Name:
Section (time):

Math 340 Quiz 10

1.) Is the following operation an inner product? Why or why not?

$$\langle x, y\rangle = \begin{cases} 0 & \text{if } x \neq y \\ 91 & \text{if } x = y. \end{cases}$$

2.) Use an inner product to find the angle between vectors $(-4, 0)$ and $(2, 0)$. (You can figure this out in a graph, but show the steps using an inner product.)

Not Quiz 10

A set of vectors is said to be an **orthogonal set of vectors** if does not contain the zero vector and for any two distinct vectors a and b, $\langle a, b\rangle = 0$. An orthogonal set is said to be **orthonormal** if in addition $\|a\| = 1$ for every vector in a in the set.

Linear combinations of orthogonal vectors: Suppose that $\{a_1, a_2, \ldots, a_3\}$ is an orthogonal set of nonzero vectors in the inner product space $\mathcal{V}$. If u is a linear combination of $a_1, \ldots, a_n$, then

$$u = \frac{\langle u, a_1\rangle}{\langle a_1, a_1\rangle} a_1 + \cdots + \frac{\langle u, a_n\rangle}{\langle a_n, a_n\rangle} a_n.$$

Gram-Schmidt Process: Given a basis $\{b_1, \ldots, b_n\}$ for an inner product space $\mathcal{W}$, we can find a orthonormal basis $\{\bar{a}_1, \ldots, \bar{a}_n\}$ using the following process.

Define $a_1, \ldots, a_n$:

$$\begin{aligned}
a_1 &= b_1 \\
a_2 &= b_2 - \frac{\langle b_2, a_1\rangle}{\langle a_1, a_1\rangle} a_1 \\
&\vdots \\
a_n &= b_n - \frac{\langle b_n, a_{n-1}\rangle}{\langle a_{n-1}, a_{n-1}\rangle} a_{n-1} - \cdots - \frac{\langle b_n, a_1\rangle}{\langle a_1, a_1\rangle} a_1.
\end{aligned}$$

This will be an orthogonal basis. We normalize the vectors by setting $\bar{a}_i = \frac{1}{\|a_i\|} a_i$ to create the orthonormal basis $\{\bar{a}_1, \ldots, \bar{a}_n\}$.

Tentative advice: Focus on understanding GS in two dimensions. Then, look at the multidimensional process as working iteratively by each 2-D subspace. That is we might find a_3 by combining orthonormalizations of b_3 and a_1, then b_3 and a_2.

a.) Is the set of vectors $\{(1, 0), (0, 1), (1, 1)\}$ orthogonal? Hint: you can answer this by counting and citing previous results.

b.) Is this an inner product? $\langle x, y\rangle = \|x\|\|y\|$?

c.) Use Gram-Schmidt to find an orthonormal basis for P_2. Hint: start with the basis $\{1, x, x^2\}$. Use $\langle f, g\rangle = \int_{[-1,1]} fg\,dx$.

d.) Write the vector $a = (1, -1, 1)$ as a linear combination of the vectors $v_1 = (1, 1, 1), v_2 = (0, 1, -1), v_3 = (-2, 1, 1)$.

e.) Orthonormalize the basis $\{(1, 2), (3, 4)\}$.

Not Quiz 10 Solutions

a.) This is not a set of linearly independent vectors, because we cannot have three vectors in $\mathbb{R}^2$ be linearly independent (at most two such vectors can be). Because we know a set of orthogonal vectors is linearly independent, these cannot be orthogonal.

b.) This is not an inner product because it fails the property $\langle x + a, y\rangle = \langle x, y\rangle + \langle a, y\rangle$. Consider $x, y, a \in \mathbb{R}^1$. Let $x = y = 1$ and $a = -1$. Then $\langle x + a, y\rangle = \langle 0, 1\rangle = 0$. However $\langle x, y\rangle + \langle a, y\rangle = 1 + 1 = 2$, meaning this cannot be an inner product because it fails to satisfy the stated property.

c.) We use $\langle f, g\rangle = \int_{[-1,1]} fg dx$.

$$\begin{aligned} a_1 &= 1 \\ a_2 &= x - \frac{\langle x, 1\rangle}{\langle 1, 1\rangle} 1 \\ a_3 &= x^2 - \frac{\langle x^2, 1\rangle}{\langle 1, 1\rangle} 1 - \frac{\langle x^2, a_2\rangle}{\langle a_2, a_2\rangle} a_2 \end{aligned}$$

We can calculate

$$\begin{aligned} \langle x, 1\rangle &= \int_{-1}^{1} x dx = 0 \\ \langle 1, 1\rangle &= \int_{-1}^{1} 1 dx = 2 \end{aligned}$$

So that $a_2 = x$. We proceed to find a_3.

$$\begin{aligned} \langle x^2, 1\rangle &= \int_{-1}^{1} x^2 dx = \frac{2}{3} \\ \langle x^2, a_2\rangle &= \int_{-1}^{1} x^3 dx = 0 \\ \langle a_2, a_2\rangle &= \int_{-1}^{1} x^2 dx = \frac{2}{3} \end{aligned}$$

Thus $a_1 = 1, a_2 = x$, and $a_3 = x^2 - \frac{\frac{2}{3}}{2} a_1 - 0 a_2 = x^2 - \frac{1}{3}$.

d.) v_1, v_2, v_3 are orthogonal. therefore we must have

$a = \frac{\langle a, v_1\rangle}{\langle v_1, v_1\rangle} v_1 + \frac{\langle a, v_2\rangle}{\langle v_2, v_2\rangle} v_2 + \frac{\langle a, v_3\rangle}{\langle v_3, v_3\rangle} v_3$.

Computation gives $a = \frac{1}{3}v_1 - 1v_2 - \frac{1}{3}v_3$.

e.) We use the standard inner product. Row vectors are used in place of the usual column vectors because they are easier to type. Please don't be mad.

$$\begin{aligned} a_1 &= (1,2) \\ a_2 &= (3,4) - \frac{\langle (3,4),(1,2)\rangle}{\langle (1,2),(1,2)\rangle}(1,2) \end{aligned}$$

We reduce the last line,

$$a_2 = (3,4) - \frac{11}{5}(1,2) = (4/5, -2/5)$$

.

We can verify these are orthogonal $(1,2) \cdot (4/5, -2/5) = 4/5 - 4/5 = 0$.

Now let's normalize to vectors $\frac{1}{\sqrt{5}}(1,2)$ and $\frac{1}{\sqrt{\frac{4}{5}}}(4/5, -2/5)$.