Сашко Петров.
Домашка #2

Notebook з моделюваннями
можна знайти за адресою (GitHub):
github.com/alexandervpetrov
і далі у репозиторії $\boxed{\text{kau-data-analysis}}$

1. Див. [GitHub].

2. $X_1, \dots, X_n \sim U(0;1)$; $[a;b] \subset [0;1]$

$$Y_k = \begin{cases} 1, & X_k \in [a;b] \\ 0, & \text{—"—} \end{cases}$$

ⓐ $EY_k = 1 \cdot P(X_k \in [a;b]) = b-a$

ⓑ Нехай $\xi_{[a;b]} = \begin{cases} 1, & \text{якщо } \exists k: X_k \in [a;b] \\ 0, & \text{—"—} \end{cases}$

$$E\xi_{[a;b]} = 1 \cdot P(\exists k : X_k \in [a;b]) = 1 - P(\forall k : X_k \notin [a;b]) =$$

$$= 1 - \left[P(X_n \notin [a;b])\right]^n = 1 - (1-(b-a))^n$$

Для розбиття $[0;\frac{1}{m}), [\frac{1}{m};\frac{2}{m}), \dots, [\frac{m-1}{m};1]$

нехай $\eta = \sum_{k=1}^{m} \xi_{[\frac{k-1}{m};\frac{k}{m}]}$

Тоді середня кількість інтервалів без точок буде:

$$E(m-\eta) = m - E\eta = m - \sum_{k=1}^{m} E\xi_{[\frac{k-1}{m};\frac{k}{m}]} =$$

$$= m - \sum_{k=1}^{m}\left[1-\left(1-\frac{1}{m}\right)^n\right] = m\left(1-\frac{1}{m}\right)^n =: f(m)$$

~~Треба знайти~~ $f(\cancel{m}) \uparrow$.

Треба знайти min m: $f(m) > 0.02$.

Далі див. [GitHub].

3. $EY_n \to a$; $DY_n = EY_n^2 - (EY_n)^2 \longrightarrow 0$

$$\Downarrow$$

$$EY_n^2 = (EY_n)^2 + o(n)$$

$$E(Y_n - a)^2 = EY_n^2 - 2aEY_n + a^2 = \underbrace{\underset{\downarrow a}{(EY_n)^2} - 2a\underset{\downarrow a}{EY_n} + a^2}_{\downarrow 0} + \underset{\downarrow 0}{o(n)}$$

---

4. $U(\theta, 2\theta)$; $\theta > 0$

(a) $$\mathcal{L}(\bar{x}|\theta) = \prod_{k=1}^{n} P(x_k|\theta) = \prod_{k=1}^{n} \frac{1}{\theta} \cdot \mathbb{1}_{x_k \in [\theta, 2\theta]} =$$

$$= \frac{1}{\theta^n} \cdot \mathbb{1}_{(\theta \le \min(X) \wedge \max(X) \le 2\theta)}$$

$\frac{1}{\theta^n} \searrow$, $\Rightarrow \mathcal{L}(\bar{x}|\theta) \to \max$ коли θ - найменше можливе.

$$\frac{1}{2}\max(X) \le \theta \le \min(X)$$

$$\theta^* = \begin{cases} \frac{1}{2}\max(X), & \text{якщо } \frac{1}{2}\max(X) \le \min(X) \\ \text{не існує}, & \text{інакше} \end{cases}$$

не знаю як дослідити на консистентність тут

(б) $E\theta^* = \frac{1}{2}E(\max X) \overset{?}{=} \frac{1}{2}\frac{n}{n+1}\theta$ (з лекції)

за умови $\left(\frac{1}{2}\max(X) \le \min(X)\right)$?

$$E\theta^{*2} = \frac{1}{4}E(\max X)^2 = \frac{1}{2}\int_{-\infty}^{+\infty} x^2 \, p_{\max x_k}(x)\, dx =$$

$$= \int_0^\theta x^2 \cdot \frac{n \cdot x^{n+1}}{\theta^n} dx = \frac{n}{\theta^n}\int_0^\theta x^{n+1} dx = \frac{n}{n+2}\frac{x^{n+2}}{\theta^n}\Big|_0^\theta =$$

$$= \frac{n}{n+2}\theta^2 ; \quad D\theta^* = \frac{1}{4}\left(\frac{n}{n+2} - \frac{n^2}{(n+1)^2}\right)\theta^2 =$$

$$= \frac{\theta^2}{4}\frac{n(2n+1)}{(n+2)(n+1)^2} \longrightarrow 0 \quad \text{консист.?}$$

(б) $D\theta_n^* \to 0$, $n \to \infty$ $\Rightarrow$ консист.

---

5 $U(a;b)$; $p(x|a,b) = \begin{cases} \frac{1}{b-a} ; & x \in [a;b] \\ 0 , & \text{—и—} \end{cases}$

$$L(\bar{x}|a,b) = \frac{1}{(b-a)^n} \prod_{k=1}^{n} \mathbb{1}_{x_k \in [a;b]} = \frac{1}{(b-a)^n} \mathbb{1}_{a \le \min X} \cdot \mathbb{1}_{b \ge \max X}$$

$\frac{1}{(b-a)^n} \nearrow$ по $\underline{\underline{a}}$ и $\searrow$ по b.

$a^* = \min X$

$b^* = \max X$.

---

6 $\alpha > 0$.

(a) $p(x|\alpha) = \alpha \cdot x^{\alpha-1}$, $x \in [0;1]$

$$L(\bar{x}|\alpha) = \prod_{k=1}^{n} \alpha \cdot x_k^{\alpha-1} \cdot \mathbb{1}_{x_k \in [0;1]}$$

$$\ln L(\bar{x}|\alpha) = \sum_{k=1}^{n} \left(\ln \alpha + (\alpha-1) \ln x_k \right)$$

$$\frac{\partial}{\partial \alpha}(..) = \sum_{k} \frac{1}{\alpha} + \ln x_k = 0 \; ;$$

$$\boxed{\alpha^* = -\frac{n}{\sum_{k=1}^{n} \ln x_k}}$$

; якщо $x_k \sim \alpha x^{\alpha-1} \mathbb{1}_{[0;1]}(x)$ то $y_k = \ln x_k \sim$?

$Y = f(X) = \ln X$; $f^{-1}(Y) = e^y$

$$p_Y(y) = p\left(f^{-1}(y)\right) \cdot \frac{1}{f'(f^{-1}(y))}$$

~~$\ln x$~~ $x \in [0;1] \to \ln x \in (-\infty; 0]$

$$\ln X \sim \alpha (e^y)^{\alpha-1} \cdot \frac{1}{e^y} = \boxed{\alpha \cdot e^{y(\alpha-2)} = p_{\ln X}(y)}$$

$$E(\ln X) = \int_{-\infty}^{0} y\,\alpha \cdot e^{y(\alpha-2)}\,dy = -\frac{\alpha}{(\alpha-2)^2} \underbrace{\int_{0}^{+\infty} t\,e^{-t}\,dt}_{=1} = -\frac{\alpha}{(\alpha-2)^2}$$

Значить:

$$\frac{1}{n}\sum_{k=1}^{n} \ln x_k \xrightarrow[n\to\infty]{} -\frac{\alpha}{(\alpha-2)^2}$$

$$\alpha_n^* = \frac{n}{\sum_n \ln x_k} \xrightarrow[n\to\infty]{} -\frac{(\alpha-2)^2}{\alpha} = \alpha - 4 + \frac{4}{\alpha}$$ не консистентна

---

6) $X \sim p(x|\alpha) = \alpha \cdot x^{-(\alpha+1)}; \; x \geq 1$

$$\ln L(x|\alpha) = \sum_{k=1}^{n} \left(\ln\alpha - (\alpha+1)\ln x_k\right)$$

$$\frac{\partial}{\partial\alpha}(\ldots) = \frac{n}{\alpha} - \sum_{k=1}^{n} \ln x_k = 0, \quad \alpha_n^* = \frac{n}{\sum_{k=1}^{n} \ln x_k}$$

$$\ln X \sim \frac{\alpha}{(e^y)^{\alpha+1}} \cdot \frac{1}{e^y} = \alpha \cdot e^{-y(\alpha+2)}$$

$$E(\ln X) = \int_{0}^{+\infty} y \cdot \alpha \cdot e^{-y(\alpha+2)}\,dy = \ldots = \frac{\alpha}{(\alpha+2)^2}$$

$$\alpha_n^* = \frac{n}{\sum_{k=1}^{n} \ln X_k} \xrightarrow[n\to\infty]{} \frac{(\alpha+2)^2}{\alpha} = \alpha + 4 + \frac{1}{\alpha}.$$

---

7) $X_k \sim X \sim p(x)$: $p(x) = 0$, якщо $x < a$

$\exists\delta > 0$: $p(x) > 0$ для $x \in [a; a+\delta]$

$\min X_k \xrightarrow{P} a$?

$\varepsilon > 0$: $P(|\min x_k - a| \geq \varepsilon) = \left[\begin{array}{l}\text{якщо } \min X_k < a \\ \text{то } P(\ldots) = 0\end{array}\right] =$

$= P(\min x_k \geq \varepsilon + a) = \left(P(x_1 \geq \varepsilon + a)\right)^n =$

$$= \Big[\underbrace{\int_{a+\varepsilon}^{+\infty} p(t)\,dt}_{<1}\Big]^n \xrightarrow[n\to\infty]{} 0$$

тому що $p(x) > 0$ $[a; a+\min(\varepsilon,\delta)]$

8 $\alpha > 0$.

a: $p(x|\alpha, x_0) = \alpha \cdot e^{-\alpha(x-x_0)} \cdot \mathbb{1}(x \geq x_0)$

$$L(\bar{x}|\alpha, x_0) = \prod_{k=1}^{n} \alpha \cdot e^{-\alpha(x_k-x_0)} \cdot \mathbb{1}_{(x_k \geq x_0)} = \underbrace{\alpha^n \cdot e^{-\alpha\sum_{k=1}^{n}(x_k-x_0)}}_{\uparrow \text{ по } x_0} \mathbb{1}_{(x_0 \leq \min_k x_k)}$$

$$\boxed{x^*_{0(n)} = \min x_k} \Leftarrow$$

$$\frac{\partial}{\partial\alpha}(\ldots) = \left(n \cdot \alpha^{n-1} + \alpha^n\left(-n(\bar{x}_n - x_0)\right)\right) \cdot e^{-(\ldots)} \cdot (\ldots) = 0$$

$$1 - \alpha(\bar{x}_n - x_0) = 0.$$

$$\boxed{\alpha^*_n = \frac{1}{\bar{x}_n - x_0} = \frac{1}{\frac{1}{n}\sum_k x_k - \min(x_k)}}$$

x^*_0 — консистентна за зад. #7

$$E(X) = \int_{x_0}^{+\infty} x \cdot \alpha \cdot e^{-\alpha(x-x_0)} dx = \frac{1}{\alpha}\underbrace{\int_{x_0}^{+\infty} \alpha(x-x_0) e^{-\alpha(x-x_0)} d\alpha(x-x_0)}_{=1} + \int_{x_0}^{+\infty} \alpha x_0 e^{-\alpha(x-x_0)} dx =$$

$$= \frac{1}{\alpha} + x_0 \cdot \int_0^{+\infty} e^{-t} dt = \frac{1}{\alpha} + x_0$$

$$\bar{x}_n \longrightarrow EX = \frac{1}{\alpha} + x_0 \Rightarrow \alpha^*_n \longrightarrow \alpha$$

б: $X \sim p(x|\alpha, x_0) = \dfrac{\alpha x_0^\alpha}{x^{\alpha+1}}$; $x \geq x_0$

$Y = \ln X$; $X = e^Y$

$$p_Y(y) = \frac{\alpha x_0^\alpha}{e^{y(\alpha+1)}} \cdot \frac{1}{\frac{1}{e^y}} = \alpha x_0^\alpha e^{-y\alpha} = \begin{bmatrix} y_0 = \ln x_0 \\ x_0 = e^{y_0} \end{bmatrix} =$$

$$= \alpha e^{\alpha y_0} \cdot e^{-y\alpha} = \alpha \cdot e^{-\alpha(y-y_0)}$$

Тогда из a $\Rightarrow$ $\min(\ln x_k) \rightarrow \min(\ln x_0)$

$$\boxed{x^*_0 = \min_k x_k}$$

$$\boxed{\alpha^*_n = \frac{1}{\frac{1}{n}\sum \ln x_k - \min \ln x_k}}$$

9

(а) $X \sim B(m, p)$, m - відоме, p - невідоме

$$P(X=k) = C_k^m p^k \cdot (1-p)^{m-k}$$

$$\ln \mathcal{L}(\bar{x}|p) = \sum_{k=1}^{n} \left[\ln C_{x_k}^m + x_k \ln p + (m - x_k) \ln(1-p) \right]$$

$$\frac{\partial}{\partial p}(..) = \sum_{k=1}^{n} \left(x_k \frac{1}{p} + (m - x_k) \frac{-1}{1-p} \right) = \sum_{k=1}^{n} \left(x_k \left(\frac{1}{p} + \frac{1}{1-p} \right) - \frac{m}{1-p} \right) =$$

$$= \frac{n \bar{x}_n}{p(1-p)} - \frac{n \cdot m}{1-p} = 0 \; ; \quad \frac{\bar{x}_n}{p} - m = 0 .$$

$$\boxed{p_n^* = \frac{1}{m} \cdot \bar{x}_n}$$

$$\bar{X}_n \longrightarrow EX = n \cdot p \; ; \quad \bar{X}_n = n \cdot p + o(n)$$

$$p_n^* = \frac{\bar{X}_n}{m} = \underbrace{\frac{n \cdot p}{m}}_{\to \infty} + o(n).$$

не консист.

(б) $X \sim Pois(\lambda)$, $EX = DX = \lambda$.

$$P(X=k) = \frac{\lambda^k}{k!} \cdot e^{-\lambda} \; ; \; k \in \mathbb{N}_0$$

$$\mathcal{L}(\bar{x}|\lambda) = \prod_{k=1}^{n} \frac{\lambda^{x_k}}{x_k!} e^{-\lambda} = e^{-n\lambda} \cdot \lambda^{\sum x_k} \cdot \frac{1}{\prod_{k=1}^{n} x_k!}$$

$$\ln \mathcal{L}(\bar{x}|\lambda) = -n\lambda + n \cdot \bar{x}_n \ln \lambda - \sum_{k=1}^{n} \ln(x_k!)$$

$$\frac{\partial}{\partial \lambda}(..) = -n + \frac{n \bar{x}}{\lambda} = 0$$

$$\lambda_n^* = \bar{x}_n \longrightarrow \lambda$$ консист

(6) $X \sim G(p)$: $P(X=k) = (1-p)^k \cdot p$

$EX = \frac{1}{p}$; $DX = \frac{1-p}{p^2}$

$$L(\bar{x}|p) = \prod_{k=1}^{n} (1-p)^{x_k} p = p^n \cdot (1-p)^{n \cdot \bar{x}_n}$$

$$\ln L(\bar{x}|p) = n \cdot \ln p + n\bar{x}_n \ln(1-p)$$

$$\frac{\partial}{\partial p}(=) = \frac{n}{p} - \frac{n\bar{x}_n}{1-p} = 0 \; ; \quad 1-p = p\bar{x}_n$$

$p^*_n = \frac{1}{1+\bar{x}_n}$ – конзистентна як ф-ція моментів.

---

(10)

X	P(x)
-1	α
0	α
1	$1-2\alpha$

$EX = 1 - 3\alpha$

$EX^2 = 1 - \alpha$

$DX = 5\alpha - 9\alpha^2$

$$L(\bar{x}|\alpha) = \prod_{k=1}^{n} P(X=x_k) = \alpha^{\sum \mathbb{1}_{x_k=0,-1}} \cdot (1-2\alpha)^{\sum \mathbb{1}_{x_k=1}}$$

Нехай $M = \sum_k \mathbb{1}_{x_k=0;-1}$, $N = \sum_k \mathbb{1}_{x_k=1}$; $M+N=n$

$$\ln L(\bar{x}|\alpha) = M \cdot \ln\alpha + N \cdot \ln(1-2\alpha)$$

$$\frac{\partial}{\partial\alpha}(..) = \frac{M}{\alpha} - \frac{2N}{1-2\alpha} = 0 \; ; \quad M(1-2\alpha) = 2N\alpha$$

$$M = 2N\alpha + 2M\alpha$$

$$\boxed{\alpha^*_n = \frac{1}{2} \cdot \frac{M}{M+N} = \frac{1}{2n} \sum_{k=1}^{n} \mathbb{1}_{x_k=0,-1}}$$

$$E\alpha^*_n = \frac{1}{2n} \sum_k \left(P(X_k=0) + P(X_k=-1)\right) = \alpha = \frac{1}{3}(1-EX)$$

$\Downarrow$ конзистентна

$$E(\alpha^*)^2 = \frac{1}{4n^2}\left(n \cdot 2\alpha + n(n-1) \cdot (2\alpha)^2\right) =$$

$$= \alpha^2 + \frac{\alpha(1-2\alpha)}{2n} = D\alpha^*$$

11 $\quad X \sim p(x \mid x_0, \gamma) = \frac{1}{\pi} \cdot \frac{\gamma}{(x-x_0)^2+\gamma^2}$

$$F(x) = \int\limits_{-\infty}^{x} \frac{1}{\pi\gamma} \frac{1}{1+\left(\frac{t-x_0}{\gamma}\right)^2} dt = \frac{1}{\pi} \int\limits_{-\infty}^{x} \frac{1}{1+\left(\frac{t-x_0}{\gamma}\right)^2} d\left(\frac{t-x_0}{\gamma}\right) =$$

$$= \frac{1}{\pi} \operatorname{arctg} \frac{t-x_0}{\gamma} \Bigg|_{-\infty}^{x} = \frac{1}{\pi} \operatorname{arctg} \frac{x-x_0}{\gamma} + \frac{1}{2}$$

Медиана: $F(Z_{\frac{1}{2}}) = \frac{1}{2}$: $\frac{1}{\pi} \operatorname{arctg} \frac{x-x_0}{\gamma} = 0$; $\boxed{Z_{\frac{1}{2}} = x_0^*}$

Нижний квартиль: $F(Z_{\frac{1}{4}}) = \frac{1}{4}$: $\operatorname{arctg} \frac{x-x_0}{\gamma} = -\frac{\pi}{4}$: $\frac{x-x_0}{\gamma} = -1$

$Z_{\frac{1}{4}} - x_0 = -\gamma$

$$\boxed{\gamma^* = Z_{\frac{1}{2}} - Z_{\frac{1}{4}}}$$

$$\boxed{\begin{aligned} x_{0n}^* &= X_{([\frac{n}{2}])} \\ \gamma_n^* &= X_{([\frac{n}{2}])} - X_{([\frac{n}{4}])}. \end{aligned}}$$

---

12 $\quad p(x \mid a) = \frac{1}{2} e^{-|x-a|}$

$$\ln \mathcal{L}(\bar{x} \mid a) = \underbrace{-n \cdot \ln 2}_{const} + \sum_{k=1}^{n} |x_k - a| \xrightarrow[a]{} \max$$

$$\boxed{a_n^* = \underset{a}{\operatorname{argmin}} \sum_{k=1}^{n} |x_k - a|}$$

Дальше искала численно. см. [GitHub].

Остальные № см. [GitHub].

17 $u_1, u_2 \sim U(0;1)$

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \xmapsto{f} \begin{pmatrix} \sqrt{-2\ln u_1}\,\cos(2\pi u_2) \\ \sqrt{-2\ln u_1}\cdot\sin(2\pi u_2) \end{pmatrix} =: \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\begin{cases} x_1^2 + x_2^2 = -2\ln u_1 \\ \dfrac{x_2}{x_1} = \operatorname{tg}(2\pi u_2) \end{cases}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \xrightarrow{f^{-1}} \begin{pmatrix} \exp\left(-\dfrac{x_1^2+x_2^2}{2}\right) \\ \dfrac{1}{2\pi}\operatorname{arctg}\left(\dfrac{x_2}{x_1}\right) \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$\left\|\frac{\partial f_i}{\partial u_j}\right\| = \begin{pmatrix} \dfrac{-2\frac{1}{u_1}}{2\sqrt{-2\ln u_1}}\cos(2\pi u_2) & -\dfrac{1}{u_1\sqrt{-2\ln u_1}}\sin(2\pi u_2) \\ \sqrt{-2\ln u_1}\,(-2\pi)\cdot\sin(2\pi u_2) & \sqrt{-2\ln u_1}\cdot 2\pi\cdot\cos(2\pi u_2) \end{pmatrix}$$

$$\left|\det\left\|\frac{\partial f_i}{\partial u_j}\right\|\right|(u) = \left|\frac{2\pi}{u_1}\right| \Big| (f^{-1}(x)) = \left(\frac{1}{2\pi}\exp\left(-\frac{x_1^2+x_2^2}{2}\right)\right)^{-1}$$

$$p_{\bar{x}}(\bar{x}) = p_{\bar{u}}(f^{-1}(\bar{x}))\cdot\frac{1}{|\det \partial f(f^{-1}(\bar{x}))|} = \frac{1}{2\pi}\exp\left(-\frac{x_1^2+x_2^2}{2}\right).$$

$\Downarrow$

$\bar{x}$ - гауссовская с. в. в $\mathbb{R}^2$ с $E\bar{X} = (0;0)$, $\operatorname{cov}(..) = \|\delta_{ij}\|$.

$\Downarrow$

координаты тоже гауссовские $N(0;1)$