

## COSC-373: HOMEWORK 4

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**Question 1.** Each node  $v$  executes as follows.

Each round:

- (1) Pick a uniformly random color  $c(v)$  from the range  $1, 2, \dots, 2 \deg(v)$
- (2) Send message  $c(v)$  to all neighbors
- (3) If  $c(v) \neq c(u)$  for all neighbors  $u$ , output  $c(v)$  and halt

(Correctness) At each round a node  $v$  picks a color  $c(v)$  uniformly at random from the range  $1, 2, \dots, 2 \deg(v)$ . Then, node  $v$  sends its chosen color  $c(v)$  to all its neighbors. Similarly, node  $v$  would also receive colors from its active neighbors. If one of  $v$ 's neighbors has already halted,  $v$  uses the last color it received from the neighbor. For each neighbor  $u$  of node  $v$ , if  $c(v) \neq c(u)$ , the condition that neighboring vertices are assigned different colors is satisfied, so node  $v$  outputs  $c(v)$  and halts. Therefore, this protocol produces a proper  $2\Delta$  coloring of  $G$ .

(Congestion) Since the protocol only requires nodes to send colors from the range  $1, 2, \dots, 2\Delta$  to neighboring nodes, only messages of size  $O(1)$  bits are being sent. Thus, the protocol is in the CONGEST model.

(Runtime) To show that the protocol produces a proper  $2\Delta$  coloring of  $G$  in  $O(\log n)$  rounds, first define the random variable  $X_v$  for whether a node  $v$  halts in a given round. Specifically,  $X_v$  is 1 if node  $v$  halts in the given round, and 0 otherwise. For a given round, since node  $v$  has  $\deg(v)$  neighbors, its neighbors can choose at most  $\deg(v)$  distinct colors from each other. We also know that node  $v$  chooses uniformly at random from  $2 \deg(v)$  colors. As such, the probability that node  $v$  chooses the same color as one of its neighbors, is at most  $\frac{\deg(v)}{2 \deg(v)} = \frac{1}{2}$ . It follows that the probability  $\mathbb{P}(X_v)$  that node  $v$  halts in a given round (i.e., choosing a different color from its neighbors) is

$$\begin{aligned} \mathbb{P}(X_v) &\geq 1 - \frac{1}{2} \\ &= \frac{1}{2}. \end{aligned}$$

Hence, the expected value  $\mathbb{E}(X_v)$  of node  $v$  halting in a given round is

$$\begin{aligned} \mathbb{E}(X_v) &\geq 1 \cdot \frac{1}{2} + 0 \cdot \frac{1}{2} \\ &= \frac{1}{2}. \end{aligned}$$

Next, define the random variable  $X$  for the number of nodes that halt in a given round. Given the set of nodes  $V$  for graph  $G$ , we have that

$$X = \sum_{v \in V} X_v .$$

Taking the expected value on both sides, we get

$$\begin{aligned}\mathbb{E}(X) &= \mathbb{E}\left(\sum_{v \in V} X_v\right) \\ &= \sum_{v \in V} \mathbb{E}(X_v) \\ &\geq \sum_{v \in V} \frac{1}{2} \\ &= \frac{1}{2}n\end{aligned}$$

Thus, in expectation, at least  $\frac{1}{2}$  of all nodes halt in each round. If  $n = n_0, n_1, \dots, n_k$  are the number of active nodes after rounds  $0, 1, \dots, k$ , we have that

$$\mathbb{E}(n_k) \leq \frac{1}{2^k} \cdot n.$$

Taking  $k = 4 \log n$ , we get

$$\begin{aligned}\mathbb{E}(n_k) &\leq \frac{1}{2^{4 \log n}} \cdot n \\ &= \frac{1}{n^4} \cdot n \\ &= \frac{1}{n^3}.\end{aligned}$$

By Markov's Inequality, it follows that  $\mathbb{P}(n_k \geq 1) \leq \frac{1}{n^3}$ . That is, after  $4 \log n$  rounds, the probability that there are one or more active nodes is at most  $\frac{1}{n^3}$ . Therefore, the protocol produces a proper  $2\Delta$  coloring of  $G$  in  $O(\log n)$  rounds with high probability.

**Question 2.** For  $F = \{(A, A^c) \mid A \subseteq \{1, 2, \dots, N\}\}$  to be a fooling set for function DISJ, we must prove the following conditions:

- (1) For all  $i, j$ , we have  $\text{DISJ}(A_i, A_i^c) = \text{DISJ}(A_j, A_j^c)$ ; and
- (2) For all  $i \neq j$ , we have  $\text{DISJ}(A_i, A_i^c) \neq \text{DISJ}(A_i, A_j^c)$  or  $\text{DISJ}(A_i, A_i^c) \neq \text{DISJ}(A_j, A_i^c)$ .

To prove condition (1), let  $(A_i, A_i^c) \in F$ . Since  $A_i^c$  is the complement of  $A_i$ , it follows that  $A_i$  and  $A_i^c$  are disjoint since  $A_i \cap A_i^c = \emptyset$ . Thus,  $\text{DISJ}(A_i, A_i^c) = 1$ . Notice that  $(A_i, A_i^c)$  is any arbitrary element in  $F$ . Therefore, for all  $i, j$  where  $(A_i, A_i^c), (A_j, A_j^c) \in F$ , we have that  $\text{DISJ}(A_i, A_i^c) = 1 = \text{DISJ}(A_j, A_j^c)$ .

To prove condition (2), let  $(A_i, A_i^c), (A_j, A_j^c) \in F$ , where  $i \neq j$ . Three cases are possible.

Case 1:  $A_j \subset A_i$  (and  $A_j \not\subset A_i^c$ ). This implies that  $A_i \cap A_j^c \neq \emptyset$  and  $A_j \cap A_i^c = \emptyset$ . Thus, we have that  $\text{DISJ}(A_i, A_j^c) = 0$  and  $\text{DISJ}(A_j, A_i^c) = 1$ . Since  $\text{DISJ}(A_i, A_i^c) = 1 \neq 0 = \text{DISJ}(A_i, A_j^c)$ , condition (2) is satisfied for this case.

Case 2:  $A_j \subset A_i^c$  (and  $A_j \not\subset A_i$ ). This implies that  $A_i \cap A_j^c \neq \emptyset$  and  $A_j \cap A_i^c \neq \emptyset$ . Hence, we have that  $\text{DISJ}(A_i, A_j^c) = 0$  and  $\text{DISJ}(A_j, A_i^c) = 0$ . Since  $\text{DISJ}(A_i, A_i^c) = 1 \neq 0 = \text{DISJ}(A_i, A_j^c)$ , condition (2) is also satisfied for this case.

Case 3:  $A_j \not\subset A_i$  and  $A_j \not\subset A_i^c$ . This implies that  $A_i \cap A_j^c \neq \emptyset$  and  $A_j \cap A_i^c \neq \emptyset$ . It follows that  $\text{DISJ}(A_i, A_j^c) = 0$  and  $\text{DISJ}(A_j, A_i^c) = 0$ . Since  $\text{DISJ}(A_i, A_i^c) = 1 \neq 0 = \text{DISJ}(A_i, A_j^c)$ , condition (2) is satisfied for this case, and the proof is complete.

**Question 3.** TODO