MATH 355: HOMEWORK 4

ALEXANDER LEE

Exercise 1 (2.3.3)**.** Given $x_n \leq y_n \leq z_n$ for all $n \in \mathbb{N}$ and $\lim x_n = \lim z_n = l$, we know from the Order Limit Theorem (ii) that $l \leq \lim y_n \leq l$ for all $n \in \mathbb{N}$. Thus, we have that $\lim y_n = l$ as well.

Exercise 2 (2.3.4)**.** (a)

$$\begin{aligned}
\lim\left(\frac{1+2a_n}{1+3a_n-4{a_n}^2}\right) &= \frac{\lim 1 + 2\lim a_n}{\lim 1 + 3\lim a_n - 4\lim a_n \lim a_n} \\
&= \frac{1+2\cdot 0}{1+3\cdot 0-4\cdot 0\cdot 0} \\
&= 1
\end{aligned}$$

(b)

$$\begin{aligned}
\lim\left(\frac{(a_n+2)^2-4}{a_n}\right) &= \frac{\lim(a_n+2)\lim(a_n+2)-\lim 4}{\lim a_n} \\
&= \frac{(\lim a_n + \lim 2)(\lim a_n + \lim 2) - 4}{\lim a_n} \\
&= \frac{(\lim a_n + 2)(\lim a_n + 2) - 4}{\lim a_n} \\
&= \frac{(\lim a_n)^2 + 4\lim a_n + 4 - 4}{\lim a_n} \\
&= \frac{\lim a_n(\lim a_n + 4)}{\lim a_n} \\
&= \lim a_n + 4 \\
&= 0 + 4 \\
&= 4
\end{aligned}$$

(c)

$$\begin{aligned}
\lim\left(\frac{\frac{2}{a_n}+3}{\frac{1}{a_n}+5}\right) &= \frac{\frac{\lim 2}{\lim a_n}+\lim 3}{\frac{\lim 1}{\lim a_n}+\lim 5} \\
&= \frac{\frac{2}{\lim a_n}+3}{\frac{1}{\lim a_n}+5} \\
&= \frac{\frac{2+3\lim a_n}{\lim a_n}}{\frac{1+5\lim a_n}{\lim a_n}} \\
&= \frac{2+3\lim a_n}{1+5\lim a_n} \\
&= \frac{2+3\cdot 0}{1+5\cdot 0} \\
&= \frac{2}{1} \\
&= 1
\end{aligned}$$

Exercise 3 (2.3.7). (a) Consider the divergent sequences $(x_n) = (-1, 1, -1, 1, \ldots)$ and $(y_n) = (1, -1, 1, -1, \ldots)$. Then, we have $(x_n + y_n) = (0, 0, 0, 0, \ldots)$, which converges to 0.

(b) TODO

(c) Impossible, by the Algebraic Limit Theorem (iv).

(d) TODO

(e) TODO

Exercise 4 (2.3.10). (a) Given $\lim(a_n - b_n) = 0$, we have that $\lim a_n - \lim b_n = 0$ by the Algebraic Limit Theorem (ii). Thus, we also have that $\lim a_n = \lim b_n$.

(b) TODO

(c) Given $(a_n) \to a$ and $(b_n - a_n) \to 0$, we have that $\lim a_n = a$ and $\lim(b_n - a_n) = 0$. The later limit implies that $\lim b_n = \lim a_n$ by (a). Thus, we have that $\lim b_n = \lim a_n = a$ and so $(b_n) \to a$.

(d) TODO

Exercise 5 (2.3.12). (a) True

(b) True.

(c) TODO

Exercise 6 (2.4.1). (a) We first show by induction that (x_n) is decreasing (i.e., $x_n \geq x_{n+1}$). Since $x_1 = 3$ and $x_2 = \frac{1}{4-3} = 1$, we have that $x_n \geq x_{n+1}$. Next, suppose that $3 = x_1 \geq x_2 \geq \cdots \geq x_n \geq x_{n+1}$. We want to show that $x_{n+1} \geq x_{n+2}$. Consider

$$\begin{aligned}
x_{n+1} - x_{n+2} &= \frac{1}{4-x_n} - \frac{1}{4-x_{n+1}} \\
&= \frac{4-x_{n+1}-(4-x_n)}{(4-x_n)(4-x_{n+1})} \\
&= \frac{x_n - x_{n+1}}{(4-x_n)(4-x_{n+1})}.
\end{aligned}$$

Since $x_n \geq x_{n+1}$ by our inductive hypothesis, we know that $x_n - x_{n+1} \geq 0$. Furthermore, by our inductive hypothesis, we also know that $3 \geq x_n \geq x_{n+1} \implies -3 \leq -x_n \leq -x_{n+1} \implies 1 = 4 - 3 \leq 4 - x_n \leq 4 - x_{n+1}$. Hence, we have that $x_{n+1} - x_{n+2} \implies x_{n+1} \geq x_{n+2}$. Therefore, (x_n) is decreasing. Next, we show that (x_n) is bounded. Since $x_n < 4$ for all $n \in \mathbb{N}$, we have that $x_{n+1} = \frac{1}{4-x_n} > 0$. Thus, (x_n) is bounded. As such, (x_n) converges by the Monotone Convergence Theorem.

(b) Suppose $\lim x_n = l$. By definition, given any $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|x_n - l| < \epsilon$. Since $x_{n+1} \leq x_n$ for all $n \in \mathbb{N}$, we have $|x_{n+1} - l| \leq |x_n - l| < \epsilon$. Thus, $\lim x_{n+1} = l = \lim_{x_n}$.

(c) Suppose $\lim x_n = \lim x_{n+1} = l$. Then,

$$\begin{aligned}
\lim x_{n+1} = \lim \left(\frac{1}{4 - x_n} \right) &\implies l = \frac{1}{4 - l} \\
&\implies 4l - l^2 = 1 \\
&\implies 0 = l^2 - 4l + 1 \\
&\implies 0 = (l-2)^2 - 3 \\
&\implies 3 = (l-2)^2 \\
&\implies \pm\sqrt{3} = l - 2 \\
&\implies l = 2 \pm \sqrt{3}.
\end{aligned}$$

By the Order Limit Theorem (iii), we know that $\lim x_n = l \leq x_n = 3$. Thus, $\lim x_n = 2 - \sqrt{3}$.

Exercise 7 (2.4.8).

Exercise 8 (2.4.9).