 6
8.	$\neg p$	$\neg$i, 5–7
9.	p	$\rightarrow$ e, 8, 4
10.	$\perp$	$\neg$e, 9, 8
11.	$\neg((p \rightarrow \neg p) \wedge (\neg p \rightarrow p))$	$\neg$i, 2–11
12.	$\perp$	$\neg$e, 11, 1
13.	$\neg\neg(\neg((p \rightarrow \neg p) \wedge (\neg p \rightarrow p)))$	$\neg$i, 1–12
14.	$\neg((p \rightarrow \neg p) \wedge (\neg p \rightarrow p))$	$\neg\neg$e, 13

(c) (ii) Prove: $p \to (q \vee r) \vdash (p \to q) \vee (p \to r)$

1.	$p \to (q \vee r)$	assumption
2.	$\neg((p \to q) \vee (p \to r))$	assumption
3.	$p \to q$	assumption
4.	$(p \to q) \vee (p \to r)$	assumption
5.	$\bot$	$\neg$e, 4, 5
6.	$\neg(p \to q)$	$\neg$i, 3–5
7.	$p \wedge \neg q$	$\neg \to$ e, 6
8.	p	$\wedge e_1$, 7
9.	$(q \vee r)$	$\to$ e, 8, 1
10.	$\neg q$	$\wedge e_2$, 7
11.	$p \to r$	assumption
12.	$(p \to q) \vee (p \to r)$	$\vee i_2$, 11
13.	$\bot$	$\neg$e, 12, 2
14.	$\neg(p \to r)$	$\neg$i, 11-13
15.	$p \wedge \neg r$	$\neg \to$ e, 14
16.	$\neg r$	$\wedge e_2$, 15
17.	q	assumption
18.	$\neg q$	copy 10
19.	$\bot$	$\neg$e, 17, 18
20.	r	assumption
21.	$\neg r$	copy 16
22.	$\bot$	$\neg$e, 16
23.	$\bot$	$\vee$e, 9, 17–9, 20-22
24.	$\neg\neg((p \to q) \vee (p \to r))$	$\neg$i, 2–23
25.	$(p \to q) \vee (p \to r)$	$\neg\neg$e, 24

2017 Q1

(c) (i) Prove: $p \to r \vdash \neg p \vee r$

1.	$p \to r$	premise
2.	$\neg(\neg p \vee r)$	assumption
3.	p	assumption
4.	r	$\to$ e, 3, 1
5.	$\neg p \vee r$	$\vee i_2$, 4
6.	$\bot$	$\neg$e, 5, 2
7.	$\neg p$	$\neg$i, 3–6
8.	$\neg p \vee r$	$\vee i_1$, 7
9.	$\bot$	$\neg$e, 8, 2
10.	$\neg\neg(\neg p \vee r)$	$\neg$i, 2–9
11.	$\neg p \vee r$	$\neg\neg$e, 10

(c) (ii) Prove: $(\neg p \to r), (r \to p), (p \to s) \vdash p \land s$

1.	$\neg p \to r$	premise
2.	$r \to p$	premise
3.	$p \to s$	premise
4.	$\neg(p \land s)$	assumption
5.	p	assumption
6.	s	$\to$ e, 5, 3
7.	$p \land s$	$\land$i, 5, 6
8.	$\perp$	$\neg$e, 7, 4
9.	$\neg p$	$\neg$i, 5–8
10.	r	$\to$ e, 9, 1
11.	p	$\to$ e, 10, 2
12.	$\perp$	$\neg$e, 11, 9
13.	$\neg\neg(p \land s)$	$\neg$i, 4–12
14.	$p \land s$	$\neg\neg$e, 13

2016 Q1

I will prove Morgan's Theorem using natural deduction to use as a helper rule for the problem (ii).
I will call this rule DM_2
Prove: $\neg(p \lor q) \vdash \neg p \land \neg q$

1.	$\neg(p \lor q)$	premise
2.	$\neg(\neg p \land \neg q)$	assumption
3.	p	assumption
4.	$p \lor q$	$\lor$i$_1$, 3
5.	$\perp$	$\neg$e, 4, 1
6.	$\neg p$	$\neg$i, 3–5
7.	q	assumption
8.	$p \lor q$	$\lor$i$_2$, 7
9.	$\perp$	$\neg$e, 8, 1
10.	$\neg q$	$\neg$i, 7–9
11.	$\neg p \land \neg q$	$\land$i, 6, 10
12.	$\perp$	$\neg$e, 11, 2
13.	$\neg\neg(\neg p \land \neg q)$	$\neg$i, 2–12
14.	$\neg p \land \neg q$	$\neg\neg$e, 13

(d) (i) Prove: $p \vee (q \wedge r) \vdash (p \vee q) \wedge (p \vee r)$

1.	$p \vee (q \wedge r)$	premise
2.	$\neg((p \vee q) \wedge (p \vee r))$	assumption
3.	$p \vee q$	assumption
4.	$p \vee r$	assumption
5.	$(p \vee q) \wedge (p \vee r)$	$\wedge$i, 3, 4
6.	$\perp$	$\neg$e, 5, 2
7.	$\neg(p \vee r)$	$\neg$i, 4–6
8.	$\neg p \wedge \neg r$	DM_2, 7
9.	$\neg p$	$\wedge e_1$, 8
10.	$\neg r$	$\wedge e_2$, 8
11.	p	assumption
12.	$\neg p$	copy 9
13.	$\perp$	$\neg$e, 11, 12
14.	$(q \vee r)$	assumption
15.	r	$\wedge e_2$, 14
16.	$\perp$	$\neg$e, 15, 10
17.	$\perp$	$\vee$e, 1, 11–13, 14–16
18.	$\neg(p \vee q)$	$\neg$i, 2–17
19.	$\neg p \wedge \neg q$	DM_2, 18
20.	$\neg p$	$\wedge e_1$, 19
21.	$\neg q$	$\wedge e_2$, 19
22.	p	assumption
23.	$\perp$	$\neg$e, 22, 20
24.	$(q \wedge r)$	assumption
25.	q	$\wedge e_1$, 24
26.	$\perp$	$\neg$e, 24, 21
27.	$\perp$	$\vee$e, 1, 22–23, 24–26
28.	$\neg\neg((p \vee q) \wedge (p \vee r))$	$\neg$i, 2–27
29.	$(p \vee q) \wedge (p \vee r)$	$\neg\neg$e, 28

(d) (ii) Prove: $\phi \wedge \neg\psi \to \neg\phi \vdash \phi \to \psi$

1.	$\phi \wedge \neg\psi \to \neg\phi$	premise
2.	$\neg(\phi \to \psi)$	assumption
3.	$\phi \wedge \neg\psi$	$\neg \to$ e, 2
4.	$\neg\phi$	$\to$ e, 3, 1
5.	ϕ	$\wedge e_1$, 3
6.	$\perp$	$\neg$e, 4, 5
7.	$\neg\neg(\phi \to \psi)$	$\neg$i, 2–6
8.	$\phi \to \psi$	$\neg\neg$e, 7

2015 Q1

(e) (i) Prove: $(\phi_1 \to \psi_1), (\neg\psi_2 \to \neg\phi_2) \vdash (\phi_1 \wedge \phi_2) \to (\phi_1 \vee \phi_2)$

1.	$(\phi_1 \to \psi_1)$	premise
2.	$(\neg\psi_2 \to \neg\phi_2)$	premise
3.	$\neg((\phi_1 \wedge \phi_2) \to (\psi_1 \vee \psi_2))$	assumption
4.	$\phi_1 \wedge \phi_2$	assumption
5.	ϕ_1	$\wedge e_1$, 4
6.	ψ_1	$\to$ e, 5, 1
7.	$\psi_1 \vee \psi_2$	$\vee$i, 6
8.	$(\phi_1 \wedge \phi_2) \to \psi_1 \vee \psi_2$	$\to$ i, 4–7
9.	$\perp$	$\neg$e, 8, 3
10.	$\neg\neg((\phi_1 \wedge \phi_2) \to (\psi_1 \vee \psi_2))$	$\neg$i, 3–9
11.	$(\phi_1 \wedge \phi_2) \to (\psi_1 \vee \psi_2)$	$\neg\neg$e, 10

(e) (ii) Prove: $\phi \to \neg\psi \vdash \psi \to \neg\phi$

1.	$\phi \to \neg\phi$	premise
2.	$\neg(\psi \to \neg\phi)$	assumption
3.	$\psi \wedge \neg\neg\phi$	$\neg \to$ e, 2
4.	$\neg\neg\phi$	$\wedge e_2$, 3
5.	ϕ	$\neg\neg$e, 4
6.	$\neg\phi$	$\to$ e, 5, 1
7.	$\perp$	$\neg$e, 5, 6
8.	$\neg\neg(\psi \to \neg\psi)$	$\neg$i, 2–7
9.	$\psi \to \neg\psi$	$\neg\neg$e, 8