MATH 355: HOMEWORK 8

ALEXANDER LEE

Exercise 1 (4.2.10). (a) (Right-hand limit) Let $f : A \to \mathbb{R}$, and let a be a limit point of the domain A. We say that $\lim_{x\to a^+} f(x) = L$ provided that, for all $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $0 < x - a < \delta$ (and $x \in A$) it follows that $|f(x) - L| < \epsilon$.

(Left-hand limit) Let $f : A \to \mathbb{R}$, and let a be a limit point of the domain A. We say that $\lim_{x\to a^-} f(x) = M$ provided that, for all $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $0 < a - x < \delta$ (and $x \in A$) it follows that $|f(x) - M| < \epsilon$.

(b) ($\Longrightarrow$) Suppose that $\lim_{x\to a} f(x) = L$. By the definition of a functional limit, for all $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $0 < |x - c| < \delta$ (and $x \in A$) it follows that $|f(x) - L| < \epsilon$. Thus, for this chosen δ, we have that $0 < x-c < \delta$ (and $x \in A$) implies $|f(x) - L| < \epsilon$, and $0 < c-x < \delta$ (and $x \in A$) implies $|f(x) - L| < \epsilon$. Therefore, $\lim_{x\to c^+} f(x) = \lim_{x\to c^-} f(x) = L$ (i.e., both the right and left-hand limits equal L).

($\Longleftarrow$) Suppose $\lim_{x\to c^+} f(x) = \lim_{x\to c^-} f(x) = L$. Since we have that $\lim_{x\to c^+} f(x) = L$, for all $\epsilon > 0$, there exists a $\delta_1 > 0$ such that $0 < x - c < \delta_1$ (and $x \in A$) implies $|f(x) - L| < \epsilon$. Similarly, since we have that $\lim_{x\to c^-} f(x) = L$, for all $\epsilon > 0$, there exists a $\delta_2 > 0$ such that $0 < c - x < \delta_2$ (and $x \in A$) implies $|f(x) - L| < \epsilon$. Let $\delta = \min\{\delta_1, \delta_2\}$. Thus, for all $\epsilon > 0$, we have that $0 < x - c < \delta \le \delta_1$ (and $x \in A$) implies $|f(x) - L| < \epsilon$ and $0 < c - x < \delta \le \delta_2$ (and $x \in A$) implies $|f(x) - L| < \epsilon$. It follows immediately that $0 < |x - c| < \delta$ (and $x \in A$) implies $|f(x) - L| < \epsilon$. Therefore, $\lim_{x\to a} f(x) = L$.

Exercise 2 (4.2.11). Since $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} h(x) = L$, by the Sequential Criterion for Functional Limits, we know that for all sequences $(x_n) \subseteq A$ satisfying $x_n \ne c$ and $\lim x_n = c$, it follows that $\lim f(x_n) = L$ and $\lim h(x_n) = L$. By assumption, we have $f(x_n) \le g(x_n) \le h(x_n)$. Applying the Squeeze Theorem for sequences, it follows that $\lim g(x_n) = L$, implying that $\lim_{x\to c} g(x) = L$ as well.

Exercise 3 (4.3.2). (a) Consider the function $f(x) = k$, for some $k \in \mathbb{R}$.

(b) Consider the function $f(x) = x$.

(c) Consider the function $f(x) = 2x$.

(d) Every lesstinuous function is continuous, since choosing $0 < \delta < \epsilon$ implies choosing a $\delta > 0$. Every continuous function is also lesstinuous, since if we choose a δ, where $0 < \epsilon \le \delta$, which satisfies the definition for continuity, we can choose a δ' such that $0 < \delta' < \epsilon \le \delta$.

Exercise 4 (4.3.5). Suppose c is an isolated point of $A \subseteq \mathbb{R}$. Let $\epsilon > 0$ be arbitrary. Since c is an isolated point, there exists a δ-neighborhood $V_\delta(c)$ of c that only intersects A at c. That is, there exists a $\delta > 0$ such that the only $x \in A$ where

$|x - c| < \delta$ is $x = c$. Thus, with the chosen δ, whenever $|x - c| < \delta$, it follows that $|f(x) - f(c)| = |f(c) - f(c)| = 0 < \epsilon$.

Exercise 5 (4.3.6). (a) Consider functions

$$f(x) = \begin{cases} x+1 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases},$$

neither of which is continuous at 0. We have that $f(x)g(x) = 0$ and $f(x) + g(x) = x + 1$, both of which are continuous at 0.

(b) Impossible. Given that $f(x)$ and $f(x) + g(x)$ are continuous at 0, $g(x) = [f(x)+g(x)] - f(x)$ must also be continuous at 0 by the Algebraic Continuity Theorem.

(c) Consider functions

$$f(x) = x \quad \text{and} \quad g(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}.$$

Observe that $f(x)$ is continuous at 0 and $g(x)$ is not continuous at 0. However, we have that $f(x)g(x) = 0$, which is continuous at 0.

(d) Consider the function

$$f(x) = \begin{cases} 2+\sqrt{3} & \text{if } x \neq 0 \\ 2-\sqrt{3} & \text{if } x = 0 \end{cases},$$

which is not continuous at 0. When $x \neq 0$, it holds that

$$\begin{aligned} f(x) + \frac{1}{f(x)} &= 2 + \sqrt{3} + \frac{1}{2+\sqrt{3}} \\ &= 2 + \sqrt{3} + \frac{1}{2+\sqrt{3}} \cdot \frac{2-\sqrt{3}}{2-\sqrt{3}} \\ &= 2 + \sqrt{3} + 2 - \sqrt{3} \\ &= 4. \end{aligned}$$

Similarly, when $x = 0$, it holds that

$$\begin{aligned} f(x) + \frac{1}{f(x)} &= 2 - \sqrt{3} + \frac{1}{2-\sqrt{3}} \\ &= 2 - \sqrt{3} + \frac{1}{2-\sqrt{3}} \cdot \frac{2+\sqrt{3}}{2+\sqrt{3}} \\ &= 2 - \sqrt{3} + 2 + \sqrt{3} \\ &= 4. \end{aligned}$$

Therefore, we have that $f(x) + \frac{1}{f(x)} = 4$, which is continuous at 0.

(e) TODO

Exercise 6 (4.3.8). (a) True. Observe that g is continuous at 1. Let $x_n = 1 - \frac{1}{n}$. Clearly, $(x_n) \to 1$. By the Characterizations of Continuity, it follows that $g(x_n) \to g(1)$. Since $g(x_n) \geq 0$ for all $n \in \mathbb{N}$, we also have $g(1) \geq 0$ by the Order Limit Theorem.

(b) True. Given $x \in \mathbb{R}$, there exists a sequence of rational numbers (r_n) such that $(r_n) \to x$, by the Density of $\mathbb{Q}$ in $\mathbb{R}$. Since g is continuous at x, the Characterizations of Continuity says that $g(r_n) \to g(x)$. However, $g(r_n) = 0$ for all $n \in \mathbb{N}$. Therefore, $g(x) = 0$.

(c) True. Let $c = g(x_0) > 0$. Since g is continuous at x_0, there exists a $\delta > 0$ such that $|x - x_0| < \delta$ implies $|g(x) - g(x_0)| < c$. Thus, whenever $|x - x_0| < \delta$, it follows that

$$\begin{aligned} |g(x) - g(x_0) < c| &\implies -c < g(x) - g(x_0) < c \\ &\implies -c < g(x) - c < c \\ &\implies 0 < g(x) < 2c \end{aligned}$$

Thus, for all $x \in V_\delta(x_0)$, we have that $g(x) > 0$. Observe that there are uncountably many points in $V_\delta(x_0)$.

Exercise 7 (4.4.11)**.** ($\Longrightarrow$) Given $c \in g^{-1}(O)$, it follows that $g(c) \in O$. Because O is open, there exists a $\epsilon > 0$ such that $V_\epsilon(g(c)) \subseteq O$. Given this ϵ and the fact that g is continuous at c, we know there exists a neighborhood $V_\delta(c)$ with the property that $x \in V_\delta(c)$ implies $g(x) \in V_\epsilon(f(c)) \subseteq O$. This implies that $V_\delta(c) \subseteq g^{-1}(O)$, proving that $g^{-1}(O)$ is open.

($\Longleftarrow$) Let $c \in \mathbb{R}$, $\epsilon > 0$, and set $O = V_\epsilon(g(c))$. Since O is open, $g^{-1}(O)$ is open by assumption. Because $c \in g^{-1}(O)$, it follows that there exists a neighborhood $V_\delta(c) \subseteq g^{-1}(O)$. That is, for every $x \in V_\delta(c)$, we have that $g(x) \in O = V_\epsilon(g(c))$. Therefore g is continuous.

Exercise 8 (4.4.12)**.** (a) False. Consider $B = \{1\}$ and $f : \mathbb{R} \to B$, where $f(x) = 1$. In this case, B is finite but $f^{-1}(B) = \mathbb{R}$ is not finite.

(b) False. Consider $K = \{1\}$ and $f : \mathbb{R} \to K$, where $f(x) = 1$. In this case, K is compact but $f^{-1}(K) = \mathbb{R}$ is not compact.

(c) False. Consider $A = \{1\}$ and $f : \mathbb{R} to A$, where $f(x) = 1$. In this case, A is bounded, but $f^{-1}(A) = \mathbb{R}$ is not bounded.

(d) True. We want to show that $f^{-1}(F)$ is closed whenever F is closed. To do so, we show that $f^{-1}(F)^c$ is open whenever F is closed. First, we show that $f^{-1}(F)^c = f^{-1}(F^c)$. Consider

$$\begin{aligned} a \in f^{-1}(F)^c &\iff a \notin f^{-1}(F) \\ &\iff f(a) \notin F \\ &\iff f(a) \in F^c \\ &\iff a \in f^{-1}(F^c). \end{aligned}$$

Therefore, $f^{-1}(F)^c = f^{-1}(F^c)$. Since F is assumed to be closed, we know that F^c is open. Thus, By Exercise 4.4.11, $f^{-1}(F)^c = f^{-1}(F^c)$ is also open since f is continuous.