MATH 355: HOMEWORK 9

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Exercise 1 (4.4.1). (a) Given $c \in \mathbb{R}$, we have

$$|f(x) - f(c)| = \left|x^3 - c^3\right| = |x - c|\left|x^2 + xc + c^2\right|.$$

Choosing $\delta \leq 1$, we thus have $x \in (c-1, c+1)$. Hence,

$$\left|x^2 + xc + c^2\right| < (c+1)^2 + (c+1)^2 + c^2 < 3(c+1)^2.$$

Now, let $\delta = \min\{1, \epsilon/(3(c+1))^2\}$. Then, $|x - c| < \delta$ implies

$$|f(x) - f(c)| < \left(\frac{\epsilon}{3(c+1)^2}\right) 3(c+1)^2 = \epsilon.$$

(b) Choose $x_n = n$ and $y_n = n + 1/n$. Observe that $|x_n - y_n| = 1/n \to 0$ and

$$|f(x_n) - f(y_n)| = \left|n^3 - \left(n + \frac{1}{n}\right)^3\right| = 3n + \frac{3}{n} + \frac{1}{n^3} \geq 3.$$

(c) Suppose A is bounded by M. Given $x, c \in A$, we have that $\left|x^2 + xc + c^2\right| \leq 3M^2$. For any $\epsilon > 0$, we can choose $\delta = \epsilon/(3M)^2$. If $|x - c| < \delta$, it follows that

$$|f(x) - f(c)| < \left(\frac{\epsilon}{3M^2}\right) 3M^2 = \epsilon.$$

Exercise 2 (4.4.3). Observe that

$$|f(x) - f(y)| = \left|\frac{1}{x^2} - \frac{1}{y^2}\right| = \left|\frac{y^2 - x^2}{x^2 y^2}\right| = |y - x| \left(\frac{y + x}{x^2 y^2}\right).$$

If $x, y \in [1, \infty)$, then we have

$$\frac{y + x}{x^2 y^2} = \frac{1}{x^2 y} + \frac{1}{x y^2} \leq 1 + 1 = 2.$$

Given $\epsilon > 0$, let $\delta = \epsilon/2$ and it follows that $|f(x) - f(y)| < (\epsilon/2)2 = \epsilon$ whenever $|x - y| < \delta$. Therefore, f is uniformly continuous on $[1, \infty)$.

If $x, y \in (0, 1]$, then set $x_n = 1/\sqrt{n}$ and $y_n = 1/\sqrt{n+1}$. Then, $|x_n - y_n| \to 0$ and

$$|f(x_n) - f(y_n)| = |n - (n+1)| = 1.$$

By the Sequential Criterion for Absence of Uniform Continuity, f is not continuous on $(0, 1]$.

Exercise 3 (4.4.7). We first show that $f(x) = \sqrt{x}$ is uniformly continuous on $[1, \infty)$. Let $x, y \in [1, \infty)$. It follows that

$$|f(x) - f(y)| = \left|\sqrt{x} - \sqrt{y}\right| = \left|\frac{x - y}{\sqrt{x} + \sqrt{y}}\right| \leq |x - y| \frac{1}{2}.$$

Given $\epsilon > 0$, let $\delta = 2\epsilon$. It follows that $|f(x) - f(y)| < (2\epsilon)\frac{1}{2} = \epsilon$ whenever $|x - y| < \delta$. Thus, $f(x) = \sqrt{x}$ is uniformly continuous on $[1, \infty)$.

We also know that $f(x) = \sqrt{x}$ is continuous on $[0, 1]$ and $[0, 1]$ is a compact set, so f is also uniformly continuous on $[0, 1]$. By Exercise 4.4.5, we thus conclude that f is uniformly continuous on $[0, \infty)$.

Exercise 4 (4.5.2).

Exercise 5 (4.5.4).

Exercise 6 (4.5.7).

Exercise 7 (5.2.2).

Exercise 8 (5.2.6).