

Name:

Section (time):

Math 340 Quiz 11

- 1.) Consider \mathbb{R}^2 with the standard inner product. Let $v = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$. Find the orthogonal complement of $\text{span}(\{v\})$. Use the standard inner product.
- 2.) Let $S = \{t + 3, t^2\}$ be a basis for the subspace W of the space P_2 . Find an orthonormal basis for W . Use inner product $\langle f, g \rangle = \int_{-1}^1 f(t)g(t)dt$. You don't have to reduce fractions.

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- 2.) Let $S = \{\frac{1}{2}, t - 1\}$ be a basis for the subspace W of the space P_2 . Find an orthonormal basis for W . Use inner product $\langle f, g \rangle = \int_{-1}^1 f(t)g(t)dt$.

GS Process

$$v_1 = t + 3$$

$$v_2 = t^2 - \frac{\langle t^2, t+3 \rangle}{\langle t+3, t+3 \rangle} (t + 3)$$

$$v_2 = t^2 - \frac{2}{56/3} (t + 3)$$

It doesn't matter if you normalize as you go or all at the end when using Gram-Schmidt.

You might proceed one of two ways. You might normalize as you go or all at the end. These are equivalent. Consider the following illustration.

Suppose we have two vectors in a basis $\{a_1, a_2\}$ and we want to convert it into an orthonormal basis $\{v_1, v_2\}$.

Option 1: First we create an orthogonal basis $\{u_1, u_2\}$.

$$\begin{aligned} u_1 &= a_1 \\ u_2 &= a_2 - \frac{\langle u_1, a_2 \rangle}{\langle u_1, u_1 \rangle} u_1 \end{aligned}$$

Then let $v_i = \frac{u_i}{\|u_i\|}$.

Option 2: Normalizing as you go.

$$\begin{aligned} v_1 &= \frac{a_1}{\|a_1\|} \\ u_2 &= a_2 - \langle v_1, a_2 \rangle v_1 \\ v_2 &= \frac{u_2}{\|u_2\|} \end{aligned}$$

Now, I want to show you that $a_2 - \langle v_1, a_2 \rangle v_1 = a_2 - \frac{\langle u_1, a_2 \rangle}{\langle u_1, u_1 \rangle} u_1$, meaning that it doesn't matter if you normalize the vector $i - 1$ before calculating vector i or not.

Observe $v_1 = \frac{u_1}{\|u_1\|}$.

So,

$$a_2 - \langle v_1, a_2 \rangle v_1 = a_2 - \left\langle \frac{u_1}{\|u_1\|}, a_2 \right\rangle \frac{u_1}{\|u_1\|}.$$

Now we can pull the scalar out of the inner product,

$$= a_2 - \frac{1}{\|u_1\|} \langle u_2, a_2 \rangle \frac{u_1}{\|u_1\|} = a_2 - \frac{1}{\|u_1\|^2} \langle u_2, a_2 \rangle u_1,$$

or

$$= a_2 - \frac{\langle u_1, a_2 \rangle}{\langle u_1, u_1 \rangle} u_1,$$

which is what we had for the normalize-just-at-the-end approach.

Not Quiz 11

Projections: Let $\{v_1, \dots, v_n\}$ be an orthonormal basis for W . Then, the projection of a vector u onto W is

$$\text{proj}_W u = \sum_{i=1}^n \langle u, v_i \rangle v_i.$$

Orthogonal Complement: Let \mathcal{V} be an inner product space and \mathcal{S} is a subset of \mathcal{V} . The orthogonal complement of \mathcal{S} is defined by

$$\mathcal{S}^\perp = \{a \in \mathcal{V} \mid \langle a, b \rangle = 0, \text{ for all } b \in \mathcal{S}\}.$$

P.S. Every vector $v \in \mathcal{V}$ can be written as $v = s + s^\star$ where $s \in \mathcal{S}$ and $s^\star \in \mathcal{S}^\perp$.

Linearity:

A linear transformation $T : \mathcal{V} \rightarrow \mathcal{W}$ satisfies

1. $T(x + y) = T(x) + T(y)$, for all $x, y \in \mathcal{V}$
2. $T(\alpha x) = \alpha T(x)$ for all $\alpha \in \mathbb{R}$ and $x \in \mathcal{V}$.

The **standard matrix representing a linear transformation** $L : \mathbb{R}^n \rightarrow \mathbb{R}^m$, given the natural basis $\{e_1, \dots, e_n\}$ for \mathbb{R}^n is A where the j^{th} column is $L(e_j)$.

- a.) Can the orthogonal complement of a plane (two-dimensional subspace) in \mathbb{R}^3 be another plane?
- b.) Find the orthogonal complements of the subspaces

$$\mathcal{W} = \{\mathbf{x} \in \mathbb{R}^3 \mid x_1 + 2x_2 + 3x_3 = 0\}$$

$$\mathcal{S} = \{\mathbf{x} \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 0 \text{ and } x - y + z = 0\}$$

- c.) Find a basis for the orthogonal complement of the subspace

$$\mathcal{W} = \{\mathbf{x} \in \mathbb{R}^3 \mid \alpha x_1 + \beta x_2 + \gamma x_3 = 0\}$$

- d.) What is the orthogonal complement of $\mathbf{0} \in \mathbb{R}^n$?
 - e.) Show $\text{proj}_W(\text{proj}_W u) = \text{proj}_W u$.
 - f.) Show $R(x, y) = (x \cos \theta - y \sin \theta, x \sin \theta + y \cos \theta)$ is a linear transformation.
- HW11.) Find the standard matrix representing each given linear transformation.

$$\text{i. } L\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) = \begin{bmatrix} u_2 \\ u_1 \end{bmatrix}$$

$$\text{ii. } L\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) = \begin{bmatrix} u_1 - 3u_2 \\ 2u_1 - u_2 \\ 2u_2 \end{bmatrix}$$

$$\text{iii. } L\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\right) = \begin{bmatrix} u_1 + 4u_2 \\ -u_3 \\ u_2 + u_3 \end{bmatrix}$$

HW15.) Let $L : P_2 \rightarrow P_3$ be a linear transformation for which we know that $L(1) = 1$, $L(t) = t^2$, and $L(t^2) = t^3 + t$. Find $L(2t^2 - 5t + 3)$ and $L(at^2 + bt + c)$.

Not Quiz 11 Solutions

a.) No, two planes would intersect at infinitely many points. But $W \cap W^\perp = \{0\}$.

b.) \mathcal{W}^\perp will be a line because \mathcal{W} is a plane. Let $\{(a, b, c)\}$ be a basis for \mathcal{W}^\perp and $(x, y, z) \in \mathcal{W}$.

By properties of the orthogonal complement,

$$\langle (a, b, c), (x, y, z) \rangle = ax + by + cz = 0.$$

For $(x, y, z) \in \mathcal{W}$, $ax + by + cz = 0$ when $a = 1, b = 2, c = 3$. therefore, $(1, 2, 3) \in \mathcal{W}^\perp$. Because \mathcal{W}^\perp is 1 dimensional, the orthogonal complement is the span of $(1, 2, 3)$.

Now, we find \mathcal{S}^\perp . As the intersection of two planes, \mathcal{S} is a line. Its orthogonal complement will therefore be a plane. A basis for \mathcal{S} is $\{(1, 0, -1)\}$, since it lies on both planes defining \mathcal{S} .

The orthogonal complement must therefore be the plane containing points x such that $(1, 0, -1) \cdot x = 0$. That means

$$\mathcal{S}^\perp = \{x \in \mathbb{R}^3 \mid x_1 - x_3 = 0\}.$$

c.) See the first part of b. The basis is a set consisting of (α, β, γ) .

d.) The orthogonal complement is \mathbb{R}^n . Because the original space is 0-dimensional, the complement must be n dimensional. Further, we might realize that any vector $x \in \mathbb{R}^n$ must be represented as $x = 0 + x^*$, where $x^* \in \{0\}^\perp$. We find $x = x^*$, and because x is arbitrary, this means that $\{0\}^\perp = \mathbb{R}^n$.

e.) This is going to be annoying. Let an orthonormal basis for W be $\{w_1, \dots, w_n\}$.

$$\text{proj}_W u = \sum_{i=1}^n \langle w_i, u \rangle w_i.$$

So,

$$\text{proj}_W(\text{proj}_W u) = \sum_{i=1}^n \langle w_i, \text{proj}_W u \rangle w_i.$$

$$\begin{aligned} &= \sum_{i=1}^n \langle w_i, \sum_{j=1}^n \langle w_j, u \rangle w_j \rangle w_i \\ &= \sum_{i=1}^n \langle w_i, \langle w_1, u \rangle w_1 + \dots + \langle w_n, u \rangle w_n \rangle w_i. \end{aligned}$$

$$= \sum_{i=1}^n \langle w_1, u \rangle \langle w_i, w_1 \rangle w_i + \cdots + \langle w_n, u \rangle \langle w_i, w_n \rangle w_i.$$

Using orthogonality, terms $\langle w_i, w_j \rangle = 0$ for $i \neq j$. Normality means the inner product is 1 if $i = j$.

$$= \langle w_1, u \rangle \langle w_1, w_1 \rangle w_1 + \cdots + \langle w_n, u \rangle \langle w_n, w_n \rangle w_n.$$

$$= \langle w_1, u \rangle w_1 + \cdots + \langle w_n, u \rangle w_n.$$

$$= \sum_{i=1}^n \langle w_i, u \rangle w_i.$$

f.) Verify $R(x+a, y+b) = R(x, y) + R(a, b)$ and $R(\alpha x, \alpha y) = \alpha R(x, y)$.