MATH 355: HOMEWORK 5

ALEXANDER LEE

Exercise 1 (2.5.1).

(a) Impossible. Let (a_n) be a sequence with a subsequence (a_{n_k}), where (a_{n_k}) is bounded. By the Balzano-Weierstrass Theorem, (a_{n_k}) contains a convergent subsequence. Note that this convergent subsequence of (a_{n_k}) is also a subsequence of (a_n). Thus, (a_n) contains a convergent subsequence.

(b) Consider the sequences $a_n = \frac{1}{n}$ and $b_n = 1 + \frac{1}{n}$. Now, define the sequence $(c_n) = (a_1, b_1, a_2, b_2, a_3, b_3, \ldots)$. Notice that (a_n) and (b_n) are both subsequences of $(c)n$. Further notice that (c_n) does not contain 0 or 1 as terms but $(a_n) \to 0$ and $(b_n) \to 1$.

(c) Consider the following sequences.

$$
\begin{aligned}
(a_n^1) &= (1 + 1/1, 1 + 1/2, 1 + 1/3, \ldots) \\
(a_n^2) &= (1/2 + 1/1, 1/2 + 1/2, 1/2 + 1/3, \ldots) \\
(a_n^3) &= (1/3 + 1/1, 1/3 + 1/2, 1/3 + 1/3, \ldots) \\
&\vdots
\end{aligned}
$$

Notice that $(a_n^1) \to 1$, $(a_n^2) \to 1/2$, $(a_n^3) \to 1/3$, and so on. Now consider the sequence (a_n) constructed with the top right to bottom left diagonals across the sequences above. That is, $(a_n) = (1+1/1, 1+1/2, 1/2+1/1, 1+1/3, 1/2+1/2, 1/3+1/1, \ldots)$. By construction, (a_n) contains subsequences converging to every point in the infinite set $\{1, 1/2, 1/3, \ldots\}$.

(d) TODO

Exercise 2 (2.5.2).

(a) True. Since $(x_2, x_3, x_4, \ldots)$ converges and is a proper subsequence of (x_n), (x_n) also converges.

(b) True. Suppose towards a contradiction that (x_n) converges. Thus, we know that subsequences of (x_n) are convergent, which contradicts the assumption that (x_n) contains a divergent subsequence. Therefore, (x_n) diverges.

(c) TODO

(d) TODO

Exercise 3 (2.6.2).

(a) Consider the sequence $a_n = (-1)^n \frac{1}{n}$. (a_n) clearly converges to 0, so it is a Cauchy sequence by the Cauchy Criterion. Notice that (a_n) is not monotone as well.

(b) Impossible. Suppose (a_n) is a Cauchy sequence. Thus, (a_n) is bounded by Lemma 2.6.3. As such, every subsequence of (a_n) must be bounded as well.

(c) TODO

(d) Consider the sequence $(a_n) = (0, 1, 0, 2, 0, 3, \ldots)$. Clearly, (a_n) is unbounded. However, $(0, 0, 0, \ldots)$ is a subsequence of (a_n) that is Cauchy.

Exercise 4 (2.6.4). (a) Let $\epsilon > 0$ be arbitrary. Since (a_n) is a Cauchy sequence, there exists an $N_1 \in \mathbb{N}$ such that whenever $m, n \geq N_1$, it follows that $|a_n - a_m| < \epsilon/2$. Similarly, since (b_n) is a Cauchy sequence, there exists an $N_2 \in \mathbb{N}$ such that whenever $m, n \geq N_2$, it follows that $|b_n - b_m| < \epsilon/2$. Let $N = \max\{N_1, N_2\}$ and suppose $m, n \geq N$. Then,

$$\begin{aligned}
|c_n - c_m| &= ||a_n - b_n| - |a_m - b_m|| \\
&\leq |(a_n - b_n) - (a_m - b_m)| \text{ (by the Triangle Inequality)} \\
&= |(a_n - a_m) + (b_m - b_n)| \\
&\leq |a_n - a_m| + |b_m - b_n| \text{ (by the Triangle Inequality)} \\
&< \epsilon/2 + \epsilon/2 \\
&= \epsilon.
\end{aligned}$$

Therefore, (c_n) is a Cauchy sequence.

(b) Let $a_n = 1$. (a_n) is a Cauchy sequence, but $c_n = (-1)^n$ is divergent and thus not a Cauchy sequence.

(c) Let $a_n = (-1)^n \frac{1}{n+1}$. Then,

$$c_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ -1 & \text{if } n \text{ is odd} \end{cases},$$

which is divergent and thus not a Cauchy sequence.

Exercise 5 (2.7.1). (a) Let $\epsilon > 0$ be arbitrary. Since $(a_n) \to 0$, there exists an $N \in \mathbb{N}$ such that whenever $m \geq N$, it follows that $|a_m| < \epsilon$. Then,

$$\begin{aligned}
|s_n - s_m| &= |a_{m+1} - a_{m+2} - \cdots \pm a_n| \\
&\leq |a_{m+1}| \text{ (since } (a_n) \text{ is decreasing)} \\
&\leq |a_m| \\
&< \epsilon
\end{aligned}$$

whenever $n > m \geq N$.

Exercise 6 (2.7.2). (a) Converges. $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges because it is geometric. Also, $0 \leq \frac{1}{2^n+n} \leq \frac{1}{2^n}$ for all $n \in \mathbb{N}$, so by the Comparison Test, $\sum_{n=1}^{\infty} \frac{1}{2^n+n}$ converges.

(b) Converges. We know that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. Furthermore, since $0 \leq \frac{|\sin(n)|}{n^2} \leq \frac{1}{n^2}$ for all $n \in \mathbb{N}$, we have that $\sum_{n=1}^{\infty} \frac{|\sin(n)|}{n^2}$ converges by the Comparison Test. As such, $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$ converges as well by the Absolute Convergence Test.

(c) TODO

(d) TODO

(e) TODO

Exercise 7 (2.7.4). (a) Consider $x_n = \frac{1}{n}$ and $y_n = (-1)^n$. Since $\sum x_n$ is the harmonic series, it diverges. Since $(y_n) \not\to 0$, $\sum y_k$ diverges. However, $\sum x_n y_n = \sum (-1)^n \frac{1}{n}$ converges by the Alternating Series Test.

(b) Consider $x_n = (-1)^n \frac{1}{n}$ and $y_n = (-1)^n$. $\sum x_n$ converges by the Alternating Series Test. Clearly, (y_n) is bounded. However, $\sum x_n y_n = \sum \frac{1}{n}$ diverges since it is the harmonic series.

(c) Impossible. Since $\sum x_n$ and $\sum(x_n + y_n)$ both converge, we have that $\sum y_n = \sum(x_n+y_n)-x_n$ converges as well by the Algebraic Limit Theorem for Series.

(d) TODO

Exercise 8 (2.7.5)**.** By the Cauchy Condensation Test, the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $\sum_{n=0}^{\infty} 2^n \frac{1}{(2^n)^p}$ converges. Notice that $2^n \frac{1}{(2^n)^p} = (\frac{2}{2^p})^n$. Thus, $\sum_{n=0}^{\infty} 2^n \frac{1}{(2^n)^p}$ converges if and only if $\left|\frac{2}{2^p}\right| < 1$ by the Geometric Series Test. It follows that

$$\begin{aligned}\left|\frac{2}{2^p}\right| < 1 &\iff \frac{2}{2^p} < 1\\ &\iff 2 < 2^p\\ &\iff 1 < 2^{p-1}\\ &\iff \log_2(1) < p-1\\ &\iff 0 < p-1\\ &\iff 1 < p.\end{aligned}$$

Thus, we must have that $p > 1$.

Exercise 9 (2.7.9)**.** (a) TODO

(b) TODO

(c) TODO

Exercise 10 (3.2.3)**.** (a) $\mathbb{Q}$ is not open. Any ϵ-neighborhood of 0, is not contained in $\mathbb{Q}$. $\mathbb{Q}$ is closed.