MATH 355: HOMEWORK 4

ALEXANDER LEE

Exercise 1 (2.3.3)**.** Let $\epsilon > 0$ be arbitrary. Since $\lim x_n = l$, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, we have that $|x_n - l| < \epsilon$. Thus, we have $l - \epsilon < x_n < l + \epsilon$. Similarly, since $\lim z_n = l$, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, we have that $|z_n - l| < \epsilon$. Hence, we also have $l - \epsilon < z_n < l + \epsilon$. Because we are given that $x_n \leq y_n \leq z_n$ for all $n \in \mathbb{N}$, it follows that $l - \epsilon < x_n \leq y_n \leq z_n < l + \epsilon$, which implies that $l - \epsilon < y_n < l + \epsilon$ and $|y_n - l| < \epsilon$. Therefore, $\lim y_n = l$ as well.

Exercise 2 (2.3.4)**.** (a)

$$\begin{aligned}
\lim \left(\frac{1 + 2a_n}{1 + 3a_n - 4{a_n}^2} \right) &= \frac{\lim 1 + 2 \lim a_n}{\lim 1 + 3 \lim a_n - 4 \lim a_n \lim a_n} \\
&= \frac{1 + 2 \cdot 0}{1 + 3 \cdot 0 - 4 \cdot 0 \cdot 0} \\
&= 1
\end{aligned}$$

(b)

$$\begin{aligned}
\lim \left(\frac{(a_n + 2)^2 - 4}{a_n} \right) &= \frac{\lim(a_n + 2) \lim(a_n + 2) - \lim 4}{\lim a_n} \\
&= \frac{(\lim a_n + \lim 2)(\lim a_n + \lim 2) - 4}{\lim a_n} \\
&= \frac{(\lim a_n + 2)(\lim a_n + 2) - 4}{\lim a_n} \\
&= \frac{(\lim a_n)^2 + 4 \lim a_n + 4 - 4}{\lim a_n} \\
&= \frac{\lim a_n (\lim a_n + 4)}{\lim a_n} \\
&= \lim a_n + 4 \\
&= 0 + 4 \\
&= 4
\end{aligned}$$

(c)

$$
\begin{aligned}
\lim\left(\frac{\frac{2}{a_n}+3}{\frac{1}{a_n}+5}\right) &= \frac{\frac{\lim 2}{\lim a_n}+\lim 3}{\frac{\lim 1}{\lim a_n}+\lim 5} \\
&= \frac{\frac{2}{\lim a_n}+3}{\frac{1}{\lim a_n}+5} \\
&= \frac{\frac{2+3\lim a_n}{\lim a_n}}{\frac{1+5\lim a_n}{\lim a_n}} \\
&= \frac{2+3\lim a_n}{1+5\lim a_n} \\
&= \frac{2+3\cdot 0}{1+5\cdot 0} \\
&= \frac{2}{1} \\
&= 1
\end{aligned}
$$

Exercise 3 (2.3.7). (a) Possible. Consider the divergent sequences $(x_n) = (-1, 1, -1, 1, \ldots)$ and $(y_n) = (1, -1, 1, -1, \ldots)$. Then, we have $(x_n + y_n) = (0, 0, 0, 0, \ldots)$, which converges to 0.

(b) Impossible. Observe that $y_n = (x_n + y_n) - x_n$. Since $(x_n + y_n)$ and (x_n) converge, (y_n) must converge as well by the Algebraic Limit Theorem (ii).

(c) Impossible. Since (1) and (b_n) converge, $(1/b_n)$ must converge as well by the Algebraic Limit Theorem (iv). TODO

(d) Impossible. Since $(a_n - b_n)$ is bounded, there exists a number $M_1 > 0$ such that $|a_n - b_n| \le M_1$ for all $n \in \mathbb{N}$. Furthermore, since (b_n) converges, it is also bounded. Thus, there exists a number $M_2 > 0$ such that $|b_n| \le M_2$ for all $n \in \mathbb{N}$. As such, $|a_n| \le |a_n - b_n| + |b_n| \le M_1 + M_2$, so (a_n) must be bounded too.

(e) Impossible. TODO

Exercise 4 (2.3.10). (a) False. Consider $(a_n) = (b_n) = (1, 2, 3, 4, \ldots)$. Since $(a_n - b_n) = (0, 0, 0, 0, \ldots)$, we have that $\lim(a_n - b_n) = 0$. However, since (a_n) and (b_n) are not bounded, they do not even converge to a limit.

(b) True. Since $(b_n) \to b$, for any $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n \ge N$, we have that $|b_n - b| < \epsilon$. Since $||b_n| - |b|| \le |b_n - b|$, we also have that $||b_n| - |b|| < \epsilon$. Thus, $|b_n| \to |b|$.

(c) True. Note that $b_n = (b_n - a_n) + a_n$. Thus, by the Algebraic Limit Theorem (ii), we have $\lim b_n = \lim(b_n - a_n) + \lim a_n = 0 + a = a$.

(d) True. Since $(a_n) \to 0$, for any $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n \ge N$, we have that $|a_n| < \epsilon$, implying that $-\epsilon < a_n < \epsilon$. Since we have that $|b_n - b| \le a_n$, we therefore have $|b_n - b| \le a_n < \epsilon$. Hence, $(b_n) \to b$.

Exercise 5 (2.3.12). (a) True. Given $b \in B$, we know that $a_n \ge b$ for all $n \in N$ since a_n is an upper bound for B. By the Order Limit Theorem (iii), we have that $a = \lim a_n \ge b$. Thus, a is also an upper bound for B.

(b) True. Suppose that every a_n is in the complement of the interval $(0, 1)$. Next, suppose towards a contradiction that a is in $(0, 1)$. Since $(a_n) \to a$, there exists an ϵ-neighborhood $V_\epsilon(a) \subseteq (0, 1)$ that contains all but finitely many terms of (a_n). That is, all but finitely many terms of (a_n) are in $(0,1)$. This is a contradiction since we previously assumed that every a_n is not in $(0,1)$.

(c) False. Consider the sequence $(a_n) = (1.4, 1.41, 1.414, 1.4142, 1.41421, \ldots)$, where every consequent term is a decimal approximation of $\sqrt{2}$ that specifies an additional decimal. Each term in the sequence (a_n) is rational, however, by construction, $(a_n) \to \sqrt{2}$, which is irrational.

Exercise 6 (2.4.1). (a) We first show by induction that (x_n) is decreasing (i.e., $x_n \geq x_{n+1}$). Since $x_1 = 3$ and $x_2 = \frac{1}{4-3} = 1$, we have that $x_n \geq x_{n+1}$. Next, suppose that $3 = x_1 \geq x_2 \geq \cdots \geq x_n \geq x_{n+1}$. We want to show that $x_{n+1} \geq x_{n+2}$. Consider

$$\begin{aligned} x_{n+1} - x_{n+2} &= \frac{1}{4-x_n} - \frac{1}{4-x_{n+1}} \\ &= \frac{4 - x_{n+1} - (4 - x_n)}{(4-x_n)(4-x_{n+1})} \\ &= \frac{x_n - x_{n+1}}{(4-x_n)(4-x_{n+1})}. \end{aligned}$$

Since $x_n \geq x_{n+1}$ by our inductive hypothesis, we know that $x_n - x_{n+1} \geq 0$. Furthermore, by our inductive hypothesis, we also know that $3 \geq x_n \geq x_{n+1} \implies -3 \leq -x_n \leq -x_{n+1} \implies 1 = 4 - 3 \leq 4 - x_n \leq 4 - x_{n+1}$. Hence, we have that $x_{n+1} - x_{n+2} \implies x_{n+1} \geq x_{n+2}$. Therefore, (x_n) is decreasing. Next, we show that (x_n) is bounded. Since $x_n < 4$ for all $n \in \mathbb{N}$, we have that $x_{n+1} = \frac{1}{4-x_n} > 0$. Thus, (x_n) is bounded. As such, (x_n) converges by the Monotone Convergence Theorem.

(b) Suppose $\lim x_n = l$. By definition, given any $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|x_n - l| < \epsilon$. Since $x_{n+1} \leq x_n$ for all $n \in \mathbb{N}$, we have $|x_{n+1} - l| \leq |x_n - l| < \epsilon$. Thus, $\lim x_{n+1} = l = \lim_{x_n}$.

(c) Suppose $\lim x_n = \lim x_{n+1} = l$. Then,

$$\begin{aligned} \lim x_{n+1} = \lim\left(\frac{1}{4-x_n}\right) &\implies l = \frac{1}{4-l} \\ &\implies 4l - l^2 = 1 \\ &\implies 0 = l^2 - 4l + 1 \\ &\implies 0 = (l-2)^2 - 3 \\ &\implies 3 = (l-2)^2 \\ &\implies \pm\sqrt{3} = l - 2 \\ &\implies l = 2 \pm \sqrt{3}. \end{aligned}$$

By the Order Limit Theorem (iii), we know that $\lim x_n = l \leq x_n = 3$. Thus, $\lim x_n = 2 - \sqrt{3}$.

Exercise 7 (2.4.8).

Exercise 8 (2.4.9).