MATH 355: HOMEWORK 4

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Exercise 1 (2.3.3)**.** Let $\epsilon > 0$ be arbitrary. Since $\lim x_n = l$, there exists an $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$, we have that $|x_n - l| < \epsilon$. Thus, we have $l-\epsilon < x_n < l+\epsilon$. Similarly, since $\lim z_n = l$, there exists an $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$, we have that $|z_n - l| < \epsilon$. Hence, we also have $l - \epsilon < z_n < l + \epsilon$. Let $N = \max\{N_1, N_2\}$. Because we are given that $x_n \leq y_n \leq z_n$, it follows that for all $n \geq N$, we have $l - \epsilon < x_n \leq y_n \leq z_n < l + \epsilon$, which implies that $l - \epsilon < y_n < l + \epsilon$ and $|y_n - l| < \epsilon$. Therefore, $\lim y_n = l$ as well.

Exercise 2 (2.3.4)**.** (a)

$$\begin{aligned}
\lim \left(\frac{1 + 2a_n}{1 + 3a_n - 4{a_n}^2} \right) &= \frac{\lim 1 + 2 \lim a_n}{\lim 1 + 3 \lim a_n - 4 \lim a_n \lim a_n} \\
&= \frac{1 + 2 \cdot 0}{1 + 3 \cdot 0 - 4 \cdot 0 \cdot 0} \\
&= 1
\end{aligned}$$

(b)

$$\begin{aligned}
\lim \left(\frac{(a_n + 2)^2 - 4}{a_n} \right) &= \frac{\lim(a_n + 2) \lim(a_n + 2) - \lim 4}{\lim a_n} \\
&= \frac{(\lim a_n + \lim 2)(\lim a_n + \lim 2) - 4}{\lim a_n} \\
&= \frac{(\lim a_n + 2)(\lim a_n + 2) - 4}{\lim a_n} \\
&= \frac{(\lim a_n)^2 + 4 \lim a_n + 4 - 4}{\lim a_n} \\
&= \frac{\lim a_n (\lim a_n + 4)}{\lim a_n} \\
&= \lim a_n + 4 \\
&= 0 + 4 \\
&= 4
\end{aligned}$$

(c)

$$\begin{aligned}
\lim\left(\frac{\frac{2}{a_n}+3}{\frac{1}{a_n}+5}\right) &= \frac{\frac{\lim 2}{\lim a_n}+\lim 3}{\frac{\lim 1}{\lim a_n}+\lim 5} \\
&= \frac{\frac{2}{\lim a_n}+3}{\frac{1}{\lim a_n}+5} \\
&= \frac{\frac{2+3\lim a_n}{\lim a_n}}{\frac{1+5\lim a_n}{\lim a_n}} \\
&= \frac{2+3\lim a_n}{1+5\lim a_n} \\
&= \frac{2+3\cdot 0}{1+5\cdot 0} \\
&= \frac{2}{1} \\
&= 2
\end{aligned}$$

Exercise 3 (2.3.7). (a) Possible. Consider the divergent sequences $(x_n) = (-1, 1, -1, 1, \dots)$ and $(y_n) = (1, -1, 1, -1, \dots)$. Then, we have $(x_n + y_n) = (0, 0, 0, 0, \dots)$, which converges to 0.

(b) Impossible. Observe that $y_n = (x_n + y_n) - x_n$. Since $(x_n + y_n)$ and (x_n) converge, (y_n) must converge as well by the Algebraic Limit Theorem (ii).

(c) Possible. Consider the convergent sequence $(b_n) = (\frac{1}{n})$. Since $(1/b_n) = (n)$ is unbounded, it diverges.

(d) Impossible. Since $(a_n - b_n)$ is bounded, there exists a number $M_1 > 0$ such that $|a_n - b_n| \leq M_1$ for all $n \in \mathbb{N}$. Furthermore, since (b_n) converges, it is also bounded. Thus, there exists a number $M_2 > 0$ such that $|b_n| \leq M_2$ for all $n \in \mathbb{N}$. As such, $|a_n| \leq |a_n - b_n| + |b_n| \leq M_1 + M_2$, so (a_n) must be bounded too.

(e) Possible. Consider the sequences $(a_n) = (0, 0, 0, 0, \dots)$ and $(b_n) = (1, 2, 3, 4, \dots)$. Clearly, both $(a_n b_n) = (0 \cdot 1, 0 \cdot 2, 0 \cdot 3, 0 \cdot 4, \dots) = (0, 0, 0, 0, \dots)$ and (a_n) are convergent. However, (b_n) diverges since it is unbounded.

Exercise 4 (2.3.10). (a) False. Consider $(a_n) = (b_n) = (1, 2, 3, 4, \dots)$. Since $(a_n - b_n) = (0, 0, 0, 0, \dots)$, we have that $\lim(a_n - b_n) = 0$. However, since (a_n) and (b_n) are not bounded, they do not even converge to a limit.

(b) True. Since $(b_n) \to b$, for any $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, we have that $|b_n - b| < \epsilon$. Since $||b_n| - |b|| \leq |b_n - b|$, we also have that $||b_n| - |b|| < \epsilon$. Thus, $|b_n| \to |b|$.

(c) True. Note that $b_n = (b_n - a_n) + a_n$. Thus, by the Algebraic Limit Theorem (ii), we have $\lim b_n = \lim(b_n - a_n) + \lim a_n = 0 + a = a$.

(d) True. Since $(a_n) \to 0$, for any $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, we have that $|a_n| < \epsilon$, implying that $-\epsilon < a_n < \epsilon$. Since we have that $|b_n - b| \leq a_n$, we therefore have $|b_n - b| \leq a_n < \epsilon$. Hence, $(b_n) \to b$.

Exercise 5 (2.3.12). (a) True. Given $b \in B$, we know that $a_n \geq b$ for all $n \in \mathbb{N}$ since a_n is an upper bound for B. By the Order Limit Theorem (iii), we have that $a = \lim a_n \geq b$. Thus, a is also an upper bound for B.

(b) True. Suppose that every a_n is in the complement of the interval $(0, 1)$. Next, suppose towards a contradiction that a is in $(0, 1)$. Since $(a_n) \to a$, there exists an ϵ-neighborhood $V_\epsilon(a) \subseteq (0,1)$ that contains all but finitely many terms of (a_n). That is, all but finitely many terms of (a_n) are in $(0,1)$. This is a contradiction since we previously assumed that every a_n is not in $(0,1)$.

(c) False. Consider the sequence $(a_n) = (1.4, 1.41, 1.414, 1.4142, 1.41421, \ldots)$, where every consequent term is a decimal approximation of $\sqrt{2}$ that specifies an additional decimal. Each term in the sequence (a_n) is rational, however, by construction, $(a_n) \to \sqrt{2}$, which is irrational.

Exercise 6 (2.4.1). (a) We first show by induction that (x_n) is decreasing (i.e., $x_n \geq x_{n+1}$). Since $x_1 = 3$ and $x_2 = \frac{1}{4-3} = 1$, we have that $x_n \geq x_{n+1}$. Next, suppose that $3 = x_1 \geq x_2 \geq \cdots \geq x_n \geq x_{n+1}$. We want to show that $x_{n+1} \geq x_{n+2}$. Consider

$$\begin{aligned} x_{n+1} - x_{n+2} &= \frac{1}{4-x_n} - \frac{1}{4-x_{n+1}} \\ &= \frac{4 - x_{n+1} - (4 - x_n)}{(4-x_n)(4-x_{n+1})} \\ &= \frac{x_n - x_{n+1}}{(4-x_n)(4-x_{n+1})}. \end{aligned}$$

Since $x_n \geq x_{n+1}$ by our inductive hypothesis, we know that $x_n - x_{n+1} \geq 0$. Furthermore, by our inductive hypothesis, we also know that $3 \geq x_n \geq x_{n+1} \implies -3 \leq -x_n \leq -x_{n+1} \implies 1 = 4 - 3 \leq 4 - x_n \leq 4 - x_{n+1}$. Hence, we have that $x_{n+1} - x_{n+2} \implies x_{n+1} \geq x_{n+2}$. Therefore, (x_n) is decreasing. Next, we show that (x_n) is bounded. Since $x_n < 4$ for all $n \in \mathbb{N}$, we have that $x_{n+1} = \frac{1}{4-x_n} > 0$. Thus, (x_n) is bounded. As such, (x_n) converges by the Monotone Convergence Theorem.

(b) Suppose $\lim x_n = l$. By definition, given any $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|x_n - l| < \epsilon$. Since $x_{n+1} \leq x_n$ for all $n \in \mathbb{N}$, we have $|x_{n+1} - l| \leq |x_n - l| < \epsilon$. Thus, $\lim x_{n+1} = l = \lim_{x_n}$.

(c) Suppose $\lim x_n = \lim x_{n+1} = l$. Then,

$$\begin{aligned} \lim x_{n+1} = \lim \left(\frac{1}{4-x_n} \right) &\implies l = \frac{1}{4-l} \\ &\implies 4l - l^2 = 1 \\ &\implies 0 = l^2 - 4l + 1 \\ &\implies 0 = (l-2)^2 - 3 \\ &\implies 3 = (l-2)^2 \\ &\implies \pm\sqrt{3} = l - 2 \\ &\implies l = 2 \pm \sqrt{3}. \end{aligned}$$

By the Order Limit Theorem (iii), we know that $\lim x_n = l \leq x_n = 3$. Thus, $\lim x_n = 2 - \sqrt{3}$.

Exercise 7 (2.4.8). (a) The sequence of partial sums is defined by

$$\begin{aligned} s_m &= \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^m} \\ &= \frac{2^{m-1} + 2^{m-2} + \cdots + 1}{2^m} \\ &= \frac{2^m - 1}{2^m} \\ &= 1 - \frac{1}{2^m}. \end{aligned}$$

Since the sequence is increasing and bounded above by 1, the sequence of partial sums converges by the Monotone Convergence Theorem. Thus, the series converges.

(b) The sequence of partial sums is defined by

$$\begin{aligned} s_m &= \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n(n+1)} \\ &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= 1 - \frac{1}{n+1}. \end{aligned}$$

Since the sequence of partial sums is increasing and bounded above by 1, the it converges by the Monotone Convergence Theorem. Thus, the series converges.

(c) The sequence of partial sums is defined by

$$\begin{aligned} s_m &= \log\left(\frac{1}{2}\right) + \log\left(\frac{2}{3}\right) + \log\left(\frac{3}{4}\right) + \cdots + \log\left(\frac{n+1}{n}\right) \\ &= \log\left(\frac{1 \times 2 \times 3 \times \cdots \times n+1}{2 \times 3 \times 4 \times \cdots \times n}\right) \\ &= \log\left(\frac{n+1}{1}\right) \\ &= \log(n+1). \end{aligned}$$

Since the sequence of partial sums is unbounded, it diverges.

Exercise 8 (2.4.9). Let (s'_m) be the sequence of partial sums for the series $\sum_{n=0}^{\infty} 2^n b_{2^n}$. Since $\sum_{n=0}^{\infty} 2^n b_{2^n}$ diverges, (s'_m) diverges by definition. Furthermore, we know that (s'_m) is increasing since $b_n \geq 0$ for all $n \in \mathbb{N}$. By the Monotone Convergence Theorem, we thus know that (s'_m) is not bounded above. Let (s_m) be the sequence of partial sums for the series $\sum_{n=1}^{\infty} b_n$. We aim to show that (s_m) is also not bounded above. We show that $2s_{2^m} \geq s'_m$.

$$\begin{aligned} 2s_{2^m} &= 2b_1 + 2b_2 + 2b_3 + 2b_4 + \cdots + 2b_{2^{m-1}+1} + 2b_{2^{m-1}+2} \cdots + 2b_{2^m} \\ &\geq 2b_1 + 2b_2 + (2b_4 + 2b_4) + \cdots + (2b_{2^m} + 2b_{2^m} + \cdots + 2b_{2^m}) \\ &= 2b_1 + 2b_2 + 4b_4 + 8b_8 + \cdots + 2^m b_{2^m} \\ &\geq b_1 + 2b_2 + 4b_4 + 8b_8 + \cdots + 2^m b_{2^m} \\ &= s'_m. \end{aligned}$$

Since (s'_m) is not bounded from above, we know that (s_m) is also not bounded from above. Thus, (s_m) diverges and so does $\sum_{n=1}^{\infty} b_n$.