

CSCI 1550: PROBABILISTIC METHODS IN CS

NOTES

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1. EVENTS AND PROBABILITY

Definition. A probability space has three components:

- (1) a sample space Ω , which is the set of all possible outcomes of the random process modeled by the probability space;
- (2) a family of sets \mathcal{F} representing the allowable events, where each set in \mathcal{F} is a subset of the sample space Ω ; and
- (3) a probability function $\Pr : \mathcal{F} \rightarrow \mathbb{R}$ satisfying the following definition.

Definition. A probability function is any function $\Pr : \mathcal{F} \rightarrow \mathbb{R}$ that satisfies the following conditions:

- (1) for any event E , $0 \leq \Pr(E) \leq 1$;
- (2) $\Pr(\Omega) = 1$; and
- (3) for any finite or countably infinite sequence of pairwise mutually disjoint events E_1, E_2, E_3, \dots ,

$$\Pr \left(\bigcup_{i \geq 1} E_i \right) = \sum_{i \geq 1} \Pr(E_i).$$

Lemma. For any two events E_1 and E_2 ,

$$\Pr(E_1 \cup E_2) = \Pr(E_1) + \Pr(E_2) - \Pr(E_1 \cap E_2).$$

Lemma (Union Bound). For any finite or countably infinite sequence of events E_1, E_2, \dots ,

$$\Pr \left(\bigcup_{i \geq 1} E_i \right) \leq \sum_{i \geq 1} \Pr(E_i).$$

Lemma (1.3). Let E_1, \dots, E_n be any n events. Then

$$\begin{aligned} \Pr \left(\bigcup_{i=1}^n E_i \right) &= \sum_{i=1}^n \Pr(E_i) - \sum_{i < j} \Pr(E_i \cap E_j) + \sum_{i < j < k} \Pr(E_i \cap E_j \cap E_k) \\ &\quad - \dots + (-1)^{l+1} \sum_{i_1 < i_2 < \dots < i_l} \Pr \left(\bigcap_{r=1}^l E_{i_r} \right) + \dots. \end{aligned}$$

Definition. Two events E and F are independent if and only if

$$\Pr(E \cap F) = \Pr(E) \cdot \Pr(F).$$

More generally, events E_1, E_2, \dots, E_k are mutually independent if and only if, for any subset $I \subseteq [1, k]$,

$$\Pr\left(\bigcap_{i \in I} E_i\right) = \prod_{i \in I} \Pr(E_i).$$

Definition. The conditional probability that event E occurs given that event F occurs is

$$\Pr(E \mid F) = \frac{\Pr(E \cap F)}{\Pr(F)}.$$

The conditional probability is well-defined only if $\Pr(F) > 0$.

Theorem (Law of Total Probability). Let E_1, E_2, \dots, E_n be mutually disjoint events in the sample space Ω , and let $\bigcup_{i=1}^n E_i = \Omega$. Then

$$\Pr(B) = \sum_{i=1}^n \Pr(B \cap E_i) = \sum_{i=1}^n \Pr(B \mid E_i) \Pr(E_i).$$

Theorem (Bayes' Law). Assume that E_1, E_2, \dots, E_n are mutually disjoint events in the sample space Ω such that $\bigcup_{i=1}^n E_i = \Omega$. Then

$$\Pr(E_j \mid B) = \frac{\Pr(E_j \cap B)}{\Pr(B)} = \frac{\Pr(B \mid E_j) \Pr(E_j)}{\sum_{i=1}^n \Pr(B \mid E_i) \Pr(E_i)}.$$