COSC-373: HOMEWORK 2

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Question 1. TODO

Question 2. TODO

Question 3. (a) We proceed with devising a greedy algorithm that is guaranteed to produce a proper $\Delta + 1$ coloring of G. The algorithm executes as follows:

(1) Initialize an empty set C, which represents the set of colors.
(2) Assign any vertex in V the color 1 and add 1 to C.
(3) Choose an uncolored vertex $v \in V$. Assign v the smallest color in C that has not been assigned to v's colored neighbors. If v's colored neighbors have been assigned all the colors in C, add a new color $\max\{C\} + 1$ to C and assign it to v.
(4) Repeat step (2) until all vertices are colored.

Since the algorithm assigns an uncolored vertex $v \in V$ with the lowest color in C that has not been assigned to v's neighbors or assigns v with a new color not yet in C, no two neighboring vertices are assigned the same color. Furthermore, because the algorithm only adds a new color to C when all neighbors of v have been assigned all the colors in C and v has at most Δ neighbors, the algorithm adds at most $\Delta + 1$ colors to C. Thus, only $\Delta+1$ colors are required. Therefore, the algorithm produces a proper $\Delta+1$ coloring of G.

(b) Consider the graph G_k of $\Delta + 1$ vertices, where each vertex is connected to every other vertex. In this case, every vertex has a degree of Δ, so the maximum degree is Δ. Now, suppose towards a contradiction that we have a Δ proper coloring of G_k. Thus, at most Δ colors have been assigned to $\Delta+1$ vertices of G_k. By the Pigeonhole Principle, at least two vertices in G_k have been assigned the same color. Since every vertex is connected to every other vertex, these two vertices with the same color are neighbors. This is a contradiction since we assumed that we have a Δ proper coloring of G_k, where no two neighboring vertices are assigned the same color. Therefore, G_k does not admit a proper Δ coloring.

(c) Each phase i:

- If received message "halted" from neighbor in phase $i - 1$
  - Remove neighbor from set of neighbors
- Execute protocol Π
- If node is in the MIS found from Π
  - Output i, send message "halted" to neighbors, and halt

(Correctness) In each phase i, protocol Π finds a MIS in the network of remaining nodes that have not halted. The nodes found in the MIS from phase i output color i. Neighboring nodes will not output the same color since they belong to different maximal independent sets.

(Runtime) After each phase if a node is not in a MIS, then at least one of its neighbors becomes part of an MIS, by the maximality condition. As such, after each phase, a node's degree decreases by at least one. Since a node has at most Δ neighbors, at most Δ phases are needed so that all it's neighbors are assigned a color. Hence, each node requires at most $\Delta+1$ phases to output a color i. Since each phase runs in $T(n)$ rounds, the protocol thus computes a property $\Delta+1$ coloring of G in $O((\Delta+1)T(n)) = O(\Delta T(n) + T(n)) = O(\Delta T(n))$ rounds.

(Congestion) Since the above protocol uses the CONGEST protocol Π and apart from in protocol Π, nodes only send the messages "halted" to neighbors, which only require $O(1)$ bits, the protocol above is therefore a CONGEST protocol.