MATH 355: HOMEWORK 9

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Exercise 1 (4.4.1). (a) Given $c \in \mathbb{R}$, we have

$$|f(x) - f(c)| = \left|x^3 - c^3\right| = |x - c|\left|x^2 + xc + c^2\right|.$$

Choosing $\delta \leq 1$, we thus have $x \in (c-1, c+1)$. Hence,

$$\left|x^2 + xc + c^2\right| < (c+1)^2 + (c+1)^2 + c^2 < 3(c+1)^2.$$

Now, let $\delta = \min\{1, \epsilon/(3(c+1))^2\}$. Then, $|x - c| < \delta$ implies

$$|f(x) - f(c)| < \left(\frac{\epsilon}{3(c+1)^2}\right) 3(c+1)^2 = \epsilon.$$

(b) Choose $x_n = n$ and $y_n = n + 1/n$. Observe that $|x_n - y_n| = 1/n \to 0$ and

$$|f(x_n) - f(y_n)| = \left|n^3 - \left(n + \frac{1}{n}\right)^3\right| = 3n + \frac{3}{n} + \frac{1}{n^3} \geq 3.$$

(c) Suppose A is bounded by M. Given $x, c \in A$, we have that $\left|x^2 + xc + c^2\right| \leq 3M^2$. For any $\epsilon > 0$, we can choose $\delta = \epsilon/(3M)^2$. If $|x - c| < \delta$, it follows that

$$|f(x) - f(c)| < \left(\frac{\epsilon}{3M^2}\right) 3M^2 = \epsilon.$$

Exercise 2 (4.4.3). Observe that

$$|f(x) - f(y)| = \left|\frac{1}{x^2} - \frac{1}{y^2}\right| = \left|\frac{y^2 - x^2}{x^2 y^2}\right| = |y - x| \left(\frac{y + x}{x^2 y^2}\right).$$

If $x, y \in [1, \infty)$, then we have

$$\frac{y + x}{x^2 y^2} = \frac{1}{x^2 y} + \frac{1}{x y^2} \leq 1 + 1 = 2.$$

Given $\epsilon > 0$, let $\delta = \epsilon/2$ and it follows that $|f(x) - f(y)| < (\epsilon/2)2 = \epsilon$ whenever $|x - y| < \delta$. Therefore, f is uniformly continuous on $[1, \infty)$.

If $x, y \in (0, 1]$, then set $x_n = 1/\sqrt{n}$ and $y_n = 1/\sqrt{n+1}$. Then, $|x_n - y_n| \to 0$ and

$$|f(x_n) - f(y_n)| = |n - (n+1)| = 1.$$

By the Sequential Criterion for Absence of Uniform Continuity, f is not continuous on $(0, 1]$.

Exercise 3 (4.4.7). We first show that $f(x) = \sqrt{x}$ is uniformly continuous on $[1, \infty)$. Let $x, y \in [1, \infty)$. It follows that

$$|f(x) - f(y)| = \left|\sqrt{x} - \sqrt{y}\right| = \left|\frac{x - y}{\sqrt{x} + \sqrt{y}}\right| \leq |x - y| \frac{1}{2}.$$

Given $\epsilon > 0$, let $\delta = 2\epsilon$. It follows that $|f(x) - f(y)| < (2\epsilon)\frac{1}{2} = \epsilon$ whenever $|x - y| < \delta$. Thus, $f(x) = \sqrt{x}$ is uniformly continuous on $[1, \infty)$.

We also know that $f(x) = \sqrt{x}$ is continuous on $[0, 1]$ and $[0, 1]$ is a compact set, so f is also uniformly continuous on $[0, 1]$. By Exercise 4.4.5, we thus conclude that f is uniformly continuous on $[0, \infty)$.

Exercise 4 (4.5.2). (a) Consider

$$f(x) = \begin{cases} -1 & x \in (-2, -1) \\ x & x \in [-1, 1] \\ 1 & x \in (1, 2) \end{cases}.$$

Observe that f is continuous on the open interval $(-2, 2)$ and has range equal to the closed interval $[-1, 1]$.

(b) Impossible. If a continuous function is defined a closed interval, then the continuous function is defined on a compact set. By the Preservation of Compact Sets, the range must also be compact and thus cannot be an open interval.

(c) Consider

$$f(x) = \begin{cases} 1/x & x \in (0, 1) \\ 1 & x \in [1, 2) \end{cases}.$$

Observe that f is continuous on the open interval $(0, 2)$ with range equal to the unbounded closed set $[1, \infty) \neq \mathbb{R}$.

(d) Impossible. Suppose towards a contradiction that f is a continuous function defined on all of $\mathbb{R}$ with range equal to $\mathbb{Q}$. Then, there exists $a, b \in \mathbb{R}$ such that $f(a), f(b) \in \mathbb{Q}$. Without loss of generality, suppose $a < b$ and $f(a) < f(b)$. By the Density of Irrationals in $\mathbb{R}$, there exists an irrational number L such that $f(a) < L < f(b)$. By the Intermediate Value Theorem, there exists a point $c \in (a, b)$ where $f(c) = L$, which contradicts our assumption that f has range equal to $\mathbb{Q}$.

Exercise 5 (4.5.4). If g is one-to-one, there are no points where g fails to be one-to-one. Thus, F is empty in this case.

If g is not one-to-one, then there exist points $x, y \in A$, $x \neq y$, such that $f(x) = f(y)$. Without loss of generality, suppose $x < y$. If f is the constant function, we are done. Otherwise, choose $z \in (x, y)$ such that $f(z) \neq f(x)$. Without loss of generality, suppose $f(x) < f(z)$. By the Intermediate Value Theorem, for all $L \in (f(x), f(z))$, there exists a point $c_1 \in (x, z)$ where $f(c_1) = L$. Similarly, for all $L \in (f(x), f(z))$, there exists a point $c_2 \in (z, y)$ where $f(c_2) = L$. Therefore, for all $L \in (f(x), f(z))$, there exists a $c_1 \in (x, z)$ and $c_2 \in (z, y)$ such that $f(c_1) = L = f(c_2)$. Since $(f(x), f(z))$ is uncountable, F is uncountable as well.

Exercise 6 (4.5.7). Consider the function $g(x) = f(x) - x$ defined on the closed interval $[0, 1]$. To show that $f(x) = x$ for at least one value of $x \in [0, 1]$, we want to show that $g(x) = 0$ for at least one value of $x \in [0, 1]$. First, consider $g(0) = f(0) - 0 = f(0)$. Since f has range contained in $[0, 1]$, it must be that $g(0) = f(0) \in [0, 1]$. Similarly, $g(1) = f(1) - 1 \in [-1, 0]$. If $g(0), g(1) \neq 1$, then by the Intermediate Value Theorem, there exists a point $x \in [0, 1]$ where $g(x) = 0$, i.e., $f(x) = x$. If $g(0) = 0$, then $f(0) = 0$ and 0 is a fixed point. If $g(1) = 0$, then $f(1) = 1$ and 1 is a fixed point.

Exercise 7 (5.2.2). (a) Consider $f(x) = g(x) = |x|$. Clearly, f and g are not differentiable at zero. However, $(fg)(x) = |x| \cdot |x| = x^2$, which is differentiable at zero.

(b) Consider $f(x) = |x|$ and $g(x) = x^2$. Clearly, f is not differentiable at zero and g is differentiable at zero. However, $(fg)(x) = |x|x^2 = \left|x^3\right|$, which is differentiable at zero.

(c) Impossible. Given that g and $f+g$ are differentiable at zero, $f = (f+g)-g$ is also differentiable at zero by the Algebraic Differentiability Theorem.

(d) Consider

$$f(x) = \begin{cases} x^2 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}.$$

We first show that f is differentiable at zero. We claim that

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = 0.$$

To prove the claim, first suppose $\epsilon > 0$ and let $\delta = \epsilon$. Then, if $|x - 0| < \delta$, it follows that

$$\left| \frac{f(x) - f(0)}{x - 0} - 0 \right| = \left| \frac{f(x)}{x} \right| < |x| < \delta = \epsilon.$$

Therefore, f is differentiable at zero.

Now we show that f is not differentiable at $c \in \mathbb{R}$ where $c \neq 0$. That is, we aim to show that

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

does not exist for $c \in \mathbb{R}$ and $c \neq 0$. Consider sequences (x_n) and (y_n) where $x_n = c + 1/n$ and $y_n = c + \sqrt{2}/n$. Clearly, $(x_n), (y_n) \to c$. However, we have that

$$\lim \frac{f(y_n) - f(c)}{y_n - c} = \lim \frac{0 - f(c)}{c + \sqrt{2}/n - c} = \lim \frac{-nf(c)}{\sqrt{2}},$$

which does not exist. Thus, by the Divergence Criterion for Functional Limits, we have that f is not differentiable at $c \in \mathbb{R}$ where $c \neq 0$.

Exercise 8 (5.2.6). (a) The new definition replaces x from Definition 5.2.1 with $c + h$.

(b)

$$\begin{aligned}
\lim_{h\to 0}\frac{g(c+h)-g(c-h)}{2h} &= \frac{1}{2}\left(\lim_{h\to 0}\frac{g(c+h)-g(c)+g(c)-g(c-h)}{h}\right)\\
&= \frac{1}{2}\left(\lim_{h\to 0}\frac{g(c+h)-g(c)}{h}+\lim_{h\to 0}\frac{g(c)-g(c-h)}{h}\right)\\
&= \frac{1}{2}\left(\lim_{h\to 0}\frac{g(c+h)-g(c)}{h}+\lim_{x\to c}\frac{g(c)-g(c-(c-x))}{c-x}\right)\\
&= \frac{1}{2}\left(\lim_{h\to 0}\frac{g(c+h)-g(c)}{h}+\lim_{x\to c}\frac{g(c)-g(x)}{c-x}\right)\\
&= \frac{1}{2}\left(\lim_{h\to 0}\frac{g(c+h)-g(c)}{h}+\lim_{x\to c}\frac{g(x)-g(c)}{x-c}\right)\\
&= \frac{1}{2}(g'(c)+g'(c))\\
&= g'(c).
\end{aligned}$$