

MATH 355: NOTES

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1 THE REAL NUMBERS

1.2 Some Preliminaries.

Theorem (Triangle Inequality). *For all choices of a and b , $|a + b| \leq |a| + |b|$.*

Theorem. *Two real numbers a and b are equal if and only if for every real number $\epsilon > 0$ it follows that $|a - b| < \epsilon$.*

1.3 The Axiom of Completeness.

Axiom (Axiom of Completeness). *Every nonempty set of real numbers that is bounded above has a least upper bound.*

Definition. A set $A \subseteq \mathbb{R}$ is *bounded above* if there exists a number $b \in \mathbb{R}$ such that $a \leq b$ for all $a \in A$. The number b is called an *upper bound* for A .

Similarly, the set A is *bounded below* if there exists a *lower bound* $l \in \mathbb{R}$ satisfying $l \leq a$ for every $a \in A$.

Definition. A real number s is the *least upper bound* for a set $A \subseteq \mathbb{R}$ if it meets the following two criteria:

- (i) s is an upper bound for A ;
- (ii) if b is any upper bound for A , then $s \leq b$.

The least upper bound is also frequently called the *supremum* of the set A . We write $s = \sup(A)$ for the least upper bound.

The *greatest lower bound* or *infimum* for A is defined in a similar way and is denoted by $\inf(A)$.

Theorem. *Let $A \subseteq \mathbb{R}$ be bounded above and below. Then, the $\sup(A)$ and $\inf(A)$ are unique.*

Definition. A real number a_0 is a *maximum* of the set A if a_0 is an element of A and $a_0 \geq a$ for all $a \in A$. Similarly, a number a_1 is a *minimum* of A if $a_1 \in A$ and $a_1 \leq a$ for every $a \in A$.

Theorem. *Let $A \subseteq \mathbb{R}$ be nonempty and bounded above.*

- (i) *Let $c \in \mathbb{R}$ and define the set $c + A$ by*

$$c + A = \{c + a : a \in A\}.$$

Then $\sup(c + A) = c + \sup(A)$.

- (ii) *Let $c \in \mathbb{R}$ with $c > 0$ and define the set cA by*

$$cA = \{ca : a \in A\}.$$

Then $\sup(cA) = c \sup(A)$.

Lemma. Assume $s \in \mathbb{R}$ is an upper bound for a set $A \subseteq \mathbb{R}$. Then, $s = \sup(A)$ if and only if, for every choice of $\epsilon > 0$, there exists an element $a \in A$ satisfying $s - \epsilon < a$.

1.4 Consequences of Completeness.

Theorem (Nested Interval Property). For each $n \in \mathbb{N}$, assume we are given a closed interval $I_n = [a_n, b_n] = \{x \in \mathbb{R} : a_n \leq x \leq b_n\}$. Assume also that each I_n contains I_{n+1} . Then, the resulting nested sequence of closed intervals

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq I_4 \supseteq \cdots$$

has a nonempty intersection; that is, $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

Theorem (Archimedean Property). (i) Given any number $x \in \mathbb{R}$, there exists an $n \in \mathbb{N}$ satisfying $n > x$.

(ii) Given any real number $y > 0$, there exists an $n \in \mathbb{N}$ satisfying $1/n < y$.

Definition. A set X is dense in \mathbb{R} if for any $a, b \in \mathbb{R}$ with $a < b$, $\exists x \in X$ with $a < x < b$.

Theorem (Density of \mathbb{Q} in \mathbb{R}). For every two real numbers a and b with $a < b$, there exists a rational number r satisfying $a < r < b$.

1.5 Cardinality.

Definition. A function $f : A \rightarrow B$ is 1-1 (injective) if for all $a_1, a_2 \in A$, $f(a_1) = f(a_2)$ implies that $a_1 = a_2$.

Definition. A function $f : A \rightarrow B$ is onto (surjective) if for all $b \in B$, there exists an $a \in A$ such that $f(a) = b$.

Definition. A function $f : A \rightarrow B$ is a bijection if it is both 1-1 and onto.

Definition. Two sets A and B have the same cardinality if there exists a bijection $f : A \rightarrow B$. In this case, we write $A \sim B$.

Definition. A set A is finite if there exists an $n \in \mathbb{N}$ such that $A \sim \{1, 2, \dots, n\}$.

Definition. A set A is countable if $A \sim \mathbb{N}$.

Definition. A set which is not finite nor countable is uncountable.

Theorem. (i) The set \mathbb{Q} is countable.

(ii) The set \mathbb{R} is uncountable.

Theorem. If $A \subseteq B$ is countable, then A is either countable or finite.

Theorem. (i) If A_1, A_2, \dots, A_m are each countable sets, then the union $A_1 \cup A_2 \cup \dots \cup A_m$ is countable.

(ii) If A_n is a countable set for each $n \in \mathbb{N}$, then $\bigcup_{n=1}^{\infty} A_n$ is countable.

1.6 Cantor's Theorem.

Theorem. The open interval $(0, 1) = \{x \in \mathbb{R} : 0 < x < 1\}$ is uncountable.

Definition. Given a set A , the power set $P(A)$ refers to the collection of all subsets of A .

Theorem (Cantor's Theorem). Given any set A , there does not exist a function $f : A \rightarrow P(A)$ that is onto.

2 SEQUENCES AND SERIES

2.2 The Limit of a Sequence.

Definition. A *sequence* is a function whose domain is \mathbb{N} .

Definition (Convergence of a Sequence). A sequence (a_n) *converges* to a real number a if, for every positive number ϵ , there exists and $N \in \mathbb{N}$ such that whenever $n \geq N$, it follows that $|a_n - a| < \epsilon$.

Definition. Given a real number $a \in \mathbb{R}$ and a positive number $\epsilon > 0$, the set

$$V_\epsilon(a) = \{x \in \mathbb{R} : |x - a| < \epsilon\} = (a - \epsilon, a + \epsilon)$$

is called the ϵ -neighborhood of a .

Definition (Convergence of a Sequence: Topological Version). A sequence (a_n) converges to a if, given any ϵ -neighborhood $V_\epsilon(a)$ of a , there exists a point in the sequence after which all of the terms are in $V_\epsilon(a)$. In other words, every ϵ -neighborhood contains all but a finite number of terms of (a_n) .

Theorem (Uniqueness of Limits). *The limit of a sequence, when it exists, must be unique.*

Definition. A sequence that does not converge is said to *diverge*.

2.3 The Algebraic and Order Limit Theorems.

Definition. A sequence (x_n) is *bounded* if there exists a number $M > 0$ such that $|x_n| \leq M$ for all $n \in \mathbb{N}$.

Theorem. *Every convergent sequence is bounded.*

Theorem (Algebraic Limit Theorem). *Let $\lim a_n = a$ and $\lim b_n = b$. Then,*

- (i) $\lim(ca_n) = ca$, for all $c \in \mathbb{R}$;
- (ii) $\lim(a_n + b_n) = a + b$;
- (iii) $\lim(a_nb_n) = ab$;
- (iv) $\lim(a_n/b_n) = a/b$, provided $b \neq 0$.

Theorem (Order Limit Theorem). *Assume $\lim a_n = a$ and $\lim b_n = b$.*

- (i) *If $a_n \geq 0$ for all $n \in \mathbb{N}$, then $a \geq 0$.*
- (ii) *If $a_n \leq b_n$ for all $n \in \mathbb{N}$, then $a \leq b$.*
- (iii) *If there exists $c \in \mathbb{R}$ for which $c \leq b_n$ for all $n \in \mathbb{N}$, then $c \leq b$. Similarly, if $a_n \leq c$ for all $n \in \mathbb{N}$, then $a \leq c$.*

2.4 The Monotone Convergence Theorem and a First Look at Infinite Series.

Definition. A sequence (a_n) is *increasing* if $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$ and *decreasing* if $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$. A sequence is *monotone* if it is either increasing or decreasing.

Theorem (Monotone Convergence Theorem). *If a sequence is monotone and bounded, then it converges.*

Definition (Convergence of a Series). Let (b_n) be a sequence. An *infinite series* is a formal expression of the form

$$\sum_{n=1}^{\infty} b_n = b_1 + b_2 + b_3 + b_4 + b_5 + \cdots$$

We define the corresponding *sequence of partial sums* (s_m) by

$$s_m = b_1 + b_2 + b_3 + \cdots + b_m,$$

and say that the series $\sum_{n=1}^{\infty} b_n$ *converges to* B if the sequence (s_m) converges to B . In this case, we write $\sum_{n=1}^{\infty} b_n = B$.

Theorem (Cauchy Condensation Test). *Suppose (b_n) is decreasing and satisfies $b_n \geq 0$ for all $n \in \mathbb{N}$. Then, the series $\sum_{n=1}^{\infty} b_n$ converges if and only if the series*

$$\sum_{n=0}^{\infty} 2^n b_{2^n} = b_1 + 2b_2 + 4b_4 + 8b_8 + 16b_{16} + \cdots$$

converges.

Corollary. *The series $\sum_{n=1}^{\infty} 1/n^p$ converges if and only if $p > 1$.*

2.5 Subsequences and the Bolzano-Weierstrass Theorem.

Definition. Let (a_n) be a sequence of real numbers, and let $n_1 < n_2 < n_3 < n_4 < n_5 < \cdots$ be an increasing sequence of natural numbers. Then the sequence

$$(a_{n_1}, a_{n_2}, a_{n_3}, a_{n_4}, a_{n_5}, \dots)$$

is called a *subsequence* of (a_n) and is denoted by (a_{n_k}) , where $k \in \mathbb{N}$ indexes the subsequence.

Theorem. *Subsequences of a convergent sequence converge to the same limit as the original sequence.*

Corollary (Divergence Criterion). *Suppose that (a_n) is a sequence and (a_{n_k}) is a subsequence that diverges, then (a_n) diverges. If $(a_{n_k}^1)$ and $(a_{n_k}^2)$ converge to a^1 and a^2 with $a^1 \neq a^2$, then (a_n) diverges.*

Theorem (Bolzano-Weierstrass Theorem). *Every bounded sequence contains a convergent subsequence.*

2.6 The Cauchy Criterion.

Definition. A sequence (a_n) is called a *Cauchy sequence* if, for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that whenever $m, n \geq N$ it follows that $|a_n - a_m| < \epsilon$.

Theorem. *Every convergence sequence is a Cauchy sequence.*

Lemma. *Cauchy sequences are bounded.*

Theorem (Cauchy Criterion). *A sequence converges if and only if it is a Cauchy sequence.*

2.7 Properties of Infinite Series.

Theorem (Algebraic Limit Theorem for Series). *If $\sum_{k=1}^{\infty} a_k = A$ and $\sum_{k=1}^{\infty} b_k = B$, then*

- (i) $\sum_{k=1}^{\infty} ca_k = cA$ for all $c \in \mathbb{R}$ and
- (ii) $\sum_{k=1}^{\infty} (a_k + b_k) = A + B$.

Theorem (Cauchy Criterion for Series). *The series $\sum_{k=1}^{\infty} a_k$ converges if and only if, given $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that whenever $n > m \geq N$ it follows that*

$$|a_{m+1} + a_{m+2} + \cdots + a_n| < \epsilon.$$

Theorem (Divergence Test). *If the series $\sum_{k=1}^{\infty} a_k$ converges, then $(a_k) \rightarrow 0$. Equivalently, if $(a_k) \not\rightarrow 0$, then $\sum_{k=1}^{\infty} a_k$ diverges.*

Theorem (Comparison Test). *Assume (a_k) and (b_k) are sequences satisfying $0 \leq a_k \leq b_k$ for all $k \in \mathbb{N}$.*

- (i) *If $\sum_{k=1}^{\infty} b_k$ converges, then $\sum_{k=1}^{\infty} a_k$ converges.*
- (ii) *If $\sum_{k=1}^{\infty} a_k$ diverges, then $\sum_{k=1}^{\infty} b_k$ diverges.*

Theorem (Squeeze Theorem). *Suppose $a_n \leq b_n \leq c_n$ for all $n \in \mathbb{N}$, and if $\lim a_n = \lim c_n = l$, then $\lim b_n = l$ as well.*

Definition (Geometric Series). A series is called *geometric* if it is of the form

$$\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + ar^3 + \cdots.$$

Theorem. $\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$ if and only if $|r| < 1$.

Theorem (Absolute Convergence Test). *If the series $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges as well.*

Theorem (Alternating Series Test). *Let (a_n) be a sequence satisfying,*

- (i) $a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_n \geq a_{n+1} \geq \cdots$ and
- (ii) $(a_n) \rightarrow 0$.

Then, the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges.

Definition. If $\sum_{n=1}^{\infty} |a_n|$ converges, then we say that the original series $\sum_{n=1}^{\infty} a_n$ *converges absolutely*. If, on the other hand, the series $\sum_{n=1}^{\infty} a_n$ converges but the series of absolute values $\sum_{n=1}^{\infty} |a_n|$ does not converge, then we say that the original series $\sum_{n=1}^{\infty} a_n$ *converges conditionally*.

3 BASIC TOPOLOGY OF \mathbb{R}

3.2 Open and Closed Sets.

Definition. A set $O \subseteq \mathbb{R}$ is *open* if for all points $a \in O$ there exists an ϵ -neighborhood $V_{\epsilon}(a) \subseteq O$.

Theorem. (i) *The union of an arbitrary collection of open sets is open.*
(ii) *The intersection of a finite collection of open sets is open.*

Definition. A point x is a *limit point* of a set A if every ϵ -neighborhood $V_{\epsilon}(x)$ of x intersects the set A at some point other than x .

Theorem. A point x is a limit point of a set A if and only if $x = \lim a_n$ for some sequence (a_n) contained in A satisfying $a_n \neq x$ for all $n \in \mathbb{N}$.

Definition. A point $a \in A$ is an *isolated point* of A if it is not a limit point of A .

Definition. A set $F \subseteq \mathbb{R}$ is *closed* if it contains its limit points.

Theorem. A set $F \subset \mathbb{R}$ is closed if and only if every Cauchy sequence contained in F has a limit that is also an element of F .

Definition. Given a set $A \subseteq \mathbb{R}$, let L be the set of all limit points of A . The *closure* of A is defined to be $\bar{A} = A \cup L$.

Theorem. For any $A \subseteq \mathbb{R}$, the closure \bar{A} is a closed set and is the smallest closed set containing A .

Theorem. A set O is open if and only if O^c is closed. Likewise, a set F is closed if and only if F^c is open.

Theorem. (i) The union of a finite collection of closed sets is closed.
(ii) The intersection of an arbitrary collection of closed sets is closed.

3.3 Compact Sets.

Definition (Compactness). A set $K \subseteq \mathbb{R}$ is *compact* if every sequence in K has a subsequence that converges to a limit that is also in K .

Definition. A set $A \subseteq \mathbb{R}$ is *bounded* if there exists $M > 0$ such that $|a| \leq M$ for all $a \in A$.

Theorem (Characterization of Compactness in \mathbb{R}). A set $K \subseteq \mathbb{R}$ is compact if and only if it is closed and bounded.

Theorem (Nested Compact Set Property). If

$$K_1 \supseteq K_2 \supseteq K_3 \supseteq K_4 \supseteq \cdots$$

is a nested sequence of nonempty compact sets, then the intersection $\bigcap_{n=1}^{\infty} K_n$ is not empty.

Definition. Let $A \subseteq \mathbb{R}$. An *open cover* for A is a (possibly infinite) collection of open sets $\{O_\lambda : \lambda \in \Lambda\}$ whose union contains the set A ; that is, $A \subseteq \bigcup_{\lambda \in \Lambda} O_\lambda$. Given an open cover for A , a *finite subcover* is a finite subcollection of open sets from the original open cover whose union still manages to completely contain A .

Theorem (Heine-Borel Theorem). Let K be a subset of \mathbb{R} . All of the following statements are equivalent in the sense that any one of them implies the two others:

- (i) K is compact.
- (ii) K is closed and bounded.
- (iii) Every open cover for K has a finite subcover.

4 FUNCTIONAL LIMITS AND CONTINUITY

4.2 Functional Limits.

Definition (Functional Limit). Let $f : A \rightarrow \mathbb{R}$, and let c be a limit point of the domain A . We say that $\lim_{x \rightarrow c} f(x) = L$ provided that, for all $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $0 < |x - c| < \delta$ (and $x \in A$) it follows that $|f(x) - L| < \epsilon$.

Definition (Functional Limit: Topological Version). Let c be a limit point of the domain of $f : A \rightarrow \mathbb{R}$. We say $\lim_{x \rightarrow c} f(x) = L$ provided that, for every ϵ -neighborhood $V_\epsilon(L)$ of L , there exists a δ -neighborhood $V_\delta(c)$ around c with the property that for all $x \in V_\delta(c)$ different from c (with $x \in A$) it follows that $f(x) \in V_\epsilon(L)$.

Theorem (Sequential Criterion for Functional Limits). Given a function $f : A \rightarrow \mathbb{R}$ and a limit point c of A , the following two statements are equivalent:

- (i) $\lim_{x \rightarrow c} f(x) = L$.
- (ii) For all sequences $(x_n) \subseteq A$ satisfying $x_n \neq c$ and $(x_n) \rightarrow c$, it follows that $f(x_n) \rightarrow L$.

Corollary (Algebraic Limit Theorem for Functional Limits). Let f and g be functions defined on a domain $A \subseteq \mathbb{R}$, and assume $\lim_{x \rightarrow c} f(x) = L$ and $\lim_{x \rightarrow c} g(x) = M$ for some limit point c of A . Then,

- (i) $\lim_{x \rightarrow c} kf(x) = kL$ for all $k \in \mathbb{R}$,
- (ii) $\lim_{x \rightarrow c} [f(x) + g(x)] = L + M$,
- (iii) $\lim_{x \rightarrow c} [f(x)g(x)] = LM$, and
- (iv) $\lim_{x \rightarrow c} f(x)/g(x) = L/M$, provided $M \neq 0$.

Corollary (Divergence Criterion for Functional Limits). Let f be a function defined on A , and let c be a limit point of A . If there exists two sequences (x_n) and (y_n) in A with $x_n \neq c$ and $y_n \neq c$ and

$$\lim x_n = \lim y_n = c \quad \text{but} \quad \lim f(x_n) \neq \lim f(y_n),$$

then we can conclude that the functional limit $\lim_{x \rightarrow c} f(x)$ does not exist.

4.3 Continuous Functions.

Definition (Continuity). A function $f : A \rightarrow \mathbb{R}$ is *continuous at a point* $c \in A$ if, for all $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $|x - c| < \delta$ (and $x \in A$) it follows that $|f(x) - f(c)| < \epsilon$.

If f is continuous at every point in the domain A , then we say that f is *continuous on* A .

Theorem (Characterizations of Continuity). Let $f : A \rightarrow \mathbb{R}$, and let $c \in A$. The function f is continuous at c if and only if any one of the following three conditions is met:

- (i) For all $\epsilon > 0$, there exists a $\delta > 0$ such that $|x - c| < \delta$ (and $x \in A$) implies $|f(x) - f(c)| < \epsilon$;
- (ii) For all $V_\epsilon(f(c))$, there exists a $V_\delta(c)$ with the property that $x \in V_\delta(c)$ (and $x \in A$) implies $f(x) \in V_\epsilon(f(c))$;
- (iii) For all $(x_n) \rightarrow c$ (with $x_n \in A$), it follows that $f(x_n) \rightarrow f(c)$.
- (iv) If c is a limit point of A , then the above conditions are equivalent to $\lim_{x \rightarrow c} f(x) = f(c)$.

Corollary (Criterion for Discontinuity). Let $f : A \rightarrow \mathbb{R}$, and let $c \in A$ be a limit point of A . If there exists a sequence $(x_n) \subseteq A$ where $(x_n) \rightarrow c$ but such that $f(x_n)$ does not converge to $f(c)$, we may conclude that f is not continuous at c .

Theorem (Algebraic Continuity Theorem). Assume $f : A \rightarrow \mathbb{R}$ and $g : A \rightarrow \mathbb{R}$ are continuous at point $c \in A$. Then,

- (i) $kf(x)$ is continuous at c for all $k \in \mathbb{R}$;
- (ii) $f(x) + g(x)$ is continuous at c ;
- (iii) $f(x)g(x)$ is continuous at c ; and
- (iv) $f(x)/g(x)$ is continuous at c , provided the quotient is defined.

4.4 Continuous Functions on Compact Sets.

Theorem (Preservation of Compact Sets). *Let $f : A \rightarrow \mathbb{R}$ be continuous on A . If $K \subseteq A$ is compact, then $f(K)$ is compact as well.*

Theorem (Extreme Value Theorem). *If $f : K \rightarrow \mathbb{R}$ is continuous on a compact set $K \subseteq \mathbb{R}$, then f attains a maximum and minimum value. In other words, there exists $x_0, x_1 \in K$ such that $f(x_0) \leq f(x) \leq f(x_1)$ for all $x \in K$.*

Definition (Uniform Continuity). A function $f : A \rightarrow \mathbb{R}$ is *uniformly continuous* on A if for every $\epsilon > 0$ there exists a $\delta > 0$ such that for all $x, y \in A$, $|x - y| < \delta$ implies $f(x) - f(y) < \epsilon$.

Theorem (Sequential Criterion for Absence of Uniform Continuity). *A function $f : A \rightarrow \mathbb{R}$ fails to be uniformly continuous on A if and only if there exists a particular $\epsilon_0 > 0$ and two sequences (x_n) and (y_n) in A satisfying*

$$|x_n - y_n| \rightarrow 0 \quad \text{but} \quad |f(x_n) - f(y_n)| \geq \epsilon_0.$$

Theorem (Uniform Continuity on Compact Sets). *A function that is continuous on a compact set K is uniformly continuous on K .*