MATH 355: HOMEWORK 11

ALEXANDER LEE

Exercise 1 (6.3.1). (a) We show that (g_n) converges uniformly on $[0,1]$ to $g = \lim g_n = 0$. Given $\epsilon > 0$, choose $N \in \mathbb{N}$ such that $N > 1/\epsilon$. Then, whenever $n \geq \mathbb{N}$ and $x \in [0,1]$, it follows that

$$|g_n(x) - g(x)| = \left|\frac{x^n}{n} - 0\right| = \left|\frac{x^n}{n}\right| = \frac{x^n}{n} \leq \frac{1}{n} \leq \frac{1}{N} < \epsilon.$$

g is differentiable since it is the constant function $g(x) = 0$ for all $x \in [0,1]$. Specifically, we have that $g'(x) = 0$ for all $x \in [0,1]$.

(b) By the Algebraic Differentiability Theorem, we have that

$$g_n'(x) = \frac{n(nx^{n-1}) - x^n(0)}{n^2} = x^{n-1}.$$

It follows that (g_n') converges on [0, 1] to

$$h(x) = \lim g_n'(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x = 1 \end{cases}.$$

By the contrapositive of the Continuous Limit Theorem, since h is not continuous and g_n' is continuous on $[0,1]$, we have that (g_n') does not converge uniformly on $[0,1]$ to h. Observe that h and g' are not the same.

Exercise 2 (6.3.2). (a) The pointwise limit of (h_n) is $h(x) = x$. To show that the convergence is uniform on $\mathbb{R}$, let $\epsilon > 0$ be arbitrary and choose $N \in \mathbb{N}$ such that $N > 1/\epsilon^2$. Then, whenever $n \geq N$ and $x \in \mathbb{R}$, it follows that

$$|h_n(x) - h(x)| = \left|\sqrt{x^2 - \frac{1}{n}} - x\right| \leq \left|\sqrt{x^2} + \sqrt{\frac{1}{n}} - x\right| = \sqrt{\frac{1}{n}} \leq \sqrt{\frac{1}{N}} < \epsilon.$$

(b) By the Chain Rule, we have that

$$h_n'(x) = \frac{1}{2\sqrt{x^2 + \frac{1}{n}}} \cdot 2x = \frac{x}{\sqrt{x^2 + \frac{1}{n}}}.$$

It follows that

$$g(x) = \lim h_n'(x) = \lim \frac{x}{\sqrt{x^2 + \frac{1}{n}}} = \frac{x}{\sqrt{x^2}} = 1.$$

By the contrapositive of the Differentiable Limit Theorem, since $g(x) = 1 \neq x = h(x)$ and each h_n is differentiable, it must be that h_n' does not converge uniformly to $g(x)$ on $\mathbb{R}$, i.e., any neighborhood of zero.

Exercise 3 (6.4.2). (a) True. Given that $\sum_{n=1}^{\infty} g_n$ converges uniformly, it follows from a special case of the Cauchy Criterion for Uniform Convergence of Series that that given $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that whenever

$m \geq N$ and $x \in A$, where A is the domain of g_n, we have that $|g_{m+1}(x)| < \epsilon$. Therefore, (g_n) converges uniformly to zero.

(b) True. Given that $0 \leq f_n(x) \leq g_n(x)$ and $\sum_{n=1}^{\infty} g_n$ converges uniformly, it follows from the Cauchy Criterion for Series that given $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that whenever $n > m \geq N$, we have that

$$|f_{m+1}(x) + f_{m+2}(x) + \cdots + f_n(x)| \leq |g_{m+1}(x) + g_{m+2}(x) + \cdots + g_n(x)| < \epsilon.$$

Therefore, $\sum_{n=1}^{\infty} f_n$ converges uniformly as well.

(c) False. TODO

Exercise 4 (6.4.4)**.**

Exercise 5 (6.4.5a)**.**

Exercise 6 (6.5.1)**.**

Exercise 7 (6.5.2)**.**

Exercise 8 (6.5.4)**.**