MATH 355: NOTES

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1 The Real Numbers

1.2 Some Preliminaries.

Theorem (Triangle Inequality). *For all choices of a and b, $|a+b| \leq |a| + |b|$.*

Theorem. *Two real numbers a and b are equal if and only if for every real number $\epsilon > 0$ it follows that $|a - b| < \epsilon$.*

1.3 The Axiom of Completeness.

Axiom (Axiom of Completeness). *Every nonempty set of real numbers that is bounded above has a least upper bound.*

Definition. A set $A \subseteq \mathbb{R}$ is *bounded above* if there exists a number $b \in \mathbb{R}$ such that $a \leq b$ for all $a \in A$. The number b is called an *upper bound* for A.

Similarly, the set A is *bounded below* if there exists a *lower bound* $l \in \mathbb{R}$ satisfying $l \leq a$ for every $a \in A$.

Definition. A real number s is the *least upper bound* for a set $A \subseteq \mathbb{R}$ if it meets the following two criteria:

(i) s is an upper bound for A;
(ii) if b is any upper bound for A, then $s \leq b$.

The least upper bound is also frequently called the *supremum* of the set A. We write $s = \sup(A)$ for the least upper bound.

The *greatest lower bound* or *infimum* for A is defined in a similar way and is denoted by $\inf(A)$.

Theorem. *Let $A \subseteq \mathbb{R}$ be bounded above and below. Then, the $\sup(A)$ and $\inf(A)$ are unique.*

Definition. A real number a_0 is a *maximum* of the set A if a_0 is an element of A and $a_0 \geq a$ for all $a \in A$. Similarly, a number a_1 is a *minimum* of A if $a_1 \in A$ and $a_1 \leq a$ for every $a \in A$.

Theorem. *Let $A \subseteq \mathbb{R}$ be nonempty and bounded above.*

(i) Let $c \in \mathbb{R}$ and define the set $c + A$ by

$$c + A = \{c + a : a \in A\}.$$

Then $\sup(c + A) = c + \sup(A)$.

(ii) Let $c \in \mathbb{R}$ with $c > 0$ and define the set cA by

$$cA = \{ca : a \in A\}.$$

Then $\sup(cA) = c \sup(A)$.

Lemma. *Assume $s \in \mathbb{R}$ is an upper bound for a set $A \subseteq R$. Then, $s = \sup(A)$ if and only if, for every choice of $\epsilon > 0$, there exists an element $a \in A$ satisfying $s - \epsilon < a$.*

1.4 Consequences of Completeness.

Theorem (Nested Interval Property)**.** *For each $n \in \mathbb{N}$, assume we are given a closed interval $I_n = [a_n, b_n] = \{x \in \mathbb{R} : a_n \leq x \leq b_n\}$. Assume also that each I_n contains I_{n+1}. Then, the resulting nested sequence of closed intervals*

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq I_4 \supseteq \cdots$$

has a nonempty intersection; that is, $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

Theorem (Archimedean Property)**.** *(i) Given any number $x \in \mathbb{R}$, there exists an $n \in \mathbb{N}$ satisfying $n > x$.*
(ii) Given any real number $y > 0$, there exists an $n \in \mathbb{N}$ satisfying $1/n < y$.

Definition. A set X is *dense in* $\mathbb{R}$ if for any $a, b \in \mathbb{R}$ with $a < b$, $\exists x \in X$ with $a < x < b$.

Theorem (Density of $\mathbb{Q}$ in $\mathbb{R}$)**.** *For every two real numbers a and b with $a < b$, there exists a rational number r satisfying $a < r < b$.*

1.5 Cardinality.

Definition. A function $f : A \to B$ is *1–1 (injective)* if for all $a_1, a_2 \in A$, $f(a_1) = f(a_2)$ implies that $a_1 = a_2$.

Definition. A function $f : A \to B$ is *onto (surjective)* if for all $b \in B$, there exists an $a \in A$ such that $f(a) = b$.

Definition. A function $f : A \to B$ is a *bijection* if it is both 1–1 and onto.

Definition. Two sets A and B have the same *cardinality* if there exists a bijection $f : A \to B$. In this case, we write $A \sim B$.

Definition. A set A is *finite* if there exists an $n \in \mathbb{N}$ such that $A \sim \{1, 2, \ldots, n\}$.

Definition. A set A is *countable* if $A \sim \mathbb{N}$.

Definition. A set which is not finite nor countable is *uncountable*.

Theorem. *(i) The set $\mathbb{Q}$ is countable.*
(ii) The set $\mathbb{R}$ is uncountable.

Theorem. *If $A \subseteq B$ is countable, then A is either countable or finite.*

Theorem. *(i) If $A_1, A_2, \ldots, A_m$ are each countable sets, then the union $A_1 \cup A_2 \cup \cdots \cup A_m$ is countable.*
(ii) If A_n is a countable set for each $n \in \mathbb{N}$, then $\bigcup_{n=1}^{\infty} A_n$ is countable.

1.6 Cantor's Theorem.

Theorem. *The open interval $(0, 1) = \{x \in \mathbb{R} : 0 < x < 1\}$ is uncountable.*

Definition. Given a set A, the *power set* $P(A)$ refers to the collection of all subsets of A.

Theorem (Cantor's Theorem)**.** *Given any set A, there does not exist a function $f : A \to P(A)$ that is onto.*

2 Sequences and Series

2.2 The Limit of a Sequence.

Definition. A *sequence* is a function whose domain is $\mathbb{N}$.

Definition (Convergence of a Sequence)**.** A sequence (a_n) *converges* to a real number a if, for every positive number ϵ, there exists and $N \in \mathbb{N}$ such that whenever $n \geq N$, it follows that $|a_n - a| < \epsilon$.

Definition. Given a real number $a \in \mathbb{R}$ and a positive number $\epsilon > 0$, the set

$$V_\epsilon(a) = \{x \in \mathbb{R} : |x - a| < \epsilon\} = (a - \epsilon, a + \epsilon)$$

is called the *ϵ-neighborhood of a.*

Definition (Convergence of a Sequence: Topological Version)**.** A sequence (a_n) converges to a if, given any ϵ-neighborhood $V_\epsilon(a)$ of a, there exists a point in the sequence after which all of the terms are in $V_\epsilon(a)$. In other words, every ϵ-neighborhood contains all but a finite number of terms of (a_n).

Theorem (Uniqueness of Limits)**.** *The limit of a sequence, when it exists, must be unique.*

Definition. A sequence that does not converge is said to *diverge*.

2.3 The Algebraic and Order Limit Theorems.

Definition. A sequence (x_n) is *bounded* if there exists a number $M > 0$ such that $|x_n| \leq M$ for all $n \in \mathbb{N}$.

Theorem. *Every convergent sequence is bounded.*

Theorem (Algebraic Limit Theorem)**.** *Let* $\lim a_n = a$ *and* $\lim b_n = b$. *Then,*

(i) $\lim(ca_n) = ca$, *for all* $c \in \mathbb{R}$*;*
(ii) $\lim(a_n + b_n) = a + b$*;*
(iii) $\lim(a_n b_n) = ab$*;*
(iv) $\lim(a_n/b_n) = a/b$, *provided* $b \neq 0$.

Theorem (Order Limit Theorem)**.** *Assume* $\lim a_n = a$ *and* $\lim b_n = b$.

(i) If $a_n \geq 0$ *for all* $n \in \mathbb{N}$, *then* $a \geq 0$.
(ii) If $a_n \leq b_n$ *for all* $n \in \mathbb{N}$, *then* $a \leq b$.
(iii) If there exists $c \in \mathbb{R}$ *for which* $c \leq b_n$ *for all* $n \in \mathbb{N}$, *then* $c \leq b$. *Similarly, if* $a_n \leq c$ *for all* $n \in \mathbb{N}$, *then* $a \leq c$.

2.4 The Monotone Convergence Theorem and a First Look at Infinite Series.

Definition. A sequence (a_n) is *increasing* if $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$ and *decreasing* if $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$. A sequence is *monotone* if it is either increasing or decreasing.

Theorem (Monotone Convergence Theorem)**.** *If a sequence is monotone and bounded, then it converges.*

Definition (Convergence of a Series)**.** Let (b_n) be a sequence. An *infinite series* is a formal expression of the form

$$\sum_{n=1}^{\infty} b_n = b_1 + b_2 + b_3 + b_4 + b_5 + \cdots .$$

We definite the corresponding *sequence of partial sums* (s_m) by

$$s_m = b_1 + b_2 + b_3 + \cdots + b_m,$$

and say that the series $\sum_{n=1}^{\infty} b_n$ *converges to* B if the sequence (s_m) converges to B. In this case, we write $\sum_{n=1}^{\infty} b_n = B$.

Theorem (Cauchy Condensation Test)**.** *Suppose* (b_n) *is decreasing and satisfies* $b_n \geq 0$ *for all* $n \in \mathbb{N}$*. Then, the series* $\sum_{n=1}^{\infty} b_n$ *converges if and only if the series*

$$\sum_{n=0}^{\infty} 2^n b_{2^n} = b_1 + 2b_2 + 4b_4 + 8b_8 + 16b_{16} + \cdots$$

converges.

Corollary. *The series* $\sum_{n=1}^{\infty} 1/n^p$ *converges if and only if* $p > 1$.

2.5 Subsequences and the Bolzano-Weierstrass Theorem.

Definition. Let (a_n) be a sequence of real numbers, and let $n_1 < n_2 < n_3 < n_4 < n_5 < \ldots$ be an increasing sequence of natural numbers. Then the sequence

$$(a_{n_1}, a_{n_2}, a_{n_3}, a_{n_4}, a_{n_5}, \ldots)$$

is called a *subsequence* of (a_n) and is denoted by (a_{n_k}), where $k \in \mathbb{N}$ indexes the subsequence.

Theorem. *Subsequences of a convergent sequence converge to the same limit as the original sequence.*

Corollary (Divergence Criterion)**.** *Suppose that* (a_n) *is a sequence and* (a_{n_k}) *is a subsequence that diverges, then* (a_n) *diverges. If* $(a^1_{n_k})$ *and* $(a^2_{n_k})$ *converge to* a^1 *and* a^2 *with* $a^1 \neq a^2$*, then* (a_n) *diverges.*

Theorem (Bolzano-Weierstrass Theorem)**.** *Every bounded sequence contains a convergent subsequence.*

2.6 The Cauchy Criterion.

Definition. A sequence (a_n) is called a *Cauchy sequence* if, for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that whenever $m, n \geq N$ it follows that $|a_n - a_m| < \epsilon$.

Theorem. *Every convergence sequence is a Cauchy sequence.*

Lemma. *Cauchy sequences are bounded.*

Theorem (Cauchy Criterion)**.** *A sequence converges if and only if it is a Cauchy sequence.*

2.7 Properties of Infinite Series.

Theorem (Algebraic Limit Theorem for Series)**.** *If* $\sum_{k=1}^{\infty} a_k = A$ *and* $\sum_{k=1}^{\infty} b_k = B$*, then*

(i) $\sum_{k=1}^{\infty} ca_k = cA$ *for all* $c \in \mathbb{R}$ *and*
(ii) $\sum_{k=1}^{\infty}(a_k + b_k) = A + B$.

Theorem (Cauchy Criterion for Series)**.** *The series* $\sum_{k=1}^{\infty} a_k$ *converges if and only if, given* $\epsilon > 0$*, there exists an* $N \in \mathbb{N}$ *such that whenever* $n > m \geq N$ *it follows that*

$$|a_{m+1} + a_{m+2} + \cdots + a_n| < \epsilon.$$

Theorem (Divergence Test)**.** *If the series* $\sum_{k=1}^{\infty} a_k$ *converges, then* $(a_k) \to 0$*. Equivalently, if* $(a_k) \not\to 0$*, then* $\sum_{k=1}^{\infty} a_k$ *diverges.*

Theorem (Comparison Test)**.** *Assume* (a_k) *and* (b_k) *are sequences satisfying* $0 \leq a_k \leq b_k$ *for all* $k \in \mathbb{N}$*.*

(i) If $\sum_{k=1}^{\infty} b_k$ *converges, then* $\sum_{k=1}^{\infty} a_k$ *converges.*
(ii) If $\sum_{k=1}^{\infty} a_k$ *diverges, then* $\sum_{k=1}^{\infty} b_k$ *diverges.*

Theorem (Squeeze Theorem)**.** *Suppose* $a_n \leq b_n \leq c_n$ *for all* $n \in \mathbb{N}$*, and if* $\lim a_n = \lim c_n = l$*, then* $\lim b_n = l$ *as well.*

Definition (Geometric Series)**.** A series is called *geometric* if it is of the form

$$\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + ar^3 + \cdots .$$

Theorem. $\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$ *if and only if* $|r| < 1$*.*

Theorem (Absolute Convergence Test)**.** *If the series* $\sum_{n=1}^{\infty} |a_n|$ *converges, then* $\sum_{n=1}^{\infty} a_n$ *converges as well.*

Theorem (Alternating Series Test)**.** *Let* (a_n) *be a sequence satisfying,*

(i) $a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_n \geq a_{n+1} \geq \cdots$ *and*
(ii) $(a_n) \to 0$*.*

Then, the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ *converges.*

Definition. If $\sum_{n=1}^{\infty} |a_n|$ converges, then we say that the original series $\sum_{n=1}^{\infty} a_n$ *converges absolutely.* If, on the other hand, the series $\sum_{n=1}^{\infty} a_n$ converges but the series of absolute values $\sum_{n=1}^{\infty} |a_n|$ does not converge, then we say that the original series $\sum_{n=1}^{\infty} a_n$ *converges conditionally.*

Theorem (Ratio Test)**.** *Given a series* $\sum_{n=1}^{\infty} a_n$ *with* $a_n \neq 0$*, if* (a_n) *satisfies*

$$\lim \left| \frac{a_{n+1}}{a_n} \right| = r < 1,$$

then the series converges absolutely.

3 Basic Topology of $\mathbb{R}$

3.2 Open and Closed Sets.

Definition. A set $O \subseteq \mathbb{R}$ is *open* if for all points $a \in O$ there exists an ϵ-neighborhood $V_\epsilon(a) \subseteq O$.

Theorem. *(i) The union of an arbitrary collection of open sets is open.*
(ii) The intersection of a finite collection of open sets is open.

Definition. A point x is a *limit point* of a set A if every ϵ-neighborhood $V_\epsilon(x)$ of x intersects the set A at some point other than x.

Theorem. *A point x is a limit point of a set A if and only if $x = \lim a_n$ for some sequence (a_n) contained in A satisfying $a_n \neq x$ for all $n \in \mathbb{N}$.*

Definition. A point $a \in A$ is an *isolated point of A* if it is not a limit point of A.

Definition. A set $F \subseteq \mathbb{R}$ is *closed* if it contains its limit points.

Theorem. *A set $F \subset \mathbb{R}$ is closed if and only if every Cauchy sequence contained in F has a limit that is also an element of F.*

Definition. Given a set $A \subseteq \mathbb{R}$, let L be the set of all limit points of A. The *closure* of A is defined to be $\overline{A} = A \cup L$.

Theorem. *For any $A \subseteq \mathbb{R}$, the closure $\overline{A}$ is a closed set and is the smallest closed set containing A.*

Theorem. *A set O is open if and only if O^c is closed. Likewise, a set F is closed if and only if F^c is open.*

Theorem. *(i) The union of a finite collection of closed sets is closed.*
(ii) The intersection of an arbitrary collection of closed sets is closed.

3.3 Compact Sets.

Definition (Compactness)**.** A set $K \subseteq \mathbb{R}$ is *compact* if every sequence in K has a subsequence that converges to a limit that is also in K.

Definition. A set $A \subseteq \mathbb{R}$ is *bounded* if there exists $M > 0$ such that $|a| \leq M$ for all $a \in A$.

Theorem (Characterization of Compactness in $\mathbb{R}$)**.** *A set $K \subseteq \mathbb{R}$ is compact if and only if it is closed and bounded.*

Theorem (Nested Compact Set Property)**.** *If*

$$K_1 \supseteq K_2 \supseteq K_3 \supseteq K_4 \supseteq \cdots$$

is a nested sequence of nonempty compact sets, then the intersection $\cap_{n=1}^{\infty} K_n$ is not empty.

Definition. Let $A \subseteq \mathbb{R}$. An *open cover* for A is a (possibly infinite) collection of open sets $\{O_\lambda : \lambda \in \Lambda\}$ whose union contains the set A; that is, $A \subseteq \cup_{\lambda \in \Lambda} O_\lambda$. Given an open cover for A, a *finite subcover* is a finite subcollection of open sets from the original open cover whose union still manages to completely contain A.

Theorem (Heine-Borel Theorem)**.** *Let K be a subset of $\mathbb{R}$. All of the following statements are equivalent in the sense that any one of them implies the two others:*

(i) K is compact.
(ii) K is closed and bounded.
(iii) Every open cover for K has a finite subcover.

4 Functional Limits and Continuity

4.2 Functional Limits.

Definition (Functional Limit)**.** Let $f : A \to \mathbb{R}$, and let c be a limit point of the domain A. We say that $\lim_{x\to c} f(x) = L$ provided that, for all $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $0 < |x - c| < \delta$ (and $x \in A$) it follows that $|f(x) - L| < \epsilon$.

Definition (Functional Limit: Topological Version)**.** Let c be a limit point of the domain of $f : A \to \mathbb{R}$. We say $\lim_{x\to c} f(x) = L$ provided that, for every ϵ-neighborhood $V_\epsilon(L)$ of L, there exists a δ-neighborhood $V_\delta(c)$ around c with the property that for all $x \in V_\delta(c)$ different from c (with $x \in A$) it follows that $f(x) \in V_\epsilon(L)$.

Theorem (Sequential Criterion for Functional Limits)**.** *Given a function $f : A \to \mathbb{R}$ and a limit point c of A, the following two statements are equivalent:*

(i) $\lim_{x\to c} f(x) = L$.
(ii) For all sequences $(x_n) \subseteq A$ satisfying $x_n \neq c$ and $(x_n) \to c$, it follows that $f(x_n) \to L$.

Corollary (Algebraic Limit Theorem for Functional Limits)**.** *Let f and g be functions defined on a domain $A \subseteq \mathbb{R}$, and assume $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$ for some limit point c of A. Then,*

(i) $\lim_{x\to c} kf(x) = kL$ *for all* $k \in \mathbb{R}$,
(ii) $\lim_{x\to c}[f(x) + g(x)] = L + M$,
(iii) $\lim_{x\to c}[f(x)g(x)] = LM$, *and*
(iv) $\lim_{x\to c} f(x)/g(x) = L/M$, *provided* $M \neq 0$.

Corollary (Divergence Criterion for Functional Limits)**.** *Let f be a function defined on A, and let c be a limit point of A. If there exists two sequences (x_n) and (y_n) in A with $x_n \neq c$ and $y_n \neq c$ and*

$$\lim x_n = \lim y_n = c \quad \textit{but} \quad \lim f(x_n) \neq \lim f(y_n),$$

then we can conclude that the functional limit $\lim_{x\to c} f(x)$ does not exist.

4.3 Continuous Functions.

Definition (Continuity)**.** A function $f : A \to \mathbb{R}$ is *continuous at a point* $c \in A$ if, for all $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $|x - c| < \delta$ (and $x \in A$) it follows that $|f(x) - f(c)| < \epsilon$.

If f is continuous at every point in the domain A, then we say that f is *continuous on* A.

Theorem (Characterizations of Continuity)**.** *Let $f : A \to \mathbb{R}$, and let $c \in A$. The function f is continuous at c if and only if any one of the following three conditions is met:*

(i) For all $\epsilon > 0$, there exists a $\delta > 0$ such that $|x - c| < \delta$ (and $x \in A$) implies $|f(x) - f(c)| < \epsilon$;
(ii) For all $V_\epsilon(f(c))$, there exists a $V_\delta(c)$ with the property that $x \in V_\delta(c)$ (and $x \in A$) implies $f(x) \in V_\epsilon(f(c))$;
(iii) For all $(x_n) \to c$ (with $x_n \in A$), it follows that $f(x_n) \to f(c)$.
(iv) If c is a limit point of A, then the above conditions are equivalent to $\lim_{x\to c} f(x) = f(c)$.

Corollary (Criterion for Discontinuity). *Let $f : A \to \mathbb{R}$, and let $c \in A$ be a limit point of A. If there exists a sequence $(x_n) \subseteq A$ where $(x_n) \to c$ but such that $f(x_n)$ does not converge to $f(c)$, we may conclude that f is not continuous at c.*

Theorem (Algebraic Continuity Theorem). *Assume $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ are continuous at point $c \in A$. Then,*

(i) $kf(x)$ is continuous at c for all $k \in \mathbb{R}$;
(ii) $f(x) + g(x)$ is continuous at c;
(iii) $f(x)g(x)$ is continuous at c; and
(iv) $f(x)/g(x)$ is continuous at c, provided the quotient is defined.

4.4 Continuous Functions on Compact Sets.

Theorem (Preservation of Compact Sets). *Let $f : A \to \mathbb{R}$ be continuous on A. If $K \subseteq A$ is compact, then $f(K)$ is compact as well.*

Theorem (Extreme Value Theorem). *If $f : K \to \mathbb{R}$ is continuous on a compact set $K \subseteq \mathbb{R}$, then f attains a maximum and minimum value. In other words, there exists $x_0, x_1 \in K$ such that $f(x_0) \leq f(x) \leq f(x_1)$ for all $x \in K$.*

Definition (Uniform Continuity). A function $f : A \to \mathbb{R}$ is *uniformly continuous on A* if for every $\epsilon > 0$ there exists a $\delta > 0$ such that for all $x, y \in A$, $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$.

Theorem (Sequential Criterion for Absence of Uniform Continuity). *A function $f : A \to \mathbb{R}$ fails to be uniformly continuous on A if and only if there exists a particular $\epsilon_0 > 0$ and two sequences (x_n) and (y_n) in A satisfying*

$$|x_n - y_n| \to 0 \quad \text{but} \quad |f(x_n) - f(y_n)| \geq \epsilon_0.$$

Theorem (Uniform Continuity on Compact Sets). *A function that is continuous on a compact set K is uniformly continuous on K.*

4.5 The Intermediate Value Theorem.

Theorem (Intermediate Value Theorem). *Let $f : [a, b] \to \mathbb{R}$ be continuous. If L is a real number satisfying $f(a) < L < f(b)$ or $f(a) > L > f(b)$, then there exists a point $c \in (a, b)$ where $f(c) = L$.*

5 The Derivative

5.2 Derivatives and the Intermediate Value Property.

Definition (Differentiability). Let $g : A \to \mathbb{R}$ be a function defined on an interval A. Given $c \in A$, the *derivative of g at c* is defined by

$$g'(c) = \lim_{x \to c} \frac{g(x) - g(c)}{x - c},$$

provided this limit exists. In this case we say *g is differentiable at c*. If g' exists for all points $c \in A$, we say that *g is differentiable on A*.

Theorem. *If $g : A \to \mathbb{R}$ is differentiable at point $c \in A$, then g is continuous at c at well.*

Theorem (Algebraic Differentiability Theorem). *Let f and g be functions defined on an interval A, and assume both are differentiable at some point $c \in A$. Then,*

(i) $(f+g)'(c) = f'(c) + g'(c)$,
(ii) $(kf)'(c) = kf'(c)$, *for all* $k \in \mathbb{R}$,
(iii) $(fg)'(c) = f'(c)g(c) + f(c)g'(c)$, *and*
(iv) $(f/g)'(c) = \frac{g(c)f'(c) - f(c)g'(c)}{[g(c)]^2}$, *provided that* $g(c) \neq 0$.

Theorem (Chain Rule)**.** *Let* $f : A \to \mathbb{R}$ *and* $g : B \to \mathbb{R}$ *satisfy* $f(A) \subseteq B$ *so that the composition* $g \circ f$ *is defined. If* f *is differentiable at* $c \in A$ *and if* g *is differentiable at* $f(c) \in B$*, then* $g \circ f$ *is differentiable at* c *with* $(g \circ f)'(c) = g'(f(c)) \cdot f'(c)$.

Theorem (Interior Extremum Theorem)**.** *Let* f *be differentiable on an open interval* (a, b)*. If* f *attains a maximum value at some point* $c \in (a, b)$ *(i.e.,* $f(c) \geq f(x)$ *for all* $x \in (a, b)$*), then* $f'(c) = 0$*. The same is true if* $f(c)$ *is a minimum value.*

5.3 The Mean Value Theorems.

Theorem (Rolle's Theorem)**.** *Let* $f : [a, b] \to \mathbb{R}$ *be continuous on* $[a, b]$ *and differentiable on* (a, b)*. If* $f(a) = f(b)$*, then there exists a point* $c \in (a, b)$ *where* $f'(c) = 0$.

Theorem (Mean Value Theorem)**.** *If* $f : [a, b] \to \mathbb{R}$ *is continuous on* $[a, b]$ *and differentiable on* (a, b)*, then there exists a point* $c \in (a, b)$ *where*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Corollary. *If* $g : A \to \mathbb{R}$ *is differentiable on an interval* A *and satisfies* $g'(x) = 0$ *for all* $x \in A$*, then* $g(x) = k$ *for some constant* $k \in \mathbb{R}$.

Corollary. *If* f *and* g *are differentiable functions on an interval* A *and satisfy* $f'(x) = g'(x)$ *for all* $x \in A$*, then* $f(x) = g(x) + k$ *for some constant* $k \in \mathbb{R}$.

6 Sequences and Series of Functions

6.2 Uniform Convergence of a Sequence of Functions.

Definition (Pointwise Convergence)**.** For each $n \in \mathbb{N}$, let f_n be a function defined on a set $A \subseteq \mathbb{R}$. The sequence (f_n) of functions *converges pointwise on* A to a function f if, for all $x \in A$, the sequence of real numbers $f_n(x)$ converges to $f(x)$.

In this case, we write $f_n \to f$, $\lim f_n = f$, or $\lim_{n\to\infty} f_n(x) = f(x)$. This last expression is helpful if there is any confusion as to whether x or n is the limiting variable.

Definition (Pointwise Convergence)**.** Let (f_n) be a sequence of functions defined on a set $A \subseteq \mathbb{R}$. Then, (f_n) *converges pointwise on* A to a limit f defined on A if, for every $\epsilon > 0$ and $x \in A$, there exists an $N \in \mathbb{N}$ (perhaps dependent on x) such that $|f_n(x) - f(x)| < \epsilon$ whenever $n \geq N$.

Definition (Uniform Convergence)**.** Let (f_n) be a sequence of functions defined on a set $A \subseteq \mathbb{R}$. Then, (f_n) *converges uniformly on* A to a limit function f defined on A if, for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \epsilon$ whenever $n \geq N$ and $x \in A$.

Theorem (Cauchy Criterion for Uniform Convergence)**.** *A sequence of functions* (f_n) *defined on a set* $A \subseteq \mathbb{R}$ *converges uniformly on* A *if and only if for every* $\epsilon > 0$ *there exists an* $N \in \mathbb{N}$ *such that* $|f_n(x) - f_m(x)| < \epsilon$ *whenever* $m, n \geq N$ *and* $x \in A$.

Theorem (Continuous Limit Theorem). *Let (f_n) be a sequence of functions defined on $A \subseteq \mathbb{R}$ that converges uniformly on A to a function f. If each f_n is continuous at $c \in A$, then f is continuous at c.*

6.3 Uniform Convergence and Differentiation.

Theorem (Differentiable Limit Theorem). *Let $f_n \to f$ pointwise on the closed interval $[a, b]$, and assume that each f_n is differentiable. If (f_n') converges uniformly on $[a, b]$ to a function g, then the function is differentiable and $f' = g$.*

6.4 Series of Functions.

Definition. For each $n \in \mathbb{N}$, let f_n and f be functions defined on a set $A \subseteq \mathbb{R}$. The infinite series

$$\sum_{n=1}^{\infty} f_n(x) = f_1(x) + f_2(x) + f_3(x) + \cdots$$

converges pointwise on A to $f(x)$ if the sequence $s_k(x)$ of partial sums defined by

$$s_k(x) = f_1(x) + f_2(x) + \cdots + f_k(x)$$

converges pointwise to $f(x)$. The series *converges uniformly on* A to f if the sequence $s_k(x)$ converges uniformly on A to $f(x)$.

In either case, we write $f = \sum_{n=1}^{\infty} f_n$ or $f(x) = \sum_{n=1}^{\infty} f_n(x)$, always being explicit about the type of convergence involved.

Theorem (Term-by-term Continuity Theorem). *Let f_n be continuous functions defined on a set $A \subseteq \mathbb{R}$, and assume $\sum_{n=1}^{\infty} f_n$ converges uniformly on A to a function f. Then, f is continuous on A.*

Theorem (Term-by-term Differentiability Theorem). *Let f_n be differentiable functions defined on an interval A, and assume $\sum_{n=1}^{\infty} f_n'(x)$ converges uniformly to a limit $g(x)$ on A. If $\sum_{n=1}^{\infty} f_n(x)$ converges pointwise to $f(x)$, then $f(x)$ is differentiable and $f'(x) = g(x)$ on A.*

Theorem (Cauchy Criterion for Uniform Convergence of Series). *A series $\sum_{n=1}^{\infty} f_n$ converges uniformly on $A \subseteq \mathbb{R}$ if and only if for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that*

$$|f_{m+1}(x) + f_{m+2}(x) + f_{m+3}(x) + \cdots + f_n(x)| < \epsilon$$

whenever $n > m \geq N$ and $x \in A$.

Corollary (Weierstrass M-Test). *For each $n \in \mathbb{N}$, let f_n be a function defined on a set $A \subseteq \mathbb{R}$, and let $M_n > 0$ be a real number satisfying*

$$|f_n(x)| \leq M_n$$

for all $x \in A$. If $\sum_{n=1}^{\infty} M_n$ converges, then $\sum_{n=1}^{\infty} f_n$ converges uniformly on A.

6.5 Power Series.

Theorem. *If a power series $\sum_{n=0}^{\infty} a_n x^n$ converges at some point $x_0 \in \mathbb{R}$, then it converges absolutely for any x satisfying $|x| < |x_0|$.*

Theorem. *If a power series $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely at a point x_0, then it converges uniformly on the closed interval $[-c, c]$, where $c = |x_0|$.*

Lemma (Abel's Lemma). *Let b_n satisfy $b_1 \geq b_2 \geq b_3 \geq \cdots \geq 0$, and let $\sum_{n=1}^{\infty} a_n$ be a series for which the partial sums are bounded. In other words, assume there exists $A > 0$ such that*

$$|a_1 + a_2 + \cdots + a_n| \leq A$$

for all $n \in \mathbb{N}$. Then, for all $n \in \mathbb{N}$,

$$|a_1 b_1 + a_2 b_2 + a_3 b_3 + \cdots + a_n b_n| \leq A b_1.$$

Theorem (Abel's Theorem). *Let $g(x) = \sum_{n=0}^{\infty} a_n x^n$ be a power series that converges at the point $x = R > 0$. Then the series converges uniformly on the interval $[0, R]$. A similar result holds if the series converges at $x = -R$.*

Theorem. *If a power series converges pointwise on the set $A \subseteq \mathbb{R}$, then it converges uniformly on any compact set $K \subseteq A$.*

Theorem. *If $\sum_{n=0}^{\infty} a_n x^n$ converges for all $x \in (-R, R)$, then the differentiated series $\sum_{n=1}^{\infty} n a_n x^{n-1}$ converges at each $x \in (-R, R)$ as well. Consequently, the convergence is uniform on compact sets contained in $(-R, R)$.*

Theorem. *Assume*

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

converges on an interval $A \subseteq \mathbb{R}$. The function f is continuous on A and differentiable on any open interval $(-R, R) \subseteq A$. The derivative is given by

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}.$$

Moreover, f is infinitely differentiable on $(-R, R)$, and the successive derivatives can be obtained via term-by-term differentiable of the appropriate series.