**LINEAR ALGEBRA**

## Vector Spaces and Subspaces

**Definition.** A (real) *vector space* is a set $V$ (whose elements are called *vectors*) together with

(1) an operation called *vector addition*, which for each pair of vector $\vec{x}, \vec{y} \in V$ produces another vector in $V$ denoted $\vec{x} + \vec{y}$, and
(2) an operation called *multiplication by a scalar* (a real number), which for each vector $\vec{x} \in V$, and each scalar $c \in \mathbb{R}$ produces another vector in $V$ denoted $c\vec{x}$.

Furthermore, the two operations must satisfy the follow *axioms*:

(1) For all vectors $\vec{x}, \vec{y}$, and $\vec{z} \in V$, $(\vec{x} + \vec{y}) + \vec{z} = \vec{x} + (\vec{y} + \vec{z})$.
(2) For all vectors $\vec{x}$ and $\vec{y} \in V$, $\vec{x} + \vec{y} = \vec{y} + \vec{x}$.
(3) There exists a vector $\vec{0} \in V$ with the property that $\vec{x} + \vec{0} = \vec{x}$ for all vectors $\vec{x} \in V$.
(4) For each vector $\vec{x} \in V$, there exists a vector denoted $-\vec{x}$ with the property that $\vec{x} + -\vec{x} = \vec{0}$.
(5) For all vectors $\vec{x}$ and $\vec{y} \in V$ and all scalars $c \in \mathbb{R}$, $c(\vec{x} + \vec{y}) = c\vec{x} + c\vec{y}$.
(6) For all vectors $\vec{x} \in V$, and all scalars $c$ and $d \in \mathbb{R}$, $(c + d)\vec{x} = c\vec{x} + d\vec{x}$.
(7) For all vectors $\vec{x} \in V$, and all scalars $c$ and $d \in \mathbb{R}$, $(cd)\vec{x} = c(d\vec{x})$.
(8) For all vectors $\vec{x} \in V$, $1\vec{x} = \vec{x}$.

**Examples.** Some simple vector spaces:

- $\mathbb{R}^n$ is the vector space of ordered $n$-tuples of real numbers. Note: $\dim(\mathbb{R}^n) = n$.
- $P_n(\mathbb{R})$ is the vector space of polynomials of degree *less than or equal to* $n$. Note: $\dim(P_n(\mathbb{R})) = n + 1$.
- $M_{m\times n}(\mathbb{R})$ is the vector space of $m \times n$ matrices with real entries. Note: $\dim(M_{m\times n}(\mathbb{R})) = mn$.

**Definition.** Let $V$ be a vector space and let $W \subseteq V$ be a subset. Then $W$ is a (vector) *subspace* of $V$ if $W$ is a vector space itself under the operations of vector sum and scalar multiplication from $V$.

**Notes.** The empty set $\emptyset$ is not a vector space. Instead the smallest vector space is the trivial space, $\{\vec{0}\}$. Every vector space $V$ has two obvious subspaces: the trivial subspace $\{\vec{0}\} \subseteq V$, and the improper subspace $V \subseteq V$.

**Theorem** (Subspace Theorem)**.** Let $V$ be a vector space. A subset $W \subseteq V$ is a subspace if it satisfies the following properties:

(1) $W \neq \emptyset$
(2) For all $\vec{x}, \vec{y} \in W$ and all $c \in \mathbb{R}$, we have $c\vec{x} + \vec{y} \in W$.

**Definition.** Let $V$ be a vector space, and let $S = \{\vec{v}_1, \dots, \vec{v}_n\} \subseteq V$ be a finite set of vectors in $V$.

- A *linear combination* of elements of $S$ is an expression $a_1\vec{v}_1 + \cdots + a_n\vec{v}_n$ for some scalars $a_1, \dots, a_n \in \mathbb{R}$.
- The *span* of $S$, denoted $\mathrm{Span}(S)$, is the set of all linear combinations of elements of $S$. That is,
$$\mathrm{Span}(S) = \{a_1\vec{v}_1 + \cdots a_n\vec{v}_n \mid a_1, \dots, a_n \in \mathbb{R}\}.$$
- We define $\mathrm{Span}\emptyset = \{\vec{0}\}$.
- If $\mathrm{Span}(S) = W$, we say that $S$ spans $W$.

**Theorem.** Let $V$ be a vector space and let $S$ be any subset of $V$. Then $\mathrm{Span}(S)$ is a subspace of $V$.

**Theorem.** If $W$ is a subspace and $S \subseteq W$, then $\mathrm{Span}(S) \subseteq W$.

**Definition.** The set $S$ is *linearly dependent* if there exists scalars $a_1, \dots, a_n \in \mathbb{R}$ that are not all zero such that $a_1\vec{v}_1 + \cdots + a_n\vec{v}_n = \vec{0}$. $S$ is *linearly independent* if it is not linearly dependent. Equivalently, for any scalars $a_1, \dots, a_n \in \mathbb{R}$ such that $a_1\vec{v}_1 + \cdots a_n\vec{v}_n = \vec{0}$, we must have $a_1 = \cdots = a_n = 0$.

**Definition.** The set $S \subseteq V$ is a basis for $V$ if $S$ is linearly independent and $\mathrm{Span}(S) = V$.

**Definition.** The *dimension* of $V$ is the number $\dim(V)$ of elements in a basis for $V$. If $V$ has no finite basis, we say $\dim(V) = \infty$.

**Theorem.** Any two bases of $V$ have the same number of elements.

**Fact.** The three kinds of row reduction steps are
1. Switching two rows.
2. Multiplying a row by a nonzero scalar.
3. Adding a multiple of one row to another.

**Definition.** A matrix is in *echelon form* if it satisfies all of the following conditions:
1. If a row is not a zero row (i.e., all entries of that row are zeros), then the first nonzero entry is a 1 (called the *pivot*).
2. If a column contains a pivot, then all other entries in that column are 0.
3. If a row contains a pivot, then each row above contains a pivot further to the left. This also implies that zero rows, if any, appear at the bottom.

Variables corresponding to the pivots are called *pivot variables*. All other variables are called *free variables*.

**Definition.** A *homogeneous* system of linear equations is when all the linear combinations equal 0. A system is *inhomogeneous* otherwise.

**Definition.** The *nullspace* of a matrix $A$ is the solution set of its corresponding homogeneous system of equations. The basis of the nullspace of $A$ is the set of vectors that the free variables end up multiplied by in the solution.

**Definition.** The *column space* of a matrix $A$ is the span of its columns. If $B$ is the echelon form of $A$, then the columns of $A$ corresponding to the columns of $B$ with pivots form a basis of the column space.