MATH 355: HOMEWORK 3

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Exercise 1 (2.2.1)**.** Example: consider the sequence (x_n), where $x_n = (-1)^n$. The sequence verconges to 1 if we set $\epsilon = 3$. This sequence is also divergent. Since this sequence also verconges to -1 if we set $\epsilon = 3$, a sequence can verconge to two different values. This strange definition describes that a sequence is bounded since we have that $x - \epsilon < x_n < x + \epsilon$.

Exercise 2 (2.2.2)**.** (a) Let $\epsilon > 0$ be arbitrary. Choose $N \in \mathbb{N}$ such that $N > \frac{3}{25\epsilon} - \frac{4}{5}$. Let $n \geq N$. Then,

$$
\begin{aligned}
\left|a_n - \frac{2}{5}\right| &= \left|\frac{2n+1}{5n+4} - \frac{2}{5}\right| \\
&= \left|\frac{5(2n+1) - 2(5n+4)}{25n+20}\right| \\
&= \left|\frac{10n+5-10n-8}{25n+20}\right| \\
&= \left|\frac{-3}{25n+20}\right| \\
&= \frac{3}{25n+20} \\
&\leq \frac{3}{25N+20} \\
&< \frac{3}{25(\frac{3}{25\epsilon} - \frac{4}{5}) + 20} \\
&= \frac{3}{\frac{3}{\epsilon} - 20 + 20} \\
&= \epsilon.
\end{aligned}
$$

Hence, $\left|a_n - \frac{2}{5}\right| < \epsilon$.

(b) Let $\epsilon > 0$ be arbitrary. Choose $N \in \mathbb{N}$ such that $N > \frac{2}{\epsilon}$. Let $n \geq N$. Then,

$$\begin{aligned}
|a_n - 0| &= \left|\frac{2n^2}{n^3+3}\right| \\
&= \frac{2n^2}{n^3+3} \\
&< \frac{2n^2}{n^3} \\
&= \frac{2}{n} \\
&\leq \frac{2}{N} \\
&< \frac{2}{\frac{2}{\epsilon}} \\
&= \epsilon.
\end{aligned}$$

Hence, $|a_n - 0| < \epsilon$.

(c) Let $\epsilon > 0$ be arbitrary. Choose $N \in \mathbb{N}$ such that $N > \frac{1}{\epsilon^3}$. Let $n \geq N$. Then,

$$\begin{aligned}
|a_n - 0| &= \left|\frac{\sin(n^2)}{\sqrt[3]{n}}\right| \\
&\leq \frac{1}{\sqrt[3]{n}} \\
&\leq \frac{1}{\sqrt[3]{N}} \\
&< \frac{1}{\sqrt[3]{\frac{1}{\epsilon^3}}} \\
&= \epsilon.
\end{aligned}$$

Hence, $|a_n - 0| < \epsilon$.

Exercise 3 (2.2.4). (a) Consider the sequence (a_n), where $a_n = (-1)^n$. (a_n) has an infinite number of ones, but does not converge to one since it diverges.

(b) Impossible. Suppose towards a contradiction that our sequence $(a_n) \to a \neq 1$. Let $\epsilon = \frac{1}{2}|a - 1|$. Then, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|a_n - a| < \epsilon$. Since we have $a_n \in V_\epsilon(a)$ and $1 \notin V_\epsilon(a)$. Thus, at most N terms in (a_n) are not in $V_\epsilon(a)$. However, this is a contradiction since we assumed that we have an infinite number of ones, which are not in $V_\epsilon(a)$.

(c) Consider the divergent sequence $(a_n) = (-1, 1, -1, 1, 1, -1, 1, 1, 1, \ldots)$. For every $n \in \mathbb{N}$ it is possible to find n consecutive ones somewhere in the sequence.

Exercise 4 (2.2.5). (a) Let $a_n = [[5/n]]$. We claim that $\lim a_n = 0$. Let $\epsilon > 0$ be arbitrary. Choose $N \in \mathbb{N}$ such that $N = 6$. Let $n \geq N$. Then,

$$\begin{aligned} |a_n - 0| &= |[[5/n]]| \\ &= [[5/n]] \\ &= 0 \text{ (since for } n \geq 6,\ 0 < 5/n < 1) \\ &< \epsilon. \end{aligned}$$

Hence, $|a_n - 0| < \epsilon$.

(b) Let $a_n = [[(12+4n)/3n]]$. We claim that $\lim a_n = 1$. Let $\epsilon > 0$ be arbitrary. Choose $N \in \mathbb{N}$ such that $N = 7$. Let $n \geq N$. Then,

$$\begin{aligned} |a_n - 1| &= |[[(12 + 4n)/3n]] - 1| \\ &= [[(12 + 4n/3n)]] - 1 \text{ (since } (12 + 4n)/3n = 12/3n + 4/3 > 1) \\ &= 1 - 1 \text{ (since for } n \geq 7,\ 1 < (12 + 4n)/3n < 2) \\ &= 0 \\ &< \epsilon. \end{aligned}$$

Hence, $|a_n - 1| < \epsilon$.

Exercise 5 (2.2.7). (a) The sequence $(-1)^n$ is frequently in the set $\{1\}$.

(b) The definition of eventually is stronger than that of frequently, since eventually implies frequently.

(c) A sequence (a_n) converges to a if, given any ϵ-neighborhood $V_\epsilon(a)$ of a, (a_n) is eventually in the set $V_\epsilon(a)$. Eventually is the term we want.

(d) (x_n) is not necessarily eventually in the interval $(1.9, 2.1)$. For instance, consider the sequence $(1, 2, 1, 2, \ldots)$. However, (x_n) is frequently in $(1.9, 2.1)$.