MATH 355: HOMEWORK 1

ALEXANDER LEE

Exercise 1 (1.3.8).

(a) Supremum: 1. Infimum: 0.
(b) Supremum: 1. Infimum: -1.
(c) Supremum: 1. Infimum: $\frac{1}{4}$.
(d) Supremum: 1. Infimum: 0.

Exercise 2 (1.3.9).

(a) Since $\sup(A) < \sup(B)$, we have $\frac{\sup(B)-\sup(A)}{2} > 0$. Set $m = \frac{\sup(B)-\sup(A)}{2}$. By Lemma 1.3.8, we know $\sup(B) - m < b$ for some $b \in B$. Because

$$\begin{aligned}\sup(B) - m &= \sup(B) - \frac{\sup(B) - \sup(A)}{2} \\ &= \frac{2\sup(B) - \sup(B) + \sup(A)}{2} \\ &= \frac{\sup(B) + \sup(A)}{2} \\ &= \frac{\sup(B) - \sup(A)}{2} + \sup(A) \\ &> \sup(A),\end{aligned}$$

we know that $\sup(B) - m$ is an upper bound for A. Since $\sup(B) - m < b$, b is also an upper bound for A. Observe that $b \in B$, so we are done.

(b) Let $A = B = (0, 1)$. We have $\sup(A) = \sup(B) = 1$, but all upper bounds for A are not elements of B.

Exercise 3 (1.4.1).

(a) Given $a, b \in \mathbb{Q}$, we can write $a = \frac{p_1}{q_1}$ and $b = \frac{p_2}{q_2}$ for some $p_1, q_1, p_2, q_2 \in \mathbb{Z}$ with $q_1, q_2 \neq 0$. Then, $ab = (\frac{p_1}{q_1})(\frac{p_2}{q_2}) = \frac{p_1 p_2}{q_1 q_2}$. Since $p_1 p_2, q_1 q_2 \in \mathbb{Z}$ and $q_1 q_2 \neq 0$ because $q_1, q_2 \neq 0$, we have that $ab \in \mathbb{Q}$. Next, we can assume that if a or b are negative, the numerator is negative and the denominator is positive. Then, $a + b = \frac{p_1}{q_1} + \frac{p_2}{q_2} = \frac{p_1 q_2 + p_2 q_1}{q_1 + q_2}$. Since $p_1 q_2 + p_2 q_1 \in \mathbb{Z}$ and $q_1 + q_2 \neq 0$ because $q_1, q_2 > 0$, we have that $a + b \in \mathbb{Q}$.

(b) Given $a \in \mathbb{Q}$, we can write $a = \frac{p}{q}$ for some $p, q \in \mathbb{Z}$ with $q \neq 0$. Since $a \neq 0$, we have $p \neq 0$ as well. Suppose $t \in \mathbb{I}$. Then, $a + t = \frac{p}{q} + \frac{qt}{q} = \frac{p+qt}{q}$. Since $p, q \neq 0$, we have $p + qt \neq 0$. Furthermore, because $qt \notin \mathbb{Z}$, we have $p + qt \notin \mathbb{Z}$. Therefore, $a + t \notin \mathbb{Q}$, implying that $a + t \in \mathbb{I}$. Next, consider $at = \frac{p}{q}(t) = \frac{pt}{q}$. Since $p \neq 0$, we have that $pt \neq 0$. Furthermore, because $pt \neq \mathbb{Z}$, we have $at \notin \mathbb{Q}$, which implies that $at \in \mathbb{I}$.

(c) Given $s, t \in \mathbb{I}$, we have $s + t, st \in \mathbb{I}$.

Exercise 4 (1.4.3). Suppose towards a contradiction that $\bigcap_{n=1}^{\infty}(0, 1/n) \neq \emptyset$. Set $A = \bigcap_{n=1}^{\infty}(0, 1/n)$. For all $n \in \mathbb{N}$, $1/n$ is thus an upper bound for A. Since $A \neq \emptyset$, there exists an $a \in A$ such that $a > 0$. By the Archimedean Property, there exists an $n \in \mathbb{N}$ such that $1/n < a$, which is a contradiction since our assumption implied that for all $n \in \mathbb{N}$, $1/n$ is an upper bound for A. Therefore, $\bigcap_{n=1}^{\infty}(0, 1/n) = A = \emptyset$.

Exercise 5 (1.4.4)**.** By the construction of T, b is an upper bound for T since $b \geq t$ for all $t \in T$. Given an arbitrary upper bound $u \in \mathbb{R}$ for T, suppose towards a contradiction that $u < b$. By the density of $\mathbb{Q}$ in $\mathbb{R}$, there exists an $r \in \mathbb{Q}$ such that $u < r < b$. Since $u < r$ and $r \in T$, u is therefore not an upper bound for T, which contradicts our previous assumption that u is an upper bound for T. Hence, it must be that $b \leq u$, implying that $\sup(T) = b$.

Exercise 6 (1.5.4)**.** (a) We know that $(-1, 1) \sim \mathbb{R}$ with the function $f(x) = x/(x^2 - 1)$. Therefore, what is left to show is that $(-1, 1) \sim (a, b)$. Let $f : (-1, 1) \to (a, b)$ be given by $f(x) = \frac{b-a}{2}x + \frac{a+b}{2}$. (1–1) Suppose $f(x_1) = f(x_2)$ for some $x_1, x_2 \in (-1, 1)$. Then,

$$\begin{aligned} f(x_1) = f(x_2) &\implies \frac{b-a}{2}x_1 + \frac{a+b}{2} = \frac{b-a}{2}x_2 + \frac{a+b}{2} \\ &\implies \frac{b-a}{2}x_1 = \frac{b-a}{2}x_2 \\ &\implies x_1 = x_2. \end{aligned}$$

(Onto) Given $y \in (a, b)$, let $x = \frac{2y-a-b}{b-a}$. Then,

$$\begin{aligned} f(x) &= f\left(\frac{2y-a-b}{b-a}\right) \\ &= \frac{b-a}{2} \cdot \frac{2y-a-b}{b-a} + \frac{a+b}{2} \\ &= \frac{2y-a-b}{2} + \frac{a+b}{2} \\ &= \frac{2y}{2} \\ &= y. \end{aligned}$$

Since f is 1–1 and onto, we have $(-1, 1) \sim (a, b)$. Thus, $(a, b) \sim \mathbb{R}$.

(b) TODO

(c) TODO

Exercise 7 (1.5.5)**.** (a) $A \sim A$ for every set A because the identity function $f : A \to A$ given by $f(x) = x$ is always 1–1 and onto.

(b) Given sets A and B, $A \sim B$ is equivalent to asserting $B \sim A$ because any 1–1 and onto function $f : A \to B$ has an inverse function $f^{-1} : B \to A$ that is also 1–1 and onto.

(c) Given three sets A, B, and C, suppose $A \sim B$ and $B \sim C$. Thus, there exist 1–1 and onto functions $f : A \to B$ and $g : B \to C$. Since $g \circ f : A \to C$ is also 1–1 and onto, we have that $A \sim C$.

Exercise 8 (1.6.5)**.** (a) $P(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$

(b) We proceed with induction on n. Our base case is $n = 0$. That is, $A = \emptyset$, which only has one subset, namely $\emptyset$. Since $2^0 = 1$, the statement is true for the base case. Now, assume that if A is finite with $n = k$ elements, then $P(A)$ has 2^k elements. We want to show that the statement is true if A is finite with $n = k+1$ elements. Suppose A is finite with $k+1$ elements. Then, there exists a set B such that $A = B \cup \{a\}$ with $a \notin B$. Notice that $|B| = k$ and $B \subseteq A$. Further observe that $P(A) = P(B) \cup \{x \cup \{a\} : x \in P(B)\}$. By the inductive hypothesis, $P(B) = 2^k$. Also, $|\{x \cup \{a\} : x \in P(B)\}| = P(B) = 2^k$. Since $P(B)$ and $\{x \cup \{a\} : x \in P(B)\}$ are disjoint, we have $|P(A)| = 2^k + 2^k = 2(2^k) = 2^{k+1}$.