MATH 355: HOMEWORK 11

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Exercise 1 (6.3.1). (a) We show that (g_n) converges uniformly on $[0,1]$ to $g = \lim g_n = 0$. Given $\epsilon > 0$, choose $N \in \mathbb{N}$ such that $N > 1/\epsilon$. Then, whenever $n \geq \mathbb{N}$ and $x \in [0,1]$, it follows that

$$|g_n(x) - g(x)| = \left|\frac{x^n}{n} - 0\right| = \left|\frac{x^n}{n}\right| = \frac{x^n}{n} \leq \frac{1}{n} \leq \frac{1}{N} < \epsilon.$$

g is differentiable since it is the constant function $g(x) = 0$ for all $x \in [0,1]$. Specifically, we have that $g'(x) = 0$ for all $x \in [0,1]$.

(b) By the Algebraic Differentiability Theorem, we have that

$$g_n'(x) = \frac{n(nx^{n-1}) - x^n(0)}{n^2} = x^{n-1}.$$

It follows that (g_n') converges on [0, 1] to

$$h(x) = \lim g_n'(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x = 1 \end{cases}.$$

By the contrapositive of the Continuous Limit Theorem, since h is not continuous and g_n' is continuous on $[0,1]$, we have that (g_n') does not converge uniformly on $[0,1]$ to h. Observe that h and g' are not the same.

Exercise 2 (6.3.2). (a) The pointwise limit of (h_n) is $h(x) = \lim \sqrt{x^2 + \frac{1}{n}} = \sqrt{x^2} = |x|$. To show that the convergence is uniform on $\mathbb{R}$, let $\epsilon > 0$ be arbitrary and choose $N \in \mathbb{N}$ such that $N > 1/\epsilon^2$. Then, whenever $n \geq N$ and $x \in \mathbb{R}$, it follows that

$$\begin{aligned} |h_n(x) - h(x)| &= \left|\sqrt{x^2 + \frac{1}{n}} - |x|\right| \leq \left|\sqrt{x^2} + \sqrt{\frac{1}{n}} - |x|\right| \\ &= \left||x| + \sqrt{\frac{1}{n}} - |x|\right| = \sqrt{\frac{1}{n}} \leq \sqrt{\frac{1}{N}} < \epsilon. \end{aligned}$$

(b) By the Chain Rule, we have that

$$h_n'(x) = \frac{1}{2\sqrt{x^2 + \frac{1}{n}}} \cdot 2x = \frac{x}{\sqrt{x^2 + \frac{1}{n}}}.$$

For $x \neq 0$, it follows that

$$g(x) = \lim h_n'(x) = \lim \frac{x}{\sqrt{x^2 + \frac{1}{n}}} = \frac{x}{\sqrt{x^2}} = \frac{x}{|x|} = \begin{cases} -1 & x < 0 \\ 1 & x > 0 \end{cases}.$$

For $x = 0$, we have

$$g(0) = \lim h_n'(x) = \lim \frac{0}{\sqrt{0^2 + \frac{1}{n}}} = 0.$$

Thus, $g(x)$ exists for all x. Observe that $g(x)$ is not continuous at $x = 0$ but each $h_n'(x)$ is continuous at $x = 0$. Therefore, by the contrapositive of the Continuous Limit Theorem, it follows that $h_n'(x)$ does not converge uniformly on any neighborhood of zero.

Exercise 3 (6.4.2)**.** (a) True. Given that $\sum_{n=1}^{\infty} g_n$ converges uniformly, it follows from a special case of the Cauchy Criterion for Uniform Convergence of Series that that given $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that whenever $m \geq N$ and $x \in A$, where A is the domain of g_n, we have that $|g_{m+1}(x)| < \epsilon$. Therefore, (g_n) converges uniformly to zero.

(b) True. Given that $0 \leq f_n(x) \leq g_n(x)$ and $\sum_{n=1}^{\infty} g_n$ converges uniformly, it follows from the Cauchy Criterion for Series that given $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that whenever $n > m \geq N$, we have that

$$|f_{m+1}(x) + f_{m+2}(x) + \cdots + f_n(x)| \leq |g_{m+1}(x) + g_{m+2}(x) + \cdots + g_n(x)| < \epsilon.$$

Therefore, $\sum_{n=1}^{\infty} f_n$ converges uniformly as well.

(c) False. Consider $f_n(x) = \frac{1}{n^2}$ defined on $\mathbb{R}$. Clearly, $\sum_{n=1}^{\infty} f_n$ converges uniformly on $\mathbb{R}$. Choose $M_n = \frac{1}{n}$. Observe that $\left|\frac{1}{n^2}\right| \leq \frac{1}{n}$ for all $x \in \mathbb{R}$, but $\sum_{n=1}^{\infty} M_n$ diverges.

Exercise 4 (6.4.4)**.** Let $g_n(x) = \frac{x^{2n}}{(1+x^{2n})}$. Observe that if $|x| \geq 1$, then $\lim g_n(x) \neq 0$, and so $g(x)$ is not converge for $|x| \geq 1$. Now for any $[-c, c]$ with $c \in (0, 1)$, choose $M_n = \frac{c^{2n}}{1+c^{2n}}$. It follows from a variant of the geometric series that

$$\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{c^{2n}}{1 + c^{2n}} < \sum_{n=1}^{\infty} c^{2n} = \frac{c^2}{1 - c^2}.$$

Note that $|g_n(x)| \leq M_n$ for $x \in [-c, c]$. By the Weierstrass M-Test, it follows that $g(x) = \sum_{n=1}^{\infty} g_n(x)$ converges uniformly on $[-c, c]$. Thus, we have that $g(x)$ converges on $(-1, 1)$. For continuity, since $g(x)$ converges uniformly on $[-c, c]$ and each $g_n(x)$ is continuous on $[-c, c]$, it follows from the Term-by-term Continuity Theorem that $g(x)$ is continuous on $[-c, c]$. Hence, $g(x)$ is continuous on $(-1, 1)$.

Exercise 5 (6.4.5a)**.** Observe that each $\frac{x^n}{n^2}$ is continuous on $[-1, 1]$. Also note that for each $n \in \mathbb{N}$, $\left|\frac{x^n}{n^2}\right| \leq \frac{1}{n^2}$ for all $x \in [-1, 1]$. Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, it follows from the Weierstrass M-Test that $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$ converges uniformly on $[-1, 1]$. Therefore, $h(x)$ is continuous on $[-1, 1]$ by the Term-by-term Continuity Theorem.

Exercise 6 (6.5.1)**.** (a) g can be rewritten as the power series

$$g(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}.$$

If $x = 1$, then $g(x)$ converges by the Alternating Series Test. By Theorem 6.5.1, it follows that $g(x)$ converges absolutely for any $x \in (-1, 1)$. Therefore, g is converges and hence defined on $(-1, 1)$.

Since g converges on $(-1, 1)$, it follows from Theorem 6.5.7 that g is continuous on $(-1, 1)$.

Since $g(x)$ converges at the point $x = 1$, it follows from Abel's Theorem that g converges uniformly on the interval $[0, 1]$. We established previously that g also converges on $(-1, 1)$. Therefore, we can conclude that g converges on $(-1, 1]$ and is thus defined on this set.

Since $g(x)$ converges on $(-1, 1]$, it follows from Theorem 6.5.7 that g is continuous on this set.

$g(x)$ is undefined for $x = -1$ since this value of x yields the harmonic series, which does not converge. Thus, g is not defined on $[-1, 1]$ and so cannot even be continuous on this set.

The power series for $g(x)$ cannot possibly converge for any other points $|x| > 1$ because $g(x)$ would be unbounded.

(b) $g'(x)$ is defined on $(-1, 1)$, where

$$g'(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{nx^{n-1}}{n} = \sum_{n=1}^{\infty} (-1)^{n+1} x^{n-1},$$

by Theorem 6.5.7.

Exercise 7 (6.5.2). (a) Consider $a_n = \frac{1}{n!}$. We have that

$$\lim \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \lim \left| \frac{x}{n+1} \right| = 0 < 1.$$

By the Ratio Test, it follows that $\sum a_n x^n$ converges.

(b) Consider $a_n = n!$. Since $\lim a_n x^n \neq 0$ for all $x \in \mathbb{R}$, it follows that $\sum a_n x^n$ diverges.

(c) Consider

$$a_n = \begin{cases} 0 & \text{if } n \text{ odd} \\ 1/(\frac{n}{2})^2 & \text{if } n \text{ even} \end{cases}.$$

Then, $\sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} \frac{x^{2n}}{n^2}$, which converges absolutely absolutely for all $x \in [-1, 1]$ and diverges off this set.

(d) Impossible. If the power series $\sum a_n x^n$ converges conditionally at $x = -1$, then we know that $\sum |a_n (-1)^n| = \sum |a_n 1^n|$ diverges. Therefore, the power series cannot converge absolutely at $x = 1$.

(e) Consider

$$a_n = \begin{cases} 0 & \text{if } n \text{ odd} \\ (-1)^{\frac{n}{2}} / \frac{n}{2} & \text{if } n \text{ even} \end{cases}.$$

Then, $\sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{n}$, which converges conditionally at both $x = -1$ and $x = 1$.

Exercise 8 (6.5.4). (a) We first show that $F(x)$ is defined on $(-R, R)$. Observe that

$$0 \leq \left| \frac{a_n}{n+1} x^{n+1} \right| = \left| \frac{a_n}{n+1} \right| |x||x^n| \leq |a_n||x||x^n| = |x||a_n x^n|.$$

We now show that $\sum_{n=0}^{\infty} |x||a_n x^n|$ converges. Recall that $f(x)$ converges on $(-R, R)$. Thus, by Theorem 6.5.1, $f(x)$ converges absolutely on $(-R, R)$. Since $\sum_{n=0}^{\infty} |a_n x^n|$ converges, it follows that $\sum_{n=0}^{\infty} |x||a_n x^n|$ converges. By the Comparison Test, it follows that $\sum_{n=0}^{\infty} \left| \frac{a_n}{n+1} x^{n+1} \right|$ converges. Therefore, $F(x)$ converges on $(-R, R)$ as well by the Absolute Convergence Test.

Since $F(x)$ converges on $(-R, R)$, we have that F is differentiable and $F'(x) = f(x)$ by Theorem 6.5.7.

(b) $g(x) = F(x) + c = (\sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}) + c$, for some $c \in \mathbb{R}$.