MATH 355: HOMEWORK 6

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Exercise 1 (3.2.2). (a) Limit points of A: $\{-1, 1\}$. Limit points of B: $[0, 1]$.
(b) A is neither open nor closed. B is neither open nor closed.
(c) A contains isolated points. B does not contain isolated points.
(d) $\overline{A} = A \cup \{-1\}$. $\overline{B} = [0, 1]$.

Exercise 2 (3.2.4). (a) If $s \in A$, then $s \in \overline{A}$ and we are done. Now suppose $s \notin A$. By Lemma 1.3.8, for every $\epsilon > 0$, there exists an $a \in A$ ($a \neq s$) such that $s - \epsilon < a$. Since $s = \sup(A)$, we also know that $a < s$. Thus, every ϵ-neighborhood $V_\epsilon(s)$ intersects A at some point other than s. That is, s is a limit point of A, so $s \in \overline{A}$ in this case as well.
(b) An open set O cannot contain its supremum $s = \sup(O)$ since every ϵ-neighborhood $V_\epsilon(s)$ of s is not be a subset of O. Specifically, this is because for any $\epsilon > 0$ and $a \in O$, we have that $a < s + \epsilon$ since $s = \sup(O)$.

Exercise 3 (3.2.6). (a) False. Consider the open set $\mathbb{R} \setminus \{\sqrt{2}\}$.
(b) False. Consider the closed sets of the form $C_n = [n, \infty)$ for $n \in \mathbb{N}$. Observe that $C_n \subseteq C_{n+1}$ and $\cup_{n=1}^{\infty} C_n = \emptyset$.
(c) True. Given a nonempty open set O, we know that for $a \in O$, there exists an ϵ-neighborhood $V_\epsilon(a) \subseteq O$. By the Density of $\mathbb{Q}$ in $\mathbb{R}$, there exists a rational number $r \in V_\epsilon(a)$. Thus, we have that $r \in O$.
(d) False. Consider the bounded infinite closed set $F = \{\sqrt{2} + 1/n : n \in \mathbb{N}\} \cup \{\sqrt{2}\}$. Observe that F does not contain any rational number.
(e) True. The Canter set is defined as $C = \cap_{n=0}^{\infty} C_n$. Since each C_n is the union of a finite collection of closed sets, each C_n is closed. The intersection of an arbitrary collection of closed sets is closed, so the Cantor set C is closed.

Exercise 4 (3.2.8). (a) Definitely closed.
(b) Definitely open.
(c) Definitely open.
(d) Both.
(e) Neither.

Exercise 5 (3.2.9). (a) TODO
(b) Suppose $\{E_\lambda : \lambda \in \Lambda\}$ is a finite collection of open sets. Each E_λ^c is thus a closed set by Theorem 3.2.13. As such, $\cup_{\lambda \in \Lambda} E_\lambda^c$ is the union of a finite collection of closed sets. By Theorem 3.2.3, $\cap_{\lambda \in \Lambda} E_\lambda$ is open. It follows that $(\cap_{\lambda \in \Lambda} E_\lambda)^c$ is closed by Theorem 3.2.13. Since $(\cap_{\lambda \in \Lambda} E_\lambda)^c = \cup_{\lambda \in \Lambda} E_\lambda^c$, the union of a finite collection of closed sets is therefore closed.

Suppose $\{E_\lambda : \lambda \in \Lambda\}$ is an arbitrary collection of open sets. As such, $\cap_{\lambda \in \Lambda} E_\lambda^c$ is the intersection of an arbitrary collection of closed sets. By Theorem 3.2.3, $\cup_{\lambda \in \Lambda} E_\lambda$ is open. It follows that $(\cup_{\lambda \in \Lambda} E_\lambda)^c$ is closed by

Theorem 3.2.13. Since $\left(\cup_{\lambda\in\Lambda} E_\lambda\right)^c = \cap_{\lambda\in\Lambda} E_\lambda^c$, the intersection of an arbitrary collection of closed sets is closed.

Exercise 6 (3.2.10)**.**

Exercise 7 (3.2.13)**.**

Exercise 8 (3.2.14)**.**