

MATH 355: HOMEWORK 6

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- Exercise 1** (3.2.2). (a) Limit points of A : $\{-1, 1\}$. Limit points of B : $[0, 1]$.
 (b) A is neither open nor closed. B is neither open nor closed.
 (c) A contains isolated points. B does not contain isolated points.
 (d) $\overline{A} = A \cup \{-1\}$. $\overline{B} = [0, 1]$.

- Exercise 2** (3.2.4). (a) If $s \in A$, then $s \in \overline{A}$ and we are done. Now suppose $s \notin A$. By Lemma 1.3.8, for every $\epsilon > 0$, there exists an $a \in A$ ($a \neq s$) such that $s - \epsilon < a$. Since $s = \sup(A)$, we also know that $a < s$. Thus, every ϵ -neighborhood $V_\epsilon(s)$ intersects A at some point other than s . That is, s is a limit point of A , so $s \in \overline{A}$ in this case as well.
 (b) An open set O cannot contain its supremum $s = \sup(O)$ since every ϵ -neighborhood $V_\epsilon(s)$ of s is not a subset of O . Specifically, this is because for any $\epsilon > 0$ and $a \in O$, we have that $a < s + \epsilon$ since $s = \sup(O)$.

- Exercise 3** (3.2.6). (a) False. Consider the open set $\mathbb{R} \setminus \{\sqrt{2}\}$.
 (b) False. Consider the closed sets of the form $C_n = [n, \infty)$ for $n \in \mathbb{N}$. Observe that $C_n \subseteq C_{n+1}$ and $\bigcup_{n=1}^{\infty} C_n = \emptyset$.
 (c) True. Given a nonempty open set O , we know that for $a \in O$, there exists an ϵ -neighborhood $V_\epsilon(a) \subseteq O$. By the Density of \mathbb{Q} in \mathbb{R} , there exists a rational number $r \in V_\epsilon(a)$. Thus, we have that $r \in O$.
 (d) False. Consider the bounded infinite closed set $F = \{\sqrt{2} + 1/n : n \in \mathbb{N}\} \cup \{\sqrt{2}\}$. Observe that F does not contain any rational number.
 (e) True. The Cantor set is defined as $C = \bigcap_{n=0}^{\infty} C_n$. Since each C_n is the union of a finite collection of closed sets, each C_n is closed. The intersection of an arbitrary collection of closed sets is closed, so the Cantor set C is closed.

- Exercise 4** (3.2.8). (a) Definitely closed.
 (b) Definitely open.
 (c) Definitely open.
 (d) Both.
 (e) Neither.

- Exercise 5** (3.2.9). (a) We first show that $(\bigcup_{\lambda \in \Lambda} E_\lambda)^c = \bigcap_{\lambda \in \Lambda} E_\lambda^c$.

$$\begin{aligned} x \in (\bigcup_{\lambda \in \Lambda} E_\lambda)^c &\iff x \notin \bigcup_{\lambda \in \Lambda} E_\lambda \\ &\iff \forall \lambda \in \Lambda, x \notin E_\lambda \\ &\iff \forall \lambda \in \Lambda, x \in E_\lambda^c \\ &\iff x \in \bigcap_{\lambda \in \Lambda} E_\lambda^c. \end{aligned}$$

Next, we show that $(\cap_{\lambda \in \Lambda} E_\lambda)^c = \cup_{\lambda \in \Lambda} E_\lambda^c$.

$$\begin{aligned} x \in (\cap_{\lambda \in \Lambda} E_\lambda)^c &\iff x \notin \cap_{\lambda \in \Lambda} E_\lambda \\ &\iff \exists \lambda \in \Lambda \text{ s.t. } x \notin E_\lambda \\ &\iff \exists \lambda \in \Lambda \text{ s.t. } x \in E_\lambda^c \\ &\iff x \in \cup_{\lambda \in \Lambda} E_\lambda^c. \end{aligned}$$

- (b) Suppose $\{E_\lambda : \lambda \in \Lambda\}$ is a finite collection of open sets. Each E_λ^c is thus a closed set by Theorem 3.2.13. As such, $\cup_{\lambda \in \Lambda} E_\lambda^c$ is the union of a finite collection of closed sets. By Theorem 3.2.3, $\cap_{\lambda \in \Lambda} E_\lambda$ is open. It follows that $(\cap_{\lambda \in \Lambda} E_\lambda)^c$ is closed by Theorem 3.2.13. Since $(\cap_{\lambda \in \Lambda} E_\lambda)^c = \cup_{\lambda \in \Lambda} E_\lambda^c$, the union of a finite collection of closed sets is therefore closed.

Suppose $\{E_\lambda : \lambda \in \Lambda\}$ is an arbitrary collection of open sets. As such, $\cap_{\lambda \in \Lambda} E_\lambda^c$ is the intersection of an arbitrary collection of closed sets. By Theorem 3.2.3, $\cup_{\lambda \in \Lambda} E_\lambda$ is open. It follows that $(\cup_{\lambda \in \Lambda} E_\lambda)^c$ is closed by Theorem 3.2.13. Since $(\cup_{\lambda \in \Lambda} E_\lambda)^c = \cap_{\lambda \in \Lambda} E_\lambda^c$, the intersection of an arbitrary collection of closed sets is closed.

- Exercise 6** (3.2.10). (i) Such a set cannot exist. Let $A \subseteq [0, 1]$ be a countable set. Since A is countable, there exists a bijection $f : \mathbb{N} \rightarrow A$. We can use the function f to define a sequence (a_n) where $a_n = f(n)$ for all $n \in \mathbb{N}$. Because $(a_n) \subseteq A \subseteq [0, 1]$, (a_n) is bounded. By the Bolzano-Weierstrass Theorem, (a_n) has a convergent subsequence $(a_{n_k}) \rightarrow a$. Since f is a bijection, all the terms of (a_n) are distinct, so at most one term in (a_{n_k}) can be equal to a . Let (b_n) be a subsequence of (a_{n_k}) without the term a if it exists. It follows that $(b_n) \rightarrow a$, so a is a limit point.
- (ii) Consider the set $\mathbb{Q} \cap [0, 1]$.
- (iii) TODO

Exercise 7 (3.2.13). TODO

Exercise 8 (3.2.14). (a) We first show that E is closed if and only if $\overline{E} = E$.

Let L be the set of all limit points of E . Then,

$$\begin{aligned} E \text{ is closed} &\iff L \subseteq E \\ &\iff E \cup L = E \\ &\iff \overline{E} = E. \end{aligned}$$

Next, we show that E is open if and only if $E^\circ = E$.

$$\begin{aligned} E \text{ is open} &\iff \forall x \in E, \exists V_\epsilon(x) \subseteq E \\ &\iff E^\circ = E. \end{aligned}$$

- (b) We begin with showing that $\overline{E^c} = (E^c)^\circ$. Let L be the set of all limit points of E . Then,

$$\begin{aligned} x \in \overline{E^c} &\iff x \in (E \cup L)^c \\ &\iff x \in E^c \cap L^c \text{ (by DeMorgan's Laws)} \\ &\iff x \in E^c \wedge x \text{ is an isolated point} \\ &\iff x \in E^c \text{ s.t. } \exists V_\epsilon(x) \subseteq E^c \\ &\iff x \in (E^c)^\circ. \end{aligned}$$

Next, we show that $(E^\circ)^c = \overline{E^c}$.

$$\begin{aligned}(E^\circ)^c &= (((E^c)^c)^\circ)^c \\ &= (\overline{E^c})^c \text{ (since } \overline{E^c} = (E^c)^\circ) \\ &= \overline{E^c}.\end{aligned}$$