MATH 355: NOTES

ALEXANDER LEE

The Real Numbers

Some Preliminaries.

Theorem. *Two real numbers* a *and* b *are equal if and only if for every real number* $\epsilon > 0$ *it follows that* $|a - b| < \epsilon$.

The Axiom of Completeness.

Axiom (Axiom of Completeness)**.** *Every nonempty set of real numbers that is bounded above has a least upper bound.*

Definition. A set $A \subseteq R$ is *bounded above* if there exists a number $b \in \mathbb{R}$ such that $a \leq b$ for all $a \in A$. The number b is called an *upper bound* for A.

Similarly, the set A is *bounded below* if there exists a *lower bound* $l \in R$ satisfying $l \leq a$ for every $a \in A$.

Definition. A real number s is the *least upper bound* for a set $A \subseteq \mathbb{R}$ if it meets the following two criteria:

(1) s is an upper bound for A;
(2) if b is any upper bound for A, then $s \leq b$.

The least upper bound is also frequently called the *supremum* of the set A. We write $s = \sup A$ for the least upper bound.

The *greatest lower bound* or *infimum* for A is defined in a similar way and is denoted by $\inf A$.

Definition. A real number a_0 is a *maximum* of the set A if a_0 is an element of A and $a_0 \geq a$ for all $a \in A$. Similarly, a number a_1 is a *minimum* of A if $a_1 \in A$ and $a_1 \leq a$ for every $a \in A$.

Lemma. *Assume* $s \in \mathbb{R}$ *is an upper bound for a set* $A \subseteq R$*. Then,* $s = \sup A$ *if and only if, for every choice of* $\epsilon > 0$*, there exists an element* $a \in A$ *satisfying* $s - \epsilon < a$.