MATH 355: HOMEWORK 10

ALEXANDER LEE

Exercise 1 (5.3.2)**.** Suppose $f(a) = f(b)$ for some $a, b \in A$. Suppose towards a contradiction that $a \neq b$. Without loss of generality, suppose $a < b$. By the Mean Value Theorem, since $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b), there exists a point $c \in (a, b)$ where

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

However, since $f(a) = f(b)$ and $b \neq a$, we have that $f'(c) = \frac{0}{b-a} = 0$. This is a contradiction, since we assumed that $f'(x) \neq 0$ for all $x \in A$. Thus, it must be that $a = b$ and therefore f is one-to-one on A.

To show that the converse statement need not be true, consider the differentiable function $f(x) = x^3$ on the interval $A = (-1, 1)$. Here, f is clearly one-to-one, but $f'(0) = 0$.

Exercise 2 (5.3.3 skip (c))**.**

(a) Consider the function $f(x) = h(x) - x$ defined on the interval $[0, 3]$. To argue that there exists a point $d \in [0, 3]$ where $h(d) = d$, we show that there exists a point $d \in [0, 3]$ where $f(d) = 0$. First, consider

$$f(0) = h(0) - 0 = 1 - 0 = 1.$$

Then, consider

$$f(3) = h(3) - 3 = 2 - 3 = -1.$$

Also notice that because h is differentiable on $[0, 3]$, h is also continuous on $[0, 3]$. It follows from the Algebraic Continuity Theorem that $f(x) = h(x) - x$ is also continuous on $[0, 3]$. Since f is continuous on $[0, 3]$ and $f(3) = -1 < 0 < 1 = f(0)$, there exists a point $d \in (0, 3)$ where $f(d) = 0$ by the Intermediate Value Theorem. It follows that there exists a point $d \in [0, 3]$ where $h(d) = d$.

(b) Since h is differentiable on $[0, 3]$, h is thus continuous on $[0, 3]$ and differentiable on $(0, 3)$. Then, by the Mean Value Theorem, there exists a point $c \in (0, 3)$ where

$$h'(c) = \frac{h(3) - h(0)}{3 - 0} = \frac{2 - 1}{3} = \frac{1}{3}.$$

Exercise 3 (5.3.4)**.**

(a) Because f is differentiable on A, f is also continuous on A. By the Characterizations on Continuity, since $(x_n) \to 0$, it follows that $f(x_n) \to f(0)$. Because $f(x_n) = 0$ for all $n \in \mathbb{N}$, it must be that $f(0) = 0$. Furthermore, since f is differentiable at zero, we have that

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0}.$$

By the Sequential Criterion for Functional Limits, we also have that

$$f'(0) = \lim \frac{f(x_n) - f(0)}{x_n - 0} = \lim \frac{0 - 0}{x_n} = \lim \frac{0}{x_n} = 0.$$

(b) Without loss of generality, suppose that $x_n > 0$ for all $n \in \mathbb{N}$. Then, by the Mean Value Theorem, there exists a point $c_n \in (0, x_n)$ such that

$$f'(c_n) = \frac{f(x_n) - f(0)}{x_n - 0} = \frac{0 - 0}{x_n} = \frac{0}{x_n} = 0$$

for all $n \in \mathbb{N}$. Since f is twice-differentiable at zero, we have that

$$f''(0) = \lim_{x \to 0} \frac{f'(x) - f'(0)}{x - 0}.$$

By the Squeeze Theorem, we have that $(c_n) \to 0$. Thus, it follows from the Sequential Criterion for Functional Limits that

$$f''(0) = \lim \frac{f'(c_n) - f'(0)}{c_n - 0} = \frac{0 - 0}{c_n} = \frac{0}{c_n} = 0.$$

Exercise 4 (6.2.1). (a) Given $x \in (0, \infty)$, we have that

$$\lim f_n(x) = \lim \frac{nx}{1 + nx^2} = \lim \frac{x}{1/n + x^2} = \frac{x}{x^2} = \frac{1}{x}$$

by the Algebraic Limit Theorem. Thus, the pointwise limit of (f_n) for all $x \in (0, \infty)$ is $f(x) = \frac{1}{x}$.

(b) No. Observe that

$$|f_n(x) - f(x)| = \left| \frac{nx}{1 + nx^2} - \frac{1}{x} \right| = \left| \frac{nx^2 - (1 + nx^2)}{x(1 + nx^2)} \right| = \frac{1}{x(1 + nx^2)}.$$

Then given $N \in \mathbb{N}$, choose $n \geq \mathbb{N}$ and $x = \frac{1}{n} \in (0, \infty)$. It follows that

$$|f_n(x) - f(x)| = \frac{1}{x(1 + nx^2)} = \frac{n}{1 + n \cdot \frac{1}{n^2}} = \frac{n}{1 + \frac{1}{n}} = \frac{n^2}{n + 1} \geq \frac{1}{2}.$$

Therefore, the convergence is not uniform on $(0, \infty)$.

(c) No. Same reasoning as (b). Observe that the choice of $x = \frac{1}{n}$ is also in $(0, 1)$.

(d) Yes. Let $\epsilon > 0$ be arbitrary. Choose $N \in \mathbb{N}$ such that $N > \frac{1}{\epsilon}$. Then, whenever $n \geq N$ and $x \in (1, \infty)$, it follows that

$$|f_n(x) - f(x)| = \frac{1}{x(1 + nx^2)} < \frac{1}{1 + n} < \frac{1}{n} \leq \frac{1}{N} < \epsilon.$$

Exercise 5 (6.2.2). (a) Each f_n is continuous at zero. To see why, let $\epsilon > 0$ be arbitrary and choose $\delta < \frac{1}{n}$. Then, if $|x| < \delta$, it follows that

$$|f_n(x) - f_n(0)| = |f_n(x) - 0| = |f_n(x)| = f_n(x) = 0 < \epsilon.$$

We first show that f is not continuous at zero. Since $x_n = \frac{1}{n}$ converges to 0 but $f(x_n)$ converges to $1 \neq 0 = f(0)$, it follows from the Criterion for Discontinuity that f is not continuous at 0.

We now show that $f_n \to f$ not uniformly on $\mathbb{R}$. By the contrapositive of the Continuous Limit Theorem, since f is not continuous at zero and each f_n is continuous at zero, it must be that $f_n \to f$ not uniformly on $\mathbb{R}$.

(b) Each f_n is continuous at zero by the same argument from (a).

To show that $g_n \to g$ uniformly on $\mathbb{R}$, Let $\epsilon > 0$ be arbitrary, choose $N \in \mathbb{N}$ and let $n \geq N$. If $x = \frac{1}{k}$ for some $k \in \mathbb{N}$ such that $k \leq n$, then

$$|g_n(x) - g(x)| = \left|\frac{1}{k} - \frac{1}{k}\right| = 0 < \epsilon.$$

Otherwise, then

$$|g_n(x) - g(x)| = |0 - 0| = 0 < \epsilon.$$

We now show that f is continuous at zero. Let $\epsilon > 0$ be arbitrary and choose $\delta = \epsilon$. If $|x| < \delta$, it follows that

$$|g(x) - g(0)| = |g(x) - 0| = |g(x)| \leq |x| < \delta = \epsilon.$$

(c) Each f_n is continuous at zero by the same argument from (a).

TODO