COSC-373: HOMEWORK 4

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Question 1. Each node v executes as follows.
Each round:

(1) Pick a uniformly random color $c(v)$ from the range $1, 2, \ldots, 2\deg(v)$
(2) Send message $c(v)$ to all neighbors
(3) If $c(v) \neq c(u)$ for all neighbors u, output $c(v)$ and halt

(Runtime) To show that the procedure produces a proper 2Δ coloring of G in $O(\log n)$, first define the random variable X_v for whether a given node v halts in a given round. Specifically, X_v is 1 if node v halts in the given round, and 0 otherwise. The probability $\mathbb{P}(X_v)$ that node v halts in a given round, is the probability $\frac{1}{2\deg(v)}$ that node v chooses color $c(v)$ from the $2\deg(v)$ colors times the probability $\frac{\deg(v)}{2\deg(v)-1}$ that each of node v's $\deg(v)$ neighbors do not choose the color $c(v)$ from the $2\deg(v)$ colors. That is,

$$\begin{aligned}\mathbb{P}(X_v) &= \frac{1}{2\deg(v)} \cdot \frac{\deg(v)}{2\deg(v)-1} \\ &= \frac{1}{4\deg(v)-2} \quad .\end{aligned}$$

It follows that the expected value $\mathbb{E}(X_v)$ of node v halting in a given round is

$$\begin{aligned}\mathbb{E}(X_v) &= 1 \cdot \frac{1}{4\deg(v)-2} + 0 \cdot (1 - \frac{1}{4\deg(v)-2}) \\ &= \frac{1}{4\deg(v)-2} \quad .\end{aligned}$$

Next, define the random variable X for the number of nodes that halt in a given round. Given the set of nodes V for graph G, we have that

$$X = \sum_{v \in V} X_v \quad .$$

Taking the expected value on both sides, we get

$$\begin{aligned}
\mathbb{E}(X) &= \mathbb{E}\left(\sum_{v\in V} X_v\right) \\
&= \sum_{v\in V} \mathbb{E}(X_v) \\
&= \sum_{v\in V} \frac{1}{4\deg(v)-2} \\
&\geq \sum_{v\in V} \frac{1}{4\Delta-2} \\
&> \sum_{v\in V} \frac{1}{4\Delta} \\
&= \frac{1}{4\Delta} n \ .
\end{aligned}$$

In expectation, $\frac{1}{4\Delta}$ nodes thus halt in each round, and the procedure therefore produces a proper 2Δ coloring of G in $O(\log n)$ rounds with high probability.

Question 2. TODO

Question 3. TODO