MATH 355: HOMEWORK 6

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Exercise 1 (3.2.2). (a) Limit points of A: $\{-1, 1\}$. Limit points of B: $[0, 1]$.
(b) A is neither open nor closed. B is neither open nor closed.
(c) A contains isolated points. B does not contain isolated points.
(d) $\overline{A} = A \cup \{-1\}$. $\overline{B} = [0, 1]$.

Exercise 2 (3.2.4). (a) If $s \in A$, then $s \in \overline{A}$ and we are done. Now suppose $s \notin A$. By Lemma 1.3.8, for every $\epsilon > 0$, there exists an $a \in A$ ($a \neq s$) such that $s - \epsilon < a$. Since $s = \sup(A)$, we also know that $a < s$. Thus, every ϵ-neighborhood $V_\epsilon(s)$ intersects A at some point other than s. That is, s is a limit point of A, so $s \in \overline{A}$ in this case as well.
(b) An open set O cannot contain its supremum $s = \sup(O)$ since every ϵ-neighborhood $V_\epsilon(s)$ of s is not be a subset of O. Specifically, this is because for any $\epsilon > 0$ and $a \in O$, we have that $a < s + \epsilon$ since $s = \sup(O)$.

Exercise 3 (3.2.6). (a) False. Consider the open set $\mathbb{R} \setminus \{\sqrt{2}\}$.
(b) False. Consider the closed sets of the form $C_n = [n, \infty)$ for $n \in \mathbb{N}$. Observe that $C_n \subseteq C_{n+1}$ and $\cup_{n=1}^{\infty} C_n = \emptyset$.
(c) True. Given a nonempty open set O, we know that for $a \in O$, there exists an ϵ-neighborhood $V_\epsilon(a) \subseteq O$. By the Density of $\mathbb{Q}$ in $\mathbb{R}$, there exists a rational number $r \in V_\epsilon(a)$. Thus, we have that $r \in O$.
(d) False. Consider the bounded infinite closed set $F = \{\sqrt{2} + 1/n : n \in \mathbb{N}\} \cup \{\sqrt{2}\}$. Observe that F does not contain any rational number.
(e) True. The Canter set is defined as $C = \cap_{n=0}^{\infty} C_n$. Since each C_n is the union of a finite collection of closed sets, each C_n is closed. The intersection of an arbitrary collection of closed sets is closed, so the Cantor set C is closed.

Exercise 4 (3.2.8). (a) Definitely closed.
(b) Definitely open.
(c) Definitely open.
(d) Both.
(e) Neither.

Exercise 5 (3.2.9). (a) We first show that $(\cup_{\lambda \in \Lambda} E_\lambda)^c = \cap_{\lambda \in \Lambda} E_\lambda^c$.

$$\begin{aligned}
x \in (\cup_{\lambda \in \Lambda} E_\lambda)^c &\iff x \notin \cup_{\lambda \in \Lambda} E_\lambda \\
&\iff \forall \lambda \in \Lambda,\ x \notin E_\lambda \\
&\iff \forall \lambda \in \Lambda,\ x \in E_\lambda^c \\
&\iff x \in \cap_{\lambda \in \Lambda} E_\lambda^c.
\end{aligned}$$

Next, we show that $(\cap_{\lambda \in \Lambda} E_\lambda)^c = \cup_{\lambda\in\Lambda} E_\lambda^c$.

$$\begin{aligned} x \in (\cap_{\lambda\in\Lambda} E_\lambda)^c &\iff x \notin \cap_{\lambda \in} E_\lambda \\ &\iff \exists \lambda \in \mathbf{\Lambda} \text{ s.t. } x \notin E_\lambda \\ &\iff \exists \lambda \in \mathbf{\Lambda} \text{ s.t. } x \in E_\lambda^c \\ &\iff x \in \cup_{\lambda\in\Lambda} E_\lambda^c. \end{aligned}$$

(b) Suppose $\{E_\lambda : \lambda \in \Lambda\}$ is a finite collection of open sets. Each E_λ^c is thus a closed set by Theorem 3.2.13. As such, $\cup_{\lambda\in\Lambda} E_\lambda^c$ is the union of a finite collection of closed sets. By Theorem 3.2.3, $\cap_{\lambda\in\Lambda} E_\lambda$ is open. It follows that $(\cap_{\lambda\in\Lambda} E_\lambda)^c$ is closed by Theorem 3.2.13. Since $(\cap_{\lambda\in\Lambda} E_\lambda)^c = \cup_{\lambda\in\Lambda} E_\lambda^c$, the union of a finite collection of closed sets is therefore closed.

Suppose $\{E_\lambda : \lambda \in \Lambda\}$ is an arbitrary collection of open sets. As such, $\cap_{\lambda\in\Lambda} E_\lambda^c$ is the intersection of an arbitrary collection of closed sets. By Theorem 3.2.3, $\cup_{\lambda\in\Lambda} E_\lambda$ is open. It follows that $(\cup_{\lambda\in\Lambda} E_\lambda)^c$ is closed by Theorem 3.2.13. Since $(\cup_{\lambda\in\Lambda} E_\lambda)^c = \cap_{\lambda\in\Lambda} E_\lambda^c$, the intersection of an arbitrary collection of closed sets is closed.

Exercise 6 (3.2.10). (i) Such a set cannot exist. Let $A \subseteq [0,1]$ be a countable set. Since A is countable, there exists a bijection $f : \mathbb{N} \to A$. We can use the function f to define a sequence (a_n) where $a_n = f(n)$ for all $n \in \mathbb{N}$. Because $(a_n) \subseteq A \subseteq [0,1]$, (a_n) is bounded. By the Bolzano-Weierstrass Theorem, (a_n) has a convergent subsequence $(a_{n_k}) \to a$. Since f is a bijection, all the terms of (a_n) are distinct, so at most one term in (a_{n_k}) can be equal to a. Let (b_n) be a subsequence of (a_{n_k}) without the term a if it exists. It follows that $(b_n) \to a$, so a is a limit point.

(ii) Consider the set $\mathbb{Q} \cap [0,1]$.

(iii) TODO

Exercise 7 (3.2.13). TODO

Exercise 8 (3.2.14). (a) TODO

(b) TODO