CSU44004 Formal Verification: First Order Logic Natural Deduction Solutions

By: Alexander Sepelenco

Note

This document will contain 9 years of first order logic natural deduction solutions ranging from 2022 to 2015.

I will create helper proofs for future questions:

$\neg \rightarrow$ e
$\neg\forall x.A \rightarrow \exists x.\neg A$,
$\neg\exists x.A \rightarrow \forall x.\neg A$,

Prove: $\neg(p \rightarrow q) \vdash p \wedge \neg q$

1.	$\neg(p \rightarrow q)$	premise
2.	$\neg(p \wedge \neg q)$	assumption
3.	p	assumption
4.	$\neg q$	assumption
5.	$p \wedge \neg q$	$\wedge$i, 3, 4
6.	$\bot$	$\neg$e, 5, 2
7.	$\neg\neg q$	$\neg$i, 4–6
8.	q	$\neg\neg$e, 6
9.	$p \rightarrow q$	$\rightarrow$ i, 3–7
10.	$\bot$	$\neg$e, 9, 1
11.	$\neg\neg(p \wedge \neg q)$	$\neg$i, 2–10
12.	$p \wedge \neg q$	$\neg\neg$e, 12

Prove: $\neg\forall x.A \vdash \exists x.\neg A$

1.	$\neg\forall x.A$	premise
2.	$\neg\exists x.\neg A$	assumption
3.	xo	
4.	$\neg A$	assumption
5.	$\exists x.\neg A$	$\exists$i, 4
6.	$\bot$	$\neg$e, 5, 2
7.	A	PBC, 4–6
8.	$\forall x.A$	$\forall$i, 3–7
9.	$\bot$	$\neg$e, 8, 1
10.	$\exists x.\neg A$	PBC, 2–9

Prove: $\neg\exists x.A \vdash \forall x.\neg A$

1.	$\neg\exists x.A$	
2.	$\neg\forall x.\neg A$	assumption
3.	xo	
4.	A	assumption
5.	$\exists x.A$	$\exists$i, 4
6.	$\perp$	$\neg$e, 5,1
7.	$\neg A$	$\neg$i, 4–6
8.	$\forall x.\neg A$	$\forall$i, 3–7
9.	$\perp$	$\neg$e, 8, 2
10.	$\forall x.\neg A$	PBC, 2–9

2022 Q2

(c) Prove: $\neg\exists x.\forall y.S(x,y) \vdash \forall x.\exists y.\neg S(x,y)$

1.	$\neg\exists x.\forall y.S(x,y)$	premise
2.	$\forall x.\neg\forall y.S(x,y)$	$\neg\exists x.A \rightarrow \forall x.\neg A$, 1
3.	xo	
4.	$\neg\forall y.S(xo,y)$	$\forall$e, 2
5.	$\exists y.\neg S(xo,y)$	$\neg\forall x.A \rightarrow \exists x.\neg A$, 4
6.	$\forall x.\exists y.\neg S(x,y)$	$\forall$i, 3–5

Here's a slightly outdated inlined version just so you can laugh at how horrible of a proof this is
Prove: $\neg\exists x.\forall y.S(x,y) \vdash \forall x.\exists y.\neg S(x,y)$

1.	$\neg\exists x.\forall y.S(x,y)$	premise
2.	$\neg\forall x.\neg\forall y.S(x,y)$	assumption
3.	$\neg\exists x.\neg\neg\forall y.S(x,y)$	assumption
4.	xo	
5.	$\neg\neg\forall y.S(xo,y)$	assumption
6.	$\exists x.\neg\neg\forall y.S(x,y)$	$\exists$i, 5
7.	$\perp$	$\neg$e, 6, 3
8.	$\neg\forall y.S(xo,y)$	PBC, 5–7
9.	$\exists x.\neg\forall y.S(x,y)$	$\exists$i, 8
10.	$\perp$	$\neg$e, 9, 1
11.	$\forall x.\perp$	$\forall$i, 4–10
12.	$\perp$	$\forall$e, 11
13.	$\exists x.\neg\neg\forall y.S(x,y)$	PBC, 3–12
14.	xo	
15.	$\neg\neg\forall y.S(xo,y)$	assumption
16.	$\forall y.S(xo,y)$	$\neg\neg$e, 13
17.	$\exists x.\forall y.S(x,y)$	$\exists$i, 16
18.	$\perp$	$\neg$e, 17, 1
19.	$\perp$	$\exists$e, 13, 14–18
20.	$\forall x.\neg\forall y.S(x,y)$	PBC, 2–19
21.	xo	
22.	$\neg\forall y.S(xo,y)$	$\forall$e, 20
23.	$\neg\exists y.\neg S(xo,y)$	assumption
24.	zo	
25.	$\neg S(xo,zo)$	assumption
26.	$\exists y.\neg S(xo,y)$	$\exists$i, 25
27.	$\perp$	$\neg$e, 26, 23
28.	$S(xo,zo)$	PBC, 25–27
29.	$\forall y.S(xo,y)$	$\forall$i, 24–28
30.	$\perp$	$\neg$e, 29, 22
31.	$\exists y.\neg S(xo,y)$	PBC, 23–30
32.	$\forall x.\exists y.\neg S(x,y)$	$\forall$i, 21–31

2021 Q2

(c) This question has been done, view blackboard for solution

2020 Q2

(c) Prove: $\forall x.\exists y.R(x,y) \vdash \neg\forall y.\forall z.(R(a,y) \to \neg R(y,z))$

1.	$\forall x.\exists y.R(x,y)$	premise
2.	$\neg\neg\forall y.\forall z(R(a,y) \to \neg R(y,z))$	assumption
3.	$\forall y.\forall z.(R(a,y) \to \neg R(y,z))$	$\neg\neg$e, 2
4.	xo	
5.	$\exists y.R(xo,y)$	$\forall$e, 1
6.	$\forall z.(R(a,xo) \to \neg R(xo,z))$	$\forall$e, 3
7.	yo	
8.	$R(xo,yo)$	assumption
9.	$R(a,xo) \to \neg R(xo,yo)$	$\forall$e, 6
10.	$\neg R(a,xo)$	MT, 8, 9
11.	$\exists y.\neg R(a,y)$	$\exists$i, 10
12.	$\exists y.\neg R(a,y)$	$\exists$e, 5, 7–11
13.	zo	
14.	$R(xo,zo)$	assumption
15.	ao	
16.	$\neg R(a,ao)$	assumption
17.	$R(xo,zo)$	copy, 14
18.	$R(a,ao)[ao/zo][a/xo]$	$=$ e, 17, $=$ e, 17
19.	$\bot$	$\neg$e, 18, 16
20.	$\bot$	$\exists$e, 12, 15–19
21.	$\bot$	$\exists$e, 5, 13–20
22.	$\forall x.\bot$	$\forall$i, 4–21
23.	$\bot$	$\neg$e, $\forall$e, 22
24.	$\neg\forall y.\forall z.(R(a,y) \to \neg R(y,z))$	PBC, 2–23

2019 Q2

(c)

(i) Prove: $\neg\forall x.D(x) \vdash \exists y.\neg D(y)$

1.	$\neg\forall x.D(x)$	premise
2.	$\neg\exists y.\neg D(y)$	assumption
3.	yo	
4.	$\neg D(yo)$	assumption
5.	$\exists y.\neg D(y)$	$\exists$i, 4
6.	$\bot$	$\neg$e, 5, 2
7.	$D(yo)$	PBC, 4–6
8.	$\forall x.D(x)$	$\forall$i, 3–7
9.	$\bot$	$\neg$e, 8, 1
10.	$\exists y.\neg D(y)$	PBC, 2–9

(ii) Prove: $\vdash \exists x.(D(x) \to \forall y.D(y))$

1.	$\neg\exists x.(D(x) \to \forall y.D(y))$	assumption
2.	xo	
3.	$D(xo) \to \forall y.D(y)$	assumption
4.	$\exists x.D(x) \to \forall y.D(y)$	$\exists$i, 3, 1
5.	$\perp$	$\neg$e, 4, 1
6.	$\neg(D(xo) \to \forall y.D(y))$	$\neg$i, 3–5
7.	$\forall z.D(z)$	assumption
8.	$D(xo)$	assumption
9.	$\forall y.D(y)$	$\forall$i, 8
10.	$D(xo) \to \forall y.D(y)$	$\to$ i, 8–9
11.	$\perp$	$\neg$e, 10, 6
12.	$\neg\forall z.D(z)$	$\neg$e, 7–11
13.	$\exists y.\neg D(y)$	$\neg\forall x.D(x) \to \exists y.\neg D(y)$, 12
14.	$y1$	
15.	$\neg D(y1)$	assumption
16.	$\neg D(y1) \vee \forall y.D(y)$	$\vee$i$_1$, 15
17.	$D(y1) \to \forall y.D(y)$	MI, 16
18.	$\exists x.(D(x) \to \forall y.D(y))$	$\exists$i, 17
19.	$\perp$	$\neg$e, 18, 1
20.	$\perp$	$\exists$e, 13, 14–19
21.	$\forall x.\perp$	$\forall$i, 2–20
22.	$\perp$	$\forall$e, 21
23.	$\exists x.(D(x) \to \forall y.D(y))$	PBC, 1–22

2018 January Q2

(c)

(i) Prove: $\exists x.S(x) \vee \exists x.T(x) \vdash \exists x.(S(x) \vee T(x))$

1.	$\exists x.S(x) \vee \exists x.T(x)$	premise
2.	$\exists x.S(x)$	assumption
3.	xo	
4.	$S(xo)$	assumption
5.	$S(xo) \vee T(xo)$	$\vee\mathrm{i}_1$, 4
6.	$\exists x.S(x) \vee T(x)$	$\exists$i, 5
7.	$\exists x.S(x) \vee T(x)$	$\exists$e, 2, 3–6
8.	$\exists x.T(x)$	assumption
9.	yo	
10.	$T(yo)$	assumption
11.	$S(yo) \vee T(yo)$	$\vee\mathrm{i}_2$, 10
12.	$\exists x.S(x) \vee T(x)$	$\exists$i, 11
13.	$\exists x.S(x) \vee T(x)$	$\exists$e, 8, 9–12
14.	$\exists x.(S(x) \vee T(x))$	$\vee$e, 1, 2–7, 8–13

(ii) Prove: $\exists x.\exists y.(P(x,y) \vee P(y,x)), \neg\exists z.P(z,z) \vdash \neg\forall x.\forall y.(x = y)$

1.	$\exists x.\exists y.(P(x,y) \vee P(y,x))$	premise
2.	$\neg\exists z.P(z,z)$	premise
3.	$\neg\neg\forall x.\forall y.(x = y)$	assumption
4.	$\forall x.\forall y.(x = y)$	$\neg\neg$e, 3
5.	xo	
6.	$\exists y.(P(xo,y) \vee P(y,xo))$	assumption
7.	$\forall y.(xo = y)$	$\forall$e, 4
8.	yo	
9.	$P(xo,yo) \vee P(yo,xo)$	assumption
10.	$(xo = yo)$	$\forall$e, 7
11.	$P(xo,xo) \vee P(xo,xo)$	$=$ e, 10, 9
12.	$P(xo,xo)$	assumption
13.	$\exists z.P(z,z)$	$\exists$i, 11
14.	$\perp$	$\neg$e, 12, 2
15.	$P(xo,xo)$	assumption
16.	$\exists z.P(z,z)$	$\exists$i, 13
17.	$\perp$	$\neg$e, 15, 2
18.	$\perp$	$\vee$e, 11, 12–14, 15–17
19.	$\perp$	$\exists$e, 6, 8–18
20.	$\perp$	$\exists$e, 1, 5–19
21.	$\neg\forall x.\forall y.(x = y)$	PBC, 3–20

2018 January Q2

(c)

(i) Prove: $\forall x.\forall y.(R(x,y) \land \neg(x = y) \to \neg R(y,x)) \vdash \forall x.\forall y.(R(x,y) \land R(y,x) \to x = y)$

1.	$\forall x.\forall y.(R(x,y) \land \neg(x = y) \to \neg R(y,x))$	premise
2.	xo	
3.	$\forall y.(R(xo,y) \land \neg(xo = y) \to \neg R(y,xo))$	$\forall$e, 1
4.	yo	
5.	$(R(xo,yo) \land \neg(xo = yo)) \to \neg R(yo,xo)$	$\forall$e, 3
6.	$\neg(R(xo,yo) \land \neg(xo = yo)) \lor \neg R(yo,xo)$	MI, 5
7.	$R(xo,yo) \land R(yo,xo)$	assumption
8.	$R(xo,yo)$	$\land$e$_1$, 7
9.	$\neg(R(xo,yo) \land \neg(xo = yo))$	assumption
10.	$\neg R(xo,yo) \lor \neg\neg(xo = yo)$	$TAUT2$, 9
11.	$\neg R(xo,yo)$	assumption
12.	$\bot$	$\neg$e, 11, 8
13.	$xo = yo$	$\bot$e, 12
14.	$\neg\neg(xo = yo)$	assumption
15.	$xo = yo$	$\neg\neg$e, 14
16.	$xo = yo$	$\lor$e, 10, 11–13, 14–15
17.	$\neg R(yo,xo)$	assumption
18.	$R(yo,xo)$	$\land$e$_2$, 7
19.	$\bot$	$\neg$e, 18, 17
20.	$xo = yo$	$\bot$e, 19
21.	$xo = yo$	$\lor$e, 6, 9–16, 17–20
22.	$R(xo,yo) \land R(yo,xo) \to xo = yo$	$\to$ i, 7–21
23.	$\forall y.(R(xo,y) \land R(y,xo) \to xo = y)$	$\forall$i, 4–22
24.	$\forall x.\forall y.(R(x,y) \land R(y,x) \to x = y)$	$\forall$i, 2–23

(ii) Prove: $\exists x.P(x) \to \exists y.(Q(y) \wedge R(y)), \exists z.(R(z) \vee S(z)) \to \forall w.T(w) \vdash \forall v.(P(v) \to T(v))$

1.	$\exists x.P(x) \to \exists y.(Q(y) \wedge R(y))$	premise
2.	$\exists z.(R(z) \vee S(z)) \to \forall w.T(w)$	premise
3.	$\neg\forall v.(P(v) \to T(v))$	assumption
4.	$\exists v.\neg(P(v) \to T(v))$	$\neg\forall x.A \to \exists x.\neg A$, 3
5.	vo	
6.	$\neg(P(vo) \to T(vo))$	assumption
7.	$P(vo) \wedge \neg T(vo)$	$\neg \to$ e, 6
8.	$P(vo)$	$\wedge e_1$, 7
9.	$\neg T(vo)$	$\wedge e_2$, 7
10.	$\exists x.P(x)$	$\exists$i, 8
11.	$\exists y.(Q(y) \wedge R(y))$	$\to$ e, 10, 1
12.	yo	
13.	$Q(yo) \wedge R(yo)$	assumption
14.	$R(yo)$	$\wedge e_2$, 13
15.	$R(yo) \vee S(yo)$	$\vee i_1$, 14
16.	$\exists z.(R(z) \vee S(z))$	$\exists$i, 15
17.	$\forall w.T(w)$	$\to$ e, 16, 2
18.	wo	
19.	$T(wo)$	$\forall$e, 17
20.	$\perp$	$\neg$e, 19, 9
21.	$\forall x.\perp$	$\forall$i, 18–20
22.	$\perp$	$\forall$e, 21
23.	$\perp$	$\exists$e, 11, 12–22
24.	$\perp$	$\exists$e, 4, 5–23
25.	$\forall v.(P(v) \to T(v))$	PBC, 3–24

2017

(i) Prove: $\exists x.(P(x) \wedge \forall y.Q(x,y)) \vdash \exists z.Q(z,z)$

1.	$\exists x.(P(x) \wedge \forall y.Q(x,y))$	
2.	$\neg\exists z.Q(z,z)$	assumption
3.	$\forall z.\neg Q(z,z)$	$\neg\exists z.A \to \forall z.\neg A$, 2
4.	xo	
5.	$P(xo) \wedge \forall y.Q(xo,y)[xo/z]$	assumption
6.	$\forall y.Q(xo,y)$	$\wedge e_2$, 5
7.	$Q(xo,xo)[xo/y]$	$\forall e$, 6
8.	$\neg Q(xo,xo)[xo/z]$	$\forall e$, 3
9.	$\perp$	$\neg e$, 7, 8
10.	$\perp$	$\exists e$, 1, 4–9
11.	$\exists z.Q(z,z)$	PBC, 2–10

(i) Prove: $\forall x.\forall y.(R(x,y) \to (x=y)) \vdash \forall z.\neg\exists w.(R(z,w) \wedge \neg(z=w))$

1.	$\forall x.\forall y.(R(x,y) \to (x=y))$	premise
2.	$\neg\forall z.\neg\exists w.(R(z,w) \wedge \neg(z=w))$	assumption
3.	$\exists z.\neg\neg\exists w.(R(z,w) \wedge \neg(z=w))$	$\neg\forall x.A \to \exists x.\neg A$, 2
4.	zo	
5.	$\neg\neg\exists w.(R(zo,w) \wedge \neg(zo=w))$	assumption
6.	$\exists w.(R(zo,w) \wedge \neg(zo=w))$	$\neg\neg e$, 5
7.	$\forall y.(R(zo,y) \to (zo=y))$	$\forall e$, 1
8.	wo	
9.	$R(zo,wo) \wedge \neg(zo=wo)$	assumption
10.	$R(zo,wo) \to (zo=wo)$	$\forall e$, 7
11.	$\neg R(zo,wo) \vee (zo=wo)$	MI, 10
12.	$\neg R(zo,wo)$	assumption
13.	$R(zo,wo)$	$\wedge e_1$, 9
14.	$\perp$	$\neg e$, 13, 12
15.	$(zo=yo)$	assumption
16.	$\neg(zo=wo)$	$\wedge e_2$, 9
17.	$\perp$	$\neg e$, 15, 16
18.	$\perp$	$\vee e$, 11, 12–14, 15–17
19.	$\perp$	$\exists e$, 6, 8–18
20.	$\perp$	$\exists e$, 3, 4–19
21.	$\forall z.\neg\exists w.(R(z,w) \wedge \neg(z=w))$	PBC, 2–20

2016

(c)

(i) Prove: $\forall x.\forall y.(P(y) \to Q(x)) \vdash \exists y.P(y) \to \forall x.Q(x)$

1.	$\forall x.\forall y.(P(y) \to Q(x))$	premise
2.	$\neg(\exists y.P(y) \to \forall x.Q(x))$	assumption
3.	$\exists y.P(y) \wedge \neg\forall x.Q(x)$	$\neg \to$ e, 2
4.	$\exists y.P(y)$	$\wedge e_1$, 3
5.	$\neg\forall x.Q(x)$	$\wedge e_2$, 3
6.	$\exists x.\neg Q(x)$	$\neg\forall x.A \to \exists x.\neg A$, 5
7.	yo	
8.	$P(yo)$	assumption
9.	xo	
10.	$\forall y.(P(y) \to Q(xo))$	$\forall$e, 1
11.	$P(yo) \to Q(xo)$	$\forall$e, 10
12.	$Q(xo)$	$\to$ e, 9, 11
13.	$\forall x.Q(x)$	$\forall$i, 8–12
14.	$\perp$	$\neg$e, 13, 5
15.	$\perp$	$\exists$e, 4, 7–14
16.	$\exists y.P(y) \to \forall x.Q(x)$	PBC, 2–15

(ii) Prove: $\exists x.\exists y.(H(x,y) \vee H(y,x)), \neg\exists z.H(z,z) \vdash \exists x.\exists y.\neg(x = y)$

1.	$\exists x.\exists y.(H(x,y) \vee H(y,x))$	premise
2.	$\neg\exists z.H(z,z)$	premise
3.	$\forall z.\neg H(z,z)$	$\neg\exists x.A \to \forall x.\neg A$, 2
4.	$\neg\exists x.\exists y.\neg(x = y)$	assumption
5.	$\forall x.\neg\exists y.\neg(x = y)$	$\neg\exists x.A \to \forall x.\neg A$, 4
6.	xo	
7.	$\exists y.(H(xo,y) \vee H(y,xo))$	assumption
8.	$\neg H(xo,xo)$	$\forall$e, 3
9.	$\neg\exists y.\neg(xo = y)$	$\forall$e, 5
10.	$\forall y.\neg\neg(xo = y)$	$\neg\exists x.A \to \forall x.\neg A$, 9
11.	yo	
12.	$H(xo,yo) \vee H(yo,xo)$	assumption
13.	$\neg\neg(xo = yo)$	$\forall$e, 10
14.	$xo = yo$	$\forall$e, 10
15.	$yo = xo$	$= sym$, 14
16.	$H(xo,xo) \vee H(xo,xo)[xo/yo]$	$=$ e, 15, 12
17.	$H(xo,xo)$	assumption
18.	$\perp$	$\neg$e, 17, 8
19.	$H(xo,xo)$	assumption
20.	$\perp$	$\neg$e, 19, 8
21.	$\perp$	$\vee$e, 16, 17–18, 19–20
22.	$\perp$	$\exists$e, 7, 11–21
23.	$\perp$	$\exists$e, 1, 6–22
24.	$\exists x.\exists y.\neg(x = y)$	PBC, 4–24

2015

(d) Prove: $\neg\forall x.\neg P(x) \dashv\vdash \exists x.P(x)$

I will prove this without using my proofs from the start ($\neg\forall x.A \to \exists x.\neg A$)

First prove: $\neg\forall x.\neg P(x) \vdash \exists x.P(x)$

1.	$\neg\forall x.\neg P(x)$	premise
2.	$\neg\exists x.P(x)$	assumption
3.	xo	
4.	$P(xo)$	assumption
5.	$\exists x.P(x)$	$\exists$i, 4
6.	$\perp$	$\neg$e, 5, 2
7.	$\neg P(xo)$	$\neg$i, 4–6
8.	$\forall x.\neg P(x)$	$\forall$i, 3–7
9.	$\perp$	$\neg$e, 8, 1
10.	$\exists x.P(x)$	PBC, 2–9

Lastly prove: $\neg\forall x.\neg P(x) \dashv \exists x.P(x)$

1.	$\exists x.P(x)$	premise
2.	$\neg\neg\forall x.\neg P(x)$	assumption
3.	$\forall x.\neg P(x)$	$\neg\neg$e, 2
4.	xo	
5.	$P(xo)$	assumption
6.	$\neg P(xo)$	$\forall$e, 3
7.	$\perp$	$\neg$e, 5, 6
8.	$\perp$	$\exists$e, 1, 4–7
9.	$\neg\forall x.\neg P(x)$	PBC, 2–8