MATH 355: HOMEWORK 5

ALEXANDER LEE

Exercise 1 (2.5.1). (a) Impossible. Let (a_n) be a sequence with a subsequence (a_{n_k}), where (a_{n_k}) is bounded. By the Bolzano-Weierstrass Theorem, (a_{n_k}) contains a convergent subsequence. Note that this convergent subsequence of (a_{n_k}) is also a subsequence of (a_n). Thus, (a_n) contains a convergent subsequence.

(b) Consider the sequences $a_n = \frac{1}{n}$ and $b_n = 1 + \frac{1}{n}$. Now, define the sequence $(c_n) = (a_1, b_1, a_2, b_2, a_3, b_3, \ldots)$. Notice that (a_n) and (b_n) are both subsequences of $(c)n$. Further notice that (c_n) does not contain 0 or 1 as terms but $(a_n) \to 0$ and $(b_n) \to 1$.

(c) Consider the following sequences.

$$\begin{aligned}
(a_n^1) &= (1 + 1/1, 1 + 1/2, 1 + 1/3, \ldots) \\
(a_n^2) &= (1/2 + 1/1, 1/2 + 1/2, 1/2 + 1/3, \ldots) \\
(a_n^3) &= (1/3 + 1/1, 1/3 + 1/2, 1/3 + 1/3, \ldots) \\
&\vdots
\end{aligned}$$

Notice that $(a_n^1) \to 1$, $(a_n^2) \to 1/2$, $(a_n^3) \to 1/3$, and so on. Now consider the sequence (a_n) constructed with the top right to bottom left diagonals across the sequences above. That is, $(a_n) = (1+1/1, 1+1/2, 1/2+1/1, 1+1/3, 1/2+1/2, 1/3+1/1, \ldots)$. By construction, (a_n) contains subsequences converging to every point in the infinite set $\{1, 1/2, 1/3, \ldots\}$.

(d) Impossible. Let (a_n) be a sequence that contains subsequences converging to every point in the infinite set $\{1, 1/2, 1/3, 1/4, 1/5, \ldots\}$. Thus, the subsequence $(a_{n_k}) \to \frac{1}{M}$ for all $M \in \mathbb{N}$. For all $M \in \mathbb{N}$, let $x_M \in V_\epsilon(\frac{1}{M})$. Since $0 \le x_M \le \frac{1}{M}$, we have that $(x_M) \to 0$ by the Squeeze Theorem. Notice that (x_M) is also a subsequence of (a_n) and 0 is not in the infinite set.

Exercise 2 (2.5.2). (a) True. Since $(x_2, x_3, x_4, \ldots)$ converges and is a proper subsequence of (x_n), (x_n) also converges.

(b) True. Suppose towards a contradiction that (x_n) converges. Thus, we know that subsequences of (x_n) are convergent, which contradicts the assumption that (x_n) contains a divergent subsequence. Therefore, (x_n) diverges.

(c) True. Suppose that (x_n) is bounded and diverges. By the Bolzano-Weierstrass Theorem, since (x_n) is bounded, it contains a convergent subsequence $(x_{n_k}) \to x$. Since (x_n) diverges, there exists an $\epsilon > 0$ such that for all $N \in \mathbb{N}$, there exists an $n \ge N$ such that $|x_n - x| \ge \epsilon$ There are infinitely many such x_n's, so throw them into a subsequence (x_{m_k}). Since (x_{m_k}) is also bounded, $(x_{m_k}) \to y$ also converges. $x \ne y$ since the terms in (x_{m_k}) are all greater than or equal to an ϵ distance away from x by construction.

(d) True. Suppose (x_n) is monotone and contains a convergent subsequence $(x_{n_k}) \to x$. Without loss of generality, suppose that (x_n) is increasing. Since $(x_{n_k}) \to x$, we have that for any $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n_k \geq N$, it follows that $|x_{n_k} - x| < \epsilon$. Take $n_k = N$ such that $|x_N - x| < \epsilon$. Next take $n_k = M \geq N$ such that $|x_M - x| < \epsilon$. There exists a term x_n in (x_n) such that $x_N \leq x_n \leq x_M$ because (x_n) is increasing. Thus, we have that

$$\epsilon > x - x_N \geq x - x_n \geq x - x_M > -\epsilon.$$

Therefore, $|x - x_n| < \epsilon$ and (x_n) converges.

Exercise 3 (2.6.2). (a) Consider the sequence $a_n = (-1)^n \frac{1}{n}$. (a_n) clearly converges to 0, so it is a Cauchy sequence by the Cauchy Criterion. Notice that (a_n) is not monotone as well.

(b) Impossible. Suppose (a_n) is a Cauchy sequence. Thus, (a_n) is bounded by Lemma 2.6.3. As such, every subsequence of (a_n) must be bounded as well.

(c) Impossible. A divergent monotone sequence is unbounded. Thus, any subsequence must also be unbounded. Since the subsequence is unbounded, it cannot converge and so it is not a Cauchy subsequence.

(d) Consider the sequence $(a_n) = (0, 1, 0, 2, 0, 3, \ldots)$. Clearly, (a_n) is unbounded. However, $(0, 0, 0, \ldots)$ is a subsequence of (a_n) that is Cauchy.

Exercise 4 (2.6.4). (a) Let $\epsilon > 0$ be arbitrary. Since (a_n) is a Cauchy sequence, there exists an $N_1 \in \mathbb{N}$ such that whenever $m, n \geq N_1$, it follows that $|a_n - a_m| < \epsilon/2$. Similarly, since (b_n) is a Cauchy sequence, there exists an $N_2 \in \mathbb{N}$ such that whenever $m, n \geq N_2$, it follows that $|b_n - b_m| < \epsilon/2$. Let $N = \max\{N_1, N_2\}$ and suppose $m, n \geq N$. Then,

$$\begin{aligned}
|c_n - c_m| &= ||a_n - b_n| - |a_m - b_m|| \\
&\leq |(a_n - b_n) - (a_m - b_m)|| \text{ (by the Triangle Inequality)} \\
&= |(a_n - a_m) + (b_m - b_n)| \\
&\leq |a_n - a_m| + |b_m - b_n| \text{ (by the Triangle Inequality)} \\
&< \epsilon/2 + \epsilon/2 \\
&= \epsilon.
\end{aligned}$$

Therefore, (c_n) is a Cauchy sequence.

(b) Let $a_n = 1$. (a_n) is a Cauchy sequence, but $c_n = (-1)^n$ is divergent and thus not a Cauchy sequence.

(c) Let $a_n = (-1)^n \frac{1}{n+1}$. Then,

$$c_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ -1 & \text{if } n \text{ is odd} \end{cases},$$

which is divergent and thus not a Cauchy sequence.

Exercise 5 (2.7.1). (a) Let $\epsilon > 0$ be arbitrary. Since $(a_n) \to 0$, there exists an $N \in \mathbb{N}$ such that whenever $m \geq N$, it follows that $|a_m| < \epsilon$. Then,

$$\begin{aligned} |s_n - s_m| &= |a_{m+1} - a_{m+2} - \cdots \pm a_n| \\ &\leq |a_{m+1}| \text{ (since } (a_n) \text{ is decreasing)} \\ &\leq |a_m| \\ &< \epsilon \end{aligned}$$

whenever $n > m \geq N$.

Exercise 6 (2.7.2). (a) Converges. $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges because it is geometric. Also, $0 \leq \frac{1}{2^n+n} \leq \frac{1}{2^n}$ for all $n \in \mathbb{N}$, so by the Comparison Test, $\sum_{n=1}^{\infty} \frac{1}{2^n+n}$ converges.

(b) Converges. We know that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. Furthermore, since $0 \leq \frac{|\sin(n)|}{n^2} \leq \frac{1}{n^2}$ for all $n \in \mathbb{N}$, we have that $\sum_{n=1}^{\infty} \frac{|\sin(n)|}{n^2}$ converges by the Comparison Test. As such, $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$ converges as well by the Absolute Convergence Test.

(c) Diverges. The series can be expressed as $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n+1}{2n}$. Because $\lim \left|(-1)^{n+1} \frac{n+1}{2n}\right| = \lim \frac{n+1}{2n} \neq 0 = |0|$, we have that $\lim (-1)^{n+1} \frac{n+1}{2n} \neq 0$ by the contrapositive of Exercise 2.3.10 (b). As such, by Theorem 2.7.3, $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n+1}{2n}$ diverges.

(d) Diverges.

(e) Diverges.

Exercise 7 (2.7.4). (a) Consider $x_n = \frac{1}{n}$ and $y_n = (-1)^n$. Since $\sum x_n$ is the harmonic series, it diverges. Since $(y_n) \not\to 0$, $\sum y_k$ diverges. However, $\sum x_n y_n = \sum (-1)^n \frac{1}{n}$ converges by the Alternating Series Test.

(b) Consider $x_n = (-1)^n \frac{1}{n}$ and $y_n = (-1)^n$. $\sum x_n$ converges by the Alternating Series Test. Clearly, (y_n) is bounded. However, $\sum x_n y_n = \sum \frac{1}{n}$ diverges since it is the harmonic series.

(c) Impossible. Since $\sum x_n$ and $\sum (x_n + y_n)$ both converge, we have that $\sum y_n = \sum (x_n + y_n) - x_n$ converges as well by the Algebraic Limit Theorem for Series.

(d) Consider

$$x_n = \begin{cases} 1/n & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

$\sum (-1)^n x_n = 1/2 + 1/4 + 1/6 + \cdots = \sum \frac{1}{2n}$. Since $\sum \frac{1}{n}$ diverges, $\sum \frac{1}{2n} = 2 \sum \frac{1}{n}$ also diverges.

Exercise 8 (2.7.5). By the Cauchy Condensation Test, the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $\sum_{n=0}^{\infty} 2^n \frac{1}{(2^n)^p}$ converges. Notice that $2^n \frac{1}{(2^n)^p} = \left(\frac{2}{2^p}\right)^n$. Thus, $\sum_{n=0}^{\infty} 2^n \frac{1}{(2^n)^p}$ converges if and only if $\left|\frac{2}{2^p}\right| < 1$ by the Geometric Series Test. It follows that

$$\begin{aligned}\left|\frac{2}{2^p}\right| < 1 &\iff \frac{2}{2^p} < 1 \\ &\iff 2 < 2^p \\ &\iff 1 < 2^{p-1} \\ &\iff \log_2(1) < p - 1 \\ &\iff 0 < p - 1 \\ &\iff 1 < p.\end{aligned}$$

Thus, we must have that $p > 1$.

Exercise 9 (2.7.9). (a) By assumption, we know that for all $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, it follows that $\left|\left|\frac{a_{n+1}}{a_n}\right| - r\right| < \epsilon$. Since $r < r' < 1$, we can let $\epsilon = r' - r$. Thus, we have $-r' + r < \left|\frac{a_{n+1}}{a_n}\right| - r < r' - r \implies \left|\frac{a_{n+1}}{a_n}\right| < r' \implies \frac{|a_{n+1}|}{|a_n|} < r' \implies |a_{n+1}| < |a_n|r' \implies |a_{n+1}| \leq |a_n|r'$.

(b) Since $\lim\left|\frac{a_{n+1}}{a_n}\right| = r$ and $\left|\frac{a_{n+1}}{a_n}\right| \geq 0$, we have that $0 \leq r < r' < 1$ by the Order Limit Theorem. Thus, $|r'| < 1$. By the Geometric Series Test, we hence know that $|a_N| \sum (r')^n$ converges.

(c) Suppose $N \in \mathbb{N}$ and $n \geq N$. Then, notice that

$$\frac{|a_{N+n}|}{(r')^{n-1}} \leq \cdots \leq \frac{|a_{N+3}|}{(r')^2} \leq \frac{|a_{N+2}|}{(r')} \leq |a_{N+1}| \leq |a_N|r'.$$

Thus, we have that $0 \leq |a_{N+n}| \leq |a_N|(r')^n$. Since we know that $\sum|a_N|(r')^n$ converges from part (b), $\sum|a_n| = \sum|a_{N+n}|$ converges by the Comparison Test. Therefore, $\sum a_n$ converges too.

Exercise 10 (3.2.3). (a) $\mathbb{Q}$ is not open. For example, no $V_\epsilon(0)$ is contained in $\mathbb{Q}$ since we can always find an irrational number in $V_\epsilon(0)$. $\mathbb{Q}$ is not closed. For example, $\sqrt{2}$ is a limit point of $\mathbb{Q}$ but $\sqrt{2} \notin \mathbb{Q}$.

(b) $\mathbb{N}$ is not open. For example, no $V_\epsilon(1)$ is contained in $\mathbb{N}$ since we can always find a non-natural (e.g., rational) number in $V_\epsilon(1)$. $\mathbb{N}$ is closed since it has no limit points.

(c) $\{x \in \mathbb{R} : x \neq 0\}$ is open since it is the union of two open sets $(-\infty, 0)$ and $(0, \infty)$. $\{x \in \mathbb{R} : x \neq 0\}$ is closed since it contains its limit points.

(d) $\{1 + 1/4 + 1/9 + \cdots + 1/n^2 : n \in \mathbb{N}\}$ is not open. For example, no $V_\epsilon(1)$ is contained in the set since we can always find an irrational number in $V_\epsilon(1)$. $\{1+1/4+1/9+\cdots+1/n^2 : n \in \mathbb{N}\}$ is not closed since $\pi^2/6$ is a limit point that is not in the set.

(e) $\{1 + 1/2 + 1/3 + \cdots + 1/n : n \in \mathbb{N}\}$ is not open. For example, no $V_\epsilon(1)$ is contained in the set since we can always find an irrational number in $V_\epsilon(1)$. $\{1 + 1/2 + 1/3 + \cdots + 1/n : n \in \mathbb{N}\}$ is closed since it does not contain any limit points.