MATH 355: HOMEWORK 5

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Exercise 1 (2.5.1).
(a) TODO
(b) TODO
(c) TODO
(d) TODO

Exercise 2 (2.5.2).
(a) TODO
(b) True. Suppose towards a contradiction that (x_n) converges. Thus, we know that subsequences of (x_n) are convergent, which contradicts the assumption that (x_n) contains a divergent subsequence. Therefore, (x_n) diverges.
(c) TODO
(d) TODO

Exercise 3 (2.6.2).
(a) Consider the sequence $a_n = (-1)^n \frac{1}{n}$. (a_n) clearly converges to 0, so it is a Cauchy sequence by the Cauchy Criterion. Notice that (a_n) is not monotone as well.
(b) Impossible. Suppose (a_n) is a Cauchy sequence. Thus, (a_n) is bounded by Lemma 2.6.3. As such, every subsequence of (a_n) must be bounded as well.
(c) TODO
(d) Consider the sequence $(a_n) = (0, 1, 0, 2, 0, 3, \ldots)$. Clearly, (a_n) is unbounded. However, $(0, 0, 0, \ldots)$ is a subsequence of (a_n) that is Cauchy.

Exercise 4 (2.6.4).
(a) Let $\epsilon > 0$ be arbitrary. Since (a_n) is a Cauchy sequence, there exists an $N_1 \in \mathbb{N}$ such that whenever $m, n \geq N_1$, it follows that $|a_n - a_m| < \epsilon/2$. Similarly, since (b_n) is a Cauchy sequence, there exists an $N_2 \in \mathbb{N}$ such that whenever $m, n \geq N_2$, it follows that $|b_n - b_m| < \epsilon/2$. Let $N = \max\{N_1, N_2\}$ and suppose $m, n \geq N$. Then,

$$\begin{aligned}
|c_n - c_m| &= ||a_n - b_n| - |a_m - b_m|| \\
&\leq |(a_n - b_n) - (a_m - b_m)| \text{ (by the Triangle Inequality)} \\
&= |(a_n - a_m) + (b_m - b_n)| \\
&\leq |a_n - a_m| + |b_m - b_n| \text{ (by the Triangle Inequality)} \\
&< \epsilon/2 + \epsilon/2 \\
&= \epsilon.
\end{aligned}$$

Therefore, (c_n) is a Cauchy sequence.
(b) Let $a_n = 1$. (a_n) is a Cauchy sequence, but $c_n = (-1)^n$ is divergent and thus not a Cauchy sequence.
(c) Let $a_n = (-1)^n \frac{1}{n}$. Then,

$$c_n = \begin{cases} 0 & \text{if } n \text{ is odd} \\ -1 & \text{if } n \text{ is even} \end{cases},$$

which is divergent and thus not a Cauchy sequence.

Exercise 5 (2.7.1). (a) Let $\epsilon > 0$ be arbitrary. Since $(a_n) \to 0$, there exists an $N \in \mathbb{N}$ such that whenever $m \geq N$, it follows that $|a_m| < \epsilon$. Then,

$$\begin{aligned} |s_n - s_m| &= |a_{m+1} - a_{m+2} - \cdots \pm a_n| \\ &\leq |a_{m+1}| \text{ (since } (a_n) \text{ is decreasing)} \\ &\leq |a_m| \\ &< \epsilon \end{aligned}$$

whenever $n > m \geq N$.

Exercise 6 (2.7.2).

Exercise 7 (2.7.4).

Exercise 8 (2.7.5).

Exercise 9 (2.7.9).

Exercise 10 (3.2.3).