Name:
Section (time):

Math 340 Quiz 2

1.) Solve the following system of equations by writing it in the form $Ax = b$ and solve using matrix multiplication.

$$2x + y = 4$$
$$x + 2y = 5$$

2.) Suppose there is a function $f : \mathbb{R}^n \to \mathbb{R}^2$ defined by $f(x) = Ax$.

We know $Au = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $Av = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$.

Show that $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is in the image of the function $f(x) = Ax$.

Row operations and reduced row echelon form.

1.) Assuming invertibility, show $(ABCD)^{-1} = D^{-1}C^{-1}B^{-1}A^{-1}$

2.) True or false? For any elementary matrix E, $E = E^{-1}$.

3.) Which of these matrices are in reduced row echelon form? Reduce the matrices not already in reduced form.

$$A = \begin{bmatrix} 1 & 0 & 3 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 1 & 0 & 3 & 4 \\ 0 & 0 & 1 & 5 & 6 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

4.) Let A be a square matrix. Show that if a sequence of elementary row operations that when applied successively to A yield the identity matrix, then the same operations applied in the same order to I yield A^{-1}.

5.) Show that if a matrix A is row equivalent to an inverse matrix, then it must be nonsingular.

Solutions

1.) Assuming invertibility, show $(ABCD)^{-1} = D^{-1}C^{-1}B^{-1}A^{-1}$

We say $Y = X^{-1}$ *if* $XY = I$.

We can check the product, $ABCD(D^{-1}C^{-1}B^{-1}A^{-1})$. *This will collapse from the inside.*

$$ABCD(D^{-1}C^{-1}B^{-1}A^{-1}) = ABCIC^{-1}B^{-1}A^{-1} = \cdots = I.$$

Note this also works if we reverse the order of multiplication.

$$(D^{-1}C^{-1}B^{-1}A^{-1})ABCD = D^{-1}C^{-1}B^{-1}IBCD = \cdots = I.$$

Therefore, $D^{-1}C^{-1}B^{-1}A^{-1}$ *must be the inverse of* $ABCD$. *We can also show this by applying the rule* $(XY)^{-1} = Y^{-1}X^{-1}$ *twice. Let* $X = AB$ *and* $Y = CD$. *We obtain an inverse* $(CD)^{-1}(AB)^{-1} = D^{-1}C^{-1}B^{-1}A^{-1}$.

2.) True or false? For any elementary matrix E, $E = E^{-1}$.

False. Swap matrices can be their own inverse, but this isn't true for the matrices we use to represent multiplying a row by a constant.

3.) Which of these matrices are in reduced row echelon form? Reduce the matrices not already in reduced form.

$$A = \begin{bmatrix} 1 & 0 & 3 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 1 & 0 & 3 & 4 \\ 0 & 0 & 1 & 5 & 6 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

C *and* D *are already reduced.* A *and* B *must be reduced further.*

$$A_{r_1 - 3r_2 \to r_1} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We need several steps for B. First we replace row 1 with row 1 minus row 2, $BB_{r_1-r_2\to r_1}$.

$$W = B_{r_1-r_2\to r_1} = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Then we substract row 3 from row 1 to eliminate the nonzero above a leading one in the fourth column.

$$X = W_{r_1-r_3\to r_1} = \begin{bmatrix} 1 & 1 & 0 & 0 & -1 & -2 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Let's get rid of the 2, entry $(3,5)$, *by substracting 2 times row 4 from frow 3*

$$Y = X_{r_3-2r_4\to r_3} = \begin{bmatrix} 1 & 1 & 0 & 0 & -1 & -2 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$Z = Y_{r_1+r_4\to r_1} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Now we have to get rid of the entries in the last column $(1,6), (3,6),$ *and,* $(4,6)$. *Let's combine a few steps. We can do that by adding or subtracting the last row with the others.*

$$B_{rref} = Z_{r_1+2r_5\to r_1, r_3-r_5\to r_3, r_4-r_5\to r_4} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

4.) Let A be a square matrix. Show that if a sequence of elementary row operations that when applied successively to A yield the identity matrix, then the same operations applied in the same order to I yield A^{-1}.

We know $E_1 \cdots E_n A = I$.

We want something like $E_1 \cdots E_n I = A^{-1}$.

If you wanted to work backwards, you could multiply both sides, from the right, by A.

$$E_1 \cdots E_n I A = A^{-1} A = I$$

And that's exactly what we wanted. The above reduces to $E_1 \cdots E_n A = I$

5.) Show that if a matrix A is row equivalent to an inverse matrix, then it must be nonsingular.

$A = \hat{E} B^{-1} = E_1 \cdots E_n B^{-1}$

We invert the right hand side.

$\left[\hat{E} B^{-1}\right]^{-1} = B E_n^{-1} \cdots E_1^{-1}$

We know each of these inverses exist because elementary matrices are invertible. Therefore, A^{-1} *exists and* $A^{-1} = B E_n^{-1} \cdots E_1^{-1}$.