

# MATH 355: HOMEWORK 11

ALEXANDER LEE

**Exercise 1 (6.3.1).** (a) We show that  $(g_n)$  converges uniformly on  $[0, 1]$  to  $g = \lim g_n = 0$ . Given  $\epsilon > 0$ , choose  $N \in \mathbb{N}$  such that  $N > 1/\epsilon$ . Then, whenever  $n \geq N$  and  $x \in [0, 1]$ , it follows that

$$|g_n(x) - g(x)| = \left| \frac{x^n}{n} - 0 \right| = \left| \frac{x^n}{n} \right| = \frac{x^n}{n} \leq \frac{1}{n} \leq \frac{1}{N} < \epsilon.$$

$g$  is differentiable since it is the constant function  $g(x) = 0$  for all  $x \in [0, 1]$ . Specifically, we have that  $g'(x) = 0$  for all  $x \in [0, 1]$ .

(b) By the Algebraic Differentiability Theorem, we have that

$$g'_n(x) = \frac{n(nx^{n-1}) - x^n(0)}{n^2} = x^{n-1}.$$

It follows that  $(g'_n)$  converges on  $[0, 1]$  to

$$h(x) = \lim g'_n(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x = 1 \end{cases}.$$

By the contrapositive of the Continuous Limit Theorem, since  $h$  is not continuous and  $g'_n$  is continuous on  $[0, 1]$ , we have that  $(g'_n)$  does not converge uniformly on  $[0, 1]$  to  $h$ . Observe that  $h$  and  $g'$  are not the same.

**Exercise 2 (6.3.2).** (a) The pointwise limit of  $(h_n)$  is  $h(x) = x$ . To show that the convergence is uniform on  $\mathbb{R}$ , let  $\epsilon > 0$  be arbitrary and choose  $N \in \mathbb{N}$  such that  $N > 1/\epsilon^2$ . Then, whenever  $n \geq N$  and  $x \in \mathbb{R}$ , it follows that

$$|h_n(x) - h(x)| = \left| \sqrt{x^2 - \frac{1}{n}} - x \right| \leq \left| \sqrt{x^2} + \sqrt{\frac{1}{n}} - x \right| = \sqrt{\frac{1}{n}} \leq \sqrt{\frac{1}{N}} < \epsilon.$$

(b) By the Chain Rule, we have that

$$h'_n(x) = \frac{1}{2\sqrt{x^2 + \frac{1}{n}}} \cdot 2x = \frac{x}{\sqrt{x^2 + \frac{1}{n}}}.$$

It follows that

$$g(x) = \lim h'_n(x) = \lim \frac{x}{\sqrt{x^2 + \frac{1}{n}}} = \frac{x}{\sqrt{x^2}} = 1.$$

By the contrapositive of the Differentiable Limit Theorem, since  $g(x) = 1 \neq x = h(x)$  and each  $h_n$  is differentiable, it must be that  $h'_n$  does not converge uniformly to  $g(x)$  on  $\mathbb{R}$ , i.e., any neighborhood of zero.

**Exercise 3 (6.4.2).** (a) True. Given that  $\sum_{n=1}^{\infty} g_n$  converges uniformly, it follows from a special case of the Cauchy Criterion for Uniform Convergence of Series that that given  $\epsilon > 0$ , there exists an  $N \in \mathbb{N}$  such that whenever

$m \geq N$  and  $x \in A$ , where  $A$  is the domain of  $g_n$ , we have that  $|g_{m+1}(x)| < \epsilon$ . Therefore,  $(g_n)$  converges uniformly to zero.

- (b) True. Given that  $0 \leq f_n(x) \leq g_n(x)$  and  $\sum_{n=1}^{\infty} g_n$  converges uniformly, it follows from the Cauchy Criterion for Series that given  $\epsilon > 0$ , there exists an  $N \in \mathbb{N}$  such that whenever  $n > m \geq N$ , we have that

$$|f_{m+1}(x) + f_{m+2}(x) + \cdots + f_n(x)| \leq |g_{m+1}(x) + g_{m+2}(x) + \cdots + g_n(x)| < \epsilon.$$

Therefore,  $\sum_{n=1}^{\infty} f_n$  converges uniformly as well.

- (c) False. TODO

**Exercise 4** (6.4.4). TODO

**Exercise 5** (6.4.5a). Observe that each  $\frac{x^n}{n^2}$  is continuous on  $[-1, 1]$ . Also note that for each  $n \in \mathbb{N}$ ,  $|\frac{x^n}{n^2}| \leq \frac{1}{n^2}$  for all  $x \in [-1, 1]$ . Since  $\sum_{n=1}^{\infty} \frac{1}{n^2}$  converges, it follows from the Weierstrass M-Test that  $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$  converges uniformly on  $[-1, 1]$ . Therefore,  $h(x)$  is continuous on  $[-1, 1]$  by the Term-by-term Continuity Theorem.

**Exercise 6** (6.5.1).

**Exercise 7** (6.5.2).

**Exercise 8** (6.5.4).