Matching models I: Theory[1]

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[1]Slides provided by Thimo De Schouwer – Based on Chiappori (2017) and Chade *et al.* (2017)
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Introduction – Matching models

- Previous classes studied (discrete) choice models:
  - consumers choose which yogurts to buy
  - commuters choose their mode of transportation
- In many markets agents on both sides of the market have preferences:
  - labor market: firms have preferences over workers and vice versa
  - marriage market: potential partners have preferences over one another
  - credit market: banks have preferences over customers and vice versa
- The following two lectures discuss a tractable class of models to study these markets

Introduction – Matching models (cont.)

- In this lecture we get familiar with main theoretical insights of matching models:
  - properties of the optimal assignment and the competitive equilibrium
    - studied since Koopmans and Beckmann (1957) and Shapley and Shubik (1971)
  - and sorting patterns and how these relate to the match surplus
    - studied in this setting since Becker (1973)

- Next lecture we'll look into an empirical application of the model:
  - starting with Becker (1973) the toolbox of economics has been applied to marriage
  - recent advances in the econometrics of matching models following Choo and Siow (2006)
  - e.g. to study changes in matching patterns between men and women

Introduction – Matching models (cont.)

- Example: assortative matching and inequality
  - see Burtless (1999) and Ciscato and Weber (2020)
- Changing correlation of husband's and wife's earnings has reinforced inequality:
  - over last fifty years inequality has increased drastically in the United States (and elsewhere)
  - up to 1/3 of $\uparrow$ household level inequality related to rise of single adult households
  - up to 1/6 is due to an increase in *assortative matching* (i.e. likes marrying likes)
- We can address the question of *why* this correlation changed using matching models

Introduction – Matching models (cont.)

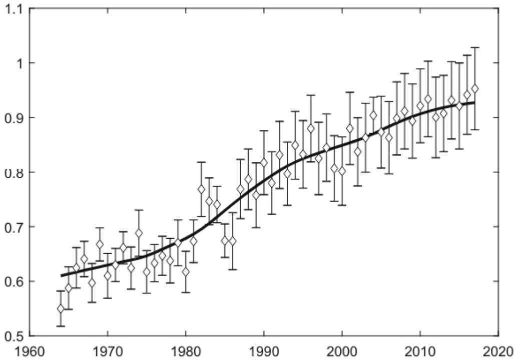

Fig. 6 Assortativeness in education. Sample used: baseline A. The figure displays the estimated trend of the diagonal element of the marital preference parameter matrix A capturing the interaction between husband's and wife's schooling levels. We observe an increase in education complementarity

Figure: Changes in Assortative Matching in the United States – taken from Ciscato and Weber (2020)

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Transferable Utility models

- We'll study one particular class of models $\rightarrow$ one-to-one models with transferable utility

1. One individual from a population matches with one from another (*one-to-one*)
   - as opposed to several individuals matching to one firm
   - or people matching within a population (e.g. college room mates or same-sex marriage)
2. We assume no search frictions or market power (*perfect competition*)
   - as opposed to many models in labor economics that assume search frictions
   - or models of risk sharing and information frictions
3. We assume existence of a frictionless transfer technology (*transferable utility*)
   - as opposed to situations where transfers are taxed (wages in the labor market)
   - or situations where favors are exchanged that may not be valued equally

Other models

- These models are suited to some – but certainly not all – applications:
  - some markets lack transfers or competition (e.g. organ donation, public housing)
  - many markets better modeled with search frictions or as many-to-one (e.g. labor market)
- We study a simple model today – but this has been extended in numerous ways:
  - allowing for frictions on transfers or in the matching process
  - generalizing agents' preferences to allow for risk-aversion
  - modeling markets with uni-partite or many-to-one matching
- Recent reviews on matching models can be found in:
  - see Chiappori (2017) for models without frictions
  - see Chade *et al.* (2017) for models with frictions

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Intuition

- The main intuition behind the model is as follows:
  - we have two heterogeneous populations
  - individuals from each population can match one to one or remain single
  - the gain generated by matching is match-specific
- To make things concrete we'll use the terminology of marriage markets:
  - i.e. we study how men and women match to form heterosexual couples
  - note that we can always replace women with 'workers' and men with 'firms' etc.

Populations

- There is a collection of men of type $x = \{1, ..., X\}$ and women of type $y = \{1, ..., Y\}$
  - assume that these types are discrete – not necessary but simpler continuous types
  - types usually contain variables such as age and education
    - or more exotic traits like political preference or personality traits

$\rightarrow$ suppose we have young/old and low/high educated people – then $|X| = 4$

- Let $f(x)$ be the total number of type x men and $g(y)$ the total number of type y women

Surplus

- When a type x man and a type y woman match this generates a surplus $\mathcal{S}(x, y)$
  - interpret this as how much the man and woman mutually like each-other
  - e.g. a man may like a woman but she may not like him all that much
  - the surplus could be low or high depending on relative strength of preferences
- Pairs of men and women can freely transfer utility between them to divide this surplus
  - this is why we call these transferable utility models ($=$ TU models)
  - there is no friction or loss on these transfers ($=$ ITU models)

Surplus (cont.)

- In TU models, the gain generated by the matching satisfies an important property:

$$u(x) + v(y) = \mathcal{S}(x, y) \tag{1}$$

where the payoff functions $u(x), v(y)$ represent post-transfer $t(x, y)$ utilities:

$$u(x) = \max\{\max_y\{u(x, y) - t(x, y)\}, 0\}$$
$$v(y) = \max\{\max_x\{v(x, y) + t(x, y)\}, 0\}$$

- Transfers $t(x, y)$ are an endogenous object (or *outcome*) of the model
  - therefore part of the solution to the matching problem
  - not necessary to think in terms of money – also e.g. in-kind favors

Matching

- A matching $\mu(x, y)$ defines the number of type x men that are matched to type y women
  - the element (x, y) of this matrix contains the number of (x, y) matches
  - we allow for agents to be single by matching them with an outside option $\{0\}$
  - i.e. we denote unmatched men of type x by $\mu(x, 0)$ and women of type y by $\mu(0, y)$
- This matrix and the number of singles have to satisfy the following marginals restrictions:

$$\sum_y \mu(x, y) + \mu(x, 0) = f(x) \text{ for all } x \in X \tag{2}$$

$$\sum_x \mu(x, y) + \mu(0, y) = g(y) \text{ for all } y \in Y \tag{3}$$

Equilibrium matching

- We solve the matching problem by looking for a stable equilibrium matching:
  - a matching is stable iff it is robust w.r.t. uni- and bilateral deviations
    - the matching is *individually* rational – no one that is matched would prefer to be single
    - the matching features no blocking *pairs* – no two unmatched agents prefer being matched to each-other over their current partners
- We can characterize stability through:

$$u(x) + v(y) \geq \mathcal{S}(x, y) \text{ for all } (x, y) \in X \times Y \tag{4}$$

Summary

- To summarize, a bipartite TU matching problem is defined by
  - two sets X and Y, together with their measures f and g
  - a surplus function $\mathcal{S} : X \times Y \to \mathbb{R}$, characterizing the gains from the matching
- The solution to the matching problem is given by
  - a *matching* μ on $X \times Y$ satisfying the marginals restrictions (2) and (3)
  - two payoff functions $u : X \to \mathbb{R}$ and $v : Y \to \mathbb{R}$ satisfying the TU property (1)
- A stable matching further satisfies (4)

Stable equilibrium – Example

- Consider a set of men $x = 1, 2, 3$ and women $y = 1, 2, 3$
  - suppose these represent three levels of height: 170cm – 180cm – 190cm
  - note that this imposes a natural ordering
- Let the match surplus function be $\mathcal{S}(x, y) = xy$
  - we can write this in a matrix as:

Table: Example based on Eeckhout (2018)

	x_1	x_2	x_3
y_1	1	2	3
y_2	2	4	6
y_3	3	6	9

Stable equilibrium – Example (cont.)

- The optimality condition $u(x) + v(y) \geq \mathcal{S}(x, y)$ implies a system of nine inequalities
- These hold with equality along the equilibrium allocation: $\tilde{\mu}(1, 2, 3) = (1, 2, 3)$
- Note that this allocation satisfies the two properties we imposed:
  - individually rational and no blocking pairs
- The equilibrium transfers t_{xy} are characterized by the following set of inequalities:

$$
\begin{aligned}
1 &\leq t_{22} - t_{11} \leq 2 \\
2 &\leq t_{33} - t_{11} \leq 6 \\
2 &\leq t_{33} - t_{22} \leq 3
\end{aligned}
$$

Equilibrium – Properties

- The previous example showed that finding an equilibrium allocation was not trivial
- It turns out that TU models have a unique equilibrium
  - often referred to as *assignment games*
  - stable outcome coincides with the core (Shapley and Shubik 1971)
  - in fact stability is equivalent to maximization of the total surplus
  - uniqueness thus also generically established
- Numerical optimization of the surplus provides a practical way of computing the equilibrium outcome
  - Class of linear optimization problems referred to as optimal transportation problems
  - Particularly apparent with continuous types optimal transportation
- What else can we say about the equilibrium allocations?

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Assortativeness

- The previous example – where $\tilde{\mu}(1,2,3) = (1,2,3)$ – is one specific case of sorting:
    - the tallest men x is matched to the tallest women y and so on
    - we call this *positive assortative matching* (PAM)
    - the opposite is called *negative assortative matching* (NAM)
- What determines whether the equilibrium will exhibit PAM or NAM?
    - in these models only the surplus function $\mathcal{S}(x, y)$

Supermodularity

- Matchings are positive assortative if the surplus function supermodular:

$$\mathcal{S}(x, y) + \mathcal{S}(x', y') > \mathcal{S}(x, y') + \mathcal{S}(x', y) \quad (5)$$

  for all $x, x' \in X$ and $y, y' \in Y$
    - this is the crucial insight of Becker (1973)
    - flipping the sign naturally yields the condition for NAM (submodularity)
- When $\mathcal{S}(x, y)$ is differentiable, PAM is equivalent to positive cross-partial derivatives
    - Going back to our example – we can check the cross partial derivative $\mathcal{S}_{xy}$
- Note that the opposite direction is not necessarily true
    - possible to construct examples where matching exhibits PAM but surplus is submodular

Supermodularity – Intuition

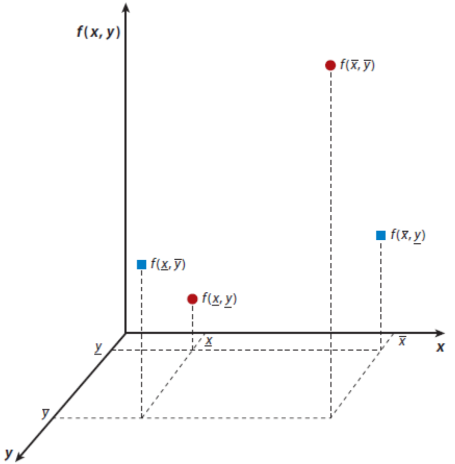

Figure 1

Supermodularity. A function $f(x, y)$ is supermodular if the sum of its value at the extremes (*red circles*) exceeds that at the intermediates (*blue squares*).

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Positive Assortative Matching – Age

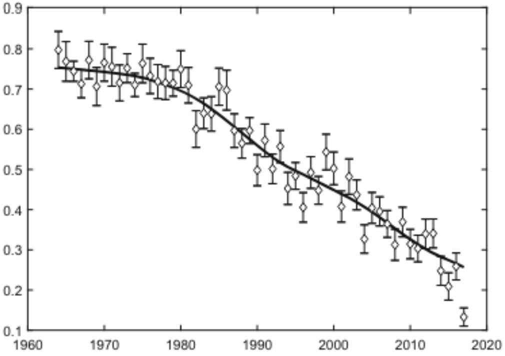

Fig. 5 Assortativeness in age. Sample used: baseline A. The figure displays the estimated trend of the diagonal element of the marital preference parameter matrix A capturing the interaction between husband's and wife's ages. We observe a decrease in age complementarity

Figure: Changes in Assortative Matching in the United States – taken from Ciscato and Weber (2020)

Negative/Positive Assortative Matching – Hours worked

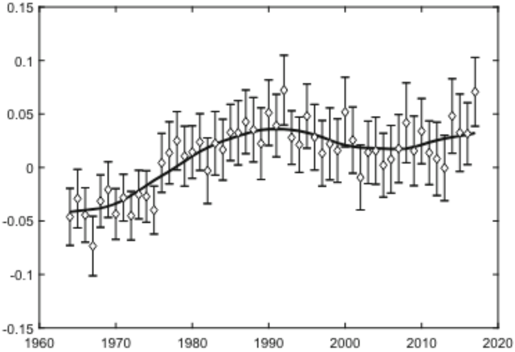

Fig. 8 Assortativeness in hours of work. Sample used: baseline A. The estimated trend of the diagonal element of the marital preference parameter matrix A capturing the interaction between husband's and wife's hours worked. We observe a possible rising of a relatively weak complementarity in hours worked which was not observed in the early waves

Figure: Changes in Assortative Matching in the United States – taken from Ciscato and Weber (2020)

Negative/Positive Assortative Matching – Productivity

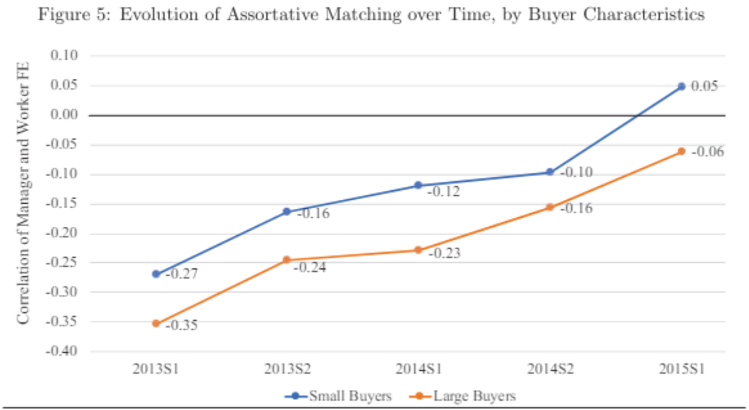

Figure: Changes in Assortative Matching – taken from Adhvaryu *et al.* (forthcoming)

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Conclusion

- Properties such as PAM or NAM are useful frameworks to think about problems
- But of course they will never hold exactly in any real data
  - however, this is what the theoretical model would predict ...
- Next lecture we'll see how to extend the model for empirical applications
  - discrete types model of Choo and Siow (2006)
  - continuous types model of Dupuy and Galichon (2014)

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Appendix I – Continuous types

- The main intuition remains the same, but we now assume that types are continuous
  - male characteristics x are drawn from $X \subset \mathbb{R}^n$ according to $F(X)$
  - female characteristics y are drawn from $Y \subset \mathbb{R}^m$ according to $G(Y)$
    - characteristics can be multidimensional ($n, m \geq 1$), but finite-dimensional
    - $F(x)$ and $G(y)$ provide the continuous analogues to $f(x)$ and $g(y)$
    - i.e., the *'number'* of type x men and type y women, respectively

Appendix I – Continuous types (cont.)

- A matching $\mu(x, y)$ is now defined as a measure on $X \times Y$
  - can be interpreted as the *probability* that type x men are matched to type y women
  - still allow for agents to be single by matching them with dummy partners $\{0_x\}$ and $\{0_y\}$
  - i.e., $X := \bar{X} \cup \{0_x\}$ and $Y := \bar{Y} \cup \{0_y\}$

- The marginals restrictions now become:

$$\int_{y \in Y} d\mu(x, y) = F(x) \tag{2'}$$

$$\int_{x \in X} d\mu(x, y) = G(y) \tag{3'}$$

Appendix I – Continuous types (cont.)

- Characterization of a stable equilibrium matching still boils down to:

$$u(x) + v(y) \geq \mathcal{S}(x, y) \text{ for all } (x, y) \in X \times Y \tag{4'}$$

- In fact, a stable equilibrium must satisfy that:

$$\begin{aligned} &u(x) + v(y) = \mathcal{S}(x, y) \text{ for all } (x, y) \in \operatorname{supp}(\mu) \\ &u(x) + v(y) > \mathcal{S}(x, y) \text{ otherwise} \end{aligned}$$

Appendix I – Continuous types (cont.)

- Stable matching can be obtained as the solution to optimal transportation or assignment problem:

$$\max_{\mu} \int_{X \times Y} \mathcal{S}(x, y) d\mu(x, y) \tag{OTP}$$

subject to the marginals conditions (2') and (3')

back