

## MATH 355: HOMEWORK 7

ALEXANDER LEE

**Exercise 1** (3.3.1). Since  $K$  is compact, it is closed and bounded. It follows from the Axiom of Completeness that because  $K$  is nonempty and bounded (above and below),  $\sup(K)$  and  $\inf(K)$  both exist. Since  $K$  is closed,  $K = \overline{K}$ , so  $\sup(K) \in \overline{K} = K$  by Exercise 3.2.4 (a). By similar reasoning, it follows that  $\inf(K) \in K$ .

**Exercise 2** (3.3.2, except (c)). (a) Not compact. Consider the sequence  $(a_n)$  where  $a_n = n$ .

- (b) Not compact. We can construct a sequence  $(a_n)$  that converges to  $1/\sqrt{2} \notin \mathbb{Q} \cap [0, 1]$ . All subsequences of  $(a_n)$  must also converge to  $1/\sqrt{2}$ .
- (d) Not compact. Consider the sequence of partial sums  $(s_m)$  where  $s_m = \sum_{n=1}^m 1/n^2$ .
- (e) Compact. The set is closed since 1 is its only limit point, which is contained in the set, and the set is also bounded.

**Exercise 3** (3.3.4). (a) The set is definitely closed since  $K$  is compact and thus closed, and the intersection of an arbitrary collection of closed sets is closed. The set is definitely compact since  $K \cap F$  is closed as explained previously and is bounded because  $K$  is bounded.

- (b) The set is definitely closed since the closure of any set is closed. The set is not definitely compact.
- (c) Neither.
- (d) The set is definitely closed since the closure of any set is closed. The set is definitely compact since it is closed as explained previously and is bounded since  $K \cap F^c$  is bounded and the closure of a bounded set is also bounded.

**Exercise 4** (3.3.5). (a) True. The arbitrary intersection of compact sets is closed since each compact set is closed and the arbitrary intersection of closed sets is closed. The arbitrary intersection of compact sets is bounded since each compact set is bounded and the intersection of bounded sets must be bounded.

- (b) False. Consider  $\cup_{n=1}^{\infty} [n, n+1]$ . This union of compact sets is unbounded, and thus not compact.
- (c) False. Consider  $A = (0, 1)$  and  $K = [0, 1]$ .  $A \cap K = (0, 1)$  is open, and thus not compact.
- (d) False. Consider  $\cap_{n=1}^{\infty} [n, \infty) = \emptyset$ .

**Exercise 5** (3.3.11).  $\mathbb{N}$ : consider the open cover  $\{V_1(n) : n \in \mathbb{N}\}$ .

$\mathbb{Q} \cap [0, 1]$ : consider the open cover TODO

$\{1 + 1/2^2 + 1/3^2 + \dots + 1/n^2 : n \in \mathbb{N}\}$ : consider the open cover TODO

**Exercise 6** (4.2.2). (a) TODO

- (b) TODO
- (c) TODO

(d) TODO

**Exercise 7 (4.2.5).** (a) Let  $\epsilon > 0$ . Notice that

$$|3x + 4 - 10| = |3x + 6| = 3|x - 2|.$$

Thus, if we choose  $\delta = \epsilon/3$ , then  $0 < |x - 2| < \delta$  implies  $|3x + 4 - 10| < 3(\epsilon/3) = \epsilon$ .

(b) Let  $\epsilon > 0$ . Notice that

$$|x^3 - 0| = |x^3| = x^2|x|.$$

Choose a  $\delta$ -neighborhood around  $c = 0$  to with radius no bigger than  $\delta = 1$ . Then, we get the upper bound  $x^2 \leq 1$  for all  $x \in V_\delta(c)$ . Now, choose  $\delta = \min\{1, \epsilon\}$ . If  $0 < |x - 0| < \delta$ , then it follows that

$$|x^3 - 0| = x^2|x| < 1(\epsilon) = \epsilon.$$

(c) Let  $\epsilon > 0$ . Notice that

$$|x^2 + x - 1 - 5| = |x^2 + x - 6| = |x + 3||x - 2|.$$

Choose a  $\delta$ -neighborhood around  $c = 2$  with radius no bigger than  $\delta = 1$ . Then, we get the upper bound  $|x + 3| \geq |3 + 3| = 6$  for all  $x \in V_\delta(c)$ . Now, choose  $\delta = \min\{1, \epsilon/6\}$ . If  $0 < |x - 2| < \delta$ , then it follows that

$$|x^2 + x - 1 - 5| = |x + 3||x - 2| < 6(\epsilon/6) = \epsilon.$$

(d) Let  $\epsilon > 0$ . Notice that

$$|1/x - 1/3| = \left| \frac{3 - x}{3x} \right| = \frac{|3 - x|}{|3x|}.$$

Choose a  $\delta$ -neighborhood around  $c = 3$  with radius no bigger than  $\delta = 1$ . Then, we get the lower bound  $|3x| \geq |3 \cdot 2| = 6$ . Now, choose  $\delta = \min\{1, 6/\epsilon\}$ . If  $0 < |x - 3| < \delta$ , then it follows that

$$|1/x - 1/3| = \frac{|3 - x|}{|3x|} < \frac{6}{\epsilon} \cdot \frac{1}{6} = \epsilon.$$

**Exercise 8 (4.2.6).** (a) True. If a particular  $\delta$  has been constructed as a suitable response to a particular  $\epsilon$  challenge, then we know that for all  $x \in V_\delta(c)$  different from  $c$  (with  $x \in A$ ) it follows that  $f(x) \in V_\epsilon(L)$ . Therefore, any smaller positive  $\delta$  will also suffice since all  $x$ 's will still be in the  $\delta$ -neighborhood.

(b) True. TODO

(c) True. Given  $\lim_{x \rightarrow a} f(x) = L$ , we have that

$$\lim_{x \rightarrow a} 3[f(x) - 2]^2 = 3[\lim_{x \rightarrow a} f(x) - 2]^2 = 3(L - 2)^2$$

by the Algebraic Limit Theorem for Functional Limits.

(d) False.  $\lim_{x \rightarrow a} g(x)$  may not exist.