MATH 355: HOMEWORK 7

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Exercise 1 (3.3.1)**.** Since K is compact, it is closed and bounded. It follows from the Axiom of Completeness that because K is nonempty and bounded (above and below), $\sup(K)$ and $\inf(K)$ both exist. Since K is closed, $K = \overline{K}$, so $\sup(K) \in \overline{K} = K$ by Exercise 3.2.4 (a). By similar reasoning, it follows that $\inf(K) \in K$.

Exercise 2 (3.3.2, except (c))**.**
(a) Not compact. Consider the sequence (a_n) where $a_n = n$.
(b) Not compact. We can construct a sequence (a_n) that converges to $1/\sqrt{2} \notin \mathbb{Q} \cap [0,1]$. All subsequences of (a_n) must also converge to $1/\sqrt{2}$.
(d) Not compact. Consider the sequence of partial sums (s_m) where $s_m = \sum_{n=1}^{m} 1/n^2$.
(e) Compact. The set is closed since 1 is its only limit point, which is contained in the set, and the set is also bounded.

Exercise 3 (3.3.4)**.**
(a) The set is definitely closed since K is compact and thus closed, and the intersection of an arbitrary collection of closed sets is closed. The set is definitely compact since $K \cap F$ is closed as explained previously and is bounded because K is bounded.
(b) The set is definitely closed since the closure of any set is closed. The set is not definitely compact.
(c) Neither.
(d) The set is definitely closed since the closure of any set is closed. The set is definitely compact since it is closed as explained previously and is bounded since $K \cap F^c$ is bounded and the closure of a bounded set is also bounded.

Exercise 4 (3.3.5)**.**
(a) True. The arbitrary intersection of compact sets is closed since each compact set is closed and the arbitrary intersection of closed sets is closed. The arbitrary intersection of compact sets is bounded since each compact set is bounded and the intersection of bounded sets must be bounded.
(b) False. Consider $\cup_{n=1}^{\infty}[n, n+1]$. This union of compact sets is unbounded, and thus not compact.
(c) False. Consider $A = (0,1)$ and $K = [0,1]$. $A \cap K = (0,1)$ is open, and thus not compact.
(d) False. Consider $\cap_{n=1}^{\infty}[n, \infty) = \emptyset$.

Exercise 5 (3.3.11)**.**
- $\mathbb{N}$: Consider the open cover $\{V_1(n) : n \in \mathbb{N}\}$.
- $\mathbb{Q} \cap [0,1]$: Let (a_n) be a sequence of increasing irrational numbers such that $(a_n) \to 1/\sqrt{2}$ and $a_1 = -1/\sqrt{2}$. Now, consider the open cover $\{(a_n, a_{n+1}) : n \in \mathbb{N}\} \cup \{(1/\sqrt{2}, 2)\}$.

- $\{1+1/2^2+1/3^2+\cdots+1/n^2 : n \in \mathbb{N}\}$: Let (s_n) be the sequence of partial sums $s_n = 1+1/2^2+1/3^2+\cdots+1/n^2$. Now, consider the open cover $\{(\frac{s_{n-1}+s_n}{2}, \frac{s_n+s_{n+1}}{2}) : n \in \mathbb{N}\}$, where s_0 is defined as 0.

Exercise 6 (4.2.2). (a) Notice that

$$\begin{aligned} |(5x-6)-9| < 1 &\implies |5x-15| < 1 \\ &\implies 5|x-3| < 1 \\ &\implies |x-3| < 1/5 \\ &\implies -1/5 < x-3 < 1/5 \\ &\implies 14/5 < x < 16/5. \end{aligned}$$

Thus, if $x \in (14/5, 16/5)$, then $5x-6 \in (8,10)$. It follows that

$$\begin{aligned} \delta &= \min\{|14/5-3|, |16/5-3|\} \\ &= \min\{1/5, 1/5\} \\ &= 1/5. \end{aligned}$$

(b) Notice that

$$\begin{aligned} \left|\sqrt{x}-2\right| < 1 &\implies -1 < \sqrt{x}-2 < 1 \\ &\implies 1 < \sqrt{x} < 3 \\ &\implies 1 < x < 9. \end{aligned}$$

Thus, if $x \in (1,9)$, then $\sqrt{x} \in (1,3)$. It follows that

$$\begin{aligned} \delta &= \min\{|1-4|, |9-4|\} \\ &= \min\{3,5\} \\ &= 3. \end{aligned}$$

(c) Notice that

$$\begin{aligned} |[[x]]-3| < 1 &\implies -1 < [[x]]-3 < 1 \\ &\implies 2 < [[x]] < 4 \\ &\implies 3 < x < 4 \end{aligned}$$

Thus, if $x \in (3,4)$, then $[[x]] \in (2,4)$. It follows that

$$\begin{aligned} \delta &= \min\{|3-\pi|, |4-\pi|\} \\ &= \pi - 3. \end{aligned}$$

(d) Notice that

$$\begin{aligned} |[[x]]-3| < 0.01 &\implies -0.01 < [[x]]-3 < 0.01 \\ &\implies 2.99 < [[x]] < 3.01 \\ &\implies 3 < x < 4. \end{aligned}$$

Thus, if $x \in (3,4)$, then $[[x]] \in (2.99, 3.01)$. It follows that

$$\begin{aligned} \delta &= \min\{|3-\pi|, |4-\pi|\} \\ &= \pi - 3. \end{aligned}$$

Exercise 7 (4.2.5). (a) Let $\epsilon > 0$. Notice that

$$|3x + 4 - 10| = |3x + 6| = 3|x - 2|.$$

Thus, if we choose $\delta = \epsilon/3$, then $0 < |x - 2| < \delta$ implies $|3x + 4 - 10| < 3(\epsilon/3) = \epsilon$.

(b) Let $\epsilon > 0$. Notice that

$$\left|x^3 - 0\right| = \left|x^3\right| = x^2|x|.$$

Choose a δ-neighborhood around $c = 0$ to with radius no bigger than $\delta = 1$. Then, we get the upper bound $x^2 \leq 1$ for all $x \in V_\delta(c)$. Now, choose $\delta = \min\{1, \epsilon\}$. If $0 < |x - 0| < \delta$, then it follows that

$$\left|x^3 - 0\right| = x^2|x| < 1(\epsilon) = \epsilon.$$

(c) Let $\epsilon > 0$. Notice that

$$\left|x^2 + x - 1 - 5\right| = \left|x^2 + x - 6\right| = |x + 3||x - 2|.$$

Choose a δ-neighborhood around $c = 2$ with radius no bigger than $\delta = 1$. Then, we get the upper bound $|x + 3| \geq |3 + 3| = 6$ for all $x \in V_\delta(c)$. Now, choose $\delta = \min\{1, \epsilon/6\}$. If $0 < |x - 2| < \delta$, then it follows that

$$\left|x^3 + x - 1 - 5\right| = |x + 3||x - 2| < 6(\epsilon/6) = \epsilon.$$

(d) Let $\epsilon > 0$. Notice that

$$|1/x - 1/3| = \left|\frac{3 - x}{3x}\right| = \frac{|3 - x|}{|3x|}.$$

Choose a δ-neighborhood around $c = 3$ with radius no bigger than $\delta = 1$. Then, we get the lower bound $|3x| \geq |3 \cdot 2| = 6$. Now, choose $\delta = \min\{1, 6\epsilon\}$. If $0 < |x - 3| < \delta$, then it follows that

$$|1/x - 1/3| = \frac{|3 - x|}{|3x|} < \frac{6\epsilon}{6} = \epsilon.$$

Exercise 8 (4.2.6). (a) True. If a particular δ has been constructed as a suitable response to a particular ϵ challenge, then we know that for all $x \in V_\delta(c)$ different from c (with $x \in A$) it follows that $f(x) \in V_\epsilon(L)$. Therefore, any smaller positive δ will also suffice since all x's will still be in the δ-neighborhood.

(b) False. Let $a \in \mathbb{R}$ and consider $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x) = \begin{cases} x & \text{if } x \neq a \\ x + 1 & \text{if } x = a \end{cases}.$$

$\lim_{x \to a} f(x) = a$, however $a \neq f(a) = a + 1$.

(c) True. Given $\lim_{x \to a} f(x) = L$, we have that

$$\lim_{x \to a} 3[f(x) - 2]^2 = 3[\lim_{x \to a} f(x) - 2]^2 = 3(L - 2)^2$$

by the Algebraic Limit Theorem for Functional Limits.

(d) False. Let $a = 0$ and consider $f, g : \mathbb{R} \to \mathbb{R}$ such that $f(x) = x$ and $g(x) = 1/x$. Notice that $\lim_{x \to a} f(x) = 0$, but $\lim_{x \to a} f(x)g(x) = \lim_{x \to a} 1 = 1 \neq 0$ when $x \neq 0$.