

## MATH 355: HOMEWORK 6

ALEXANDER LEE

- Exercise 1** (3.2.2). (a) Limit points of  $A$ :  $\{-1, 1\}$ . Limit points of  $B$ :  $[0, 1]$ .  
 (b)  $A$  is neither open nor closed.  $B$  is neither open nor closed.  
 (c)  $A$  contains isolated points.  $B$  does not contain isolated points.  
 (d)  $\overline{A} = A \cup \{-1\}$ .  $\overline{B} = [0, 1]$ .

- Exercise 2** (3.2.4). (a) If  $s \in A$ , then  $s \in \overline{A}$  and we are done. Now suppose  $s \notin A$ . By Lemma 1.3.8, for every  $\epsilon > 0$ , there exists an  $a \in A$  ( $a \neq s$ ) such that  $s - \epsilon < a$ . Since  $s = \sup(A)$ , we also know that  $a < s$ . Thus, every  $\epsilon$ -neighborhood  $V_\epsilon(s)$  intersects  $A$  at some point other than  $s$ . That is,  $s$  is a limit point of  $A$ , so  $s \in \overline{A}$  in this case as well.  
 (b) An open set  $O$  cannot contain its supremum  $s = \sup(O)$  since every  $\epsilon$ -neighborhood  $V_\epsilon(s)$  of  $s$  is not a subset of  $O$ . Specifically, this is because for any  $\epsilon > 0$  and  $a \in O$ , we have that  $a < s + \epsilon$  since  $s = \sup(O)$ .

- Exercise 3** (3.2.6). (a) False. Consider the open set  $\mathbb{R} \setminus \{\sqrt{2}\}$ .  
 (b) False. Consider the closed sets of the form  $C_n = [n, \infty)$  for  $n \in \mathbb{N}$ . Observe that  $C_n \subseteq C_{n+1}$  and  $\bigcup_{n=1}^{\infty} C_n = \emptyset$ .  
 (c) True. Given a nonempty open set  $O$ , we know that for  $a \in O$ , there exists an  $\epsilon$ -neighborhood  $V_\epsilon(a) \subseteq O$ . By the Density of  $\mathbb{Q}$  in  $\mathbb{R}$ , there exists a rational number  $r \in V_\epsilon(a)$ . Thus, we have that  $r \in O$ .  
 (d) False. Consider the bounded infinite closed set  $F = \{\sqrt{2} + 1/n : n \in \mathbb{N}\} \cup \{\sqrt{2}\}$ . Observe that  $F$  does not contain any rational number.  
 (e) True. The Cantor set is defined as  $C = \bigcap_{n=0}^{\infty} C_n$ . Since each  $C_n$  is the union of a finite collection of closed sets, each  $C_n$  is closed. The intersection of an arbitrary collection of closed sets is closed, so the Cantor set  $C$  is closed.

- Exercise 4** (3.2.8). (a) Definitely closed.  
 (b) Definitely open.  
 (c) Definitely open.  
 (d) Both.  
 (e) Neither.

- Exercise 5** (3.2.9). (a) We first show that  $(\bigcup_{\lambda \in \Lambda} E_\lambda)^c = \bigcap_{\lambda \in \Lambda} E_\lambda^c$ .

$$\begin{aligned} x \in (\bigcup_{\lambda \in \Lambda} E_\lambda)^c &\iff x \notin \bigcup_{\lambda \in \Lambda} E_\lambda \\ &\iff \forall \lambda \in \Lambda, x \notin E_\lambda \\ &\iff \forall \lambda \in \Lambda, x \in E_\lambda^c \\ &\iff x \in \bigcap_{\lambda \in \Lambda} E_\lambda^c. \end{aligned}$$

Next, we show that  $(\cap_{\lambda \in \Lambda} E_\lambda)^c = \cup_{\lambda \in \Lambda} E_\lambda^c$ .

$$\begin{aligned}
 x \in (\cap_{\lambda \in \Lambda} E_\lambda)^c &\iff x \notin \cap_{\lambda \in \Lambda} E_\lambda \\
 &\iff \exists \lambda \in \Lambda \text{ s.t. } x \notin E_\lambda \\
 &\iff \exists \lambda \in \Lambda \text{ s.t. } x \in E_\lambda^c \\
 &\iff x \in \cup_{\lambda \in \Lambda} E_\lambda^c.
 \end{aligned}$$

- (b) Suppose  $\{E_\lambda : \lambda \in \Lambda\}$  is a finite collection of open sets. Each  $E_\lambda^c$  is thus a closed set by Theorem 3.2.13. As such,  $\cup_{\lambda \in \Lambda} E_\lambda^c$  is the union of a finite collection of closed sets. By Theorem 3.2.3,  $\cap_{\lambda \in \Lambda} E_\lambda$  is open. It follows that  $(\cap_{\lambda \in \Lambda} E_\lambda)^c$  is closed by Theorem 3.2.13. Since  $(\cap_{\lambda \in \Lambda} E_\lambda)^c = \cup_{\lambda \in \Lambda} E_\lambda^c$ , the union of a finite collection of closed sets is therefore closed.

Suppose  $\{E_\lambda : \lambda \in \Lambda\}$  is an arbitrary collection of open sets. As such,  $\cap_{\lambda \in \Lambda} E_\lambda^c$  is the intersection of an arbitrary collection of closed sets. By Theorem 3.2.3,  $\cup_{\lambda \in \Lambda} E_\lambda$  is open. It follows that  $(\cup_{\lambda \in \Lambda} E_\lambda)^c$  is closed by Theorem 3.2.13. Since  $(\cup_{\lambda \in \Lambda} E_\lambda)^c = \cap_{\lambda \in \Lambda} E_\lambda^c$ , the intersection of an arbitrary collection of closed sets is closed.

**Exercise 6 (3.2.10).** (i) Such a set cannot exist. Let  $A \subseteq [0, 1]$  be a countable set. Since  $A$  is countable, there exists a bijection  $f : \mathbb{N} \rightarrow A$ . We can use the function  $f$  to define a sequence  $(a_n)$  where  $a_n = f(n)$  for all  $n \in \mathbb{N}$ . Because  $(a_n) \subseteq A \subseteq [0, 1]$ ,  $(a_n)$  is bounded. By the Bolzano-Weierstrass Theorem,  $(a_n)$  has a convergent subsequence  $(a_{n_k}) \rightarrow a$ . Since  $f$  is a bijection, all the terms of  $(a_n)$  are distinct, so at most one term in  $(a_{n_k})$  can be equal to  $a$ . Let  $(b_n)$  be a subsequence of  $(a_{n_k})$  without the term  $a$  if it exists. It follows that  $(b_n) \rightarrow a$ , so  $a$  is a limit point.

(ii) Consider the set  $\mathbb{Q} \cap [0, 1]$ .

- (iii) Such a set cannot exist. Suppose  $A$  has an uncountable number of isolated points. For every isolated point  $x \in A$ , there exists an  $\epsilon_x > 0$  such that  $V_{\epsilon_x}(x) \cap A = \{x\}$ . Each neighborhood  $V_{\epsilon_x}(x)$  can intersect with at most 2 other neighborhoods, since if a neighborhood intersects with more than 2 neighborhoods, then one of the three neighborhoods in question would intersect another isolated point in  $A$ , which is not possible. By the Density of  $\mathbb{Q}$  in  $\mathbb{R}$ , we can choose a rational number  $r$  in each neighborhood. Since each neighborhood intersects with at most 2 other neighborhoods, the same  $r$  can be chosen for at most 3 neighborhoods. Regardless, we are still have uncountably many rational numbers since each rational number  $r$  corresponds to at most 3 neighborhoods and we have uncountably many neighborhoods. This is a contradiction since  $\mathbb{Q}$  is countable.

**Exercise 7 (3.2.13).** TODO

**Exercise 8 (3.2.14).** (a) We first show that  $E$  is closed if and only if  $\overline{E} = E$ . Let  $L$  be the set of all limit points of  $E$ . Then,

$$\begin{aligned}
 E \text{ is closed} &\iff L \subseteq E \\
 &\iff E \cup L = E \\
 &\iff \overline{E} = E.
 \end{aligned}$$

Next, we show that  $E$  is open if and only if  $E^\circ = E$ .

$$\begin{aligned} E \text{ is open} &\iff \forall x \in E, \exists V_\epsilon(x) \subseteq E \\ &\iff E^\circ = E. \end{aligned}$$

- (b) We begin with showing that  $\overline{E^c} = (E^\circ)^\circ$ . Let  $L$  be the set of all limit points of  $E$ . Then,

$$\begin{aligned} x \in \overline{E^c} &\iff x \in (E \cup L)^c \\ &\iff x \in E^c \cap L^c \text{ (by DeMorgan's Laws)} \\ &\iff x \in E^c \wedge x \text{ is not a limit point of } E \\ &\iff x \in E^c \text{ s.t. } \exists V_\epsilon(x) \subseteq E^c \\ &\iff x \in (E^c)^\circ. \end{aligned}$$

Next, we show that  $(E^\circ)^c = \overline{E^c}$ .

$$\begin{aligned} (E^\circ)^c &= (((E^c)^\circ)^\circ)^c \\ &= (\overline{E^c})^c \text{ (since } \overline{E^c} = (E^c)^\circ) \\ &= \overline{E^c}. \end{aligned}$$