# MATH 355: HOMEWORK 6

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**Exercise 1** (3.2.2). (a) Limit points of $A$: $\{-1, 1\}$. Limit points of $B$: $[0, 1]$.

(b) $A$ is neither open nor closed. $B$ is neither open nor closed.

(c) $A$ contains isolated points. $B$ does not contain isolated points.

(d) $\overline{A} = A \cup \{-1\}$. $\overline{B} = [0, 1]$.

**Exercise 2** (3.2.4). (a) If $s \in A$, then $s \in \overline{A}$ and we are done. Now suppose $s \notin A$. By Lemma 1.3.8, for every $\epsilon > 0$, there exists an $a \in A$ ($a \neq s$) such that $s - \epsilon < a$. Since $s = \sup(A)$, we also know that $a < s$. Thus, every $\epsilon$-neighborhood $V_\epsilon(s)$ intersects $A$ at some point other than $s$. That is, $s$ is a limit point of $A$, so $s \in \overline{A}$ in this case as well.

(b) An open set $O$ cannot contain its supremum $s = \sup(O)$ since every $\epsilon$-neighborhood $V_\epsilon(s)$ of $s$ is not be a subset of $O$. Specifically, this is because for any $\epsilon > 0$ and $a \in O$, we have that $a < s + \epsilon$ since $s = \sup(O)$.

**Exercise 3** (3.2.6). (a) False. Consider the open set $\mathbb{R} \setminus \{\sqrt{2}\}$.

(b) False. Consider the closed sets of the form $C_n = [n, \infty)$ for $n \in \mathbb{N}$. Observe that $C_n \subseteq C_{n+1}$ and $\cap_{n=1}^{\infty} C_n = \emptyset$.

(c) True. Given a nonempty open set $O$, we know that for $a \in O$, there exists an $\epsilon$-neighborhood $V_\epsilon(a) \subseteq O$. By the Density of $\mathbb{Q}$ in $\mathbb{R}$, there exists a rational number $r \in V_\epsilon(a)$. Thus, we have that $r \in O$.

(d) False. Consider the bounded infinite closed set $F = \{\sqrt{2} + 1/n : n \in \mathbb{N}\} \cup \{\sqrt{2}\}$. Observe that $F$ does not contain any rational number.

(e) True. The Canter set is defined as $C = \cap_{n=0}^{\infty} C_n$. Since each $C_n$ is the union of a finite collection of closed sets, each $C_n$ is closed. The intersection of an arbitrary collection of closed sets is closed, so the Cantor set $C$ is closed.

**Exercise 4** (3.2.8). (a) Definitely closed since the closure of any set is closed.

(b) Definitely open since $A \setminus B = A \cap B^c$ and the intersection of a finite collection of open sets is open.

(c) Definitely open since $(A^c \cup B)^c = A \cap B^c$, which is closed as explained in part (b).

(d) Definitely closed since $(A \cap B) \cup (A^c \cap B) = (A \cup A^c) \cap B = \mathbb{R} \cap B = B$.

(e) Definitely open. By the definition of closure, we know $A^c \subseteq \overline{A^c}$. We also know that $\overline{A}^c \subseteq A^c$. Thus, $\overline{A}^c \subseteq \overline{A^c}$. It follows that $\overline{A}^c \cap \overline{A^c} = \overline{A}^c$, which is open since the closure of any set is closed and the complement of a closed set is open.

**Exercise 5** (3.2.9). (a) We first show that $(\cup_{\lambda\in\Lambda}E_\lambda)^c = \cap_{\lambda\in\Lambda}E_\lambda^c$.

$$\begin{aligned} x \in (\cup_{\lambda\in\Lambda}E_\lambda)^c &\iff x \notin \cup_{\lambda\in\Lambda}E_\lambda \\ &\iff \forall\lambda\in\Lambda,\ x\notin E_\lambda \\ &\iff \forall\lambda\in\Lambda,\ x\in E_\lambda^c \\ &\iff x\in \cap_{\lambda\in\Lambda}E_\lambda^c. \end{aligned}$$

Next, we show that $(\cap_{\lambda\in\Lambda}E_\lambda)^c = \cup_{\lambda\in\Lambda}E_\lambda^c$.

$$\begin{aligned} x \in (\cap_{\lambda\in\Lambda}E_\lambda)^c &\iff x \notin \cap_{\lambda\in\Lambda}E_\lambda \\ &\iff \exists\lambda\in\Lambda \text{ s.t. } x\notin E_\lambda \\ &\iff \exists\lambda\in\Lambda \text{ s.t. } x\in E_\lambda^c \\ &\iff x\in \cup_{\lambda\in\Lambda}E_\lambda^c. \end{aligned}$$

(b) Suppose $\{E_\lambda : \lambda\in\Lambda\}$ is a finite collection of open sets. Each $E_\lambda^c$ is thus a closed set by Theorem 3.2.13. As such, $\cup_{\lambda\in\Lambda}E_\lambda^c$ is the union of a finite collection of closed sets. By Theorem 3.2.3, $\cap_{\lambda\in\Lambda}E_\lambda$ is open. It follows that $(\cap_{\lambda\in\Lambda}E_\lambda)^c$ is closed by Theorem 3.2.13. Since $(\cap_{\lambda\in\Lambda}E_\lambda)^c = \cup_{\lambda\in\Lambda}E_\lambda^c$, the union of a finite collection of closed sets is therefore closed.

Suppose $\{E_\lambda : \lambda\in\Lambda\}$ is an arbitrary collection of open sets. As such, $\cap_{\lambda\in\Lambda}E_\lambda^c$ is the intersection of an arbitrary collection of closed sets. By Theorem 3.2.3, $\cup_{\lambda\in\Lambda}E_\lambda$ is open. It follows that $(\cup_{\lambda\in\Lambda}E_\lambda)^c$ is closed by Theorem 3.2.13. Since $(\cup_{\lambda\in\Lambda}E_\lambda)^c = \cap_{\lambda\in\Lambda}E_\lambda^c$, the intersection of an arbitrary collection of closed sets is closed.

**Exercise 6** (3.2.10). (i) Such a set cannot exist. Let $A \subseteq [0,1]$ be a countable set. Since $A$ is countable, there exists a bijection $f : \mathbb{N}\to A$. We can use the function $f$ to define a sequence $(a_n)$ where $a_n = f(n)$ for all $n\in\mathbb{N}$. Because $(a_n)\subseteq A\subseteq[0,1]$, $(a_n)$ is bounded. By the Bolzano-Weierstrass Theorem, $(a_n)$ has a convergent subsequence $(a_{n_k})\to a$. Since $f$ is a bijection, all the terms of $(a_n)$ are distinct, so at most one term in $(a_{n_k})$ can be equal to $a$. Let $(b_n)$ be a subsequence of $(a_{n_k})$ without the term $a$ if it exists. It follows that $(b_n)\to a$, so $a$ is a limit point.

(ii) Consider the set $\mathbb{Q}\cap[0,1]$.

(iii) Such a set cannot exist. Suppose $A$ has an uncountable number of isolated points. For every isolated point $x\in A$, there exists an $\epsilon_x > 0$ such that $V_{\epsilon_x}(x)\cap A = \{x\}$. Each neighborhood $V_{\epsilon_x}(x)$ can intersect with at most 2 other neighborhoods, since if a neighborhood intersects with more than 2 neighborhoods, then one of the three neighborhoods in question would intersect another isolated point in $A$, which is not possible. By the Density of $\mathbb{Q}$ in $\mathbb{R}$, we can choose a rational number $r$ in each neighborhood. Each $r$ can be chosen for at most 2 neighborhoods. Regardless, because we have uncountably many neighbors, we therefore have uncountably many rational numbers since each rational number $r$ corresponds to at most 2 neighborhoods. This is a contradiction since $\mathbb{Q}$ is countable.

**Exercise 7** (3.2.13). Suppose $A$ is an open and closed set such that $A\subset\mathbb{R}$ and $A\neq\emptyset$. Thus, there exists $x_0\in A$ and $x_1\notin A$. Without loss of generality, suppose $x_0 < x_1$. Consider the closed interval $I = [x_0, x_1]$. $A$ and $I$ are closed, so $A\cap I$ is closed as well since the intersection of an arbitrary collection of closed sets is closed.

Since $A \cap I$ is nonempty and bounded above, it has a supremum $s = \sup(A \cap I)$ by Axiom of Completeness. It follows that $s \in I \cap A \subseteq A$ because $A \cap I$ is closed. Since $A$ is also open, there exists $V_\epsilon(s) \subseteq A$. We also know that $[s, x_1] \subseteq I$. Choose $\epsilon_1 < \min\{\epsilon, x_1\}$. It follows that $s + \epsilon_1 \in A$ and $s + \epsilon_1 \in I$, so $s + \epsilon_1 \in A \cap I$. This is a contradiction since we assumed that $s = \sup(A \cap I)$. Therefore, the only sets that are both open and closed are $\mathbb{R}$ and $\emptyset$.

**Exercise 8** (3.2.14). (a) We first show that $E$ is closed if and only if $\overline{E} = E$. Let $L$ be the set of all limit points of $E$. Then,

$$\begin{aligned} E \text{ is closed} &\iff L \subseteq E \\ &\iff E \cup L = E \\ &\iff \overline{E} = E. \end{aligned}$$

Next, we show that $E$ is open if and only if $E^\circ = E$.

$$\begin{aligned} E \text{ is open} &\iff \forall x \in E,\ \exists V_\epsilon(x) \subseteq E \\ &\iff E^\circ = E. \end{aligned}$$

(b) We begin with showing that $\overline{E}^c = (E^c)^\circ$. Let $L$ be the set of all limit points of $E$. Then,

$$\begin{aligned} x \in \overline{E}^c &\iff x \in (E \cup L)^c \\ &\iff x \in E^c \cap L^c \text{ (by DeMorgan's Laws)} \\ &\iff x \in E^c \wedge x \text{ is not a limit point of E} \\ &\iff x \in E^c \text{ s.t. } \exists V_\epsilon(x) \subseteq E^c \\ &\iff x \in (E^c)^\circ. \end{aligned}$$

Next, we show that $(E^\circ)^c = \overline{E^c}$.

$$\begin{aligned} (E^\circ)^c &= (((E^c)^c)^\circ)^c \\ &= (\overline{E^c}^c)^c \text{ (since } \overline{E}^c = (E^c)^\circ) \\ &= \overline{E^c}. \end{aligned}$$