MATH 355: HOMEWORK 1

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Exercise 1 (1.3.8). (a) Supremum: 1. Infimum: 0.
(b) Supremum: 1. Infimum: -1.
(c) Supremum: $\frac{1}{3}$. Infimum: $\frac{1}{4}$.
(d) Supremum: 1. Infimum: 0.

Exercise 2 (1.3.9). (a) Since $\sup(A) < \sup(B)$, we have $\frac{\sup(B)-\sup(A)}{2} > 0$. Set $m = \frac{\sup(B)-\sup(A)}{2}$. By Lemma 1.3.8, we know $\sup(B) - m < b$ for some $b \in B$. Because

$$\begin{aligned} \sup(B) - m &= \sup(B) - \frac{\sup(B) - \sup(A)}{2} \\ &= \frac{2\sup(B) - \sup(B) + \sup(A)}{2} \\ &= \frac{\sup(B) + \sup(A)}{2} \\ &= \frac{\sup(B) - \sup(A)}{2} + \sup(A) \\ &> \sup(A), \end{aligned}$$

we know that $\sup(B) - m$ is an upper bound for A. Since $\sup(B) - m < b$, b is also an upper bound for A. Observe that $b \in B$, so we are done.
(b) Let $A = B = (0, 1)$. We have $\sup(A) = \sup(B) = 1$, but all upper bounds for A are not elements of B.

Exercise 3 (1.4.1). (a) Given $a, b \in \mathbb{Q}$, we can write $a = \frac{p_1}{q_1}$ and $b = \frac{p_2}{q_2}$ for some $p_1, q_1, p_2, q_2 \in \mathbb{Z}$ with $q_1, q_2 \neq 0$. Then, $ab = (\frac{p_1}{q_1})(\frac{p_2}{q_2}) = \frac{p_1 p_2}{q_1 q_2}$. Since $p_1 p_2, q_1 q_2 \in \mathbb{Z}$ and $q_1 q_2 \neq 0$ because $q_1, q_2 \neq 0$, we have that $ab \in \mathbb{Q}$. Next, we can assume that if a or b are negative, the numerator is negative and the denominator is positive. Then, $a + b = \frac{p_1}{q_1} + \frac{p_2}{q_2} = \frac{p_1 q_2 + p_2 q_1}{q_1 q_2}$. Since $p_1 q_2 + p_2 q_1 \in \mathbb{Z}$ and $q_1 q_2 \neq 0$ because $q_1, q_2 \neq 0$, we have that $a + b \in \mathbb{Q}$.
(b) Let $a \in \mathbb{Q}$ with $a \neq 0$ and $t \in \mathbb{I}$. Suppose towards a contradiction that $a+t \in \mathbb{Q}$. Since $-a \in \mathbb{Q}$ and $\mathbb{Q}$ is closed under addition, $t = (a + t) + (-a) \in \mathbb{Q}$. This is a contradiction since we assumed that $t \in \mathbb{I}$. Therefore, $a + t \in \mathbb{I}$. Next, suppose towards a contradiction that $at \in \mathbb{I}$. Since $a \neq 0$, we have $\frac{1}{a} \in \mathbb{Q}$. Because $\mathbb{Q}$ is closed under multiplication $t = (\frac{1}{a})(at) \in \mathbb{Q}$. This is a contradiction since we assumed $t \in \mathbb{I}$. Therefore, $at \in \mathbb{I}$.
(c) $\mathbb{I}$ is not closed under addition and multiplication. To show that $\mathbb{I}$ is not closed under addition, consider $s = \sqrt{2}$ and $t = 1-\sqrt{2}$. We thus have $s+t = \sqrt{2} + 1 - \sqrt{2} = 1 \notin \mathbb{I}$. To show that $\mathbb{I}$ is not closed under multiplication, consider $s = t = \sqrt{2}$. Then, $st = \sqrt{2} \cdot \sqrt{2} = 2 \notin \mathbb{I}$.

Exercise 4 (1.4.3)**.** Suppose towards a contradiction that $\bigcap_{n=1}^{\infty}(0,1/n) \neq \emptyset$. Set $A = \bigcap_{n=1}^{\infty}(0,1/n)$. For all $n \in \mathbb{N}$, $1/n$ is thus an upper bound for A. Since $A \neq \emptyset$, there exists an $a \in A$ such that $a > 0$. By the Archimedean Property, there exists an $n \in \mathbb{N}$ such that $1/n < a$, which is a contradiction since our assumption implied that for all $n \in \mathbb{N}$, $1/n$ is an upper bound for A. Therefore, $\bigcap_{n=1}^{\infty}(0,1/n) = A = \emptyset$.

Exercise 5 (1.4.4)**.** By the construction of T, b is an upper bound for T since $b \geq t$ for all $t \in T$. Given an arbitrary upper bound $u \in \mathbb{R}$ for T, suppose towards a contradiction that $u < b$. By the density of $\mathbb{Q}$ in $\mathbb{R}$, there exists an $r \in \mathbb{Q}$ such that $u < r < b$. Since $u < r$ and $r \in T$, u is therefore not an upper bound for T, which contradicts our previous assumption that u is an upper bound for T. Hence, it must be that $b \leq u$, implying that $\sup(T) = b$.

Exercise 6 (1.5.4)**.**

(a) We know that $(-1,1) \sim \mathbb{R}$ with the function $f(x) = x/(x^2-1)$. Therefore, what is left to show is that $(-1,1) \sim (a,b)$. Let $f : (-1,1) \to (a,b)$ be given by $f(x) = \frac{b-a}{2}x + \frac{a+b}{2}$. (1–1) Suppose $f(x_1) = f(x_2)$ for some $x_1, x_2 \in (-1,1)$. Then,

$$\begin{aligned} f(x_1) = f(x_2) &\implies \frac{b-a}{2}x_1 + \frac{a+b}{2} = \frac{b-a}{2}x_2 + \frac{a+b}{2} \\ &\implies \frac{b-a}{2}x_1 = \frac{b-a}{2}x_2 \\ &\implies x_1 = x_2. \end{aligned}$$

(Onto) Given $y \in (a,b)$, let $x = \frac{2y-a-b}{b-a}$. Then,

$$\begin{aligned} f(x) &= f\left(\frac{2y-a-b}{b-a}\right) \\ &= \frac{b-a}{2} \cdot \frac{2y-a-b}{b-a} + \frac{a+b}{2} \\ &= \frac{2y-a-b}{2} + \frac{a+b}{2} \\ &= \frac{2y}{2} \\ &= y. \end{aligned}$$

Since f is 1–1 and onto, we have $(-1,1) \sim (a,b)$. Thus, $(a,b) \sim \mathbb{R}$.

(b) Consider the function $f : (a,\infty) \to \mathbb{R}$ given by $f(x) = \ln(x-a)$. (1–1) Suppose $f(x_1) = f(x_2)$ for some $x_1, x_2 \in (a,\infty)$. Then,

$$\begin{aligned} f(x_1) = f(x_2) &\implies \ln(x_1 - a) = \ln(x_2 - a) \\ &\implies x_1 - a = x_2 - a \\ &\implies x_1 = x_2. \end{aligned}$$

(Onto) Given $y \in \mathbb{R}$, let $x = e^y + a$. Then,

$$\begin{aligned} f(x) &= f(e^y + a) \\ &= \ln(e^y + a - a) \\ &= \ln(e^y) \\ &= y. \end{aligned}$$

Since f is 1–1 and onto, we have $(a,\infty) \sim \mathbb{R}$.

(c) Let $T = \mathbb{Q} \cap (0,1) = \{r_1, r_2, r_2, \ldots\}$. Next, consider the function $f : [0,1) \to (0,1)$ given by

$$f(x) = \begin{cases} r_1 & x = 0 \\ r_{i+1} & x = r_i \ , i \in \mathbb{N}. \\ x & x \notin T \end{cases}$$

(1–1) Suppose $f(x_1) = f(x_2) = y$ for some $x_1, x_2 \in [0,1)$. If $y = r_1$, then $x_1 = 0 = x_2$. If $y = r_{i+1}$ for some $i \in \mathbb{N}$, then $x_1 = r_i = x_2$. If $y = x \notin T$, then $x_1 = x = x_2$. (Onto) Suppose $y \in (0,1)$. If $y = r_1$, then $f(0) = r_1 = y$. If $y = r_{i+1}$, then $f(r_i) = r_{i+1} = y$. If $y = x \notin T$, then $f(x) = x = y$. Therefore, since f is 1–1 and onto, we have $[0,1) \sim (0,1)$.

Exercise 7 (1.5.5). (a) $A \sim A$ for every set A because the identity function $f : A \to A$ given by $f(x) = x$ is always 1–1 and onto.

(b) Given sets A and B, $A \sim B$ is equivalent to asserting $B \sim A$ because any 1–1 and onto function $f : A \to B$ has an inverse function $f^{-1} : B \to A$ that is also 1–1 and onto.

(c) Given three sets A, B, and C, suppose $A \sim B$ and $B \sim C$. Thus, there exist 1–1 and onto functions $f : A \to B$ and $g : B \to C$. Since $g \circ f : A \to C$ is also 1–1 and onto, we have that $A \sim C$.

Exercise 8 (1.6.5). (a) $P(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$

(b) We proceed with induction on n. Our base case is $n = 0$. That is, $A = \emptyset$, which only has one subset, namely $\emptyset$. Since $2^0 = 1$, the statement is true for the base case. Now, assume that if A is finite with $n = k$ elements, then $P(A)$ has 2^k elements. We want to show that the statement is true if A is finite with $n = k+1$ elements. Suppose A is finite with $k+1$ elements. Then, there exists a set B such that $A = B \cup \{a\}$ with $a \notin B$. Notice that $|B| = k$ and $B \subseteq A$. Further observe that $P(A) = P(B) \cup \{x \cup \{a\} : x \in P(B)\}$. By the inductive hypothesis, $|P(B)| = 2^k$. Also, $|\{x \cup \{a\} : x \in P(B)\}| = |P(B)| = 2^k$. Since $P(B)$ and $\{x \cup \{a\} : x \in P(B)\}$ are disjoint, we have $|P(A)| = 2^k + 2^k = 2(2^k) = 2^{k+1}$.