The Polyphonic Portfolio: A Regime-Sensitive Expected Free Energy Framework for Multi-Objective Allocation

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1 Overview

The Polyphonic Portfolio is a regime-sensitive, multi-objective portfolio construction framework grounded in:

- Latent regime inference via Gaussian Hidden Markov Models (HMMs),
- Belief-conditioned predictive return modelling,
- Multi-step Monte Carlo planning,
- Polyphonic coordination of partially competing objectives.

The framework unifies statistical regime detection with a Free Energy–inspired multi-objective planning mechanism.

2 Market as a Latent Regime Process

We model market returns $\mathbf{r}_t \in \mathbb{R}^D$ as generated by a discrete latent state $z_t \in \{1, \ldots, K\}$.

2.1 State Dynamics

$$P(z_t = j \mid z_{t-1} = i) = A_{ij} \tag{1}$$

where A is a row-stochastic transition matrix with strong diagonal mass ("stickiness").

2.2 Emission Model

Conditioned on regime k:

$$\mathbf{r}_t \mid z_t = k \sim \mathcal{N}(\boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k) \tag{2}$$

In implementation, $\boldsymbol{\Sigma}_k$ is diagonal:

$$\boldsymbol{\Sigma}_k = \mathrm{diag}(\boldsymbol{\sigma}_k^2) \tag{3}$$

3 Online Belief Filtering

Let $\mathbf{q}_{t-1}$ be the posterior belief over regimes at time $t-1$.

Prediction step:

$$\tilde{\mathbf{q}}_t = \mathbf{q}_{t-1} A \tag{4}$$

Update step:

$$q_t(k) \propto \tilde{q}_t(k)\, p(\mathbf{r}_t \mid z_t = k) \tag{5}$$

Normalised via softmax:

$$\mathbf{q}_t = \operatorname{softmax}\left(\log \tilde{\mathbf{q}}_t + \log p(\mathbf{r}_t \mid z_t)\right) \tag{6}$$

4 Belief-Conditioned Predictive Moments

Using moment-matching over regime mixture:

4.1 Predictive Mean

$$\boldsymbol{\mu}_{\text{pred}} = \sum_{k=1}^{K} q_t(k) \boldsymbol{\mu}_k \tag{7}$$

4.2 Predictive Covariance

$$\boldsymbol{\Sigma}_{\text{pred}} = \sum_{k=1}^{K} q_t(k) \left(\boldsymbol{\Sigma}_k + \boldsymbol{\mu}_k \boldsymbol{\mu}_k^\top\right) - \boldsymbol{\mu}_{\text{pred}} \boldsymbol{\mu}_{\text{pred}}^\top \tag{8}$$

This incorporates both within-regime variance and between-regime uncertainty.

5 Multi-Step Planning via Monte Carlo Rollout

At rebalance time t, candidate weights $\mathbf{w}$ are evaluated over horizon H.

5.1 Regime Path Sampling

Sample regime trajectories:

$$z_{t:t+H} \sim P(z \mid \mathbf{q}_t, A) \tag{9}$$

5.2 Return Sampling

$$\mathbf{r}_{t+h} \sim \mathcal{N}(\boldsymbol{\mu}_{z_{t+h}}, \boldsymbol{\Sigma}_{z_{t+h}}) \tag{10}$$

5.3 Portfolio Returns

$$r^p_{t+h} = \mathbf{w}^\top \mathbf{r}_{t+h} \tag{11}$$

6 Polyphonic Objectives (Voices)

Each candidate is evaluated under four partially competing voices.

6.1 Growth Voice

Maximise expected discounted log wealth:

$$J_{\text{growth}} = -\mathbb{E}\left[\sum_{h=0}^{H-1} \gamma^h \log\bigl(1 + r^p_{t+h}\bigr)\right] \tag{12}$$

6.2 Risk Voice

Penalise pathwise variance:

$$J_{\text{risk}} = \operatorname{Var}\left(\sum_{h=0}^{H-1} \gamma^h r^p_{t+h}\right) \tag{13}$$

6.3 Drawdown Voice

Let wealth evolve:

$$W_{h+1} = W_h(1 + r^p_{t+h}) \tag{14}$$

Maximum drawdown:

$$\text{MDD} = \max_h \frac{\max_{s\le h} W_s - W_h}{\max_{s\le h} W_s} \tag{15}$$

$$J_{\text{drawdown}} = \mathbb{E}[\text{MDD}] \tag{16}$$

6.4 Turnover Voice

$$J_{\text{turnover}} = \|\mathbf{w} - \mathbf{w}_{t-1}\|_1 + c\|\mathbf{w} - \mathbf{w}_{t-1}\|_1 \tag{17}$$

7 Polyphonic Negotiation

Let cost vector for candidate i:

$$\mathbf{J}_i = \begin{bmatrix} J_{\text{growth}} \\ J_{\text{risk}} \\ J_{\text{drawdown}} \\ J_{\text{turnover}} \end{bmatrix} \tag{18}$$

Robust scaling per voice:

$$\tilde{J}_{ij} = \frac{J_{ij} - \text{median}(J_{\cdot j})}{\text{MAD}(J_{\cdot j})} \tag{19}$$

Voice weights:

$$\boldsymbol{\pi}_t = \frac{\max(\boldsymbol{\pi}_t, \pi_{\min})}{\sum \max(\boldsymbol{\pi}_t, \pi_{\min})} \tag{20}$$

Negotiated score:

$$S_i = \tilde{\mathbf{J}}_i^\top \boldsymbol{\pi}_t \tag{21}$$

Chosen portfolio:

$$\mathbf{w}_t = \arg\min_i S_i \tag{22}$$

8 Adaptive Voice Dynamics

Voice weights evolve toward the relative prominence of realised costs:

$$\boldsymbol{\pi}_{t+1} = (1-\eta)\boldsymbol{\pi}_t + \eta \frac{\max(\mathbf{J}_t, \pi_{\min})}{\sum \max(\mathbf{J}_t, \pi_{\min})} \tag{23}$$

This enforces non-dominance while allowing context-sensitive emphasis shifts.

9 Rebalancing Schedule

Weights are updated only at specified frequency (monthly or quarterly):

$$\mathbf{w}_{t+1} = \begin{cases} \text{negotiated solution} & \text{if rebalance date} \\ \mathbf{w}_t & \text{otherwise} \end{cases} \tag{24}$$

10 Conceptual Interpretation

The Polyphonic Portfolio can be viewed as:

- A partially observable Markov decision process,
- A multi-objective expected free energy minimiser,
- A non-dominating coordination architecture.

Unlike scalarised mean-variance optimisation, this framework preserves structural tension between competing financial objectives and resolves them dynamically.

11 Relation to Expected Free Energy

Classical Expected Free Energy decomposes into:

$$G = \text{Risk} - \text{Information Gain} \tag{25}$$

Here, we generalise:

$$G_{\text{polyphonic}} = \sum_{v=1}^{V} \pi_v J_v \tag{26}$$

Each J_v corresponds to a domain-specific risk functional (growth loss, volatility, drawdown, turnover).

Thus the portfolio minimises a polyphonic free energy functional.

12 Summary

The Polyphonic Portfolio integrates:

1. Regime inference via Gaussian HMM,
2. Belief-weighted predictive modelling,
3. Multi-step Monte Carlo planning,
4. Multi-objective negotiation,
5. Adaptive coordination dynamics.

It is a structured, regime-sensitive, dynamically coordinated alternative to classical static allocation methods.

13 POMDP Formalisation

The Polyphonic Portfolio can be formally cast as a Partially Observable Markov Decision Process (POMDP):

13.1 Latent States

Hidden market regime:

$$z_t \in \{1, \ldots, K\} \tag{27}$$

State dynamics:

$$P(z_t \mid z_{t-1}) = A \tag{28}$$

13.2 Observations

Observed asset returns:

$$\mathbf{o}_t \equiv \mathbf{r}_t \tag{29}$$

Emission likelihood:

$$P(\mathbf{o}_t \mid z_t) = \mathcal{N}(\boldsymbol{\mu}_{z_t}, \boldsymbol{\Sigma}_{z_t}) \tag{30}$$

13.3 Beliefs

Posterior belief over latent regimes:

$$q_t(z_t) \approx P(z_t \mid \mathbf{o}_{1:t}) \tag{31}$$

Belief updating follows Bayesian filtering:

$$q_t(z_t) \propto P(\mathbf{o}_t \mid z_t) \sum_{z_{t-1}} P(z_t \mid z_{t-1}) q_{t-1}(z_{t-1}) \tag{32}$$

13.4 Actions

Portfolio weights:

$$\mathbf{a}_t \equiv \mathbf{w}_t \in \Delta^{D-1} \tag{33}$$

where Δ^{D-1} is the simplex.

13.5 State–Action Coupling

Unlike classical control, actions do not influence regime transitions directly:

$$P(z_{t+1} \mid z_t, \mathbf{a}_t) = P(z_{t+1} \mid z_t) \tag{34}$$

However, actions determine realised wealth:

$$W_{t+1} = W_t(1 + \mathbf{w}_t^\top \mathbf{r}_{t+1}) \tag{35}$$

Thus the POMDP structure is:

$$(z_t) \to \mathbf{o}_t \quad \text{and} \quad \mathbf{w}_t \to W_{t+1} \tag{36}$$

The control problem is therefore belief-space optimisation.

14 Free Energy Principle Derivation

We now derive the polyphonic objective from variational free energy principles.

14.1 Variational Free Energy

Define approximate posterior $q(z_{1:T})$ over latent regimes.
The variational free energy is:

$$F = \mathbb{E}_q \left[\log q(z_{1:T}) - \log P(z_{1:T}, \mathbf{o}_{1:T})\right] \tag{37}$$

Minimising F yields approximate Bayesian filtering (implemented via HMM EM + online update).

14.2 Expected Free Energy for Planning

In Active Inference, policies π are selected to minimise Expected Free Energy:

$$G(\pi) = \mathbb{E}_{q(\mathbf{o}_{t+1:T}, z_{t+1:T} \mid \pi)} \left[\log q(z_{t+1:T} \mid \pi) - \log P(\mathbf{o}_{t+1:T}, z_{t+1:T})\right] \tag{38}$$

Under common decompositions:

$$G(\pi) = \underbrace{\mathbb{E}[D_{\mathrm{KL}}(q(z) \| p(z \mid \mathbf{o}))]}_{\text{Epistemic Value}} + \underbrace{\mathbb{E}[-\log P(\mathbf{o})]}_{\text{Risk / Extrinsic Value}} \tag{39}$$

14.3 Mapping to Portfolio Context

In financial allocation:

- Epistemic value (information gain) is minimal because actions do not change regime inference.
- Extrinsic value dominates: portfolios are selected to minimise expected future loss.

Thus Expected Free Energy reduces to a risk-sensitive objective over predicted returns:

$$G(\mathbf{w}) \approx \mathbb{E}\left[-\log P(\mathbf{o}_{t+1:T} \mid \mathbf{w})\right] \tag{40}$$

Under Gaussian predictive assumptions and log-wealth utility, this becomes:

$$G(\mathbf{w}) \sim -\mathbb{E}\left[\sum_{h=0}^{H-1} \gamma^h \log\Big(1 + \mathbf{w}^\top \mathbf{r}_{t+h}\Big)\right] \tag{41}$$

which corresponds directly to the growth voice.

14.4 Polyphonic Extension of Expected Free Energy

Instead of a single scalar extrinsic value, we define multiple domain-specific risk functionals:

$$\mathbf{J}(\mathbf{w}) = \begin{bmatrix} J_{\text{growth}} \\ J_{\text{risk}} \\ J_{\text{drawdown}} \\ J_{\text{turnover}} \end{bmatrix} \tag{42}$$

We then define Polyphonic Free Energy:

$$G_{\text{poly}}(\mathbf{w}) = \boldsymbol{\pi}_t^\top \mathbf{J}(\mathbf{w}) \tag{43}$$

where $\boldsymbol{\pi}_t$ is a coordination vector over objectives.
This differs from classical scalarisation because:

- $\boldsymbol{\pi}_t$ is adaptive,
- Each voice is normalised separately,
- No objective can collapse to zero influence (enforced floor $\pi_{\min}$).

Thus the Polyphonic Portfolio implements:

$$\mathbf{w}_t = \arg\min_{\mathbf{w}} G_{\text{poly}}(\mathbf{w}) \tag{44}$$

14.5 Interpretation

The architecture can be understood as:

- A belief-space controller,
- Minimising Expected Free Energy under regime uncertainty,
- With a non-dominating coordination layer over heterogeneous risk functionals.

Unlike classical mean–variance optimisation, this formulation:

- Is explicitly forward-looking,
- Integrates latent regime inference,
- Separates objective voices rather than collapsing them a priori.