MATH 355: HOMEWORK 8

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Exercise 1 (4.2.10). (a) (Right-hand limit) Let $f : A \to \mathbb{R}$, and let A be a limit point of the domain A. We say that $\lim_{x\to a^+} f(x) = L$ provided that, for all $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $0 < a - x < \delta$ (and $x \in A$) it follows that $|f(x) - L| < \epsilon$.

(Left-hand limit) Let $f : A \to \mathbb{R}$, and let A be a limit point of the domain A. We say that $\lim_{x\to a^-} f(x) = M$ provided that, for all $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $0 < x - a < \delta$ (and $x \in A$) it follows that $|f(x) - M| < \epsilon$.

(b) ($\implies$) Suppose that $\lim_{x\to a} f(x) = L$. By the definition of a functional limit, for all $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $0 < |x - c| < \delta$ (and $x \in A$) it follows that $|f(x) - L| < \epsilon$. Thus, for this chosen δ, we have that $0 < c-x < \delta$ (and $x \in A$) implies $|f(x) - L| < \epsilon$, and $0 < x-c < \delta$ (and $x \in A$) implies $|f(x) - L| < \epsilon$. Therefore, $\lim_{x\to c^+} f(x) = \lim_{x\to c^-} f(x) = L$ (i.e., both the right and left-hand limits equal L).

($\impliedby$) Suppose $\lim_{x\to c^+} f(x) = \lim_{x\to c^-} f(x) = L$. Since we have that $\lim_{x\to c^+} f(x) = L$, for all $\epsilon > 0$, there exists a $\delta_1 > 0$ such that $0 < c - x < \delta_1$ (and $x \in A$) implies $|f(x) - L| < \epsilon$. Similarly, since we have that $\lim_{x\to c^-} f(x) = L$, for all $\epsilon > 0$, there exists a $\delta_2 > 0$ such that $0 < x - c < \delta_2$ (and $x \in A$) implies $|f(x) - L| < \epsilon$. Let $\delta = \min\{\delta_1, \delta_2\}$. Thus, for all $\epsilon > 0$, we have that $0 < c - x < \delta \le \delta_1$ (and $x \in A$) implies $|f(x) - L| < \epsilon$ and $0 < x - c < \delta \le \delta_2$ (and $x \in A$) implies $|f(x) - L| < \epsilon$. It follows immediately that $0 < |x - c| < \delta$ (and $x \in A$) implies $|f(x) - L| < \epsilon$. Therefore, $\lim_{x\to a} f(x) = L$.

Exercise 2 (4.2.11). Since $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} h(x) = L$, by the Sequential Criterion for Functional Limits, we know that for all sequences $(x_n) \subseteq A$ satisfying $x_n \neq c$ and $\lim x_n = c$, it follows that $\lim f(x_n) = L$ and $\lim h(x_n) = L$. By assumption, we have $f(x_n) \le g(x_n) \le h(x_n)$. Applying the Squeeze Theorem for sequences, it follows that $\lim g(x_n) = L$, implying that $\lim_{x\to c} g(x) = L$ as well.

Exercise 3 (4.3.2).

Exercise 4 (4.3.5).

Exercise 5 (4.3.6).

Exercise 6 (4.3.8).

Exercise 7 (4.4.11).

Exercise 8 (4.4.12).