MATH 355: NOTES

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1 The Real Numbers

1.2 Some Preliminaries.

Theorem (Triangle Inequality)**.** *For all choices of a and b, $|a + b| \leq |a| + |b|$.*

Theorem. *Two real numbers a and b are equal if and only if for every real number $\epsilon > 0$ it follows that $|a - b| < \epsilon$.*

1.3 The Axiom of Completeness.

Axiom (Axiom of Completeness)**.** *Every nonempty set of real numbers that is bounded above has a least upper bound.*

Definition. A set $A \subseteq \mathbb{R}$ is *bounded above* if there exists a number $b \in \mathbb{R}$ such that $a \leq b$ for all $a \in A$. The number b is called an *upper bound* for A.

Similarly, the set A is *bounded below* if there exists a *lower bound* $l \in \mathbb{R}$ satisfying $l \leq a$ for every $a \in A$.

Definition. A real number s is the *least upper bound* for a set $A \subseteq \mathbb{R}$ if it meets the following two criteria:

(i) s is an upper bound for A;
(ii) if b is any upper bound for A, then $s \leq b$.

The least upper bound is also frequently called the *supremum* of the set A. We write $s = \sup(A)$ for the least upper bound.

The *greatest lower bound* or *infimum* for A is defined in a similar way and is denoted by $\inf A$.

Theorem. *Let $A \subseteq \mathbb{R}$ be bounded above and below. Then, the $\sup(A)$ and $\inf A$ are unique.*

Definition. A real number a_0 is a *maximum* of the set A if a_0 is an element of A and $a_0 \geq a$ for all $a \in A$. Similarly, a number a_1 is a *minimum* of A if $a_1 \in A$ and $a_1 \leq a$ for every $a \in A$.

Theorem. *Let $A \subseteq \mathbb{R}$ be nonempty and bounded above.*

(i) Let $c \in \mathbb{R}$ and define the set $c + A$ by

$$c + A = \{c + a : a \in A\}.$$

Then $\sup(c + A) = c + \sup(A)$.

(ii) Let $c \in \mathbb{R}$ with $c > 0$ and define the set cA by

$$cA = \{ca : a \in A\}.$$

Then $\sup(cA) = c \sup(A)$.

Lemma. *Assume $s \in \mathbb{R}$ is an upper bound for a set $A \subseteq R$. Then, $s = \sup(A)$ if and only if, for every choice of $\epsilon > 0$, there exists an element $a \in A$ satisfying $s - \epsilon < a$.*

1.4 Consequences of Completeness.

Theorem (Nested Interval Property). *For each $n \in \mathbb{N}$, assume we are given a closed interval $I_n = [a_n, b_n] = \{x \in \mathbb{R} : a_n \leq x \leq b_n\}$. Assume also that each I_n contains I_{n+1}. Then, the resulting nested sequence of closed intervals*

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq I_4 \supseteq \cdots$$

has a nonempty intersection; that is, $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

Theorem (Archimedean Property). *(i) Given any number $x \in \mathbb{R}$, there exists an $n \in \mathbb{N}$ satisfying $n > x$.*
(ii) Given any real number $y > 0$, there exists an $n \in \mathbb{N}$ satisfying $1/n < y$.

Definition. A set X is *dense in* $\mathbb{R}$ if for any $a, b \in \mathbb{R}$ with $a < b$, $\exists x \in X$ with $a < x < b$.

Theorem (Density of $\mathbb{Q}$ in $\mathbb{R}$). *For every two real numbers a and b with $a < b$, there exists a rational number r satisfying $a < r < b$.*

1.5 Cardinality.

Definition. A function $f : A \to B$ is *1–1 (injective)* if for all $a_1, a_2 \in A$, $f(a_1) = f(a_2)$ implies that $a_1 = a_2$.

Definition. A function $f : A \to B$ is *onto (surjective)* if for all $b \in B$, there exists an $a \in A$ such that $f(a) = b$.

Definition. A function $f : A \to B$ is a *bijection* if it is both 1–1 and onto.

Definition. Two sets A and B have the same *cardinality* if there exists a bijection $f : A \to B$. In this case, we write $A \sim B$.

Definition. A set A is *finite* if there exists an $n \in \mathbb{N}$ such that $A \sim \{1, 2, \ldots, n\}$.

Definition. A set A is *countable* if $A \sim \mathbb{N}$.

Definition. A set which is not finite nor countable is *uncountable.*

Theorem. *(i) The set $\mathbb{Q}$ is countable.*
(ii) The set $\mathbb{R}$ is uncountable.