MATH 355: HOMEWORK 1

ALEXANDER LEE

Exercise 1 (1.2.2)**.** Suppose towards a contradiction that there is a rational number $r \in \mathbf{Q}$ satisfying $2^r = 3$. Since $r \in \mathbf{Q}$, we can write $r = \frac{p}{q}$ for some $p, q \in \mathbf{Z}$ with $q \neq 0$. Thus, we have $2^{\frac{p}{q}} = 3 \Rightarrow 2^p = 3^q$. TODO

Exercise 2 (1.2.3)**.** (a) False. Consider infinite set of the form $A_n = [0, \frac{1}{n}]$ for $n \in \mathbf{N}$. Our definition of A_n satisfies $A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4 \cdots$. However, notice that $\bigcap_{n=1}^{\infty} A_n = \{0\}$, which is not an infinite set.

(b) True.

(c) Let $A = \{0\}$, $B = \{0, 1\}$, and $C = \{2, 3\}$. Then,

$$A \cap (B \cup C) = \{0\} \cap (\{0, 1\} \cup \{2, 3\}) = \{0\},$$

but

$$(A \cap B) \cup C = (\{0\} \cap \{0, 1\}) \cup \{2, 3\} = \{0, 2, 3\}.$$

Here, $A \cap (B \cup C) \neq (A \cap B) \cup C$.

(d) True.

(e) True.

Exercise 3 (1.2.6)**.** (a) Suppose $a, b \in \mathbf{R}$ where $a, b > 0$. We have that $|a + b| = |a| + |b|$. We also have that $|-a + (-b)| = |-(a + b)| = |a + b| = |-a| + |-b| = |a| + |b|$. Thus, the triangle inequality holds when a and b have the same sign.

(b) Given $a, b \in \mathbf{R}$,

$$\begin{aligned}(a+b)^2 &= a^2 + 2ab + b^2 \\ &\leq |a|^2 + 2|a||b| + |b|^2 \\ &= (|a| + |b|)^2. \quad \square\end{aligned}$$

(c) Given $a, b, c, d \in \mathbf{R}$,

$$\begin{aligned}|a - b| &= |(a - c) + (c - d) + (d - b)| \\ &\leq |a - c| + |(c - d) + (d - b)| \text{ (by the triangle inequality)} \\ &\leq |a - c| + |c - d| + |d - b| \text{ (by the triangle inequality).} \quad \square\end{aligned}$$

(d) Given $a, b \in \mathbf{R}$,

$$\begin{aligned}||a| - |b|| &= ||a - b + b| - |b|| \\ &\leq ||a - b| + |b| - |b|| \text{ (by the triangle inequality)} \\ &= ||a - b|| \\ &= |a - b|. \quad \square\end{aligned}$$

Exercise 4 (1.2.8)**.** (a) Impossible.

(b) Let $f : \mathbf{N} \to \mathbf{N}$ be defined by $f(a) = |a|$, $a \in \mathbf{N}$.

(c) Impossible.

Exercise 5 (1.2.10)**.**

Exercise 6 (1.3.2)**.**

Exercise 7 (1.3.3)**.**

Exercise 8 (1.3.6)**.**