Math 340 Compilation

Sections: 328 and or 329
TA: Zander

December 2017

Hi. This document includes quizzes (no solutions), handouts, handout solutions, and some solution sketches for the pages I graded on the midterms.

Contents

1 Discussion Stuff

Name:
Section (time):

Math 340 Quiz 1

1.) $G = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$.

2.) Write the following system of equations in the form $Ax = b$

$$\begin{gathered} 10x_1 + 9x_2 + x_3 - 9 = 0 \\ 9x_2 + x_3 + 1 = 0 \\ 10x_1 - x_3 = 8 \end{gathered}$$

Math 340 9/14

i.) Consider $A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 8 & 0 & 1 & 777 & 1 & 1 & 0 \\ 1 & 0 & 0 & 9 & 0 & 1 & 1 & 0 \\ 1 & 5 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 27 & 1 & 0 \\ 1 & 0 & 44 & 1 & 0 & 1 & 3 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$.

Let B = 2A and $C = AB$ and $D = C^T$.

Find d_{81}.

ii.) $A = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$

and $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

Find AB. Find $C = BA$. Find C^2.

iii.) Suppose $AB = 0$ and $C = BA$. What can you say about C^2? Support your claim.

iv.) The following is an example of a *shift matrix*,

$$S = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Why is it called that? Try computing SA and AS for

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}.$$

Try also

$$B = \begin{bmatrix} r & s \\ t & u \\ v & w \end{bmatrix}.$$

Math 340 9/14 Solutions

i.) $d_{81} = 0$. That is, entry $(8, 1)$ of D is zero.

We know that matrix D is the transposition of matrix C.

Therefore, we are trying to compute $c_{18} = d_{81}$.

We are given $C = AB = A2A = 2A^2$.

Therefore, $c_{18} = 2a^2_{18}$.

Recall the entry (i, j) from a matrix that is the product of two matrices will be equal to the i^{th} *row* of the first matrix dot producted with the j^{th} *column* of the second matrix. So, we take the first row of A and dot that with the last column. Note the last column is all zeros. Therefore the dot product will be zero, and $a^2_{18} = 0$.

Now, $d_{81} = c_{18} = 2a^2_{18} = 2 \times 0 = 0$.

ii.) C^2 is a 2×2 matrix of all zeros.

iii.) This is an exercise in squaring matrices and using associativity of matrix multiplication.

We are given $C = BA$ and asked to compute C^2.

$C^2 = BABA$, *note that* $C^2 \neq B^2A^2$.

Using associativity, $C^2 = B(AB)A = B0A = 0$.

iv.)

$$SA = \begin{bmatrix} d & e & f \\ g & h & i \\ 0 & 0 & 0 \end{bmatrix}$$

$$AS = \begin{bmatrix} 0 & a & b \\ 0 & d & e \\ 0 & g & h \end{bmatrix}$$

$$SB = \begin{bmatrix} t & u \\ v & w \\ 0 & 0 \end{bmatrix}$$

Please note BS is not computable. B is of dimension 3×2 and S is of dimension 3×3. The first matrix in a product must have the same number of columns and the second matrix has rows. This ensures that the dot products featured in matrix multiplication will be defined.

Name:
Section (time):

Math 340 Quiz 2a

Consider the matrices

$$B_1 = \begin{bmatrix} 2 & 3 \\ 4 & -2 \end{bmatrix} \text{ and } B_2 = \begin{bmatrix} 4 & 6 \\ 8 & -4 \end{bmatrix}.$$

1.) Find $B_1 B_2$.

2.) Write the below system in the form $Ax = b$ and solve **using matrix multiplication.**

$$\begin{aligned} 2x + 3y &= 8 \\ 4x - 2y &= 16 \end{aligned}$$

Name:
Section (time):

Math 340 Quiz 2

1.) Solve the following system of equations by writing it in the form $Ax = b$ and solve using matrix multiplication.

$$\begin{aligned} 2x + y &= 4 \\ x + 2y &= 5 \end{aligned}$$

2.) Suppose there is a function $f : \mathbb{R}^n \to \mathbb{R}^2$ defined by $f(x) = Ax$.

We know $Au = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $Av = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$.

Show that $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is in the image of the function $f(x) = Ax$.

Row operations and reduced row echelon form.

1.) Assuming invertibility, show $(ABCD)^{-1} = D^{-1}C^{-1}B^{-1}A^{-1}$

2.) True or false? For any elementary matrix E, $E = E^{-1}$.

3.) Which of these matrices are in reduced row echelon form? Reduce the matrices not already in reduced form.

$$A = \begin{bmatrix} 1 & 0 & 3 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 1 & 0 & 3 & 4 \\ 0 & 0 & 1 & 5 & 6 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

4.) Let A be a square matrix. Show that if a sequence of elementary row operations that when applied successively to A yield the identity matrix, then the same operations applied in the same order to I yield A^{-1}.

5.) Show that if a matrix A is row equivalent to an inverse matrix, then it must be nonsingular.

Solutions

1.) Assuming invertibility, show $(ABCD)^{-1} = D^{-1}C^{-1}B^{-1}A^{-1}$

We say $Y = X^{-1}$ *if* $XY = I$.

We can check the product, $ABCD(D^{-1}C^{-1}B^{-1}A^{-1})$. *This will collapse from the inside.*

$$ABCD(D^{-1}C^{-1}B^{-1}A^{-1}) = ABCIC^{-1}B^{-1}A^{-1} = \cdots = I.$$

Note this also works if we reverse the order of multiplication.

$$(D^{-1}C^{-1}B^{-1}A^{-1})ABCD = D^{-1}C^{-1}B^{-1}IBCD = \cdots = I.$$

Therefore, $D^{-1}C^{-1}B^{-1}A^{-1}$ *must be the inverse of* $ABCD$. *We can also show this by applying the rule* $(XY)^{-1} = Y^{-1}X^{-1}$ *twice. Let* $X = AB$ *and* $Y = CD$. *We obtain an inverse* $(CD)^{-1}(AB)^{-1} = D^{-1}C^{-1}B^{-1}A^{-1}$.

2.) True or false? For any elementary matrix E, $E = E^{-1}$.

False. Swap matrices can be their own inverse, but this isn't true for the matrices we use to represent multiplying a row by a constant.

3.) Which of these matrices are in reduced row echelon form? Reduce the matrices not already in reduced form.

$$A = \begin{bmatrix} 1 & 0 & 3 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 1 & 0 & 3 & 4 \\ 0 & 0 & 1 & 5 & 6 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

C *and* D *are already reduced.* A *and* B *must be reduced further.*

$$A_{r_1 - 3r_2 \to r_1} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We need several steps for B. First we replace row 1 with row 1 minus row 2, $BB_{r_1-r_2\to r_1}$.

$$W = B_{r_1-r_2\to r_1} = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Then we substract row 3 from row 1 to eliminate the nonzero above a leading one in the fourth column.

$$X = W_{r_1-r_3\to r_1} = \begin{bmatrix} 1 & 1 & 0 & 0 & -1 & -2 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Let's get rid of the 2, entry $(3,5)$, by substracting 2 times row 4 from frow 3

$$Y = X_{r_3-2r_4\to r_3} = \begin{bmatrix} 1 & 1 & 0 & 0 & -1 & -2 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$Z = Y_{r_1+r_4\to r_1} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Now we have to get rid of the entries in the last column $(1,6),(3,6),$ and, $(4,6)$. Let's combine a few steps. We can do that by adding or subtracting the last row with the others.

$$B_{rref} = Z_{r_1+2r_5\to r_1, r_3-r_5\to r_3, r_4-r_5\to r_4} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

4.) Let A be a square matrix. Show that if a sequence of elementary row operations that when applied successively to A yield the identity matrix, then the same operations applied in the same order to I yield A^{-1}.

We know $E_1 \cdots E_n A = I$.

We want something like $E_1 \cdots E_n I = A^{-1}$.

If you wanted to work backwards, you could multiply both sides, from the right, by A.

$$E_1 \cdots E_n I A = A^{-1} A = I$$

And that's exactly what we wanted. The above reduces to $E_1 \cdots E_n A = I$

5.) Show that if a matrix A is row equivalent to an inverse matrix, then it must be nonsingular.

$A = \hat{E} B^{-1} = E_1 \cdots E_n B^{-1}$

We invert the right hand side.

$\left[\hat{E} B^{-1}\right]^{-1} = B E_n^{-1} \cdots E_1^{-1}$

We know each of these inverses exist because elementary matrices are invertible. Therefore, A^{-1} *exists and* $A^{-1} = B E_n^{-1} \cdots E_1^{-1}$.

Name:
Section (time):

Math 340 Quiz 3

1.) Represent the following system of equations as an augmented matrix.

$$\begin{aligned} 10x_1 + 9x_2 + x_3 - 9 &= 0 \\ 9x_2 + x_3 + 1 &= 0 \\ 10x_1 - x_3 &= 8 \\ x_4 &= x_1 \end{aligned}$$

2.) Reduce the following matrix to reduced row echelon form.

$$A = \begin{bmatrix} 0 & -1 & 4 \\ 4 & -2 & 8 \\ 1 & 0 & 0 \end{bmatrix}$$

Name:
Section (time):

Math 340 Quiz 4

1.) Compute the inverse of the following matrix. Show all of your work.

$$A = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{bmatrix}$$

2.)

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } C = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

What is the determinant of B? What is the determinant of C? Explain how you can calculate the determinant of C using the properties of determinants and row operations.

Not quiz 4: Elementary Ops, Inverses, but mostly Determinants

a.) Factor $G = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ into the product of elementary matrices.

b.) We discussed the determinant as a function that spits out a number after you feed it a matrix. What is the image (values the function will actually take) of this determinant function if the domain is all non-singular matrices?

c.) Let A be a special kind of matrix, $A^2 = A$. Can the determinant be zero? Can you find possible determinant value(s) for these kinds of matrices?[1]

d.) Find the determinant of:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & r & 1 \end{bmatrix} \text{ where } r \in \mathbb{R}.$$

We can say that the determinant behaves like a linear function *on the rows* of a matrix. Can you provide an example of this by expressing $\det(B)$ as a sum of two determinants?

e.) Suppose A is a square, $n \times n$ matrix. What is the determinant of tA?

f.) Consider matrices:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Billy claims both have the same determinant because B was obtained by adding row 2 to row 3 and row 3 to row 2. Is he right?$^{(no)}$ What is his mistake?

HW30.) Let A be a 3×3 matrix with $\det(A)$=3. What is the rref to which A is row equivalent? How many solutions does the homogeneous system $Ax = 0$ have?

[1] This is called an idempotent matrix, but that's not the point.

Solutions

a.) This is nonsingular so it will be row equivalent to the identity matrix.

$G = \hat{E}I = \hat{E} = E_1 \cdots E_n$.

$$H = G_{r_3+r_2\to r_2} = \begin{bmatrix} 0 & 1 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$J = H_{r_1+r_3\to r_3} = \begin{bmatrix} 0 & 1 & 1 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}$$

$$K = J_{(r_2-r_1)/2\to r_2} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

$$L = K_{r_3-r_2-r_1\to r_3} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$M = L_{r_1-r_3-\to r_1} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$N = L_{r_1\leftrightarrow r_2 \text{ then } r_2\leftrightarrow r_3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So the elementary matrices describe these operations.

$$G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$G =$ (add rows)(add rows)(subtract and scale)(subtract rows)(subtract rows)(swap)(swap).

You may find a different answer that is still equivalent.

b.) $\mathbb{R} \setminus \{0\}$, the reals except zero.

c.) The determinant must be one. The most direct way is using properties:

i.) det(AB)=det(A)det(B)

ii.) det(A^{-1})= det$(A)^{-1}$

Note $A = A^{-1}$. Then det(AA) =det(A)/det(A) = det(I) $= 1 \implies$ det(A)=1.

d.) det(A)=8

det(B)=1

$$|B| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & r & 0 \end{vmatrix} = 1 + 0.$$

e.) $t^n \det(A)$.

f.) He added rows not in sequence.

HW30.) It's homework.

Name:
Section (time):

Math 340 Quiz $\star 4 \star$

1.) For what values of $t \in \mathbb{R}$ is the following matrix invertible?

$$A = \begin{bmatrix} 1 & t \\ 2 & t \end{bmatrix}.$$

2.) (a) Using any method *except* cofactor expansion, compute the determinant of

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}.$$

(b) Using cofactor expansion, compute the determinant of

$$B = \begin{bmatrix} 1 & 0 & 0 & 9 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 7 \end{bmatrix}.$$

Name:
Section (time):

Math 340 Quiz 5

1.) For what values of $t \in \mathbb{R}$ is the following matrix invertible?

$$A = \begin{bmatrix} 1 & t \\ -t & 91 \end{bmatrix}.$$

2.) (a) Using any method, compute the determinant of

$$B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}.$$

(b) Using cofactor expansion, compute the determinant of

$$C = \begin{bmatrix} 1 & 0 & 0 & 9 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 7 \end{bmatrix}.$$

Name:
Section (time):

Math 340 Quiz 6

1.) Give an example of two vectors in $\mathbb{R}^2$ that are equal and have different tails.

2.) Let $\vec{PQ}$ have tail $(1, 0)$ and head $(2, 1)$.

Let $\vec{PS}$ have tail $(1, 0)$ and head $(2, -1)$.

Draw these vectors in a graph and argue they are not equal.

Not Quiz 6

a.) Let $\mathbb{R}_{++}$ be the set of all strictly positive real numbers and define, for $x, y \in \mathbb{R}_{++}$ a vector sum by

$$x \oplus y = x \cdot y,$$

where the product on the right is the usual product of numbers. If a is a real number and $x \in \mathbb{R}_{++}$ define

$$a \odot x = x^a.$$

Is this a vector space?

b.) Which of the following are subspaces?

1. $\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 = 0\}$
2. $\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_2 = 0\}$
3. $\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 + x_2 = 0\}$
4. $\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 + x_2 = 1\}$
5. $\{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \geq 0\}$

c.) Consider $\mathcal{U} \in \mathbb{R}^n$, the subset consisting of all vectors a with the property $a_1 + a_2 + \ldots a_n = 0$. Define $\oplus$ and $\cdot$ as the usual vector addition and scalar multiplication:

$$a \oplus b = (a_1 + b_1, \ldots, a_n + b_n),$$

$$c \odot a = (ca_1, \ldots ca_n).$$

Show this is a subspace of $\mathbb{R}^n$.

d.) (4.3 HW37) Which of the following points are on the line

$$x = 4 - 2t$$
$$y = -3 + 2t$$
$$z = 4 - 5t$$

i (0,1,-6)

ii (1,2,3)

iii (4,-3,4)

iv (0,1,-1)

e.) What is the span of $\{1 - x, 1 + x\}$ in the set of polynomials of degree one?

Solution sketches to Not Quiz 6

a.) Yes.

1. $\oplus$ is commutative because the normal multiplication $\cdot$ is.
2. $\oplus$ is associative because multiplication $\cdot$ is.
3. Here the additive identity is 1, $x \oplus 1 = x$.
4. The additive inverse of x is $\frac{1}{x}$. $x \oplus 1/x = 1$ =the additive identity.
5. $c \odot (x \oplus y) = (x\dot{y})^c = x^c y^c = c \odot x \oplus c \odot y$
6. $(c + d) \odot x = x^{c+d} = x^c \cdot x^d = c \odot x \oplus d \odot x$
7. $c \odot (d \odot x) = (x^d)^c = x^{cd} = (cd) \odot x$
8. $1 \odot x = x^1 = x$.

We've now verified the eight properties of a vector space.

b.) We check closure of addition and multiplication by a scalar. We assume the normal addition and multiplication operations.

1. yes
2. yes
3. Yes. Observe $c(x_1 + x_2) = c0 = 0$, so $c\vec{x}$ is in the proposed set for any $c \in \mathbb{R}$. If we add two vectors, x and y, then $(x_1 + y + 1) + (x_2 + y_2) = (x_1 + x_2) + (y_1 + y_2) = 0 + 0$, so the set is closed under addition.
4. No, this is not closed under scalar multiplication.
5. No, the set is not closed under scalar multiplication (of a negative).

c.) As before, we check closure under addition and scalar multiplication.

1. Let $u = a \oplus b = (a_1 + b_1, \ldots, a_n + b_n)$ Then $u_i = a_i + b_i$ for $i = 1, 2, \ldots, n$. Now we check if $\sum_{i=1}^n u_i = 0$. Observe $\sum_{i=1}^n u_i = \sum_{i=1}^n a_i + b_i = \sum_{i=1}^n a_i + \sum_{i=1}^n b_i = 0 + 0$. Thus, this space is closed under addition.
2. For $c \odot a$, We have to check if $\sum_{i=1}^n ca_i$ is equal to zero. $\sum_{i=1}^n ca_i = c\sum_{i=1}^n a_i = c0 = 0$. So, the space is closed under multiplication.

We now know that this is a subspace.

d.) HW

e.) The span is the set of all polynomials of degree one. Observe that the coefficient vectors are $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. These are linearly independent, so they would span the entire space of $\mathbb{R}^2$. This means we can pick any coefficient vector $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ in $\mathbb{R}^2$ we want and and produce it as a linear combination of $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. This corresponds to saying we can pick any polynomial $\alpha x + \beta$ and represent it as a linear combination of $1 - x$ and $1 + x$.

Name: Section (time):

Math 340 Quiz $^\star 6^\star$

1.) Is $\{\mathbf{x} \in \mathbb{R}^3 \mid x_1 + x_2 = 0\}$ a subspace of $\mathbb{R}^3$? Why or why not? Assume the usual addition and scalar multiplication operations.

2.) Is $(1, 0)$ in the span of vectors $(1, 1)$ and $(-1, -1)$? Show why or why not.

Name:
Section (time):

Math 340 Quiz 7

1.) What is the span of $\{1 - x, 1 + x\}$ in the set of polynomials of degree n?

2.) Is the following set of vectors linearly independent? Show why or why not.

$$\left\{ \begin{pmatrix} 91 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 91 \\ 10 \\ 73 \end{pmatrix} \right\}$$

Not Quiz 7

A *basis* β for a set V has properties

1. span(β)=V
2. β is linearly independent.

a.) Can a collection of $n+1$ vectors be a basis for $\mathbb{R}^n$?

b.) Show a set of vectors containing $\mathbf{0}$ is linearly dependent.

c.) Show any subspace is a linearly dependent set of vectors.

d.) Show $E = \{(1,0,0),(0,1,0),(0,0,1),(1,1,1)\}$ is linearly dependent, but that any set of three of these vectors is linearly independent.

e.) (Proofy) Show if β contains n linearly independent vectors, then it must be a basis for $\mathbb{R}^n$.

f.) Find a basis for the subspace $\mathcal{U} = \{(x,y,z) \in \mathbb{R}^3 \mid x - z = 0\}$.

HW28.) Find a basis for $\mathbb{R}^3$ that includes

i.) $\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$

ii.) $\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$

Not Quiz 7 Solutions

Preamble: Questions a.) and e.) are difficult to show formally, and I didn't quite forecast this appropriately as I constructed the handout. So, let's not worry about proving these things. Accordingly, I have not typed up full proofs. I put them on the handout because I think they are important and helpful properties to have a feel for. You should spend some time convincing yourself that the answer to a.) is no and that the statement in e.) is true.

a.) No. They would not be linearly independent.[2]

b.) To prove linear dependence of vectors $u_1, \ldots, u_n$, you need only show that there is a nontrivial solution to $\sum_{i=1}^{n} a_i u_i = 0$. Let $u_1 = \mathbf{0}$, $a_1 = 91$, and $a_i = 0$ for all $i > 1$. That's a nontrivial solution, so the set of vectors is linearly dependent.

c.) A subspace will contain a zero element $\mathbf{0}$. By part b.), it will be linearly dependent.

d.) Observe we can subtract $(1, 1, 1)$ from the sum of the standard basis vectors to obtain $\mathbf{0}$. Therefore, the entire set is linearly dependent.

There are four combinations to consider. Each will produce a nonsingular matrix. For those combinations including $(1, 1, 1)$, elementary row operations will eventually allow you to create an identity matrix. For those combinations not including $(1, 1, 1)$, the identity matrix is immediately formed by properly ordering the vectors. If the associated matrix is nonsingular, the vectors are linearly independent.

e.) See Theorem 4.12 for the whole proof. Just have some idea that n vectors can span an n-dimensional space.

f.) The condition for a vector belonging to the set $x = z$. So, we must construct one vector that features $x = z$ with z and x nonzero. Let's choose $(1, 0, 1)$. There are no restrictions on y, so a vector like $(0, 1, 0)$, when scaled and added to $(1, 0, 1)$, let's us achieve any y value we'd like.

Together, this forms a basis $\{(1, 0, 1), (0, 1, 0)\}$, though many more exist.

Name:
Section (time):

Math 340 Quiz Oct 31

1.) What is the dimension of the space spanned by the vectors below?

[2] Proving linear dependence takes a little work. I used induction.

$$\begin{pmatrix} 91 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 73 \end{pmatrix}$$

2.) Show that vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ form a basis for $\mathbb{R}^2$.

3.) Are the following vectors linearly independent?

$$\begin{pmatrix} 91 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 91 \\ 10 \\ 73 \end{pmatrix}$$

Name:
Section (time):

Math 340 Quiz 8

1.) What is the null space of the matrix below?

$$A = \begin{bmatrix} 2 & 0 \\ 4 & 2 \end{bmatrix}$$

2.) What is the dimension of the space spanned by vectors $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 7 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$?

Not Quiz 8

a.) Find a basis for P_3 other than $\{1, x, x^2, x^3\}$.

b.) Are the vectors $(1, 1, -2)$ and $(0, 3, -3)$ a basis for the subspace

$$\mathcal{V} = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\}?$$

c.) For what values of r are the vectors $(r, 1, 1), (1, r, 1), (1, 1, r)$ a basis for $\mathbb{R}^3$?

d.) Find the coordinates of $2 - x$ relative to the basis $\{1 - x, 1 + x\}$ for P_1.

e.) Suppose $\mathbf{D}$ represents the differentiation operator. For example $\mathbf{D}(x^2 + 8) = 2x$. What is the null space of $\mathbf{D}$?

f.) Show $T(x, y, z) = (y, 0, z)$ is linear. Find the null space of T and its dimension. Represent T as a matrix.

g.) Show the following are not linear transformations.

- i $T(x, y) = (x^2, y^2)$
- ii $T(x, y, z) = (x + y + z, 1)$
- iii $T(x) = (1, -1)$
- iv $T(x, y) = (xy, y, x)$

g.) Let $T : X \to Y$ be a linear transformation and let $a_1, \ldots, a_n$ be a basis for X. Show that T is one-to-one if and only if $T(a_1), T(a_2), \ldots T(a_n)$ are linearly independent.

Not Quiz 8 Solutions

a.) This is a silly question. $\{2, x, x^2, x^3\}$ works or $\{1 + x, x, x^2, x^3\}$.

b.) Yes. You can show that the two vectors are linearly independent and in the set $\mathcal{V}$. Also, the set $\mathcal{V}$ is a two-dimensional subspace because $x + y + z = 0$ describes a plane. Two linearly independent vectors must in a two dimensional space must span the entire space, making these vectors a basis.

c.) $r \neq 1, -2$.

d.) Solve $a(1 - x) + b(1 + x) = 2 - x$.

Algebra gives $a = 1.5$ and $b = .5$. Therefore the coordinates of $2 - x$ are $(1.5, 0.5)$.

e.) The null space is the set of all functions that have a derivative of zero. Thus, null($\mathbf{D}$)$= P_0$, all constant functions.

f.) Linearity of a function f requires $f(a\mathbf{x}) = af(\mathbf{x})$ and $f(\mathbf{x}+\mathbf{y}) = f(\mathbf{x})+f(\mathbf{y})$.

For our function $T(x, y, z)$, the first property will hold:

$T(ax, ay, az) = (ay, 0, az) = a \cdot (y, 0, z) = aT(x, y, z)$.

The second property also holds. Consider $(x, y, z) + (u, v, w)$.

$T(x + u, y + v, z + w) = (y + v, 0, z + w) = (y, 0, z) + (v, 0, w) = T(x, y, z) + T(u, v, w)$.

So this is linear.

The null space will be the set of all vectors where $y, z = 0$. This is the x-axis, and thus is one-dimensional space with a (one of many) basis $(1, 0, 0)$.

$$T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

g.) For each of the functions, I give just one reason why they fail to be linear. There will be many other correct answers.

i fails. Just one example of T failing linearity is the following. $T(1, 1)+T(1, 1) = (1, 1) + (1, 1) \neq T(2, 2) = (4, 4)$.

ii fails because $T(ax, ay, az) = (ax+ay+az, 1) \neq aT(x, y, z) = (ax+ay+az, a)$ for $a \neq= 1$.

iii fails because $T(x + x') = (1, -1) \neq T(x) + T(x') = (2, -2)$.

iv fails because $T(ax, ay) = (a^2xy, ay, ax) \neq aT(x, y) = (axy, ay, ax)$.

Name:
Section (time):

Math 340 Quiz 9

1.) Find a basis for the null space of the matrix A.

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \end{bmatrix}$$

2.) What is the rank of the 2×9 matrix, Π?

$$\Pi = \begin{bmatrix} 3 & 1 & 4 & 1 & 5 & 9 & 2 & 6 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Not Quiz 9

An inner product satisfies

i.) $\langle a, b\rangle = \langle b, a\rangle$

ii.) $\langle ra, b\rangle = r\langle a, b\rangle = \langle a, rb\rangle$

iii.) $\langle a, b + c\rangle = \langle a, b\rangle + \langle a, c\rangle$ and $\langle a + c, b\rangle = \langle a, b\rangle + \langle c, b\rangle$

iv.) $\langle a, a\rangle \geq 0$ and with equality if and only if $a = \mathbf{0}$.

We mostly use the dot product, which is an inner product.
Important property: $a \cdot b = \|a\|\|b\| \cos(\theta)$.

Implication: Nonzero a and b are perpendicular if $a \cdot b = 0$. If a, b form an acute (obtuse) angle, the dot product is positive (negative).

Another Property: An orthogonal (perpendicular) set of nonzero vectors in an inner product space is linearly independent.[3]

a.) Verify that $\langle f(x), g(x)\rangle = \int_{-1}^{1} f(x)g(x)dx$ is an inner product on the P_k, real valued polynomials of degree k. Verify that $f(x) = 1$ and $g(x) = x$ are "perpendicular." Show x and $2x$ have a nonzero inner product, so they are not perpendicular.

b.) *Different Bases Give Different Inner Products*

Consider $a = (1, 0)$ and $b = (0, 1)$. Show that these vectors are perpendicular, using the dot product.

Find the coordinates a and b if we use $\{(1, 0), (1, 1)\}$ as a basis for $\mathbb{R}^2$. Now find the inner product of the vectors constructed with the new coordinates.

c.) Find the angle between $(2, 1)$ and $(2, 2)$. Find the angle between $(3, 4)$ and $(5, 12)$. Find the angle between $(1, 1)$ and $(-5, -5)$.

d.) Find a vector v that is orthogonal to any vector in the subspace spanned by $(1, 1, 0)$ and $(2, 0, 0)$.

e.) Show that if $a \cdot b = 0$ and $a \cdot c = 0$, then a is orthogonal to any vector $v \in \text{Span}(b, c)$.

HW32.) Let u be a fixed vector in $\mathbb{R}^n$. Prove that the set of all vectors v such that $u \cdot v = 0$is a subspace of $\mathbb{R}^n$.

[3] This is a homework question.

Not Quiz 9 Solutions

a.) Commutativity: $f(x)g(x) = g(x)f(x)$ so, $\int f(x)g(x)dx = \int g(x)f(x)dx$.

Scalars: $r\int f(x)g(x)dx = \int rf(x)g(x)dx = \int f(x)rg(x)dx = \langle rf(x), g(x)\rangle = \langle f(x), rg(x)\rangle$.

Distribution of addition: $\langle f(x), h(x) + g(x)\rangle = \int f(x)h(x) + f(x)g(x)dx = \int f(x)h(x)dx + \int f(x)g(x)dx = \langle f(x), h(x)\rangle + \langle f(x), g(x)\rangle$.

Nonnegativity: $f(x)f(x) = f(x)^2 \geq 0$ so $\langle f(x), f(x)\rangle \geq 0$.

b.) $a\dot{b} = 1 \times 0 + 0 \times 1 = 0 \implies a \perp b$.

For any vector $v = (v_1, v_2)$, it will have new coordinates $(v_1 - v_2, v_2)$. Therefore, $a \sim (1, 0)$ and $b \sim (-1, 1)$. The new inner product is $(1, 0) \cdot (-1, 1) = -1$.

c.) $(2, 1) \cdot (2, 2) = 4 + 2 = 6 = \|(2, 1)\|\|(2, 2)\| \cos(\theta)$

We can compute $\|(2, 1)\| = \sqrt{5}$ and $\|(2, 2)\| = \sqrt{8}$.

Then $6 = \sqrt{40}\cos(\theta)$. So, $\theta = \arccos \frac{3}{\sqrt{10}}$

For $(3, 4)$ and $(5, 12)$, we can calculate respective norms 5 and 13.

Thus $(3, 4) \cdot (5, 12) = 15 + 48 = 63 = 5 \times 13 \times \cos(\theta)$.

Thus $\theta = \arccos \frac{63}{65}$.

For $(1, 1)$ and $(-5, -5)$, we might note these are parallel and form an angle of π. Let's verify.

$(1, 1) \cdot (-5, -5) = -10 = \sqrt{2}\sqrt{50}\cos(\theta)$

Then $\cos(\theta) = -1 \implies \theta = \pi$.

d.) The spanned subspace is the set of all vectors (x, y, z) with $z = 0$. Then, an orthogonal vector will be a multiple of $(0, 0, 1)$.

e.) If $u \in \text{Span}(b, c)$, Then $u = \omega_1 b + \omega_2 c$.

We check $a \cdot u$. We obtain $\sum_{i=1}^{n} a_i u_i = \sum_{i=1} a_i(\omega_1 b_i + \omega_2 c_i) = \omega_1 \sum_{i=1}^{n} a_i b_i + \omega_2 \sum_{i=1}^{n} a_i c_i = \omega_1(0) + \omega_2(0)$.

c.) Find the column space and null space of

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}.$$

Find also a basis for each space.

Null space of Π, We have hyperplane $3x_1 + 1x_2 + \cdots 6x_8 = 5$.

That will have dimension 7.

Since the first midterm, main topics have included vectors spaces and subspaces, linear independence, span, bases, null spaces, isomorphisms, and dimensionality.

Suppose you have to allocate your money across n risky assets. For example, you can spend all of your money printing Wisconsin 2018 NCAA Basketball Champions shirts. Selling those will be profitable in one state of the world, but unprofitable in the state of the world where Georgetown wins. Or you could hide your money under your mattress and hope your house doesn't burn down. Let a risky asset be represented by an $m \times 1$ vector, $\mathbf{a}$, where a_i gives the return on your investment in state i. For the hiding-your-money-under-your-mattress asset, we might let $m = 2$ and represent it as $\mathbf{a} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$, where a_1 corresponds to state 1, no fire and no loss nor gain, and a_2 corresponds to state 2, fire and a 100% loss.

a.) Suppose all n assets are linearly independent. Can you design a *riskless portfolio* $\mathbf{x} = (x_i)_{i=1}^n$, giving the share of your wealth in each asset, that assures you of a return of 0 in all m states? Note for this to be a sensible portfolio, we must have $\sum_{i=1}^n x_i = 1$.

b.) We know the null space of a matrix A is a subspace. For an asset matrix A, will the set of riskless portfolios be a subspace? What if instead of giving of defining portfolio as a vector $\mathbf{x}$ giving the fraction of wealth in each asset, we define a portfolio as vector $\mathbf{y}$ giving the dollars invested in each asset and allow portfolios to vary in size? Assume a negative investment is possible (selling the asset).

I wrote that $\langle a, b\rangle \geq 0$. This is not true! In fact, if a and b form an obtuse angle, and we use the dot product, then the inner product will be negative.

Instead, I should have written $\langle a, a\rangle \geq 0$.

Name:
Section (time):

Math 340 Quiz $\star$9$\star$

1.) Find all values of c so that $\|u\| = 3$ for

$$u = \begin{bmatrix} 1 \\ c \\ 0 \end{bmatrix}.$$

2.) Find c so that the following vectors are orthogonal.

$$u = \begin{bmatrix} 2 \\ c \end{bmatrix},\ v = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Name:
Section (time):

Math 340 Quiz 10

1.) Is the following operation an inner product? Why or why not?

$$\langle x, y\rangle = \begin{cases} 0 & \text{if } x \neq y \\ 91 & \text{if } x = y. \end{cases}$$

2.) Use an inner product to find the angle between vectors $(-4, 0)$ and $(2, 0)$. (You can figure this out in a graph, but show the steps using an inner product.)

Not Quiz 10

A set of vectors is said to be an **orthogonal set of vectors** if does not contain the zero vector and for any two distinct vectors a and b, $\langle a, b\rangle = 0$. An orthogonal set is said to be **orthonormal** if in addition $\|a\| = 1$ for every vector in a in the set.

Linear combinations of orthogonal vectors: Suppose that $\{a_1, a_2, \ldots, a_3\}$ is an orthogonal set of nonzero vectors in the inner product space $\mathcal{V}$. If u is a linear combination of $a_1, \ldots, a_n$, then

$$u = \frac{\langle u, a_1\rangle}{\langle a_1, a_1\rangle} a_1 + \cdots + \frac{\langle u, a_n\rangle}{\langle a_n, a_n\rangle} a_n.$$

Gram-Schmidt Process: Given a basis $\{b_1, \ldots, b_n\}$ for an inner product space $\mathcal{W}$, we can find a orthonormal basis $\{\bar{a}_1, \ldots, \bar{a}_n\}$ using the following process.

Define $a_1, \ldots, a_n$:

$$\begin{aligned} a_1 &= b_1 \\ a_2 &= b_2 - \frac{\langle b_2, a_1\rangle}{\langle a_1, a_1\rangle} a_1 \\ &\vdots \\ a_n &= b_n - \frac{\langle b_n, a_{n-1}\rangle}{\langle a_{n-1}, a_{n-1}\rangle} a_{n-1} - \cdots - \frac{\langle b_n, a_1\rangle}{\langle a_1, a_1\rangle} a_1. \end{aligned}$$

This will be an orthogonal basis. We normalize the vectors by setting $\bar{a}_i = \frac{1}{\|a_i\|} a_i$ to create the orthonormal basis $\{\bar{a}_1, \ldots, \bar{a}_n\}$.

Tentative advice: Focus on understanding GS in two dimensions. Then, look at the multidimensional process as working iteratively by each 2-D subspace. That is we might find a_3 by combining orthonormalizations of b_3 and a_1, then b_3 and a_2.

a.) Is the set of vectors $\{(1, 0), (0, 1), (1, 1)\}$ orthogonal? Hint: you can answer this by counting and citing previous results.

b.) Is this an inner product? $\langle x, y\rangle = \|x\|\|y\|$?

c.) Use Gram-Schmidt to find an orthonormal basis for P_2. Hint: start with the basis $\{1, x, x^2\}$. Use $\langle f, g\rangle = \int_{[-1,1]} fg dx$.

d.) Write the vector $a = (1, -1, 1)$ as a linear combination of the vectors $v_1 = (1, 1, 1), v_2 = (0, 1, -1), v_3 = (-2, 1, 1)$.

e.) Orthonormalize the basis $\{(1, 2), (3, 4)\}$.

Not Quiz 10 Solutions

a.) This is not a set of linearly independent vectors, because we cannot have three vectors in $\mathbb{R}^2$ be linearly independent (at most two such vectors can be). Because we know a set of orthogonal vectors is linearly independent, these cannot be orthogonal.

b.) This is not an inner product because it fails the property $\langle x+a, y\rangle = \langle x, y\rangle + \langle a, y\rangle$. Consider $x, y, a \in \mathbb{R}^1$. Let $x = y = 1$ and $a = -1$. Then $\langle x+a, y\rangle = \langle 0, 1\rangle = 0$. However $\langle x, y\rangle + \langle a, y\rangle = 1+1 = 2$, meaning this cannot be an inner product because it fails to satisfy the stated property.

c.) We use $\langle f, g\rangle = \int_{[-1,1]} fg dx$.

$$
\begin{aligned}
a_1 &= 1 \\
a_2 &= x - \frac{\langle x, 1\rangle}{\langle 1, 1\rangle} 1 \\
a_3 &= x^2 - \frac{\langle x^2, 1\rangle}{\langle 1, 1\rangle} 1 - \frac{\langle x^2, a_2\rangle}{\langle a_2, a_2\rangle} a_2
\end{aligned}
$$

We can calculate

$$
\begin{aligned}
\langle x, 1\rangle &= \int_{-1}^{1} x dx = 0 \\
\langle 1, 1\rangle &= \int_{-1}^{1} 1 dx = 2
\end{aligned}
$$

So that $a_2 = x$. We proceed to find a_3.

$$
\begin{aligned}
\langle x^2, 1\rangle &= \int_{-1}^{1} x^2 dx = \frac{2}{3} \\
\langle x^2, a_2\rangle &= \int_{-1}^{1} x^3 dx = 0 \\
\langle a_2, a_2\rangle &= \int_{-1}^{1} x^2 dx = \frac{2}{3}
\end{aligned}
$$

Thus $a_1 = 1, a_2 = x$, and $a_3 = x^2 - \frac{\frac{2}{3}}{2} a_1 - 0 a_2 = x^2 - \frac{1}{3}$.

d.) v_1, v_2, v_3 are orthogonal. therefore we must have

$a = \frac{\langle a, v_1\rangle}{\langle v_1, v_1\rangle} v_1 + \frac{\langle a, v_2\rangle}{\langle v_2, v_2\rangle} v_2 + \frac{\langle a, v_3\rangle}{\langle v_3, v_3\rangle} v_3$.

Computation gives $a = \frac{1}{3}v_1 - 1v_2 - \frac{1}{3}v_3$.

e.) We use the standard inner product. Row vectors are used in place of the usual column vectors because they are easier to type. Please don't be mad.

$$\begin{aligned} a_1 &= (1,2) \\ a_2 &= (3,4) - \frac{\langle (3,4),(1,2)\rangle}{\langle (1,2),(1,2)\rangle}(1,2) \end{aligned}$$

We reduce the last line,

$$a_2 = (3,4) - \frac{11}{5}(1,2) = (4/5, -2/5)$$

.

We can verify these are orthogonal $(1,2)\cdot(4/5,-2/5) = 4/5 - 4/5 = 0$.

Now let's normalize to vectors $\frac{1}{\sqrt{5}}(1,2)$ and $\frac{1}{\sqrt{\frac{4}{5}}}(4/5,-2/5)$.

Name:
Section (time):

Math 340 Quiz 11

1.) Consider $\mathbb{R}^2$ with the standard inner product. Let $v = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$. Find the orthogonal complement of $\operatorname{span}(\{v\})$. Use the standard inner product.

2.) Let $S = \{t+3, t^2\}$ be a basis for the subspace W of the space P_2. Find an orthonormal basis for W. Use inner product $\langle f, g\rangle = \int_{-1}^{1} f(t)g(t)dt$. You don't have to reduce fractions.

Name:
Section (time):

Math 340 Quiz 11

1.) Consider $\mathbb{R}^2$ with the standard inner product. Let $v = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$. Find the orthogonal complement of $\operatorname{span}(\{v\})$. Use the standard inner product.

2.) Let $S = \{\frac{1}{2}, t-1\}$ be a basis for the subspace W of the space P_2. Find an orthonormal basis for W. Use inner product $\langle f, g\rangle = \int_{-1}^{1} f(t)g(t)dt$.

GS Process

$v_1 = t + 3$

$v_2 = t^2 - \frac{\langle t^2, t+3 \rangle}{\langle t+3, t+3 \rangle}(t+3)$

$v_2 = t^2 - \frac{2}{56/3}(t+3)$

It doesn't matter if you normalize as you go or all at the end when using Gram-Schmidt.

You might proceed one of two ways. You might normalize as you go or all at the end. These are equivalent. Consider the following illustration.

Suppose we have two vectors in a basis $\{a_1, a_2\}$ and we want to convert it into an orthonormal basis $\{v_1, v_2\}$.

Option 1: First we create an orthogonal basis $\{u_1, u_2\}$.

$$\begin{aligned} u_1 &= a_1 \\ u_2 &= a_2 - \frac{\langle u_1, a_2 \rangle}{\langle u_1, u_1 \rangle} u_1 \end{aligned}$$

Then let $v_i = \frac{u_i}{\|u_i\|}$.

Option 2: Normalizing as you go.

$$\begin{aligned} v_1 &= \frac{a_1}{\|a_1\|} \\ u_2 &= a_2 - \langle v_1, a_2 \rangle v_1 \\ v_2 &= \frac{u_2}{\|u_2\|} \end{aligned}$$

Now, I want to show you that $a_2 - \langle v_1, a_2 \rangle v_1 = a_2 - \frac{\langle u_1, a_2 \rangle}{\langle u_1, u_1 \rangle} u_1$, meaning that it doesn't matter if you normalize the vector $i - 1$ before calculating vector i or not.

Observe $v_1 = \frac{u_1}{\|u_1\|}$.

So,

$$a_2 - \langle v_1, a_2 \rangle v_1 = a_2 - \langle \frac{u_1}{\|u_1\|}, a_2 \rangle \frac{u_1}{\|u_1\|}.$$

Now we can pull the scalar out of the inner product,

$$= a_2 - \frac{1}{\|u_1\|} \langle u_2, a_2 \rangle \frac{u_1}{\|u_1\|} = a_2 - \frac{1}{\|u_1\|^2} \langle u_2, a_2 \rangle u_1,$$

or

$$= a_2 - \frac{\langle u_1, a_2 \rangle}{\langle u_1, u_1 \rangle} u_1,$$

which is what we had for the normalize-just-at-the-end approach.

Not Quiz 11

Projections: Let $\{v_1, ...v_n\}$ be an orthonormal basis for W. Then, the projection of a vector u onto W is

$$\text{proj}_W u = \sum_{i=1}^{n} \langle u, v_i \rangle v_i.$$

Orthogonal Complement: Let $\mathcal{V}$ be an inner product space and $\mathcal{S}$ is a subset of $\mathcal{V}$. The orthogonal complement of $\mathcal{S}$ is defined by

$$\mathcal{S}^\perp = \{a \in \mathcal{V} \mid \langle a, b \rangle = 0, \text{ for all } b \in \mathcal{S}\}.$$

P.S. Every vector $v \in \mathcal{V}$ can be written as $v = s + s^\star$ where $s \in \mathcal{S}$ and $s^\star \in \mathcal{S}^\perp$.

Linearity:

A linear transformation $T : \mathcal{V} \to \mathcal{W}$ satisfies

1. $T(x + y) = T(x) + T(y)$, for all $x, y \in \mathcal{V}$
2. $T(\alpha x) = \alpha T(x)$ for all $\alpha \in \mathbb{R}$ and $x \in \mathcal{V}$.

The **standard matrix representing a linear transformation** $L : \mathbb{R}^n \to \mathbb{R}^m$, given the natural basis $\{e_1, \dots, e_n\}$ for $\mathbb{R}^n$ is A where the j^{th} column is $L(e_j)$.

a.) Can the orthogonal complement of a plane (two-dimensional subspace) in $\mathbb{R}^3$ be another plane?

b.) Find the orthogonal complements of the subspaces

$$\mathcal{W} = \left\{\mathbf{x} \in \mathbb{R}^3 \mid x_1 + 2x_2 + 3x_3 = 0\right\}$$

$$\mathcal{S} = \left\{\mathbf{x} \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 0 \text{ and } x - y + z = 0\right\}$$

c.) Find a basis for the orthogonal complement of the subspace

$$\mathcal{W} = \left\{\mathbf{x} \in \mathbb{R}^3 \mid \alpha x_1 + \beta x_2 + \gamma x_3 = 0\right\}$$

d.) What is the orthogonal complement of $\mathbf{0} \in \mathbb{R}^n$?

e.) Show $\text{proj}_W(\text{proj}_W u) = \text{proj}_W u$.

f.) Show $R(x, y) = (x \cos\theta - y \sin\theta, x \sin\theta + y \cos\theta)$ is a linear transformation.

HW11.) Find the standard matrix representing each given linear transformation.

i. $L\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) = \begin{bmatrix} u_2 \\ u_1 \end{bmatrix}$

ii. $L\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) = \begin{bmatrix} u_1 - 3u_2 \\ 2u_1 - u_2 \\ 2u_2 \end{bmatrix}$

iii. $L\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\right) = \begin{bmatrix} u_1 + 4u_2 \\ -u_3 \\ u_2 + u_3 \end{bmatrix}$

HW15.) Let $L : P_2 \to P_3$ be a linear transformation for which we knot that $L(1) = 1, L(t) = t^2$, and $L(t^2) = t^3 + t$. Find $L(2t^2 - 5t + 3)$ and $L(at^2 + bt + c)$.

Not Quiz 11 Solutions

a.) No, two planes would intersect at infinitely many points. But $W \cap W^{\perp} = \{0\}$.

b.) $\mathcal{W}^{\perp}$ will be a line because $\mathcal{W}$ is a plane. Let $\{(a, b, c)\}$ be a basis for $\mathcal{W}^{\perp}$ and $(x, y, z) \in \mathcal{W}$.

By properties of the orthogonal complement,

$$\langle (a, b, c), (x, y, z) \rangle = ax + by + cz = 0.$$

For $(x, y, z) \in \mathcal{W}$, $ax + by + cz = 0$ when $a = 1 =, b = 2, c = 3$. therefore, $(1, 2, 3) \in \mathcal{W}^{\perp}$. Because $\mathcal{W}^{\perp}$ is 1 dimensional, the orthogonal complement is the span of $(1, 2, 3)$.

Now, we find $\mathcal{S}^{\perp}$. As the intersection of two planes, $\mathcal{S}$ is a line. Its orthogonal complement will therefore be a line. A basis for $\mathcal{S}$ is $\{(1, 0, -1)\}$, since it lies on both planes defining $\mathcal{S}$.

The orthogonal complement must therefore be the plane containing points x such that $(1, 0, -1) \cdot x = 0$. That means

$$\mathcal{S}^{\perp} = \{x \in \mathbb{R}^3 \mid x_1 - x_3 = 0\}.$$

c.) See the first part of b. The basis is a set consisting of (α, β, γ).

d.) The orthogonal complement is $\mathbb{R}^n$. Because the original space is 0-dimensional, the complement must be n dimensional. Further, we might realize that any vector $x \in \mathbb{R}^n$ must be represented as $x = 0 + x^{\star}$, where $x^{\star} \in \{0\}^{\perp}$. We find $x = x^{\star}$, and because x is arbitrary, this means that $\{0\}^{\perp} = \mathbb{R}^n$.

e.) This is going to be annoying. Let an orthonormal basis for W be $\{w_1, \ldots, w_n\}$.

$$\text{proj}_W u = \sum_{i=1}^{n} \langle w_i, u \rangle w_i.$$

So,

$$\text{proj}_W(\text{proj}_W u) = \sum_{i=1}^{n} \langle w_i, \text{proj}_W u \rangle w_i.$$

$$= \sum_{i=1}^{n} \langle w_i, \sum_{i=1}^{n} \langle w_i, u \rangle w_i \rangle w_i.$$

$$= \sum_{i=1}^{n} \langle w_i, \langle w_1, u \rangle w_1 + \cdots + \langle w_n, u \rangle w_n \rangle w_i.$$

$$= \sum_{i=1}^{n} \langle w_1, u\rangle \langle w_i, w_1\rangle w_i + \cdots + \langle w_n, u\rangle \langle w_i, w_n\rangle w_i.$$

Using orthogonality, terms $\langle w_i, w_j\rangle = 0$ for $i \neq j$. Normality means the inner product is 1 if $i = j$.

$$= \langle w_1, u\rangle \langle w_1, w_1\rangle w_1 + \cdots + \langle w_n, u\rangle \langle w_n, w_n\rangle w_n.$$

$$= \langle w_1, u\rangle w_1 + \cdots + \langle w_n, u\rangle w_n.$$

$$= \sum_{i=1}^{n} \langle w_i, u\rangle w_i.$$

f.) Verify $R(x + a, y + b) = R(x, y) + R(a, b)$ and $R(\alpha x, \alpha y) = \alpha R(x, y)$.

Name:
Section (time):

Math 340 Quiz 12

1.) Let $\langle \cdot, \cdot\rangle$ be an inner product (do not assume it is the dot product). Is the following function a linear transformation? $L : \mathbb{R} \to \mathbb{R}$,

$$L(x) = \langle x, 91\rangle.$$

2.) Let A and B be square matrices. Show that if B is similar to A, then A is similar to B.

Not Quiz 12

Quiz before I made it a little easier.) Let $\langle \cdot, \cdot \rangle$ be an inner product. Is the following function a linear transformation? $L : \mathcal{V} \to \mathbb{R}$,

$$L_y(x) = \langle x, y \rangle.$$

a.) Is $L(x, y) = \sqrt{xy}$ a linear transformation?

b.) Is $H(x, y) = \langle x, y \rangle$ a linear transformation?

c.) Is $T(x, y, z) = 91$ a linear transformation?

d.) Let $L : P_2 \to P_1$ be the linear transformation defined by

$$L(at^2 + bt + c) = (a + b)t + (b - c).$$

Find a basis for ker and range L.

e.) *HW25 from 6.3* Let $L : \mathbb{R}^4 \to \mathbb{R}^6$ be a linear transformation. If dim ker $L = 2$, find dim range L? If dim range $L = 3$, what is dim ker L?

f.) Let $T(x, y, z) = (x+y, y+z)$. Calculate the matrix of T relative to the standard bases of $\mathbb{R}^3$ and $\mathbb{R}^2$. Then, relative to the bases $\{(1, 0, 0), (0, 0, 1), (1, -1, 1)\}$ and $\{(1, 0), (0, 1)\}$.

g.) Find the value of $T(1, 1, -1)$ for the linear transformation $T : \mathbb{R}^3$ whose matrix relative to the standard basis and $\{1, x, x^2\}$ is

$$\begin{bmatrix} 1 & 0 & -1 \\ 2 & 4 & -3 \\ 3 & 0 & 2 \end{bmatrix}.$$

Theorem 6.12 Let $L : V \to W$ be a linear transformation with matrix A. Let S and S' be ordered bases for V and T and T' be ordered bases for W. Let P and Q be the transition matrices from S to S' and T and T', respectively. Then $Q^{-1}AP$ is the representation of L with respect to S' and T'.

Definition Matrix B is similar to A if $B = P^{-1}AP$.

HW9.) Let $L : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation with matrix

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 3 & 1 & 0 \end{bmatrix}$$

with respect to $S = \{(1, 0, -1), (0, 2, 0), (1, 2, 3)\}$ and $T = \{(1, -1), (2, 0)\}$.

Find the representation of L with respect to the natural bases for $\mathbb{R}^3$ and $\mathbb{R}^2$.

h.) Let λ be the eigenvalues of A. Find the eigenvalues of A^n and $(A + cI)$. Recall $Ax = \lambda x$ for any eigenvalue λ and an associated eigenvector x.

Not Quiz 12

Quiz before I made it a little easier.) Let $\langle \cdot, \cdot \rangle$ be an inner product. Is the following function a linear transformation? $L : \mathcal{V} \to \mathbb{R}$,

$$L_y(x) = \langle x, y \rangle.$$

a.) Is $L(x, y) = \sqrt{xy}$ a linear transformation?

b.) Is $H(x, y) = \langle x, y \rangle$ a linear transformation?

c.) Is $T(x, y, z) = 91$ a linear transformation?

d.) Let $L : P_2 \to P_1$ be the linear transformation defined by

$$L(at^2 + bt + c) = (a + b)t + (b - c).$$

Find a basis for ker and range L.

e.) *HW25 from 6.3* Let $L : \mathbb{R}^4 \to \mathbb{R}^6$ be a linear transformation. If dim ker $L = 2$, find dim range L? If dim range $L = 3$, what is dim ker L?

f.) Let $T(x, y, z) = (x+y, y+z)$. Calculate the matrix of T relative to the standard bases of $\mathbb{R}^3$ and $\mathbb{R}^2$. Then, relative to the bases $\{(1, 0, 0), (0, 0, 1), (1, -1, 1)\}$ and $\{(1, 0), (0, 1)\}$.

g.) Find the value of $T(1, 1, -1)$ for the linear transformation $T : \mathbb{R}^3$ whose matrix relative to the standard basis and $\{1, x, x^2\}$ is

$$\begin{bmatrix} 1 & 0 & -1 \\ 2 & 4 & -3 \\ 3 & 0 & 2 \end{bmatrix}.$$

Theorem 6.12 Let $L : V \to W$ be a linear transformation with matrix A. Let S and S' be ordered bases for V and T and T' be ordered bases for W. Let P and Q be the transition matrices from S to S' and T and T', respectively. Then $Q^{-1}AP$ is the representation of L with respect to S' and T'.

Definition Matrix B is similar to A if $B = P^{-1}AP$.

HW9.) Let $L : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation with matrix

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 3 & 1 & 0 \end{bmatrix}$$

with respect to $S = \{(1, 0, -1), (0, 2, 0), (1, 2, 3)\}$ and $T = \{(1, -1), (2, 0)\}$.

Find the representation of L with respect to the natural bases for $\mathbb{R}^3$ and $\mathbb{R}^2$.

h.) Let λ be the eigenvalues of A. Find the eigenvalues of A^n and $(A + cI)$. Recall $Ax = \lambda x$ for any eigenvalue λ and an associated eigenvector x.

Not Quiz 12—Solution Sketches

Quiz before I made it a little easier.) Just, like the quiz, this will be a linear transformation. Note y is a parameter and not an input in the function.

a.) This is not linear. Though it is true that $\alpha L(x, y) = L(\alpha x, \alpha y)$, the function fails additivity. Note $L(0, 1) = L(1, 0) = 0$. But $L(1, 1) = 1$. Additivity would require $L(0, 1) + L(1, 0) = L(1, 1)$.

b.) This is not linear. Observe $H(\alpha x, \alpha y) = \alpha^2 H(x, y)$. Additivity would also fail.

c.) This is not linear, failing both the scalar thing (technical name is something like homogeneous of degree 1) and additivity. $T(x, y, z) + T(a, b, c) = 91 + 91 \neq T(x + a, y + b, z + c) = 91$.

d.) Kernel:

We must have $a + b = 0$ and $b - c = 0$, or $a = -b$ and $b = c$.

Thus, one vector/polynomial kernel is $-x^2 + x + 1$. We claim this is a basis.

The range will be all polynomials in P_1. Note, we can achieve any polnyomial $\beta_1 x + \beta_2$ by letting $a = \beta_1$ and $c = \beta_2$. So, our basis may include x and 1.

Note that $\dim ker + \dim range = \dim P_2$, and this holds given the bases selected above.

e.) dim range L is 2 and dim ker L 1.

f.)

$$\begin{aligned} T(1, 0, 0) &= (1, 0) \\ T(0, 1, 0) &= (1, 1) \\ T(0, 0, 1) &= (0, 1) \end{aligned}$$

so A binds these as columns as a matrix,

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Then, using the other bases, $T(1, -1, 1) = (0, 0)$.

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

g.)

$$\begin{aligned}T(1,1,-1) &= T(e_1+e_2-e_3)\\ &= T(e_1)+T(e_2)-T(e_3)\\ &= (1+2x+3x^2)+(4x)-(-1-3x+2x^2) \quad = 1+1+2x+4x+3x+3x^2-2x^2\\ &= 2+9x+x^2.\end{aligned}$$

h.) If $Ax = \lambda x$. Then $A^n x = A^{n-1}\lambda x = \lambda A^{n-1}x = \lambda^n x$.

The above is probably hazy. Try with $n = 2$, see what the above line implies. Then, successive application of that trick should get you the result.

If $Ax = \lambda x$ then $(A + cI)x = (\lambda + c)x$. So, the eigenvalues are shifted up by c .

2 Midterm Stuff

Midterm I Solution Sketches for Pages 2-4[4]

1. We are given

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \\ 3 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 3 \\ -1 & -1 \\ -5 & 4 \end{bmatrix}, \text{ and } C = \begin{bmatrix} 1 & 2 \\ 4 & -8 \end{bmatrix}.$$

a.)

$$2A - B = \begin{bmatrix} 2-1 & 4-3 \\ -2+1 & 8+1 \\ 6+5 & 2-4 \end{bmatrix}.$$

b.)

$$C^{-1} = \frac{1}{-8-8}\begin{bmatrix} -8 & 2 \\ -4 & 1 \end{bmatrix}.$$

c.) AC will have dimension 3×2,

$$AC = \begin{bmatrix} 1+8 & 2-16 \\ -1+16 & -2-32 \\ 3+4 & 6-8 \end{bmatrix} = \begin{bmatrix} 9 & -14 \\ 15 & -34 \\ 7 & -2 \end{bmatrix}.$$

2.

a.) det(D) = -1(-5-0)+1(-1-0) = 5-1=4

[4] I just made these to help me grade. I made some arithmetic errors somewhere, and I can't remember if I fixed them or not. :) :(

b.)

$$\mathrm{rref}(D) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

c.) The above shows this matrix is nonsingular. Therefore, only the trivial solution exists. That is, the solution set is $\{\mathbf{0}\}$. (Corollary 3.1 in the textbook)

d.)

$$A^T x = \begin{bmatrix} 1 & -1 & 3 \\ 2 & 4 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \mathbf{0}$$

We know that x must have dimension 3×1 to be a vector and give a valid multiplication.

This give two equations with three unknowns;

$$x_1 - x_2 + x_3 = 0,$$
$$2x_1 + 4x_2 + x_3 = 0.$$

The solution set is $\{x \in \mathbb{R}^3 \mid x_2 = \frac{-5}{13}x_1 \text{ and } x_3 = \frac{-6}{13}x_1\}$.

3.

a.) $Ax = 0$ blah blah

b.) A square matrix is singular if the determinant is zero. For these matrices, no inverse exists, so that if S is singular, then there does not exist a matrix S^{-1} such that $SS^{-1} = I$. For example,

$$S = \begin{bmatrix} 0 \end{bmatrix}.$$

c.) Two matrices are row equivalent if one is the product of a sequence of elementary row operations and the other.

Elementary row operations are row swaps, addition of rows, and scaling of a row by a constant.

[1] and [2] are row equivalent. But perhaps you would like a less trivial example:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, B = \hat{E}A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

B is obtained from A by adding row 1 to row 2 and swapping the resulting row 2 with row 1. You might also note that D and I from the previous question are row equivalent, as I=rref(D).

Midterm II Solution Sketches for Pages 4 and 5[5]

4.)

(a) The rank of a matrix is the number of LI rows or columns.

(b) set of linear combos

(c) number of vectors in a basis

(d) Subset of vector space that is closed under the addition and scalar multiplication operations.

(e)

5.) Rank + Nullity = No. of Columns

6.)

a.) We Want $\langle x+k, x^2\rangle = 0$.

$$\int_{-1}^{1}(x+k)x^2dx = \int_{-1}^{1} x^3 + kx^2dx$$

$$= \frac{1}{4}x^4 + \frac{k}{3}x^3\Big|_{-1}^{1}$$

$$= (\frac{1}{4} + \frac{k}{3}) - (\frac{1}{4} - \frac{k}{3})$$

$$= \frac{2k}{3}$$

$$\implies k = 0.$$

b.) We know x^2 is parallel to αx^2 for any scalar $\alpha \in \mathbb{R}$. So, we simply solve for α so that $\langle \alpha x^2, \alpha x^2\rangle = 1$.

$$\int_{-1}^{1} \alpha^2 x^4 dx = \alpha^2 \frac{1}{5}x^5\Big|_{-1}^{1}$$

$$= \frac{1}{5}(\alpha^2 + \alpha^2)$$

So we solve $\frac{2}{5}\alpha^2 = 1$.

[5] I just made these to help me grade. Don't expect anything polished.

$$\alpha^2 = \frac{5}{2} \implies \alpha = \pm\frac{\sqrt{5}}{\sqrt{2}}.$$

You can also try to find a polynomial so that $\cos\theta = \pm 1$.

c.) This question asks if the angle is obtuse. An angle is obtuse when the inner product is negative. So we check $\langle x+1, x^2\rangle$.

Recall that $\langle x, x^2\rangle = 0$. Using the properties of inner products,

$$\langle x+1, x^2\rangle = \underbrace{\langle x, x^2\rangle}_{=0} + \langle 1, x^2\rangle$$

So we only have to check $\int_{-1}^{1} x^2 dx$. Because $x^2 \geq 0$, $\int x^2$ is always positive over a nondegenerate interval. Therefore, the angle is acute. Therefore, the angle is less than $\frac{\pi}{2}$.