

## MATH 355: NOTES

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### 1 THE REAL NUMBERS

#### 1.2 Some Preliminaries.

**Theorem** (Triangle Inequality). *For all choices of  $a$  and  $b$ ,  $|a + b| \leq |a| + |b|$ .*

**Theorem.** *Two real numbers  $a$  and  $b$  are equal if and only if for every real number  $\epsilon > 0$  it follows that  $|a - b| < \epsilon$ .*

#### 1.3 The Axiom of Completeness.

**Axiom** (Axiom of Completeness). *Every nonempty set of real numbers that is bounded above has a least upper bound.*

**Definition.** A set  $A \subseteq \mathbb{R}$  is *bounded above* if there exists a number  $b \in \mathbb{R}$  such that  $a \leq b$  for all  $a \in A$ . The number  $b$  is called an *upper bound* for  $A$ .

Similarly, the set  $A$  is *bounded below* if there exists a *lower bound*  $l \in \mathbb{R}$  satisfying  $l \leq a$  for every  $a \in A$ .

**Definition.** A real number  $s$  is the *least upper bound* for a set  $A \subseteq \mathbb{R}$  if it meets the following two criteria:

- (i)  $s$  is an upper bound for  $A$ ;
- (ii) if  $b$  is any upper bound for  $A$ , then  $s \leq b$ .

The least upper bound is also frequently called the *supremum* of the set  $A$ . We write  $s = \sup(A)$  for the least upper bound.

The *greatest lower bound* or *infimum* for  $A$  is defined in a similar way and is denoted by  $\inf(A)$ .

**Theorem.** *Let  $A \subseteq \mathbb{R}$  be bounded above and below. Then, the  $\sup(A)$  and  $\inf(A)$  are unique.*

**Definition.** A real number  $a_0$  is a *maximum* of the set  $A$  if  $a_0$  is an element of  $A$  and  $a_0 \geq a$  for all  $a \in A$ . Similarly, a number  $a_1$  is a *minimum* of  $A$  if  $a_1 \in A$  and  $a_1 \leq a$  for every  $a \in A$ .

**Theorem.** *Let  $A \subseteq \mathbb{R}$  be nonempty and bounded above.*

- (i) *Let  $c \in \mathbb{R}$  and define the set  $c + A$  by*

$$c + A = \{c + a : a \in A\}.$$

*Then  $\sup(c + A) = c + \sup(A)$ .*

- (ii) *Let  $c \in \mathbb{R}$  with  $c > 0$  and define the set  $cA$  by*

$$cA = \{ca : a \in A\}.$$

*Then  $\sup(cA) = c \sup(A)$ .*

**Lemma.** Assume  $s \in \mathbb{R}$  is an upper bound for a set  $A \subseteq \mathbb{R}$ . Then,  $s = \sup(A)$  if and only if, for every choice of  $\epsilon > 0$ , there exists an element  $a \in A$  satisfying  $s - \epsilon < a$ .

#### 1.4 Consequences of Completeness.

**Theorem** (Nested Interval Property). For each  $n \in \mathbb{N}$ , assume we are given a closed interval  $I_n = [a_n, b_n] = \{x \in \mathbb{R} : a_n \leq x \leq b_n\}$ . Assume also that each  $I_n$  contains  $I_{n+1}$ . Then, the resulting nested sequence of closed intervals

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq I_4 \supseteq \cdots$$

has a nonempty intersection; that is,  $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$ .

**Theorem** (Archimedean Property). (i) Given any number  $x \in \mathbb{R}$ , there exists an  $n \in \mathbb{N}$  satisfying  $n > x$ .

(ii) Given any real number  $y > 0$ , there exists an  $n \in \mathbb{N}$  satisfying  $1/n < y$ .

**Definition.** A set  $X$  is dense in  $\mathbb{R}$  if for any  $a, b \in \mathbb{R}$  with  $a < b$ ,  $\exists x \in X$  with  $a < x < b$ .

**Theorem** (Density of  $\mathbb{Q}$  in  $\mathbb{R}$ ). For every two real numbers  $a$  and  $b$  with  $a < b$ , there exists a rational number  $r$  satisfying  $a < r < b$ .

#### 1.5 Cardinality.

**Definition.** A function  $f : A \rightarrow B$  is 1-1 (injective) if for all  $a_1, a_2 \in A$ ,  $f(a_1) = f(a_2)$  implies that  $a_1 = a_2$ .

**Definition.** A function  $f : A \rightarrow B$  is onto (surjective) if for all  $b \in B$ , there exists an  $a \in A$  such that  $f(a) = b$ .

**Definition.** A function  $f : A \rightarrow B$  is a bijection if it is both 1-1 and onto.

**Definition.** Two sets  $A$  and  $B$  have the same cardinality if there exists a bijection  $f : A \rightarrow B$ . In this case, we write  $A \sim B$ .

**Definition.** A set  $A$  is finite if there exists an  $n \in \mathbb{N}$  such that  $A \sim \{1, 2, \dots, n\}$ .

**Definition.** A set  $A$  is countable if  $A \sim \mathbb{N}$ .

**Definition.** A set which is not finite nor countable is uncountable.

**Theorem.** (i) The set  $\mathbb{Q}$  is countable.

(ii) The set  $\mathbb{R}$  is uncountable.

**Theorem.** If  $A \subseteq B$  is countable, then  $A$  is either countable or finite.

**Theorem.** (i) If  $A_1, A_2, \dots, A_m$  are each countable sets, then the union  $A_1 \cup A_2 \cup \dots \cup A_m$  is countable.

(ii) If  $A_n$  is a countable set for each  $n \in \mathbb{N}$ , then  $\bigcup_{n=1}^{\infty} A_n$  is countable.

#### 1.6 Cantor's Theorem.

**Theorem.** The open interval  $(0, 1) = \{x \in \mathbb{R} : 0 < x < 1\}$  is uncountable.

**Definition.** Given a set  $A$ , the power set  $P(A)$  refers to the collection of all subsets of  $A$ .

**Theorem** (Cantor's Theorem). Given any set  $A$ , there does not exist a function  $f : A \rightarrow P(A)$  that is onto.

## 2 SEQUENCES AND SERIES

## 2.2 The Limit of a Sequence.

**Definition.** A *sequence* is a function whose domain is  $\mathbb{N}$ .

**Definition** (Convergence of a Sequence). A sequence  $(a_n)$  *converges* to a real number  $a$  if, for every positive number  $\epsilon$ , there exists and  $N \in \mathbb{N}$  such that whenever  $n \geq N$ , it follows that  $|a_n - a| < \epsilon$ .

**Definition.** Given a real number  $a \in \mathbb{R}$  and a positive number  $\epsilon > 0$ , the set

$$V_\epsilon(a) = \{x \in \mathbb{R} : |x - a| < \epsilon\} = (a - \epsilon, a + \epsilon)$$

is called the  $\epsilon$ -neighborhood of  $a$ .

**Definition** (Convergence of a Sequence: Topological Version). A sequence  $(a_n)$  converges to  $a$  if, given any  $\epsilon$ -neighborhood  $V_\epsilon(a)$  of  $a$ , there exists a point in the sequence after which all of the terms are in  $V_\epsilon(a)$ . In other words, every  $\epsilon$ -neighborhood contains all but a finite number of terms of  $(a_n)$ .

**Theorem** (Uniqueness of Limits). *The limit of a sequence, when it exists, must be unique.*

**Definition.** A sequence that does not converge is said to *diverge*.

## 2.3 The Algebraic and Order Limit Theorems.

**Definition.** A sequence  $(x_n)$  is *bounded* if there exists a number  $M > 0$  such that  $|x_n| \leq M$  for all  $n \in \mathbb{N}$ .

**Theorem.** *Every convergent sequence is bounded.*

**Theorem** (Algebraic Limit Theorem). *Let  $\lim a_n = a$  and  $\lim b_n = b$ . Then,*

- (i)  $\lim(ca_n) = ca$ , for all  $c \in \mathbb{R}$ ;
- (ii)  $\lim(a_n + b_n) = a + b$ ;
- (iii)  $\lim(a_nb_n) = ab$ ;
- (iv)  $\lim(a_n/b_n) = a/b$ , provided  $b \neq 0$ .

**Theorem** (Order Limit Theorem). *Assume  $\lim a_n = a$  and  $\lim b_n = b$ .*

- (i) *If  $a_n \geq 0$  for all  $n \in \mathbb{N}$ , then  $a \geq 0$ .*
- (ii) *If  $a_n \leq b_n$  for all  $n \in \mathbb{N}$ , then  $a \leq b$ .*
- (iii) *If there exists  $c \in \mathbb{R}$  for which  $c \leq b_n$  for all  $n \in \mathbb{N}$ , then  $c \leq b$ . Similarly, if  $a_n \leq c$  for all  $n \in \mathbb{N}$ , then  $a \leq c$ .*

## 2.4 The Monotone Convergence Theorem and a First Look at Infinite Series.

**Definition.** A sequence  $(a_n)$  is *increasing* if  $a_n \leq a_{n+1}$  for all  $n \in \mathbb{N}$  and *decreasing* if  $a_n \geq a_{n+1}$  for all  $n \in \mathbb{N}$ . A sequence is *monotone* if it is either increasing or decreasing.

**Theorem** (Monotone Convergence Theorem). *If a sequence is monotone and bounded, then it converges.*

**Definition** (Convergence of a Series). Let  $(b_n)$  be a sequence. An *infinite series* is a formal expression of the form

$$\sum_{n=1}^{\infty} b_n = b_1 + b_2 + b_3 + b_4 + b_5 + \cdots$$

We define the corresponding *sequence of partial sums*  $(s_m)$  by

$$s_m = b_1 + b_2 + b_3 + \cdots + b_m,$$

and say that the series  $\sum_{n=1}^{\infty} b_n$  *converges to*  $B$  if the sequence  $(s_m)$  converges to  $B$ . In this case, we write  $\sum_{n=1}^{\infty} b_n = B$ .

**Theorem** (Cauchy Condensation Test). *Suppose  $(b_n)$  is decreasing and satisfies  $b_n \geq 0$  for all  $n \in \mathbb{N}$ . Then, the series  $\sum_{n=1}^{\infty} b_n$  converges if and only if the series*

$$\sum_{n=0}^{\infty} 2^n b_{2^n} = b_1 + 2b_2 + 4b_4 + 8b_8 + 16b_{16} + \cdots$$

*converges.*

**Corollary.** *The series  $\sum_{n=1}^{\infty} 1/n^p$  converges if and only if  $p > 1$ .*

## 2.5 Subsequences and the Bolzano-Weierstrass Theorem.

**Definition.** Let  $(a_n)$  be a sequence of real numbers, and let  $n_1 < n_2 < n_3 < n_4 < n_5 < \cdots$  be an increasing sequence of natural numbers. Then the sequence

$$(a_{n_1}, a_{n_2}, a_{n_3}, a_{n_4}, a_{n_5}, \dots)$$

is called a *subsequence* of  $(a_n)$  and is denoted by  $(a_{n_k})$ , where  $k \in \mathbb{N}$  indexes the subsequence.

**Theorem.** *Subsequences of a convergent sequence converge to the same limit as the original sequence.*

**Corollary** (Divergence Criterion). *Suppose that  $(a_n)$  is a sequence and  $(a_{n_k})$  is a subsequence that diverges, then  $(a_n)$  diverges. If  $(a_{n_k}^1)$  and  $(a_{n_k}^2)$  converge to  $a^1$  and  $a^2$  with  $a^1 \neq a^2$ , then  $(a_n)$  diverges.*

**Theorem** (Bolzano-Weierstrass Theorem). *Every bounded sequence contains a convergent subsequence.*

## 2.6 The Cauchy Criterion.

**Definition.** A sequence  $(a_n)$  is called a *Cauchy sequence* if, for every  $\epsilon > 0$ , there exists an  $N \in \mathbb{N}$  such that whenever  $m, n \geq N$  it follows that  $|a_n - a_m| < \epsilon$ .

**Theorem.** *Every convergence sequence is a Cauchy sequence.*

**Lemma.** *Cauchy sequences are bounded.*

**Theorem** (Cauchy Criterion). *A sequence converges if and only if it is a Cauchy sequence.*

## 2.7 Properties of Infinite Series.

**Theorem** (Algebraic Limit Theorem for Series). *If  $\sum_{k=1}^{\infty} a_k = A$  and  $\sum_{k=1}^{\infty} b_k = B$ , then*

- (i)  $\sum_{k=1}^{\infty} ca_k = cA$  for all  $c \in \mathbb{R}$  and
- (ii)  $\sum_{k=1}^{\infty} (a_k + b_k) = A + B$ .

**Theorem** (Cauchy Criterion for Series). *The series  $\sum_{k=1}^{\infty} a_k$  converges if and only if, given  $\epsilon > 0$ , there exists an  $N \in \mathbb{N}$  such that whenever  $n > m \geq N$  it follows that*

$$|a_{m+1} + a_{m+2} + \cdots + a_n| < \epsilon.$$

**Theorem** (Divergence Test). *If the series  $\sum_{k=1}^{\infty} a_k$  converges, then  $(a_k) \rightarrow 0$ . Equivalently, if  $(a_k) \not\rightarrow 0$ , then  $\sum_{k=1}^{\infty} a_k$  diverges.*

**Theorem** (Comparison Test). *Assume  $(a_k)$  and  $(b_k)$  are sequences satisfying  $0 \leq a_k \leq b_k$  for all  $k \in \mathbb{N}$ .*

- (i) *If  $\sum_{k=1}^{\infty} b_k$  converges, then  $\sum_{k=1}^{\infty} a_k$  converges.*
- (ii) *If  $\sum_{k=1}^{\infty} a_k$  diverges, then  $\sum_{k=1}^{\infty} b_k$  diverges.*

**Theorem** (Squeeze Theorem). *Suppose  $a_n \leq b_n \leq c_n$  for all  $n \in \mathbb{N}$ , and if  $\lim a_n = \lim c_n = l$ , then  $\lim b_n = l$  as well.*

**Definition** (Geometric Series). A series is called *geometric* if it is of the form

$$\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + ar^3 + \cdots.$$

**Theorem.**  $\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$  if and only if  $|r| < 1$ .

**Theorem** (Absolute Convergence Test). *If the series  $\sum_{n=1}^{\infty} |a_n|$  converges, then  $\sum_{n=1}^{\infty} a_n$  converges as well.*

**Theorem** (Alternating Series Test). *Let  $(a_n)$  be a sequence satisfying,*

- (i)  $a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_n \geq a_{n+1} \geq \cdots$  and
- (ii)  $(a_n) \rightarrow 0$ .

*Then, the alternating series  $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$  converges.*

**Definition.** If  $\sum_{n=1}^{\infty} |a_n|$  converges, then we say that the original series  $\sum_{n=1}^{\infty} a_n$  *converges absolutely*. If, on the other hand, the series  $\sum_{n=1}^{\infty} a_n$  converges but the series of absolute values  $\sum_{n=1}^{\infty} |a_n|$  does not converge, then we say that the original series  $\sum_{n=1}^{\infty} a_n$  *converges conditionally*.

## 3 BASIC TOPOLOGY OF $\mathbb{R}$

### 3.2 Open and Closed Sets.

**Definition.** A set  $O \subseteq \mathbb{R}$  is *open* if for all points  $a \in O$  there exists an  $\epsilon$ -neighborhood  $V_{\epsilon}(a) \subseteq O$ .

**Theorem.** (i) *The union of an arbitrary collection of open sets is open.*  
(ii) *The intersection of a finite collection of open sets is open.*

**Definition.** A point  $x$  is a *limit point* of a set  $A$  if every  $\epsilon$ -neighborhood  $V_{\epsilon}(x)$  of  $x$  intersects the set  $A$  at some point other than  $x$ .

**Theorem.** A point  $x$  is a limit point of a set  $A$  if and only if  $x = \lim a_n$  for some sequence  $(a_n)$  contained in  $A$  satisfying  $a_n \neq x$  for all  $n \in \mathbb{N}$ .

**Definition.** A point  $a \in A$  is an *isolated point* of  $A$  if it is not a limit point of  $A$ .

**Definition.** A set  $F \subseteq \mathbb{R}$  is *closed* if it contains its limit points.

**Theorem.** A set  $F \subset \mathbb{R}$  is closed if and only if every Cauchy sequence contained in  $F$  has a limit that is also an element of  $F$ .

**Definition.** Given a set  $A \subseteq \mathbb{R}$ , let  $L$  be the set of all limit points of  $A$ . The *closure* of  $A$  is defined to be  $\bar{A} = A \cup L$ .

**Theorem.** For any  $A \subseteq \mathbb{R}$ , the closure  $\bar{A}$  is a closed set and is the smallest closed set containing  $A$ .

**Theorem.** A set  $O$  is open if and only if  $O^c$  is closed. Likewise, a set  $F$  is closed if and only if  $F^c$  is open.

**Theorem.** (i) The union of a finite collection of closed sets is closed.  
(ii) The intersection of an arbitrary collection of closed sets is closed.

### 3.3 Compact Sets.

**Definition** (Compactness). A set  $K \subseteq \mathbb{R}$  is *compact* if every sequence in  $K$  has a subsequence that converges to a limit that is also in  $K$ .

**Definition.** A set  $A \subseteq \mathbb{R}$  is *bounded* if there exists  $M > 0$  such that  $|a| \leq M$  for all  $a \in A$ .

**Theorem** (Characterization of Compactness in  $\mathbb{R}$ ). A set  $K \subseteq \mathbb{R}$  is compact if and only if it is closed and bounded.

**Theorem** (Nested Compact Set Property). If

$$K_1 \supseteq K_2 \supseteq K_3 \supseteq K_4 \supseteq \cdots$$

is a nested sequence of nonempty compact sets, then the intersection  $\bigcap_{n=1}^{\infty} K_n$  is not empty.

**Definition.** Let  $A \subseteq \mathbb{R}$ . An *open cover* for  $A$  is a (possibly infinite) collection of open sets  $\{O_\lambda : \lambda \in \Lambda\}$  whose union contains the set  $A$ ; that is,  $A \subseteq \bigcup_{\lambda \in \Lambda} O_\lambda$ . Given an open cover for  $A$ , a *finite subcover* is a finite subcollection of open sets from the original open cover whose union still manages to completely contain  $A$ .

**Theorem** (Heine-Borel Theorem). Let  $K$  be a subset of  $\mathbb{R}$ . All of the following statements are equivalent in the sense that any one of them implies the two others:

- (i)  $K$  is compact.
- (ii)  $K$  is closed and bounded.
- (iii) Every open cover for  $K$  has a finite subcover.

## 4 FUNCTIONAL LIMITS AND CONTINUITY

### 4.2 Functional Limits.

**Definition** (Functional Limit). Let  $f : A \rightarrow \mathbb{R}$ , and let  $c$  be a limit point of the domain  $A$ . We say that  $\lim_{x \rightarrow c} f(x) = L$  provided that, for all  $\epsilon > 0$ , there exists a  $\delta > 0$  such that whenever  $0 < |x - c| < \delta$  (and  $x \in A$ ) it follows that  $|f(x) - L| < \epsilon$ .

**Definition** (Functional Limit: Topological Version). Let  $c$  be a limit point of the domain of  $f : A \rightarrow \mathbb{R}$ . We say  $\lim_{x \rightarrow c} f(x) = L$  provided that, for every  $\epsilon$ -neighborhood  $V_\epsilon(L)$  of  $L$ , there exists a  $\delta$ -neighborhood  $V_\delta(c)$  around  $c$  with the property that for all  $x \in V_\delta(c)$  different from  $c$  (with  $x \in A$ ) it follows that  $f(x) \in V_\epsilon(L)$ .

**Theorem** (Sequential Criterion for Functional Limits). *Given a function  $f : A \rightarrow \mathbb{R}$  and a limit point  $c$  of  $A$ , the following two statements are equivalent:*

- (i)  $\lim_{x \rightarrow c} f(x) = L$ .
- (ii) *For all sequences  $(x_n) \subseteq A$  satisfying  $x_n \neq c$  and  $(x_n) \rightarrow c$ , it follows that  $f(x_n) \rightarrow L$ .*

**Corollary** (Algebraic Limit Theorem for Functional Limits). *Let  $f$  and  $g$  be functions defined on a domain  $A \subseteq \mathbb{R}$ , and assume  $\lim_{x \rightarrow c} f(x) = L$  and  $\lim_{x \rightarrow c} g(x) = M$  for some limit point  $c$  of  $A$ . Then,*

- (i)  $\lim_{x \rightarrow c} kf(x) = kL$  for all  $k \in \mathbb{R}$ ,
- (ii)  $\lim_{x \rightarrow c} [f(x) + g(x)] = L + M$ ,
- (iii)  $\lim_{x \rightarrow c} [f(x)g(x)] = LM$ , and
- (iv)  $\lim_{x \rightarrow c} f(x)/g(x) = L/M$ , provided  $M \neq 0$ .

**Corollary** (Divergence Criterion for Functional Limits). *Let  $f$  be a function defined on  $A$ , and let  $c$  be a limit point of  $A$ . If there exists two sequences  $(x_n)$  and  $(y_n)$  in  $A$  with  $x_n \neq c$  and  $y_n \neq c$  and*

$$\lim x_n = \lim y_n = c \quad \text{but} \quad \lim f(x_n) \neq \lim f(y_n),$$

*then we can conclude that the functional limit  $\lim_{x \rightarrow c} f(x)$  does not exist.*

### 4.3 Continuous Functions.

**Definition** (Continuity). A function  $f : A \rightarrow \mathbb{R}$  is *continuous at a point*  $c \in A$  if, for all  $\epsilon > 0$ , there exists a  $\delta > 0$  such that whenever  $|x - c| < \delta$  (and  $x \in A$ ) it follows that  $|f(x) - f(c)| < \epsilon$ .

If  $f$  is continuous at every point in the domain  $A$ , then we say that  $f$  is *continuous on  $A$* .

**Theorem** (Characterizations of Continuity). *Let  $f : A \rightarrow \mathbb{R}$ , and let  $c \in A$ . The function  $f$  is continuous at  $c$  if and only if any one of the following three conditions is met:*

- (i) *For all  $\epsilon > 0$ , there exists a  $\delta > 0$  such that  $|x - c| < \delta$  (and  $x \in A$ ) implies  $|f(x) - f(c)| < \epsilon$ ;*
- (ii) *For all  $V_\epsilon(f(c))$ , there exists a  $V_\delta(c)$  with the property that  $x \in V_\delta(c)$  (and  $x \in A$ ) implies  $f(x) \in V_\epsilon(f(c))$ ;*
- (iii) *For all  $(x_n) \rightarrow c$  (with  $x_n \in A$ ), it follows that  $f(x_n) \rightarrow f(c)$ .*
- (iv) *If  $c$  is a limit point of  $A$ , then the above conditions are equivalent to  $\lim_{x \rightarrow c} f(x) = f(c)$ .*

**Corollary** (Criterion for Discontinuity). *Let  $f : A \rightarrow \mathbb{R}$ , and let  $c \in A$  be a limit point of  $A$ . If there exists a sequence  $(x_n) \subseteq A$  where  $(x_n) \rightarrow c$  but such that  $f(x_n)$  does not converge to  $f(c)$ , we may conclude that  $f$  is not continuous at  $c$ .*

**Theorem** (Algebraic Continuity Theorem). *Assume  $f : A \rightarrow \mathbb{R}$  and  $g : A \rightarrow \mathbb{R}$  are continuous at point  $c \in A$ . Then,*

- (i)  $kf(x)$  is continuous at  $c$  for all  $k \in \mathbb{R}$ ;
- (ii)  $f(x) + g(x)$  is continuous at  $c$ ;
- (iii)  $f(x)g(x)$  is continuous at  $c$ ; and
- (iv)  $f(x)/g(x)$  is continuous at  $c$ , provided the quotient is defined.

#### 4.4 Continuous Functions on Compact Sets.

**Theorem** (Preservation of Compact Sets). *Let  $f : A \rightarrow \mathbb{R}$  be continuous on  $A$ . If  $K \subseteq A$  is compact, then  $f(K)$  is compact as well.*

**Theorem** (Extreme Value Theorem). *If  $f : K \rightarrow \mathbb{R}$  is continuous on a compact set  $K \subseteq \mathbb{R}$ , then  $f$  attains a maximum and minimum value. In other words, there exists  $x_0, x_1 \in K$  such that  $f(x_0) \leq f(x) \leq f(x_1)$  for all  $x \in K$ .*

**Definition** (Uniform Continuity). A function  $f : A \rightarrow \mathbb{R}$  is *uniformly continuous* on  $A$  if for every  $\epsilon > 0$  there exists a  $\delta > 0$  such that for all  $x, y \in A$ ,  $|x - y| < \delta$  implies  $|f(x) - f(y)| < \epsilon$ .

**Theorem** (Sequential Criterion for Absence of Uniform Continuity). *A function  $f : A \rightarrow \mathbb{R}$  fails to be uniformly continuous on  $A$  if and only if there exists a particular  $\epsilon_0 > 0$  and two sequences  $(x_n)$  and  $(y_n)$  in  $A$  satisfying*

$$|x_n - y_n| \rightarrow 0 \quad \text{but} \quad |f(x_n) - f(y_n)| \geq \epsilon_0.$$

**Theorem** (Uniform Continuity on Compact Sets). *A function that is continuous on a compact set  $K$  is uniformly continuous on  $K$ .*

#### 4.5 The Intermediate Value Theorem.

**Theorem** (Intermediate Value Theorem). *Let  $f : [a, b] \rightarrow \mathbb{R}$  be continuous. If  $L$  is a real number satisfying  $f(a) < L < f(b)$  or  $f(a) > L > f(b)$ , then there exists a point  $c \in (a, b)$  where  $f(c) = L$ .*

### 5 THE DERIVATIVE

#### 5.2 Derivatives and the Intermediate Value Property.

**Definition** (Differentiability). Let  $g : A \rightarrow \mathbb{R}$  be a function defined on an interval  $A$ . Given  $c \in A$ , the *derivative of  $g$  at  $c$*  is defined by

$$g'(c) = \lim_{x \rightarrow c} \frac{g(x) - g(c)}{x - c},$$

provided this limit exists. In this case we say  $g$  is *differentiable at  $c$* . If  $g'$  exists for all points  $c \in A$ , we say that  $g$  is *differentiable on  $A$* .

**Theorem.** *If  $g : A \rightarrow \mathbb{R}$  is differentiable at point  $c \in A$ , then  $g$  is continuous at  $c$  as well.*

**Theorem** (Algebraic Differentiability Theorem). *Let  $f$  and  $g$  be functions defined on an interval  $A$ , and assume both are differentiable at some point  $c \in A$ . Then,*

- (i)  $(f + g)'(c) = f'(c) + g'(c)$ ,
- (ii)  $(kf)'(c) = kf'(c)$ , for all  $k \in \mathbb{R}$ ,
- (iii)  $(fg)'(c) = f'(c)g(c) + f(c)g'(c)$ , and
- (iv)  $(f/g)'(c) = \frac{g(c)f'(c) - f(c)g'(c)}{[g(c)]^2}$ , provided that  $g(c) \neq 0$ .



**Theorem (Chain Rule).** Let  $f : A \rightarrow \mathbb{R}$  and  $g : B \rightarrow \mathbb{R}$  satisfy  $f(A) \subseteq B$  so that the composition  $g \circ f$  is defined. If  $f$  is differentiable at  $c \in A$  and if  $g$  is differentiable at  $f(c) \in B$ , then  $g \circ f$  is differentiable at  $c$  with  $(g \circ f)'(c) = g'(f(c)) \cdot f'(c)$ .

**Theorem (Interior Extremum Theorem).** Let  $f$  be differentiable on an open interval  $(a, b)$ . If  $f$  attains a maximum value at some point  $c \in (a, b)$  (i.e.,  $f(c) \geq f(x)$  for all  $x \in (a, b)$ ), then  $f'(c) = 0$ . The same is true if  $f(c)$  is a minimum value.

### 5.3 The Mean Value Theorems.

**Theorem (Rolle's Theorem).** Let  $f : [a, b] \rightarrow \mathbb{R}$  be continuous on  $[a, b]$  and differentiable on  $(a, b)$ . If  $f(a) = f(b)$ , then there exists a point  $c \in (a, b)$  where  $f'(c) = 0$ .

**Theorem (Mean Value Theorem).** If  $f : [a, b] \rightarrow \mathbb{R}$  is continuous on  $[a, b]$  and differentiable on  $(a, b)$ , then there exists a point  $c \in (a, b)$  where

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

**Corollary.** If  $g : A \rightarrow \mathbb{R}$  is differentiable on an interval  $A$  and satisfies  $g'(x) = 0$  for all  $x \in A$ , then  $g(x) = k$  for some constant  $k \in \mathbb{R}$ .

**Corollary.** If  $f$  and  $g$  are differentiable functions on an interval  $A$  and satisfy  $f'(x) = g'(x)$  for all  $x \in A$ , then  $f(x) = g(x) + k$  for some constant  $k \in \mathbb{R}$ .

## 6 SEQUENCES AND SERIES OF FUNCTIONS

### 6.2 Uniform Convergence of a Sequence of Functions.

**Definition (Pointwise Convergence).** For each  $n \in \mathbb{N}$ , let  $f_n$  be a function defined on a set  $A \subseteq \mathbb{R}$ . The sequence  $(f_n)$  of functions converges pointwise on  $A$  to a function  $f$  if, for all  $x \in A$ , the sequence of real numbers  $f_n(x)$  converges to  $f(x)$ .

In this case, we write  $f_n \rightarrow f$ ,  $\lim f_n = f$ , or  $\lim_{n \rightarrow \infty} f_n(x) = f(x)$ . This last expression is helpful if there is any confusion as to whether  $x$  or  $n$  is the limiting variable.

**Definition (Pointwise Convergence).** Let  $(f_n)$  be a sequence of functions defined on a set  $A \subseteq \mathbb{R}$ . Then,  $(f_n)$  converges pointwise on  $A$  to a limit  $f$  defined on  $A$  if, for every  $\epsilon > 0$  and  $x \in A$ , there exists an  $N \in \mathbb{N}$  (perhaps dependent on  $x$ ) such that  $|f_n(x) - f(x)| < \epsilon$  whenever  $n \geq N$ .

**Definition (Uniform Convergence).** Let  $(f_n)$  be a sequence of functions defined on a set  $A \subseteq \mathbb{R}$ . Then,  $(f_n)$  converges uniformly on  $A$  to a limit function  $f$  defined on  $A$  if, for every  $\epsilon > 0$ , there exists an  $N \in \mathbb{N}$  such that  $|f_n(x) - f(x)| < \epsilon$  whenever  $n \geq N$  and  $x \in A$ .

**Theorem (Cauchy Criterion for Uniform Convergence).** A sequence of functions  $(f_n)$  defined on a set  $A \subseteq \mathbb{R}$  converges uniformly on  $A$  if and only if for every  $\epsilon > 0$  there exists an  $N \in \mathbb{N}$  such that  $|f_n(x) - f_m(x)| < \epsilon$  whenever  $m, n \geq N$  and  $x \in A$ .

**Theorem (Continuous Limit Theorem).** Let  $(f_n)$  be a sequence of functions defined on  $A \subseteq \mathbb{R}$  that converges uniformly on  $A$  to a function  $f$ . If each  $f_n$  is continuous at  $c \in A$ , then  $f$  is continuous at  $c$ .

### 6.3 Uniform Convergence and Differentiation.

**Theorem** (Differentiable Limit Theorem). *Let  $f_n \rightarrow f$  pointwise on the closed interval  $[a, b]$ , and assume that each  $f_n$  is differentiable. If  $(f'_n)$  converges uniformly on  $[a, b]$  to a function  $g$ , then the function is differentiable and  $f' = g$ .*

### 6.4 Series of Functions.

**Definition.** For each  $n \in \mathbb{N}$ , let  $f_n$  and  $f$  be functions defined on a set  $A \subseteq \mathbb{R}$ . The infinite series

$$\sum_{n=1}^{\infty} f_n(x) = f_1(x) + f_2(x) + f_3(x) + \cdots$$

converges pointwise on  $A$  to  $f(x)$  if the sequence  $s_k(x)$  of partial sums defined by

$$s_k(x) = f_1(x) + f_2(x) + \cdots + f_k(x)$$

converges pointwise to  $f(x)$ . The series converges uniformly on  $A$  to  $f$  if the sequence  $s_k(x)$  converges uniformly on  $A$  to  $f(x)$ .

In either case, we write  $f = \sum_{n=1}^{\infty} f_n$  or  $f(x) = \sum_{n=1}^{\infty} f_n(x)$ , always being explicit about the type of convergence involved.

**Theorem** (Term-by-term Continuity Theorem). *Let  $f_n$  be continuous functions defined on a set  $A \subseteq \mathbb{R}$ , and assume  $\sum_{n=1}^{\infty} f_n$  converges uniformly on  $A$  to a function  $f$ . Then,  $f$  is continuous on  $A$ .*

**Theorem** (Term-by-term Differentiability Theorem). *Let  $f_n$  be differentiable functions defined on an interval  $A$ , and assume  $\sum_{n=1}^{\infty} f'_n(x)$  converges uniformly to a limit  $g(x)$  on  $A$ . If  $\sum_{n=1}^{\infty} f_n(x)$  converges pointwise to  $f(x)$ , then  $f(x)$  is differentiable and  $f'(x) = g(x)$  on  $A$ .*

**Theorem** (Cauchy Criterion for Uniform Convergence of Series). *A series  $\sum_{n=1}^{\infty} f_n$  converges uniformly on  $A \subseteq \mathbb{R}$  if and only if for every  $\epsilon > 0$  there exists an  $N \in \mathbb{N}$  such that*

$$|f_{m+1}(x) + f_{m+2}(x) + f_{m+3}(x) + \cdots + f_n(x)| < \epsilon$$

whenever  $n > m \geq N$  and  $x \in A$ .

**Corollary** (Weierstrass M-Test). *For each  $n \in \mathbb{N}$ , let  $f_n$  be a function defined on a set  $A \subseteq \mathbb{R}$ , and let  $M_n > 0$  be a real number satisfying*

$$|f_n(x)| \leq M_n$$

for all  $x \in A$ . If  $\sum_{n=1}^{\infty} M_n$  converges, then  $\sum_{n=1}^{\infty} f_n$  converges uniformly on  $A$ .