

MATH 355: HOMEWORK 11

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Exercise 1 (6.3.1). (a) We show that (g_n) converges uniformly on $[0, 1]$ to $g = \lim g_n = 0$. Given $\epsilon > 0$, choose $N \in \mathbb{N}$ such that $N > 1/\epsilon$. Then, whenever $n \geq N$ and $x \in [0, 1]$, it follows that

$$|g_n(x) - g(x)| = \left| \frac{x^n}{n} - 0 \right| = \left| \frac{x^n}{n} \right| = \frac{x^n}{n} \leq \frac{1}{n} \leq \frac{1}{N} < \epsilon.$$

g is differentiable since it is the constant function $g(x) = 0$ for all $x \in [0, 1]$. Specifically, we have that $g'(x) = 0$ for all $x \in [0, 1]$.

(b) By the Algebraic Differentiability Theorem, we have that

$$g'_n(x) = \frac{n(nx^{n-1}) - x^n(0)}{n^2} = x^{n-1}.$$

It follows that (g'_n) converges on $[0, 1]$ to

$$h(x) = \lim g'_n(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x = 1 \end{cases}.$$

By the contrapositive of the Continuous Limit Theorem, since h is not continuous and g'_n is continuous on $[0, 1]$, we have that (g'_n) does not converge uniformly on $[0, 1]$ to h . Observe that h and g' are not the same.

Exercise 2 (6.3.2). (a) The pointwise limit of (h_n) is $h(x) = x$. To show that the convergence is uniform on \mathbb{R} , let $\epsilon > 0$ be arbitrary and choose $N \in \mathbb{N}$ such that $N > 1/\epsilon^2$. Then, whenever $n \geq N$ and $x \in \mathbb{R}$, it follows that

$$|h_n(x) - h(x)| = \left| \sqrt{x^2 - \frac{1}{n}} - x \right| \leq \left| \sqrt{x^2} + \sqrt{\frac{1}{n}} - x \right| = \sqrt{\frac{1}{n}} \leq \sqrt{\frac{1}{N}} < \epsilon.$$

(b) By the Chain Rule, we have that

$$h'_n(x) = \frac{1}{2\sqrt{x^2 + \frac{1}{n}}} \cdot 2x = \frac{x}{\sqrt{x^2 + \frac{1}{n}}}.$$

It follows that

$$g(x) = \lim h'_n(x) = \lim \frac{x}{\sqrt{x^2 + \frac{1}{n}}} = \frac{x}{\sqrt{x^2}} = 1.$$

By the contrapositive of the Differentiable Limit Theorem, since $g(x) = 1 \neq x = h(x)$ and each h_n is differentiable, it must be that h'_n does not converge uniformly to $g(x)$ on \mathbb{R} , i.e., any neighborhood of zero.

Exercise 3 (6.4.2). (a) True. Given that $\sum_{n=1}^{\infty} g_n$ converges uniformly, it follows from a special case of the Cauchy Criterion for Uniform Convergence of Series that that given $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that whenever

$m \geq N$ and $x \in A$, where A is the domain of g_n , we have that $|g_{m+1}(x)| < \epsilon$. Therefore, (g_n) converges uniformly to zero.

- (b) True. Given that $0 \leq f_n(x) \leq g_n(x)$ and $\sum_{n=1}^{\infty} g_n$ converges uniformly, it follows from the Cauchy Criterion for Series that given $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that whenever $n > m \geq N$, we have that

$$|f_{m+1}(x) + f_{m+2}(x) + \cdots + f_n(x)| \leq |g_{m+1}(x) + g_{m+2}(x) + \cdots + g_n(x)| < \epsilon.$$

Therefore, $\sum_{n=1}^{\infty} f_n$ converges uniformly as well.

- (c) False. TODO

Exercise 4 (6.4.4). TODO

Exercise 5 (6.4.5a). Observe that each $\frac{x^n}{n^2}$ is continuous on $[-1, 1]$. Also note that for each $n \in \mathbb{N}$, $|\frac{x^n}{n^2}| \leq \frac{1}{n^2}$ for all $x \in [-1, 1]$. Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, it follows from the Weierstrass M-Test that $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$ converges uniformly on $[-1, 1]$. Therefore, $h(x)$ is continuous on $[-1, 1]$ by the Term-by-term Continuity Theorem.

Exercise 6 (6.5.1). (a) g can be rewritten as $g(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$. If $x = 1$, then $g(x)$ converges by the Alternating Series Test. By Theorem 6.5.1, it follows that $g(x)$ converges absolutely for any $x \in (-1, 1)$. Therefore, g is defined on $(-1, 1)$.

Since g converges absolutely on $(-1, 1)$, it follows from Theorem 6.5.2 that g converge uniformly on $(-1, 1)$. Also note that $(-1)^{n+1} \frac{x^n}{n}$ is continuous on $(-1, 1)$. By the Term-by-term Continuity Theorem, we have that g is continuous on $(-1, 1)$.

Since $g(x)$ converges at the point $x = 1$, it follows from Abel's Theorem that g converges uniformly on the interval $[0, 1]$. We established previously that g also converges uniformly on $(-1, 1)$. Therefore, we can conclude that g converges uniformly on $(-1, 1]$ and is thus defined on this set.

Since $g(x)$ converges uniformly on $(-1, 1]$, we can also conclude from the Term-by-term Continuity Theorem that g is continuous on $(-1, 1]$.

$g(x)$ is not defined when $x = -1$ since this value of x yields the harmonic series, which does not converge. Thus, g is not defined on $[-1, 1]$ and so cannot even be continuous on this set.

The power series for $g(x)$ cannot possibly converge for any other points $|x| > 1$ because $g(x)$ would be unbounded.

- (b) $g'(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{nx^{n-1}}{n} = \sum_{n=1}^{\infty} (-1)^{n+1} x^{n-1}$. $g'(x)$ is defined on $(-1, 1)$.

Exercise 7 (6.5.2). TODO

Exercise 8 (6.5.4). TODO