MATH 355: NOTES

ALEXANDER LEE

1 The Real Numbers

1.2 Some Preliminaries.

Theorem (Triangle Inequality). *For all choices of a and b, $|a+b| \leq |a| + |b|$.*

Theorem. *Two real numbers a and b are equal if and only if for every real number $\epsilon > 0$ it follows that $|a - b| < \epsilon$.*

1.3 The Axiom of Completeness.

Axiom (Axiom of Completeness). *Every nonempty set of real numbers that is bounded above has a least upper bound.*

Definition. A set $A \subseteq \mathbb{R}$ is *bounded above* if there exists a number $b \in \mathbb{R}$ such that $a \leq b$ for all $a \in A$. The number b is called an *upper bound* for A.

Similarly, the set A is *bounded below* if there exists a *lower bound* $l \in \mathbb{R}$ satisfying $l \leq a$ for every $a \in A$.

Definition. A real number s is the *least upper bound* for a set $A \subseteq \mathbb{R}$ if it meets the following two criteria:

(i) s is an upper bound for A;

(ii) if b is any upper bound for A, then $s \leq b$.

The least upper bound is also frequently called the *supremum* of the set A. We write $s = \sup(A)$ for the least upper bound.

The *greatest lower bound* or *infimum* for A is defined in a similar way and is denoted by $\inf(A)$.

Theorem. *Let $A \subseteq \mathbb{R}$ be bounded above and below. Then, the $\sup(A)$ and $\inf(A)$ are unique.*

Definition. A real number a_0 is a *maximum* of the set A if a_0 is an element of A and $a_0 \geq a$ for all $a \in A$. Similarly, a number a_1 is a *minimum* of A if $a_1 \in A$ and $a_1 \leq a$ for every $a \in A$.

Theorem. *Let $A \subseteq \mathbb{R}$ be nonempty and bounded above.*

(i) Let $c \in \mathbb{R}$ and define the set $c + A$ by

$$c + A = \{c + a : a \in A\}.$$

Then $\sup(c + A) = c + \sup(A)$.

(ii) Let $c \in \mathbb{R}$ with $c > 0$ and define the set cA by

$$cA = \{ca : a \in A\}.$$

Then $\sup(cA) = c\sup(A)$.

Lemma. *Assume $s \in \mathbb{R}$ is an upper bound for a set $A \subseteq R$. Then, $s = \sup(A)$ if and only if, for every choice of $\epsilon > 0$, there exists an element $a \in A$ satisfying $s - \epsilon < a$.*

1.4 Consequences of Completeness.

Theorem (Nested Interval Property). *For each $n \in \mathbb{N}$, assume we are given a closed interval $I_n = [a_n, b_n] = \{x \in \mathbb{R} : a_n \leq x \leq b_n\}$. Assume also that each I_n contains I_{n+1}. Then, the resulting nested sequence of closed intervals*

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq I_4 \supseteq \cdots$$

has a nonempty intersection; that is, $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

Theorem (Archimedean Property). *(i) Given any number $x \in \mathbb{R}$, there exists an $n \in \mathbb{N}$ satisfying $n > x$.*
(ii) Given any real number $y > 0$, there exists an $n \in \mathbb{N}$ satisfying $1/n < y$.

Definition. A set X is *dense in* $\mathbb{R}$ if for any $a, b \in \mathbb{R}$ with $a < b$, $\exists x \in X$ with $a < x < b$.

Theorem (Density of $\mathbb{Q}$ in $\mathbb{R}$). *For every two real numbers a and b with $a < b$, there exists a rational number r satisfying $a < r < b$.*

1.5 Cardinality.

Definition. A function $f : A \to B$ is *1–1 (injective)* if for all $a_1, a_2 \in A$, $f(a_1) = f(a_2)$ implies that $a_1 = a_2$.

Definition. A function $f : A \to B$ is *onto (surjective)* if for all $b \in B$, there exists an $a \in A$ such that $f(a) = b$.

Definition. A function $f : A \to B$ is a *bijection* if it is both 1–1 and onto.

Definition. Two sets A and B have the same *cardinality* if there exists a bijection $f : A \to B$. In this case, we write $A \sim B$.

Definition. A set A is *finite* if there exists an $n \in \mathbb{N}$ such that $A \sim \{1, 2, \ldots, n\}$.

Definition. A set A is *countable* if $A \sim \mathbb{N}$.

Definition. A set which is not finite nor countable is *uncountable.*

Theorem. *(i) The set $\mathbb{Q}$ is countable.*
(ii) The set $\mathbb{R}$ is uncountable.

Theorem. *If $A \subseteq B$ is countable, then A is either countable or finite.*

Theorem. *(i) If $A_1, A_2, \ldots, A_m$ are each countable sets, then the union $A_1 \cup A_2 \cup \cdots \cup A_m$ is countable.*
(ii) If A_n is a countable set for each $n \in \mathbb{N}$, then $\bigcup_{n=1}^{\infty} A_n$ is countable.

1.6 Cantor's Theorem.

Theorem. *The open interval $(0, 1) = \{x \in \mathbb{R} : 0 < x < 1\}$ is uncountable.*

Definition. Given a set A, the *power set* $P(A)$ refers to the collection of all subsets of A.

Theorem (Cantor's Theorem). *Given any set A, there does not exist a function $f : A \to P(A)$ that is onto.*

2 Sequences and Series

2.2 The Limit of a Sequence.

Definition. A *sequence* is a function whose domain is $\mathbb{N}$.

Definition (Convergence of a Sequence). A sequence (a_n) *converges* to a real number a if, for every positive number ϵ, there exists and $N \in \mathbb{N}$ such that whenever $n \geq N$, it follows that $|a_n - a| < \epsilon$.

Definition. Given a real number $a \in \mathbb{R}$ and a positive number $\epsilon > 0$, the set

$$V_\epsilon(a) = \{x \in \mathbb{R} : |x - a| < \epsilon\} = (a - \epsilon, a + \epsilon)$$

is called the *ϵ-neighborhood of a*.

Definition (Convergence of a Sequence: Topological Version). A sequence (a_n) converges to a if, given any ϵ-neighborhood $V_\epsilon(a)$ of a, there exists a point in the sequence after which all of the terms are in $V_\epsilon(a)$. In other words, every ϵ-neighborhood contains all but a finite number of terms of (a_n).

Theorem (Uniqueness of Limits). *The limit of a sequence, when it exists, must be unique.*

Definition. A sequence that does not converge is said to *diverge*.

2.3 The Algebraic and Order Limit Theorems.

Definition. A sequence (x_n) is *bounded* if there exists a number $M > 0$ such that $|x_n| \leq M$ for all $n \in \mathbb{N}$.

Theorem. *Every convergent sequence is bounded.*

Theorem (Algebraic Limit Theorem). *Let* $\lim a_n = a$ *and* $\lim b_n = b$. *Then,*

(i) $\lim(ca_n) = ca$, *for all* $c \in \mathbb{R}$;
(ii) $\lim(a_n + b_n) = a + b$;
(iii) $\lim(a_n b_n) = ab$;
(iv) $\lim(a_n/b_n) = a/b$, *provided* $b \neq 0$.

Theorem (Order Limit Theorem). *Assume* $\lim a_n = a$ *and* $\lim b_n = b$.

(i) If $a_n \geq 0$ *for all* $n \in \mathbb{N}$, *then* $a \geq 0$.
(ii) If $a_n \leq b_n$ *for all* $n \in \mathbb{N}$, *then* $a \leq b$.
(iii) If there exists $c \in \mathbb{R}$ *for which* $c \leq b_n$ *for all* $n \in \mathbb{N}$, *then* $c \leq b$. *Similarly, if* $a_n \leq c$ *for all* $n \in \mathbb{N}$, *then* $a \leq c$.

2.4 The Monotone Convergence Theorem and a First Look at Infinite Series.

Definition. A sequence (a_n) is *increasing* if $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$ and *decreasing* if $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$. A sequence is *monotone* if it is either increasing or decreasing.

Theorem (Monotone Convergence Theorem). *If a sequence is monotone and bounded, then it converges.*

Definition (Convergence of a Series). Let (b_n) be a sequence. An *infinite series* is a formal expression of the form

$$\sum_{n=1}^{\infty} b_n = b_1 + b_2 + b_3 + b_4 + b_5 + \cdots .$$

We definite the corresponding *sequence of partial sums* (s_m) by

$$s_m = b_1 + b_2 + b_3 + \cdots + b_m,$$

and say that the series $\sum_{n=1}^{\infty} b_n$ *converges to* B if the sequence (s_m) converges to B. In this case, we write $\sum_{n=1}^{\infty} b_n = B$.

Theorem (Cauchy Condensation Test). *Suppose (b_n) is decreasing and satisfies $b_n \geq 0$ for all $n \in \mathbb{N}$. Then, the series $\sum_{n=1}^{\infty} b_n$ converges if and only if the series*

$$\sum_{n=0}^{\infty} 2^n b_{2^n} = b_1 + 2b_2 + 4b_4 + 8b_8 + 16b_{16} + \cdots$$

converges.

2.5 Subsequences and the Bolzano-Weierstrass Theorem.

Definition. Let (a_n) be a sequence of real numbers, and let $n_1 < n_2 < n_3 < n_4 < n_5 < \ldots$ be an increasing sequence of natural numbers. Then the sequence

$$(a_{n_1}, a_{n_2}, a_{n_3}, a_{n_4}, a_{n_5}, \ldots)$$

is called a *subsequence* of (a_n) and is denoted by (a_{n_k}), where $k \in \mathbb{N}$ indexes the subsequence.

Theorem. *Subsequences of a convergent sequence converge to the same limit as the original sequence.*

Corollary (Divergence Criterion). *Suppose that (a_n) is a sequence and (a_{n_k}) is a subsequence that diverges, then (a_n) diverges. If $(a^1_{n_k})$ and $(a^2_{n_k})$ converge to a^1 and a^2 with $a^1 \neq a^2$, then (a_n) diverges.*

Theorem (Bolzano-Weierstrass Theorem). *Every bounded sequence contains a convergent subsequence.*

2.6 The Cauchy Criterion.

Definition. A sequence (a_n) is called a *Cauchy sequence* if, for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that whenever $m, n \geq N$ it follows that $|a_n - a_m| < \epsilon$.

Theorem. *Every convergence sequence is a Cauchy sequence.*

Lemma. *Cauchy sequences are bounded.*

Theorem (Cauchy Criterion). *A sequence converges if and only if it is a Cauchy sequence.*

2.7 Properties of Infinite Series.

Theorem (Algebraic Limit Theorem for Series). *If $\sum_{k=1}^{\infty} a_k = A$ and $\sum_{k=1}^{\infty} b_k = B$, then*

(i) $\sum_{k=1}^{\infty} ca_k = cA$ *for all* $c \in \mathbb{R}$ *and*
(ii) $\sum_{k=1}^{\infty} (a_k + b_k) = A + B$.

Theorem (Cauchy Criterion for Series). *The series $\sum_{k=1}^{\infty} a_k$ converges if and only if, given $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that whenever $n > m \geq N$ it follows that*

$$|a_{m+1} + a_{m+2} + \cdots + a_n| < \epsilon.$$

Theorem (Divergence Test). *If the series $\sum_{k=1}^{\infty} a_k$ converges, then $(a_k) \to 0$. Equivalently, if $(a_k) \not\to 0$, then $\sum_{k=1}^{\infty} a_k$ diverges.*

Theorem (Comparison Test). *Assume (a_k) and (b_k) are sequences satisfying $0 \leq a_k \leq b_k$ for all $k \in \mathbb{N}$.*

(i) If $\sum_{k=1}^{\infty} b_k$ converges, then $\sum_{k=1}^{\infty} a_k$ converges.
(ii) If $\sum_{k=1}^{\infty} a_k$ diverges, then $\sum_{k=1}^{\infty} b_k$ diverges.

Theorem (Squeeze Theorem). *Suppose $a_n \leq b_n \leq c_n$ for all $n \in \mathbb{N}$, and if $\lim a_n = \lim c_n = l$, then $\lim b_n = l$ as well.*

Definition (Geometric Series). A series is called *geometric* if it is of the form

$$\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + ar^3 + \cdots .$$

Theorem. $\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$ *if and only if* $|r| < 1$.

Theorem (Absolute Convergence Test). *If the series $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges as well.*

Theorem (Alternating Series Test). *Let (a_n) be a sequence satisfying,*

(i) $a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_n \geq a_{n+1} \geq \cdots$ and
(ii) $(a_n) \to 0$.

Then, the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges.

Definition. If $\sum_{n=1}^{\infty} |a_n|$ converges, then we say that the original series $\sum_{n=1}^{\infty} a_n$ *converges absolutely*. If, on the other hand, the series $\sum_{n=1}^{\infty} a_n$ converges but the series of absolute values $\sum_{n=1}^{\infty} |a_n|$ does not converge, then we say that the original series $\sum_{n=1}^{\infty} a_n$ *converges conditionally.*