COSC-373: HOMEWORK 4

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Question 1. Each node v executes as follows.

Each round:

(1) Pick a uniformly random color $c(v)$ from the range $1, 2, \ldots, 2\deg(v)$
(2) Send message $c(v)$ to all neighbors
(3) If $c(v) \neq c(u)$ for all neighbors u, output $c(v)$ and halt

(Correctness) At each round a node v picks a color $c(v)$ uniformly at random from the range $1, 2, \ldots, 2\deg(v)$. Then, node v sends its chosen color $c(v)$ to all its neighbors. Similarly, node v would also receive colors from its active neighbors. If one of v's neighbors has already halted, v uses the last color it received from the neighbor. For each neighbor u of node v, if $c(v) \neq c(u)$, the condition that neighboring vertices are assigned different colors is satisfied, so node v outputs $c(v)$ and halts. Therefore, this protocol produces a proper 2Δ coloring of G.

(Congestion) Since the protocol only requires nodes to send colors from the range $1, 2, \ldots, 2\Delta$ to neighboring nodes, only messages of size $O(1)$ bits are being sent. Thus, the protocol is in the CONGEST model.

(Runtime) To show that the protocol produces a proper 2Δ coloring of G in $O(\log n)$ rounds, first define the random variable X_v for whether a node v halts in a given round. Specifically, X_v is 1 if node v halts in the given round, and 0 otherwise. For a given round, since node v has $\deg(v)$ neighbors, its neighbors can choose at most $\deg(v)$ distinct colors from each other. We also know that node v chooses uniformly at random from $2\deg(v)$ colors. As such, the probability that node v chooses the same color as one of its neighbors, is at most $\frac{\deg(v)}{2\deg(v)} = \frac{1}{2}$. It follows that the probability $\mathbb{P}(X_v)$ that node v halts in a given round (i.e., choosing a different color from its neighbors) is

$$\begin{aligned}\mathbb{P}(X_v) &\geq 1 - \frac{1}{2} \\ &= \frac{1}{2}.\end{aligned}$$

Hence, the expected value $\mathbb{E}(X_v)$ of node v halting in a given round is

$$\begin{aligned}\mathbb{E}(X_v) &\geq 1 \cdot \frac{1}{2} + 0 \cdot \frac{1}{2} \\ &= \frac{1}{2}.\end{aligned}$$

Next, define the random variable X for the number of nodes that halt in a given round. Given the set of nodes V for graph G, we have that

$$X = \sum_{v \in V} X_v \enspace .$$

Taking the expected value on both sides, we get

$$\begin{aligned}\mathbb{E}(X) &= \mathbb{E}\left(\sum_{v\in V} X_v\right)\\ &= \sum_{v\in V}\mathbb{E}(X_v)\\ &\geq \sum_{v\in V}\frac{1}{2}\\ &= \frac{1}{2}n\end{aligned}$$

Thus, in expectation, at least $\frac{1}{2}$ of all nodes halt in each round. If $n = n_0, n_1, \ldots, n_k$ are the number of active nodes after rounds $0, 1, \ldots, k$, we have that

$$\mathbb{E}(n_k) \leq \frac{1}{2^k}\cdot n.$$

Taking $k = 4\log n$, we get

$$\begin{aligned}\mathbb{E}(n_k) &\leq \frac{1}{2^{4\log n}}\cdot n\\ &= \frac{1}{n^4}\cdot n\\ &= \frac{1}{n^3}.\end{aligned}$$

By Markov's Inequality, it follows that $\mathbb{P}(n_k \geq 1) \leq \frac{1}{n^3}$. That is, after $4\log n$ rounds, the probability that there are one or more active nodes is at most $\frac{1}{n^3}$. Therefore, the protocol produces a proper 2Δ coloring of G in $O(\log n)$ rounds with high probability.

Question 2. For F = $\{(A_1,\ A_1^c)$ | A $\subseteq$ {1,2,... N}} to be a fooling set for function DISJ, we must prove the following:

(1) For all i,j we have DISJ(A_i, A_i^c) = DISJ(A_j, A_j^c)
(2) For all i,j we have DISJ(A_i, A_i^c) $\neq$ DISJ(A_i, A_j^c) OR DISJ(A_i, A_i^c) $\neq$ DISJ(A_j, A_i^c)

To prove part 1, suppose we have an arbitrary complementary pair of sets in F, (A_i, A_i^c). Since F is the set of all pairs of sets (A, A^c) where A^c is the complement of A, A^c is the set of elements from 1,2,...,N that are NOT contained in A. Therefore, A and A^c have no elements in common and for all i, any pair of sets in F would return DISJ(A_i, A_i^c) = 1. Thus, for all i and j, we have DISJ(A_i, A_i^c) = DIDJ(A_j, A_j^c) = 1.

To prove part 2, suppose we have an arbitrary complementary pair of sets in F, (A_i, A_i^c), and another arbitrary complementary pair of sets in F, (A_j, A_j^c). Thus, we have three situations that could occur:

(3) A_j shares elements with only A_i but does not equal A_i
   (a) DISJ(A_i, A_i^c) = 1, DISJ(A_i, A_j^c) = 0, DISJ(A_j, A_i^c) = 1. Thus, DISJ(A_i, A_i^c) $\neq$ DISJ(A_i, A_j^c)
(4) A_j shares elements with only A_i^c but does not equal A_i^c
   (a) DISJ(A_i, A_i^c) = 1, DISJ(A_i, A_j^c) = 0, DISJ(A_j, A_i^c) = 0. Thus, DISJ(A_i, A_i^c) $\neq$ DISJ(A_i, A_j^c) and DISJ(A_i, A_i^c) $\neq$ DISJ(A_j, A_i^c)
(5) Aj shares elements with both Ai and Aci
   (a) DISJ(A_i, A_i^c) = 1, DISJ(A_i, A_j^c) = 0, DISJ(A_j, A_i^c) = 0. Thus, DISJ(A_i, A_i^c) $\neq$ DISJ(A_i, A_j^c) and DISJ(A_i, A_i^c) $\neq$ DISJ(A_j, A_i^c)

Thus, for all i,j we have DISJ(A_i, A_i^c) $\neq$ DISJ(A_i, A_j^c) OR DISJ(A_i, A_i^c) $\neq$ DISJ(A_j, A_i^c) and we prove that F is a fooling set of the function DISJ.

Question 3. TODO