MATH 355: HOMEWORK 1

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Exercise 1 (1.2.2)**.** Suppose towards a contradiction that there is a rational number $r \in \mathbf{Q}$ satisfying $2^r = 3$. Since $r \in \mathbf{Q}$, we can write $r = \frac{p}{q}$ for some $p, q \in \mathbf{Z}$ with $q \neq 0$. Thus, we have $2^{\frac{p}{q}} = 3 \Rightarrow 2^p = 3^q$. Since $2^p = 2 \cdot 2^{p-1}$ and $2^{p-1} \in \mathbf{Z}$, 2^p is even. As such, 3^q is even as well. However, this is a contradiction since 3^q is odd because the product of odd numbers is odd. Therefore, there is no rational number r satisfying $2^r = 3$.

Exercise 2 (1.2.3)**.**
(a) False. Consider infinite set of the form $A_n = [0, \frac{1}{n}]$ for $n \in \mathbf{N}$. Our definition of A_n satisfies $A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4 \cdots$. However, notice that $\bigcap_{n=1}^{\infty} A_n = \{0\}$, which is not an infinite set.
(b) True.
(c) False. Let $A = \{0\}$, $B = \{0, 1\}$, and $C = \{2, 3\}$. Then,

$$A \cap (B \cup C) = \{0\} \cap (\{0, 1\} \cup \{2, 3\}) = \{0\},$$

but

$$(A \cap B) \cup C = (\{0\} \cap \{0, 1\}) \cup \{2, 3\} = \{0, 2, 3\}.$$

Here, $A \cap (B \cup C) \neq (A \cap B) \cup C$.
(d) True.
(e) True.

Exercise 3 (1.2.6)**.**
(a) Suppose $a, b \in \mathbf{R}$ where $a, b > 0$. We have that $|a + b| = |a| + |b|$. We also have that $|-a + (-b)| = |-(a + b)| = |a + b| = |-a| + |-b| = |a| + |b|$. Thus, the triangle inequality holds when a and b have the same sign.
(b) Given $a, b \in \mathbf{R}$,

$$\begin{aligned}
(a + b)^2 &= a^2 + 2ab + b^2 \\
&\leq |a|^2 + 2|a||b| + |b|^2 \\
&= (|a| + |b|)^2.
\end{aligned}$$

(c) Given $a, b, c, d \in \mathbf{R}$,

$$\begin{aligned}
|a - b| &= |(a - c) + (c - d) + (d - b)| \\
&\leq |a - c| + |(c - d) + (d - b)| \text{ (by the triangle inequality)} \\
&\leq |a - c| + |c - d| + |d - b| \text{ (by the triangle inequality).}
\end{aligned}$$

(d) Given $a, b \in \mathbf{R}$,

$$\begin{aligned} ||a| - |b|| &= ||a-b+b| - |b|| \\ &\leq ||a-b| + |b| - |b|| \text{ (by the triangle inequality)} \\ &= ||a-b|| \\ &= |a-b|. \end{aligned}$$

Exercise 4 (1.2.8). (a) Let $f : \mathbf{N} \to \mathbf{N}$ be defined by $f(n) = n+1$, $n \in \mathbf{N}$.

(b) Let $f : \mathbf{N} \to \mathbf{N}$ be defined by $f(n) = |n|$, $n \in \mathbf{N}$.

(c) Let $f : \mathbf{N} \to \mathbf{Z}$ be defined by $f(n) = \begin{cases} \frac{n-1}{2} & n \text{ is odd} \\ \frac{-n}{2} & n \text{ is even} \end{cases}$, $n \in \mathbf{N}$.

Exercise 5 (1.2.10). (a) False. Let $a = 1$, $b = 1$, and $\epsilon = 0.5$. We have that $1 < 1 + 0.5 = 1.5$, but it is not true that $a < b$ since $a = 1 = b$.

(b) False. The same counterexample as in part (a) can be used to show the statement is false.

(c) False. Let $a = 2$, $b = 1$, and $\epsilon = 2$. We have that $2 < 1 + 2 = 3$, but $a > b$ since $2 > 1$.

Exercise 6 (1.3.2). (a) Let $B = \{0\}$. $\inf B = 0 = \sup B$. Thus, $\inf B \geq \sup B$.

(b) Impossible.

(c) Let B be a bounded subset of $\mathbf{Q}$ where $B = \{r \in \mathbf{Q} : 0 < r \leq 1\}$. $\sup B = 1 \in B$, but $\inf B = 0 \notin B$.

Exercise 7 (1.3.3). (a) By definition, $\inf A \in B$ and $\inf A \geq b$ for all $b \in B$. Thus, $\inf A$ is the maximum of B, which implies that $\inf A = \sup B$.

(b) For every nonempty set A of real numbers that is bounded below, we can define $B = \{b \in \mathbf{R} : b \text{ is a lower bound for } A\}$. By the Axiom of Completeness, we know that if B is bounded above, B has a least upper bound $\sup B$. From (a), we showed that $\sup B = \inf A$. Therefore, if A is bounded below, A has a greatest lower bound $\inf A$, so there is no need to assert this in the Axiom of Completeness.

Exercise 8 (1.3.6). (a) Given $a \in A$ and $b \in B$, consider

$$\begin{aligned} a + b &\leq s + b \text{ (since } s = \sup A \geq a) \\ &\leq s + t \text{ (since } t = \sup B \geq b). \end{aligned}$$

Therefore, $s + t$ is an upper bound for $A + B$.

(b) Given an arbitrary upper bound u for $A + B$, $a \in A$, and $b \in B$, we have $a + b \leq u \Rightarrow b \leq u - a$. Thus, $u - a$ is an upper bound for B. Since $t = \sup B$ is the least upper bound for B, $t \leq u - a$.

(c) It follows from (b) that $a \leq u - t$. Thus, $u - t$ is an upper bound for A. As such, $\sup A \leq u - t \Rightarrow s \leq u - t \Rightarrow s + t \leq u$. Therefore, $\sup(A + B) = s + t$.

(d) Since $s = \sup A$ and $t = \sup B$, given an arbitrary $\epsilon > 0$, $a \in A$, and $b \in B$, we know $s - \frac{\epsilon}{2} < a$ and $t - \frac{\epsilon}{2} < b$ by Lemma 1.3.8. Then,

$$\begin{aligned} (s - \frac{\epsilon}{2}) + (t - \frac{\epsilon}{2}) &< a + b \\ s + t - \epsilon &< a + b. \end{aligned}$$

By Lemma 1.3.8, $\sup(A + B) = s + t$.