

## MATH 355: NOTES

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### 1 THE REAL NUMBERS

#### 1.2 Some Preliminaries.

**Theorem** (Triangle Inequality). *For all choices of  $a$  and  $b$ ,  $|a + b| \leq |a| + |b|$ .*

**Theorem.** *Two real numbers  $a$  and  $b$  are equal if and only if for every real number  $\epsilon > 0$  it follows that  $|a - b| < \epsilon$ .*

#### 1.3 The Axiom of Completeness.

**Axiom** (Axiom of Completeness). *Every nonempty set of real numbers that is bounded above has a least upper bound.*

**Definition.** A set  $A \subseteq \mathbb{R}$  is *bounded above* if there exists a number  $b \in \mathbb{R}$  such that  $a \leq b$  for all  $a \in A$ . The number  $b$  is called an *upper bound* for  $A$ .

Similarly, the set  $A$  is *bounded below* if there exists a *lower bound*  $l \in \mathbb{R}$  satisfying  $l \leq a$  for every  $a \in A$ .

**Definition.** A real number  $s$  is the *least upper bound* for a set  $A \subseteq \mathbb{R}$  if it meets the following two criteria:

- (i)  $s$  is an upper bound for  $A$ ;
- (ii) if  $b$  is any upper bound for  $A$ , then  $s \leq b$ .

The least upper bound is also frequently called the *supremum* of the set  $A$ . We write  $s = \sup(A)$  for the least upper bound.

The *greatest lower bound* or *infimum* for  $A$  is defined in a similar way and is denoted by  $\inf(A)$ .

**Theorem.** *Let  $A \subseteq \mathbb{R}$  be bounded above and below. Then, the  $\sup(A)$  and  $\inf(A)$  are unique.*

**Definition.** A real number  $a_0$  is a *maximum* of the set  $A$  if  $a_0$  is an element of  $A$  and  $a_0 \geq a$  for all  $a \in A$ . Similarly, a number  $a_1$  is a *minimum* of  $A$  if  $a_1 \in A$  and  $a_1 \leq a$  for every  $a \in A$ .

**Theorem.** *Let  $A \subseteq \mathbb{R}$  be nonempty and bounded above.*

- (i) *Let  $c \in \mathbb{R}$  and define the set  $c + A$  by*

$$c + A = \{c + a : a \in A\}.$$

*Then  $\sup(c + A) = c + \sup(A)$ .*

- (ii) *Let  $c \in \mathbb{R}$  with  $c > 0$  and define the set  $cA$  by*

$$cA = \{ca : a \in A\}.$$

*Then  $\sup(cA) = c \sup(A)$ .*

**Lemma.** Assume  $s \in \mathbb{R}$  is an upper bound for a set  $A \subseteq \mathbb{R}$ . Then,  $s = \sup(A)$  if and only if, for every choice of  $\epsilon > 0$ , there exists an element  $a \in A$  satisfying  $s - \epsilon < a$ .

#### 1.4 Consequences of Completeness.

**Theorem** (Nested Interval Property). For each  $n \in \mathbb{N}$ , assume we are given a closed interval  $I_n = [a_n, b_n] = \{x \in \mathbb{R} : a_n \leq x \leq b_n\}$ . Assume also that each  $I_n$  contains  $I_{n+1}$ . Then, the resulting nested sequence of closed intervals

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq I_4 \supseteq \cdots$$

has a nonempty intersection; that is,  $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$ .

**Theorem** (Archimedean Property). (i) Given any number  $x \in \mathbb{R}$ , there exists an  $n \in \mathbb{N}$  satisfying  $n > x$ .

(ii) Given any real number  $y > 0$ , there exists an  $n \in \mathbb{N}$  satisfying  $1/n < y$ .

**Definition.** A set  $X$  is dense in  $\mathbb{R}$  if for any  $a, b \in \mathbb{R}$  with  $a < b$ ,  $\exists x \in X$  with  $a < x < b$ .

**Theorem** (Density of  $\mathbb{Q}$  in  $\mathbb{R}$ ). For every two real numbers  $a$  and  $b$  with  $a < b$ , there exists a rational number  $r$  satisfying  $a < r < b$ .

#### 1.5 Cardinality.

**Definition.** A function  $f : A \rightarrow B$  is 1-1 (injective) if for all  $a_1, a_2 \in A$ ,  $f(a_1) = f(a_2)$  implies that  $a_1 = a_2$ .

**Definition.** A function  $f : A \rightarrow B$  is onto (surjective) if for all  $b \in B$ , there exists an  $a \in A$  such that  $f(a) = b$ .

**Definition.** A function  $f : A \rightarrow B$  is a bijection if it is both 1-1 and onto.

**Definition.** Two sets  $A$  and  $B$  have the same cardinality if there exists a bijection  $f : A \rightarrow B$ . In this case, we write  $A \sim B$ .

**Definition.** A set  $A$  is finite if there exists an  $n \in \mathbb{N}$  such that  $A \sim \{1, 2, \dots, n\}$ .

**Definition.** A set  $A$  is countable if  $A \sim \mathbb{N}$ .

**Definition.** A set which is not finite nor countable is uncountable.

**Theorem.** (i) The set  $\mathbb{Q}$  is countable.

(ii) The set  $\mathbb{R}$  is uncountable.

**Theorem.** If  $A \subseteq B$  is countable, then  $A$  is either countable or finite.

**Theorem.** (i) If  $A_1, A_2, \dots, A_m$  are each countable sets, then the union  $A_1 \cup A_2 \cup \dots \cup A_m$  is countable.

(ii) If  $A_n$  is a countable set for each  $n \in \mathbb{N}$ , then  $\bigcup_{n=1}^{\infty} A_n$  is countable.

#### 1.6 Cantor's Theorem.

**Theorem.** The open interval  $(0, 1) = \{x \in \mathbb{R} : 0 < x < 1\}$  is uncountable.

**Definition.** Given a set  $A$ , the power set  $P(A)$  refers to the collection of all subsets of  $A$ .

**Theorem** (Cantor's Theorem). Given any set  $A$ , there does not exist a function  $f : A \rightarrow P(A)$  that is onto.

## 2 SEQUENCES AND SERIES

## 2.2 The Limit of a Sequence.

**Definition.** A *sequence* is a function whose domain is  $\mathbb{N}$ .

**Definition** (Convergence of a Sequence). A sequence  $(a_n)$  *converges* to a real number  $a$  if, for every positive number  $\epsilon$ , there exists and  $N \in \mathbb{N}$  such that whenever  $n \geq N$ , it follows that  $|a_n - a| < \epsilon$ .

**Definition.** Given a real number  $a \in \mathbb{R}$  and a positive number  $\epsilon > 0$ , the set

$$V_\epsilon(a) = \{x \in \mathbb{R} : |x - a| < \epsilon\} = (a - \epsilon, a + \epsilon)$$

is called the  $\epsilon$ -neighborhood of  $a$ .

**Definition** (Convergence of a Sequence: Topological Version). A sequence  $(a_n)$  converges to  $a$  if, given any  $\epsilon$ -neighborhood  $V_\epsilon(a)$  of  $a$ , there exists a point in the sequence after which all of the terms are in  $V_\epsilon(a)$ . In other words, every  $\epsilon$ -neighborhood contains all but a finite number of terms of  $(a_n)$ .

**Theorem** (Uniqueness of Limits). *The limit of a sequence, when it exists, must be unique.*

**Definition.** A sequence that does not converge is said to *diverge*.

## 2.3 The Algebraic and Order Limit Theorems.

**Definition.** A sequence  $(x_n)$  is *bounded* if there exists a number  $M > 0$  such that  $|x_n| \leq M$  for all  $n \in \mathbb{N}$ .

**Theorem.** *Every convergent sequence is bounded.*

**Theorem** (Algebraic Limit Theorem). *Let  $\lim a_n = a$  and  $\lim b_n = b$ . Then,*

- (i)  $\lim(ca_n) = ca$ , for all  $c \in \mathbb{R}$ ;
- (ii)  $\lim(a_n + b_n) = a + b$ ;
- (iii)  $\lim(a_nb_n) = ab$ ;
- (iv)  $\lim(a_n/b_n) = a/b$ , provided  $b \neq 0$ .

**Theorem** (Order Limit Theorem). *Assume  $\lim a_n = a$  and  $\lim b_n = b$ .*

- (i) *If  $a_n \geq 0$  for all  $n \in \mathbb{N}$ , then  $a \geq 0$ .*
- (ii) *If  $a_n \leq b_n$  for all  $n \in \mathbb{N}$ , then  $a \leq b$ .*
- (iii) *If there exists  $c \in \mathbb{R}$  for which  $c \leq b_n$  for all  $n \in \mathbb{N}$ , then  $c \leq b$ . Similarly, if  $a_n \leq c$  for all  $n \in \mathbb{N}$ , then  $a \leq c$ .*

## 2.4 The Monotone Convergence Theorem and a First Look at Infinite Series.

**Definition.** A sequence  $(a_n)$  is *increasing* if  $a_n \leq a_{n+1}$  for all  $n \in \mathbb{N}$  and *decreasing* if  $a_n \geq a_{n+1}$  for all  $n \in \mathbb{N}$ . A sequence is *monotone* if it is either increasing or decreasing.

**Theorem** (Monotone Convergence Theorem). *If a sequence is monotone and bounded, then it converges.*

**Definition** (Convergence of a Series). Let  $(b_n)$  be a sequence. An *infinite series* is a formal expression of the form

$$\sum_{n=1}^{\infty} b_n = b_1 + b_2 + b_3 + b_4 + b_5 + \cdots$$

We define the corresponding *sequence of partial sums*  $(s_m)$  by

$$s_m = b_1 + b_2 + b_3 + \cdots + b_m,$$

and say that the series  $\sum_{n=1}^{\infty} b_n$  *converges to*  $B$  if the sequence  $(s_m)$  converges to  $B$ . In this case, we write  $\sum_{n=1}^{\infty} b_n = B$ .

**Theorem** (Cauchy Condensation Test). *Suppose  $(b_n)$  is decreasing and satisfies  $b_n \geq 0$  for all  $n \in \mathbb{N}$ . Then, the series  $\sum_{n=1}^{\infty} b_n$  converges if and only if the series*

$$\sum_{n=0}^{\infty} 2^n b_{2^n} = b_1 + 2b_2 + 4b_4 + 8b_8 + 16b_{16} + \cdots$$

*converges.*

**Corollary.** *The series  $\sum_{n=1}^{\infty} 1/n^p$  converges if and only if  $p > 1$ .*

## 2.5 Subsequences and the Bolzano-Weierstrass Theorem.

**Definition.** Let  $(a_n)$  be a sequence of real numbers, and let  $n_1 < n_2 < n_3 < n_4 < n_5 < \cdots$  be an increasing sequence of natural numbers. Then the sequence

$$(a_{n_1}, a_{n_2}, a_{n_3}, a_{n_4}, a_{n_5}, \dots)$$

is called a *subsequence* of  $(a_n)$  and is denoted by  $(a_{n_k})$ , where  $k \in \mathbb{N}$  indexes the subsequence.

**Theorem.** *Subsequences of a convergent sequence converge to the same limit as the original sequence.*

**Corollary** (Divergence Criterion). *Suppose that  $(a_n)$  is a sequence and  $(a_{n_k})$  is a subsequence that diverges, then  $(a_n)$  diverges. If  $(a_{n_k}^1)$  and  $(a_{n_k}^2)$  converge to  $a^1$  and  $a^2$  with  $a^1 \neq a^2$ , then  $(a_n)$  diverges.*

**Theorem** (Bolzano-Weierstrass Theorem). *Every bounded sequence contains a convergent subsequence.*

## 2.6 The Cauchy Criterion.

**Definition.** A sequence  $(a_n)$  is called a *Cauchy sequence* if, for every  $\epsilon > 0$ , there exists an  $N \in \mathbb{N}$  such that whenever  $m, n \geq N$  it follows that  $|a_n - a_m| < \epsilon$ .

**Theorem.** *Every convergence sequence is a Cauchy sequence.*

**Lemma.** *Cauchy sequences are bounded.*

**Theorem** (Cauchy Criterion). *A sequence converges if and only if it is a Cauchy sequence.*

## 2.7 Properties of Infinite Series.

**Theorem** (Algebraic Limit Theorem for Series). *If  $\sum_{k=1}^{\infty} a_k = A$  and  $\sum_{k=1}^{\infty} b_k = B$ , then*

- (i)  $\sum_{k=1}^{\infty} ca_k = cA$  for all  $c \in \mathbb{R}$  and
- (ii)  $\sum_{k=1}^{\infty} (a_k + b_k) = A + B$ .

**Theorem** (Cauchy Criterion for Series). *The series  $\sum_{k=1}^{\infty} a_k$  converges if and only if, given  $\epsilon > 0$ , there exists an  $N \in \mathbb{N}$  such that whenever  $n > m \geq N$  it follows that*

$$|a_{m+1} + a_{m+2} + \cdots + a_n| < \epsilon.$$

**Theorem** (Divergence Test). *If the series  $\sum_{k=1}^{\infty} a_k$  converges, then  $(a_k) \rightarrow 0$ . Equivalently, if  $(a_k) \not\rightarrow 0$ , then  $\sum_{k=1}^{\infty} a_k$  diverges.*

**Theorem** (Comparison Test). *Assume  $(a_k)$  and  $(b_k)$  are sequences satisfying  $0 \leq a_k \leq b_k$  for all  $k \in \mathbb{N}$ .*

- (i) *If  $\sum_{k=1}^{\infty} b_k$  converges, then  $\sum_{k=1}^{\infty} a_k$  converges.*
- (ii) *If  $\sum_{k=1}^{\infty} a_k$  diverges, then  $\sum_{k=1}^{\infty} b_k$  diverges.*

**Theorem** (Squeeze Theorem). *Suppose  $a_n \leq b_n \leq c_n$  for all  $n \in \mathbb{N}$ , and if  $\lim a_n = \lim c_n = l$ , then  $\lim b_n = l$  as well.*

**Definition** (Geometric Series). A series is called *geometric* if it is of the form

$$\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + ar^3 + \cdots.$$

**Theorem.**  $\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$  if and only if  $|r| < 1$ .

**Theorem** (Absolute Convergence Test). *If the series  $\sum_{n=1}^{\infty} |a_n|$  converges, then  $\sum_{n=1}^{\infty} a_n$  converges as well.*

**Theorem** (Alternating Series Test). *Let  $(a_n)$  be a sequence satisfying,*

- (i)  $a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_n \geq a_{n+1} \geq \cdots$  and
- (ii)  $(a_n) \rightarrow 0$ .

*Then, the alternating series  $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$  converges.*

**Definition.** If  $\sum_{n=1}^{\infty} |a_n|$  converges, then we say that the original series  $\sum_{n=1}^{\infty} a_n$  *converges absolutely*. If, on the other hand, the series  $\sum_{n=1}^{\infty} a_n$  converges but the series of absolute values  $\sum_{n=1}^{\infty} |a_n|$  does not converge, then we say that the original series  $\sum_{n=1}^{\infty} a_n$  *converges conditionally*.

## 3 BASIC TOPOLOGY OF $\mathbb{R}$

### 3.2 Open and Closed Sets.

**Definition.** A set  $O \subseteq \mathbb{R}$  is *open* if for all points  $a \in O$  there exists an  $\epsilon$ -neighborhood  $V_{\epsilon}(a) \subseteq O$ .

**Theorem.** (i) *The union of an arbitrary collection of open sets is open.*  
(ii) *The intersection of a finite collection of open sets is open.*

**Definition.** A point  $x$  is a *limit point* of a set  $A$  if every  $\epsilon$ -neighborhood  $V_{\epsilon}(x)$  of  $x$  intersects the set  $A$  at some point other than  $x$ .

**Theorem.** *A point  $x$  is a limit point of a set  $A$  if and only if  $x = \lim a_n$  for some sequence  $(a_n)$  contained in  $A$  satisfying  $a_n \neq x$  for all  $n \in \mathbb{N}$ .*

**Definition.** A point  $a \in A$  is an *isolated point* of  $A$  if it is not a limit point of  $A$ .

**Definition.** A set  $F \subseteq \mathbb{R}$  is *closed* if it contains its limit points.

**Theorem.** *A set  $F \subseteq \mathbb{R}$  is closed if and only if every Cauchy sequence contained in  $F$  has a limit that is also an element of  $F$ .*

**Definition.** Given a set  $A \subseteq \mathbb{R}$ , let  $L$  be the set of all limit points of  $A$ . The *closure* of  $A$  is defined to be  $\bar{A} = A \cup L$ .

**Theorem.** *For any  $A \subseteq \mathbb{R}$ , the closure  $\bar{A}$  is a closed set and is the smallest closed set containing  $A$ .*

**Theorem.** *A set  $O$  is open if and only if  $O^c$  is closed. Likewise, a set  $F$  is closed if and only if  $F^c$  is open.*

**Theorem.** (i) *The union of a finite collection of closed sets is closed.*  
(ii) *The intersection of an arbitrary collection of closed sets is closed.*

### 3.3 Compact Sets.

**Definition** (Compactness). A set  $K \subseteq \mathbb{R}$  is *compact* if every sequence in  $K$  has a subsequence that converges to a limit that is also in  $K$ .

**Definition.** A set  $A \subseteq \mathbb{R}$  is *bounded* if there exists  $M > 0$  such that  $|a| \leq M$  for all  $a \in A$ .

**Theorem** (Characterization of Compactness in  $\mathbb{R}$ ). *A set  $K \subseteq \mathbb{R}$  is compact if and only if it is closed and bounded.*

**Theorem** (Nested Compact Set Property). *If*

$$K_1 \supseteq K_2 \supseteq K_3 \supseteq K_4 \supseteq \cdots$$

*is a nested sequence of nonempty compact sets, then the intersection  $\bigcap_{n=1}^{\infty} K_n$  is not empty.*

**Definition.** Let  $A \subseteq \mathbb{R}$ . An *open cover* for  $A$  is a (possibly infinite) collection of open sets  $\{O_\lambda : \lambda \in \Lambda\}$  whose union contains the set  $A$ ; that is,  $A \subseteq \bigcup_{\lambda \in \Lambda} O_\lambda$ . Given an open cover for  $A$ , a *finite subcover* is a finite subcollection of open sets from the original open cover whose union still manages to completely contain  $A$ .

**Theorem** (Heine-Borel Theorem). *Let  $K$  be a subset of  $\mathbb{R}$ . All of the following statements are equivalent in the sense that any one of them implies the two others:*

- (i)  *$K$  is compact.*
- (ii)  *$K$  is closed and bounded.*
- (iii) *Every open cover for  $K$  has a finite subcover.*