Сашко Петров
Домашка #1

1

(а) $E\hat{T} = E\left(\sum_{k=1}^{n} c_k X_k\right) = \sum_{k=1}^{n} c_k EX_k =$

$= \left[EX_k = EX_1\right] = \left(\sum_{k=1}^{n} c_k\right) \cdot EX_1 = EX_1 \iff \sum c_k = 1$

(б)
$$E\left[\sum c_k X_k - EX\right]^2 = E\left[\left(\sum c_k X_k\right)^2 - 2 \cdot EX \cdot \sum c_k X_k + (EX)^2\right] =$$

$$= E\left[\sum_{k=1}^{n} c_k^2 X_k^2 + 2 \cdot \sum_{1 \le i < j \le n} c_i c_j X_i X_j - 2 EX \sum c_k X_k + (EX)^2\right] =$$

$$= \sum_{k=1}^{n} c_k^2 EX_k^2 + 2 \sum_{1 \le i < j \le n} c_i c_j E(X_i \cdot X_j) - 2 \cdot EX \cdot \sum c_k EX_k + (EX)^2 =$$

$$= \left[EX_k = EX \ ; \ E(X_i X_j) = EX_i \cdot EX_j = (EX)^2\right] =$$

$$= EX^2 \cdot \sum_{k=1}^{n} c_k^2 + 2(EX)^2 \sum_{i<j} c_i c_j - 2(EX)^2 \cdot \sum_{k=1}^{n} c_k + (EX)^2 = f(\vec{c})$$

$f(\vec{c}) \longrightarrow \min$? !

$$\frac{\partial f}{\partial c_k} = 0 \iff EX^2 \cdot \cancel{2} c_k + \cancel{2}(EX)^2 \left(\cancel{2} \cdot \sum_{i \ne k} c_i - \cancel{2}\right) = 0$$

$$EX^2 \cdot c_k + (EX)^2 \cdot \sum_{i \ne k} c_i = (EX)^2$$

$\alpha := \dfrac{EX^2}{(EX)^2}$,
$$\begin{cases} \alpha c_1 + c_2 + c_3 \ldots c_n = 1 \\ c_1 + \alpha c_2 + c_3 \ldots c_n = 1 \\ \vdots \\ c_1 + c_2 + \ldots \alpha c_n = 1 \end{cases}$$

$$A = \begin{pmatrix} \alpha & & 1 \\ & \alpha & \\ 1 & \ddots & \alpha \end{pmatrix} \ ; \quad b = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

$$\boxed{\vec{c} = A^{-1} b}$$

② $X \in \{-1; 0; +1\}$

x	$P(X=x)$
-1	α
0	α
1	$1-2\alpha$

$EX = -\alpha + 1 - 2\alpha = 1 - 3\alpha$

$EX^2 = \alpha + 1 - 2\alpha = 1 - \alpha$

ⓐ $\hat{\alpha}_n = \frac{1}{n}\sum_{k=1}^{n} 1\!\!1_{X_k=-1} \xrightarrow[n\to\infty]{} E1\!\!1_{X_k=-1} = \alpha$

незм, + консист.

$E\hat{\alpha} = \frac{1}{n}\cdot n \cdot E1\!\!1_{X_k=-1} = \alpha$

$D\hat{\alpha} = E(\hat{\alpha}-\alpha)^2 = E\hat{\alpha}^2 - \alpha^2$

незалежні

$E\hat{\alpha}^2 = E\left(\frac{1}{n}\sum_k 1\!\!1_{X_k=-1}\right)^2 = \frac{1}{n^2}\cdot E\left(\sum_k 1\!\!1^2_{X_k=-1} + 2\sum_{i<j} 1\!\!1_{X_i=-1}\cdot 1\!\!1_{X_j=-1}\right) =$

$= \frac{1}{n^2}\left(\sum_k E1\!\!1_{X_k=-1} + 2\sum_{i<j} E1\!\!1_{X_i=-1}\cdot E1\!\!1_{X_j=-1}\right) =$

$= \frac{1}{n^2}\left(n\cdot\alpha + 2\cdot\frac{n(n-1)}{2}\cdot\alpha^2\right) = \frac{1}{n}\left(\alpha^2 n + \alpha - \alpha^2\right) =$

$= \alpha^2 + \frac{\alpha(1-\alpha)}{n}$

$D\hat{\alpha} = \frac{\alpha(1-\alpha)}{n}$

ⓑ $\bar{X}_n \to EX_1 = 1 - 3\alpha$

$\hat{\alpha}_n = \frac{1}{3}(1-\bar{X}_n) \to \alpha$; $E\hat{\alpha}_n = \alpha$

$\hat{\alpha}_n^2 = \frac{1}{9}\left(1 - 2\bar{X}_n + \bar{X}_n^2\right)$

$E\hat{\alpha}_n^2 = \frac{1}{9}\left(1 - 2E\bar{X}_n + E\bar{X}_n^2\right) = \frac{1}{9}\left[1 - 2(1-3\alpha) + E\bar{X}_n^2\right]$

$E\bar{X}_n^2 = E\left(\frac{1}{n}\sum X_k\right)^2 = \frac{1}{n^2}\cdot\left(\sum_k EX_k^2 + 2\cdot\sum_{i<j} EX_i\cdot EX_j\right) =$

$= \frac{1}{n^2}\cdot\left[n\cdot(1-\alpha) + 2\,\frac{n(n-1)}{2}\cdot(1-3\alpha)^2\right] =$

$= (1-3\alpha)^2 + \frac{1}{n}\left[-(1-3\alpha)^2 + 1 - \alpha\right]$

$$\mathcal{D}\hat{\alpha}_n = E\hat{\alpha}_n^2 - (\overset{\overset{\alpha}{\parallel}}{E\hat{\alpha}_n})^2 = \frac{1}{9}\Big(\underbrace{1-2(1-3\alpha)+(1-3\alpha)^2}_{\substack{(1-(1-3\alpha))^2 \\ (3\alpha)^2}} + \frac{1}{n}\big(1-\alpha-(1-3\alpha)^2\big)\Big) - \alpha^2 =$$

$$= \frac{1}{n}\left[1-\alpha-(1-3\alpha)^2\right]$$

(б) $\nu_n = \sum_{k=1}^{n} 1\!\!1_{X_k=0}$;

аналогічно до п. (а): $\hat{\alpha}_n = \frac{\nu_n}{n} = \frac{1}{n}\sum_{k=1}^{n} 1\!\!1_{X_k=0}$

$\hat{\alpha}_n \to E 1\!\!1_{X_k=0} = \alpha$;

$E\hat{\alpha}_n = \alpha$

$\mathcal{D}\hat{\alpha}_n = \frac{\alpha(1-\alpha)}{n}$.

(г) $Y_n = \sum_{k=1}^{n} X_k^2$; $\frac{1}{n}\sum_{k=1}^{n} X_k^2 \to EX^2 = 1-\alpha$

$\boxed{\hat{\alpha}_n = 1 - \frac{1}{n}Y_n} = \boxed{1 - \frac{1}{n}\sum_k X_k^2} \to \alpha$; $E\hat{\alpha}_n = \alpha$.

$\mathcal{D}\hat{\alpha}_n = E\hat{\alpha}_n^2 - (E\hat{\alpha}_n)^2 = E\hat{\alpha}_n^2 - \alpha^2$

$$E\hat{\alpha}_n^2 = E\left(1-\frac{1}{n}\sum X_k^2\right)^2 = E\left[1 - \frac{2}{n}\sum X_k^2 + \frac{1}{n^2}\left(\sum X_k^2\right)^2\right] =$$

$$= 1 - \frac{2}{n}\sum_k EX_k^2 + \frac{1}{n^2}\left(\sum_k EX_k^4 + 2\cdot\sum_{i<j} EX_i^2\cdot EX_j^2\right) =$$

$$= \left[EX^4 = EX^2 = 1-\alpha\right] =$$

$$= 1 - \frac{2}{\cancel{n}}\cdot\cancel{n}\cdot(1-\alpha) + \frac{1}{n^2}\left(\cancel{n}\cdot(1-\alpha) + \cancel{2}\cdot\frac{n(n-1)}{\cancel{2}}\cdot(1-\alpha)^2\right) =$$

$$= 1 - 2(1-\alpha) + \frac{1}{n}\left(1-\alpha + n(1-\alpha)^2 - (1-\alpha)^2\right) =$$

$$= -1 + 2\alpha + (1-\alpha)^2 + \frac{\alpha(1-\alpha)}{n} = \alpha^2 + \frac{\alpha(1-\alpha)}{n}$$

$$\mathcal{D}\hat{\alpha}_n = \frac{\alpha(1-\alpha)}{n}$$

③ $У$ = успіх ~ p

$Н$ = невдача ~ $q = 1 - p$

X − # коли стався перший успіх.

ⓐ $P(X=k) = P(\underbrace{НН\ldots Н}_{k-1}У) \underset{\text{незалежн.}}{=} q^{k-1}\cdot p$

ⓑ Якщо $M_x(t) = E(e^{tX})$, то $M_k = \frac{d^k}{dt^k} M_x(t)\Big|_{t=0}$

$$M_x(t) = \sum_{k=1}^{\infty} e^{tk}\cdot q^{k-1}\cdot p = \cancel{p\sum_{k=1}^{\infty} e^{tk}\cdot e^{\ln q^{k-1}}} =$$

$$\cancel{= p\cdot\sum_{k=1}^{\infty} e^{tk+(k-1)\ln q} = \frac{p}{q}\cdot\sum_{k=1}^{\infty}\left(e^{t+\ln q}\right)^k} =$$

$$\cancel{= \left[\text{для малых } t\right] = \frac{p}{q}\cdot\frac{1}{1-e^{t+\ln q}}}$$

$$= \frac{p}{q}\cdot\sum_{k=1}^{\infty}(q\cdot e^t)^k = \frac{p}{q}\left[\sum_{k=0}^{\infty}(q\cdot e^t)^k - 1\right] = \left[\begin{matrix}0<q<1\\ t\approx 0\end{matrix}\right] =$$

$$= \frac{p}{q}\cdot\left[\frac{1}{1-qe^t} - 1\right].$$

$$M'_x(t) = \frac{p}{\cancel{q}}\cdot\frac{\cancel{q}e^t}{(1-qe^t)^2} = p\,\frac{e^t}{(1-qe^t)^2}\Big|_{t=0} = \frac{p}{(1-q)^2} = \frac{1}{p} = EX$$

$$M''_x(t) = \frac{p}{(1-qe^t)^{\cancel{4}3}}\left[e^t(1-qe^t)^{\cancel{2}} + 2qe^{2t}\cancel{(1-qe^t)}\right] =$$

$$= \frac{p\cdot e^t}{(1-qe^t)^3}\left[1-qe^t+2qe^t\right] = \frac{p\cdot e^t(1+qe^t)}{(1-qe^t)^3}\Big|_{t=0} =$$

$$= \frac{p(1+q)}{(1-q)^3} = \frac{1+q}{p^2}$$

$$DX = M_2 - M_1^2 = \frac{1+q}{p^2} - \frac{1}{p^2} = \frac{q}{p^2}$$

(в): $\frac{1}{n}\sum_k x_k \to EX_1 = \frac{1}{p}$

$$\boxed{\hat{p}_n = \frac{1}{\overline{X}_n}}$$

(г): $\hat{p}_n = \frac{1}{n}\sum_{k=1}^{n} 1\!\!1_{X_k=0} \xrightarrow{\ \ } E 1\!\!1_{X_k=0} = p$

$E\hat{p}_n = p$

(д) $EX^2 = \frac{2-p}{p^2}$; $\frac{1}{n}\sum_k X_k^2 \to EX_1^2 = \frac{2-p}{p^2}$

$p^2 EX^2 = 2-p = 2 - p^2(EX)$

$p^2(EX^2 + EX) = 2$

$p = \sqrt{\frac{2}{EX^2+EX}}$ ← ф-ція моментів

⇓

консист. оцінка

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(4) $Pois(\lambda)$: $EX = \lambda$; $DX = \lambda$

(а) $\hat{a}_n = \hat{\lambda}_n = \overline{X}_n = \frac{1}{n}\sum_k X_k \to EX_1 = \lambda$

$E\hat{\lambda}_n = \lambda$

(б) $\hat{b}_n = \hat{a}_n + \frac{1}{n}\xi \to \lambda$

$E\hat{b}_n = \lambda + \frac{1}{n}$

(в) $\hat{c}_n = X_1$; $E\hat{c}_n = EX_1 = \lambda$

$\hat{c}_n \not\to \lambda$

(г) $\hat{d}_n \equiv 1$

$$\nu_n = \sum_{k=1}^{n} 1\!\!1_{X_k = 0}$$

$$\frac{1}{n}\sum_{k=1}^{n} 1\!\!1_{X_k} \longrightarrow E\, 1\!\!1_{X_1=0} = \left.\frac{\lambda^k e^{-\lambda}}{k!}\right|_{k=0} = e^{-\lambda}$$

$$\frac{\nu_n}{n} \rightarrow e^{-\lambda}$$

$$\hat{\lambda}_n = -\ln\frac{\nu_n}{n} = \ln n - \ln \nu_n \longrightarrow \lambda$$

$$DX = \lambda = EX^2 - (EX)^2 = EX^2 - \lambda^2$$

$$EX^2 = \lambda^2 + \lambda = \lambda^2 + EX$$

$\lambda^2 = EX^2 - EX$ ← консист., як ф-ція моментів.

$$(\hat{\lambda^2_n}) = \frac{1}{n}\left(\sum X_k^2 - \sum X_k\right) \rightarrow \lambda^2$$

$$E(\hat{\lambda}^2_n) = \frac{1}{n}\cdot\left(n\cdot(\lambda^2+\lambda) - \lambda\right) = \lambda^2.$$

$$\left(\frac{1}{n}\sum_k X_k\right)^2 \longrightarrow (EX_1)^2 = \lambda^2,$$ консист.

$$E\left(\frac{1}{n}\sum_k X_k\right)^2 = \frac{1}{n^2}\left(\sum_k EX_k^2 + 2\sum_{i<j} EX_i EX_j\right) =$$

$$= \frac{1}{n^2}\left[n\cdot(\lambda^2+\lambda) + n(n-1)\cdot\lambda^2\right] = \lambda^2 + \frac{\lambda}{n}$$ (зміщена)

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⑤

$$\xi_k = \begin{cases} 1, & p \\ 0, & q = 1-p. \end{cases}$$ ← незалежні

$$X = \sum_{k=1}^{m} \xi_k \sim B(m,p)$$

ⓐ $P(X=k) = P\left(\sum_{i=1}^{m} \xi_i = k\right) = C_k^m \cdot p^k \cdot q^{m-k}$

k – одиниць
інші = 0

(б) $EX = \sum_{k=1}^{m} E\xi_k = m \cdot p$

$DX = \sum_{k=1}^{m} D\xi_k = m \cdot D\xi_1 = m\left(E\xi^2 - (E\xi)^2\right) = m(p - p^2) =$
$= m \cdot p \cdot q$

(в) $\hat{p}_n = \frac{1}{m}\overline{X}_n = \frac{1}{m}\cdot\frac{1}{n}\cdot\sum_{k=1}^{n} X_k \longrightarrow \frac{1}{m}\cdot mp = p$

$E\hat{p}_n = p$

(г) $$\frac{1}{n}\sum_k 1\!\!1_{X_k = m} \longrightarrow E 1\!\!1_{X_1 = m} = p^m$$

$\hat{p}_n = \left(\frac{1}{n}\sum 1\!\!1_{X_k = m}\right)^{\frac{1}{m}}$ – консистентна.

$E\hat{p}_n = ?$ (не знаю).

(д) $EX^2 - (EX)^2 = mp\cdot q = EX(1-p)$

$\cancel{EX^2 = mpq + m^2p^2 = mp(1-p) + m^2p^2 =}$
$\cancel{= mp - mp^2 + m^2p^2 = EX - EX + (EX)^2}$

$$\begin{cases} \hat{p} = 1 - \frac{1}{EX}\left[EX^2 - (EX)^2\right] \\ \hat{m} = \frac{EX}{\hat{p}} \end{cases}$$ ← ф-ція моментів

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(6)

Производящая ф-ция для $\{p_k = P(X=k)$ из $B(m,p)\}$:

$f(x) = \sum_{k=0}^{\infty} p_k x^k = \sum_{k=0}^{m} C_k^m \cdot p^k \cdot (1-p)^{m-k} x^k =$

$= (1 - p + px)^m = \left(1 + \frac{(x-1)pm}{m}\right)^m \xrightarrow[pm \to \lambda]{m \to \infty}$

$$\longrightarrow \quad e^{(x-1)\lambda} = e^{x\lambda} \cdot e^{-\lambda} = \underbrace{\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} e^{-\lambda} x^k}_{\parallel}$$

производящая ф-ция
для $\left\{ \frac{\lambda^k}{k!} e^{-\lambda} \right\}_{k=0}^{\infty}$

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7 "пожежа" = "успіх".

Нехай в місті $\underline{\underline{m}}$ квартир $(m \to \infty)$.

$$\xi_{i,k} = \begin{cases} 1, & \text{пожежа в } i\text{-ій кварт. в } k\text{-ий день} \\ 0, & \text{не було пожежі} \end{cases}$$

$X_k = \sum_{i=1}^{m} \xi_{i,k}$ – кількість пожеж в місті в k-ий день (незалежно)

$X_k \sim B(m;\ p=?) \sim Y$

$P(Y=k) \approx \frac{\lambda^k}{k!} e^{-\lambda}$; $\lambda \approx m \cdot p \approx$ ~~примерное~~ приблизна кількість пожеж за день

$\lambda \approx \frac{1825}{365} = 5$

$P(Y=5) \approx \frac{5^5}{5!} e^{-5} \approx 0{,}52$

$P(Y<4) = P(Y=0) + P(Y=1) + P(Y=2) + P(Y=3) \approx$

$\approx 0{,}27$

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8 $X \sim U(a;b)$

$$p(x \mid a;b) = \begin{cases} \frac{1}{b-a}; & x \in [a;b] \\ 0 \end{cases}$$

(а) $EX = \int_{-\infty}^{+\infty} x\, p(x|a,b)\, dx = \int_a^b \frac{x}{b-a}\, dx = \frac{x^2}{2(b-a)}\Big|_a^b =$

$= \frac{b^2-a^2}{2(b-a)} = \frac{a+b}{2}$

$EX^2 = \frac{1}{b-a}\int_a^b x^2 dx = \frac{x^3}{3(b-a)}\Big|_a^b = \frac{b^3-a^3}{3(b-a)} =$

$= \frac{1}{3}(b^2+ab+a^2)$

$DX = \frac{1}{3}(b^2+ab+a^2) - \frac{1}{4}(b+a)^2 = \frac{1}{12}\left[4(b^2+ab+a^2) - 3(b^2+2ab+a^2)\right] =$

$= \frac{1}{12}(b^2-2ab+a^2) = \frac{(b-a)^2}{12}$

(б): $\hat{a}_n = \min_k X_k$

$\hat{b}_n = \max_k X_k$

$\varepsilon > 0: \ P(|\hat{a}_n - a| \geq \varepsilon) = P(|\min_k X_k - a| \geq \varepsilon) =$

$= P(\forall k: X_k > a+\varepsilon) = [P(X_1 > a+\varepsilon)]^n =$

$= \left(\frac{\varepsilon}{b-a}\right)^n \longrightarrow 0$; конечет.

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9 $\quad U(0;a)$; $X_1, \dots, X_n$

$EX = \frac{a}{2}$; $EX^2 = \frac{a^2}{3}$

$\hat{a}_n = 2\bar{X}_n \rightarrow 2\cdot\frac{a}{2} = a$; $E\hat{a}_n = a$

$E(\hat{a}_n - a)^2 = Ea_n^2 - a^2$

$E\hat{a}_n^2 = \frac{4}{n^2}\left(\sum_k EX_k^2 + 2\sum_{i<j} EX_i EX_j\right) =$

$= \frac{4}{n^2}\left[n\cdot\frac{a^2}{3} + n(n-1)\cdot\frac{a^2}{4}\right]$

$E(\hat{a}_n - a)^2 = \frac{4a^2}{n}\left(\frac{1}{3} - \frac{1}{4}\right) = \frac{1}{n}\cdot\frac{a^3}{3}$

10

$U(a;2a)$

$\frac{1}{n}\sum_1 x_k \to EX_1 = \frac{3a}{2}$

$\hat{a}_n = \frac{2}{3}\overline{X}_n$ $\qquad EX^2 = \frac{1}{3}(a^2+ab+b^2) = \frac{7}{3}a^2$

$E(\hat{a}_n - a)^2 = E\hat{a}_n^2 - a^2$

$E\hat{a}_n^2 = \frac{4}{9n^2}\cdot\left(\cancel{n}\cdot\frac{7}{3}a^2 + \cancel{n}(n-1)\frac{9}{4}a^2\right) = a^2 + \frac{4a^2}{9n}\left(\frac{7}{3}-\frac{9}{4}\right) =$

$= a^2 + \frac{\cancel{4}a^2}{9n}\cdot\frac{1}{\cancel{12}\,3} =$

$E(\hat{a}_n - a)^2 = \frac{1}{n}\cdot\frac{a^2}{27}$

11

$X \sim Exp(\alpha)$; $\alpha > 0$.

ⓐ $EX = \int_0^\infty x\cdot\alpha\cdot e^{-\alpha x}dx = \frac{1}{\alpha}\int_0^\infty (x\alpha)\,e^{-\alpha x}d(\alpha x) = \frac{1}{\alpha}\int_0^\infty te^{-t}dt =$

$= -\frac{1}{\alpha}\int_0^\infty t\,d(e^{-t}) = -\frac{1}{\alpha}\left[\underbrace{te^{-t}\Big|_0^\infty}_{=0} - \int_0^\infty e^{-t}dt\right] = \frac{1}{\alpha}$

$EX^2 = \int_0^\infty x^2\alpha e^{-\alpha x}dx = \frac{1}{\alpha^2}\int_0^\infty t^2e^{-t}dt = -\frac{1}{\alpha^2}\int_0^\infty t^2\,d(e^{-t}) =$

$= -\frac{1}{\alpha^2}\cdot\left[\underbrace{t^2e^{-t}\Big|_0^\infty}_{=0} - 2\int_0^\infty te^{-t}dt\right] = \frac{2}{\alpha^2}$

$DX = EX^2 - (EX)^2 = \frac{1}{\alpha^2}$

ⓑ $\frac{1}{n}\sum_1 x_k \longrightarrow EX_1 = \frac{1}{\alpha}$

$\frac{1}{n}\sum_1 x_k^2 \longrightarrow EX_1^2 = \frac{2}{\alpha^2}$

$a = \frac{1}{\alpha}$; $\hat{a}_n = \overline{X}_n$

$b = \frac{1}{\alpha^2}$; $\hat{b}_n = \frac{1}{2}\frac{1}{n}\sum_{k}^{1} x_k^2$

(в) $E\left(\frac{1}{\bar{X}_n}\right) = E\left(\frac{n}{\underbrace{\sum_k X_k}_{S_n}}\right)$

$S_n \sim p(x) = \frac{\alpha^n x^{n-1}}{(n-1)!} e^{-\alpha x}$;

$\xi_n = \frac{n}{S_n}$; $S_n \in [0; +\infty) \to \xi_n \in (+\infty; 0]$

$\xi_n = x \Rightarrow S_n = \frac{n}{x}$

Как найти плотность ξ_n — ?

$$P(S_n \leqslant x) = \int_{-\infty}^{x} p_{S_n}(x)\,dx.$$

$$\parallel$$

$$P\left(\frac{n}{\xi_n} \leqslant x\right)$$

$$\parallel$$

$$P\left(\xi_n \geqslant \frac{n}{x}\right) = 1 - P\left(\xi_n \leqslant \frac{n}{x}\right)$$

$\frac{n}{x} = y$

$x = \frac{n}{y}$.

$$F_{\xi_n}(y) = 1 - F_{S_n}\left(\frac{n}{y}\right)$$

$$\int_{-\infty}^{y} p_{\xi_n}(t)\,dt = 1 - \int_{-\infty}^{n/y} p_{S_n}(t)\,dt$$

?

можно замену

не знаю

(г) для этого нужно решить (в)

$\frac{1}{\bar{X}_n}$ — буде консистентною

незміщеністю?

9: $\forall s, t > 0$

$$P(X > s+t \mid X > s) = P(X > t)$$

$$\parallel$$

$$\frac{P(X > s+t \wedge X > s)}{P(X > s)} = \frac{P(X > s+t)}{P(X > s)} \ \ominus$$

$$P(X > a) = \int_a^{+\infty} \alpha e^{-\alpha x} dx = -\int_a^{+\infty} e^{-\alpha x} d(-\alpha x) =$$

$$= -e^{-\alpha x}\Big|_a^{+\infty} = e^{-\alpha a}$$

$$\ominus \ \frac{e^{-\alpha(s+t)}}{e^{-\alpha s}} = e^{-\alpha t} = P(X > t)$$

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12 $\quad Y_k = \begin{cases} 1, & k\text{-ий прилад зламався протягом року.} \\ 0 \end{cases}$

X = час роботи прилада ; $X \sim Exp(\alpha)$

$$Y_k = \mathbb{1}_{\{X_k \le T = \text{рік}\}}$$

кількість зламаних протягом року $= \xi = \sum_{k=1}^{n} Y_k = \sum_{k=1}^{n} \mathbb{1}_{X_k \le T}$

$$E\xi = \sum_{k=1}^{n} E\mathbb{1}_{X_k \le \text{рік}} = n \cdot P(X_1 \le \text{рік}) =$$

$$= n \cdot \int_0^T \alpha e^{-\alpha x} = -n e^{-\alpha x}\Big|_0^T = n(1 - e^{-\alpha T})$$

$$\cancel{1000} \cdot (1 - \underbrace{e^{-\alpha \cdot 365}}_{\frac{1}{10}}) = \cancel{100} \Rightarrow e^{-\alpha \cdot 365} = \frac{9}{10}$$

$$\alpha = -\frac{1}{365} \ln \frac{9}{10} \approx 2{,}9 \cdot 10^{-4}$$

13 $X \sim p(x|\lambda) = \lambda \cdot x^{\lambda-1}$; $x \in [0;1]$; $\lambda > 0$

б) $$EX = \int_0^1 x \cdot \lambda x^{\lambda-1} dx = \frac{\lambda}{\lambda+1} x^{\lambda+1} \Big|_0^1 = \frac{\lambda}{\lambda+1}$$

$$\lambda \cdot EX + EX = \lambda \;; \quad \lambda = \frac{EX}{1-EX} = \frac{1}{\frac{1}{EX}-1}$$

$$\hat{\lambda}_n = \frac{1}{\frac{1}{\overline{X}_n} - 1}$$

a) $$\overline{X}_n = (0{,}1 + 0{,}2 + \ldots + 2 \cdot 0{,}9)/8 = 0{,}6 = \frac{4}{5}$$

$$\frac{1}{\overline{X}_n} = \frac{5}{4}$$

$$\hat{\lambda} = \frac{1}{\frac{5}{4}-1} = 4$$

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14 $\overline{X}_n \to EX$; $\overline{X}_n = 15{,}14$

$$\frac{1}{n}\sum_k X_k^2 \approx 229{,}33$$

$$\hat{S}_n = EX^2 - (EX)^2 \approx 0{,}1125$$

$$\check{S}_n = \frac{n}{n-1}\hat{S}_n \approx 0{,}125$$

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15 $$p(x|\alpha, x_0) = \begin{cases} \alpha e^{-\alpha(x-x_0)} & ; x \geq x_0 \\ 0 & , \end{cases}$$

$\alpha > 0$, $x_0 \in \mathbb{R}$

$$M_x(t) = E(e^{tx}) = \int_{x_0}^{+\infty} e^{tx} \cdot \alpha \cdot e^{-\alpha(x-x_0)} dx = \frac{\alpha e^{\alpha x_0}}{t-\alpha} \int_{x_0}^{\infty} e^y dy =$$

$$= \frac{\alpha e^{\alpha x_0}}{t-\alpha}\left(-e^{(t-\alpha)x_0}\right) = \frac{\alpha}{\alpha - t} \cdot e^{t x_0}$$

$$M_x'(t) = -\frac{\alpha}{(\alpha-t)^2} e^{tx_0} + \frac{\alpha}{\alpha-t} x_0 e^{tx_0} = \frac{\alpha e^{tx_0}}{(\alpha-t)^2}\left(-1 + x_0(\alpha - t)\right)$$

$$M_x'(0) = \frac{\alpha}{\alpha^2}\left(-1 + x_0\alpha\right) = \boxed{x_0 - \frac{1}{\alpha}}$$

$$M_X''(t) = \frac{\alpha}{(\alpha-t)^4}\left[\left(x_0 e^{tx_0}(x_0(\alpha-t)-1) - e^{tx_0}\cdot x_0\right)(\alpha-t)^2 + \right.$$

$$\left. + e^{tx_0}(x_0(\alpha-t)-1)\cdot 2(\alpha-t)\right]$$

$$M_X''(0) = \frac{\alpha}{\alpha^{4}}\left[\left(x_0(x_0\alpha-1)-x_0\right)\alpha^2 + 2\alpha(x_0\alpha-1)\right] =$$

$$= \frac{1}{\alpha^2}\left[\alpha x_0(x_0\alpha-2) + 2(x_0\alpha-1)\right] =$$

$$= \frac{1}{\alpha^2}\left[\alpha^2 x_0^2 \cancel{-2\alpha x_0} + \cancel{2\alpha x_0} - 2\right] = x_0^2 - \frac{2}{\alpha^2}$$

$EX = M_1 = x_0 - \frac{1}{\alpha}$

$EX^2 = M_2 = x_0^2 - \frac{2}{\alpha^2}$

~~пусть $a = x_0$, $b = \frac{1}{\alpha}$~~

можно решить относительно x_0 и α

получим ф-ции моментов = консистент. оценки.

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16. Нехай смартфон падає до "першого успіху". Тижні незалежні один від одного.

p – імовірність розбити на будь-якому тижні.

тоді кількість тижнів життя $= X \sim Geom(p)$

$$\overline{X}_n \to \frac{1}{p}$$

$$\hat{p}_n = \frac{1}{\overline{X}_n} \approx 0{,}036.$$