MATH 355: HOMEWORK 3

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Exercise 1 (2.2.1)**.** Example: consider the sequence (a_n), where $a_n = (-1)^n$. The sequence verconges to 1 if we set $\epsilon = 3$. This sequence is also divergent. Since this sequence also verconges to -1 if we set $\epsilon = 3$, a sequence can verconge to two different values. This strange definition describes that a sequence is bounded.

Exercise 2 (2.2.2)**.** (a) Let $\epsilon > 0$ be arbitrary. Choose $N \in \mathbb{N}$ such that $N > \frac{3}{25\epsilon} - \frac{4}{5}$. Let $n \geq N$. Then,

$$\begin{aligned}
\left| a_n - \frac{2}{5} \right| &= \left| \frac{2n+1}{5n+4} - \frac{2}{5} \right| \\
&= \left| \frac{5(2n+1) - 2(5n+4)}{25n+20} \right| \\
&= \left| \frac{10n+5-10n-8}{25n+20} \right| \\
&= \left| \frac{-3}{25n+20} \right| \\
&= \frac{3}{25n+20} \\
&\leq \frac{3}{25N+20} \\
&< \frac{3}{25(\frac{3}{25\epsilon} - \frac{4}{5}) + 20} \\
&= \frac{3}{\frac{3}{\epsilon} - 20 + 20} \\
&= \epsilon.
\end{aligned}$$

Hence, $\left| a_n - \frac{2}{5} \right| < \epsilon$.

(b) Let $\epsilon > 0$ be arbitrary. Choose $N \in \mathbb{N}$ such that $N > \frac{2}{\epsilon}$. Let $n \geq N$. Then,

$$\begin{aligned} |a_n - 0| &= \left| \frac{2n^2}{n^3+3} \right| \\ &= \frac{2n^2}{n^3+3} \\ &< \frac{2n^2}{n^3} \\ &= \frac{2}{n} \\ &\leq \frac{2}{N} \\ &< \frac{2}{\frac{2}{\epsilon}} \\ &= \epsilon. \end{aligned}$$

Hence, $|a_n - 0| < \epsilon$.

(c) Let $\epsilon > 0$ be arbitrary. Choose $N \in \mathbb{N}$ such that $N > \frac{1}{\epsilon^3}$. Let $n \geq N$. Then,

$$\begin{aligned} |a_n - 0| &= \left| \frac{\sin(n^2)}{\sqrt[3]{n}} \right| \\ &\leq \frac{1}{\sqrt[3]{n}} \\ &\leq \frac{1}{\sqrt[3]{N}} \\ &< \frac{1}{\sqrt[3]{\frac{1}{\epsilon^3}}} \\ &= \epsilon. \end{aligned}$$

Hence, $|a_n - 0| < \epsilon$.

Exercise 3 (2.2.4). (a) Consider the sequence (a_n), where $a_n = (-1)^n$. (a_n) has an infinite number of ones, but does not converge to one since it diverges.

(b) TODO

(c) TODO

Exercise 4 (2.2.5). (a) Let $a_n = [[5/n]]$. We claim that $\lim a_n = 0$. Let $\epsilon > 0$ be arbitrary. Choose $N \in \mathbb{N}$ such that $N > 5/\epsilon$. Let $n \geq N$. Then,

$$\begin{aligned} |a_n - 0| &= |[[5/n]]| \\ &= [[5/n]] \\ &\leq 5/n \\ &\leq 5/N \\ &< 5/(5/\epsilon) \\ &= \epsilon. \end{aligned}$$

Hence, $|a_n - 0| < \epsilon$.

(b) Let $a_n = [[(12+4n)/3n]]$. We claim that $\lim a_n = 1$. Let $\epsilon > 0$ be arbitrary. Choose $N \in \mathbb{N}$ such that $N > \frac{4}{\epsilon - \frac{1}{3}}$. Let $n \geq N$. Then,

$$\begin{aligned}
|a_n - 1| &= |[[(12+4n)/3n]] - 1| \\
&= |[[(12+4n)/3n - 1]]| \\
&= |[[(12+4n-3n)/3n]]| \\
&= |[[(12+n)/3n]]| \\
&= [[(12+n)/3n]] \\
&\leq (12+n)/3n \\
&= 4/n + 1/3 \\
&\leq 4/N + 1/3 \\
&< 4/(4/(\epsilon - 1/3)) + 1/3 \\
&= \epsilon - 1/3 + 1/3 \\
&= \epsilon.
\end{aligned}$$

Hence, $|a_n - 1| < \epsilon$.

Exercise 5 (2.2.7). (a) The sequence $(-1)^n$ is frequently in the set $\{1\}$.

(b) The definition of eventually is stronger than that of frequently, since eventually implies frequently.

(c) A sequence (a_n) converges to a if, given any ϵ-neighborhood $V_\epsilon(a)$ of a, (a_n) is eventually in the set $V_\epsilon(a)$. Eventually is the term we want.

(d) (x_n) is not necessarily eventually in the interval $(1.9, 2.1)$. For instance, consider the sequence $(1, 2, 1, 2, \ldots)$. However, (x_n) is frequently in $(1.9, 2.1)$.