MATH 355: NOTES

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The Real Numbers

Some Preliminaries.

Theorem. *Two real numbers a and b are equal if and only if for every real number $\epsilon > 0$ it follows that $|a - b| < \epsilon$.*

The Axiom of Completeness.

Axiom (Axiom of Completeness)**.** *Every nonempty set of real numbers that is bounded above has a least upper bound.*

Definition. A set $A \subseteq \mathbf{R}$ is *bounded above* if there exists a number $b \in \mathbf{R}$ such that $a \leq b$ for all $a \in A$. The number b is called an *upper bound* for A.

Similarly, the set A is *bounded below* if there exists a *lower bound* $l \in \mathbf{R}$ satisfying $l \leq a$ for every $a \in A$.

Definition. A real number s is the *least upper bound* for a set $A \subseteq \mathbf{R}$ if it meets the following two criteria:

(i) s is an upper bound for A;
(ii) if b is any upper bound for A, then $s \leq b$.

The least upper bound is also frequently called the *supremum* of the set A. We write $s = \sup A$ for the least upper bound.

The *greatest lower bound* or *infimum* for A is defined in a similar way and is denoted by $\inf A$.

Theorem. *Let $A \subseteq \mathbf{R}$ be bounded above and below. Then, the $\sup A$ and $\inf A$ are unique.*

Definition. A real number a_0 is a *maximum* of the set A if a_0 is an element of A and $a_0 \geq a$ for all $a \in A$. Similarly, a number a_1 is a *minimum* of A if $a_1 \in A$ and $a_1 \leq a$ for every $a \in A$.

Theorem. *Let $A \subseteq \mathbf{R}$ be nonempty and bounded above.*

(i) Let $c \in \mathbf{R}$ and define the set $c + A$ by

$$c + A = \{c + a : a \in A\}.$$

Then $\sup(c + A) = c + \sup A$.

(ii) Let $c \in \mathbf{R}$ with $c > 0$ and define the set cA by

$$cA = \{ca : a \in A\}.$$

Then $\sup(cA) = c \sup A$.

Lemma. *Assume $s \in \mathbf{R}$ is an upper bound for a set $A \subseteq R$. Then, $s = \sup A$ if and only if, for every choice of $\epsilon > 0$, there exists an element $a \in A$ satisfying $s - \epsilon < a$.*

Consequences of Completeness.

Theorem (Nested Interval Property). *For each $n \in \mathbf{N}$, assume we are given a closed interval $I_n = [a_n, b_n] = \{x \in \mathbf{R} : a_n \le x \le b_n\}$. Assume also that each I_n contains I_{n+1}. Then, the resulting nested sequence of closed intervals*

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq I_4 \supseteq \cdots$$

has a nonempty intersection; that is, $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

Theorem (Archimedean Property).
(i) *Given any number $x \in \mathbf{R}$, there exists an $n \in \mathbf{N}$ satisfying $n > x$.*
(ii) *Given any real number $y > 0$, there exists an $n \in \mathbf{N}$ satisfying $1/n < y$.*

Definition. A set X is *dense in* $\mathbf{R}$ if for any $a, b \in \mathbf{R}$ with $a < b$, $\exists x \in X$ with $a < x < b$.

Theorem (Density of $\mathbf{Q}$ in $\mathbf{R}$). *For every two real numbers a and b with $a < b$, there exists a rational number r satisfying $a < r < b$.*