LINEAR ALGEBRA

Vector Spaces and Subspaces

Definition. A (real) *vector space* is a set V (whose elements are called *vectors*) together with

(1) an operation called *vector addition*, which for each pair of vector $\vec{x}, \vec{y} \in V$ produces another vector in V denoted $\vec{x} + \vec{y}$, and
(2) an operation called *multiplication by a scalar* (a real number), which for each vector $\vec{x} \in V$, and each scalar $c \in \mathbb{R}$ produces another vector in V denoted $c\vec{x}$.

Furthermore, the two operations must satisfy the follow *axioms*:

(1) For all vectors $\vec{x}, \vec{y}$, and $\vec{z} \in V$, $(\vec{x} + \vec{y}) + \vec{z} = \vec{x} + (\vec{y} + \vec{z})$.
(2) For all vectors $\vec{x}$ and $\vec{y} \in V$, $\vec{x} + \vec{y} = \vec{y} + \vec{x}$.
(3) There exists a vector $\vec{0} \in V$ with the property that $\vec{x} + \vec{0} = \vec{x}$ for all vectors $\vec{x} \in V$.
(4) For each vector $\vec{x} \in V$, there exists a vector denoted $-\vec{x}$ with the property that $\vec{x} + -\vec{x} = \vec{0}$.
(5) For all vectors $\vec{x}$ and $\vec{y} \in V$ and all scalars $c \in \mathbb{R}$, $c(\vec{x} + \vec{y}) = c\vec{x} + c\vec{y}$.
(6) For all vectors $\vec{x} \in V$, and all scalars c and $d \in \mathbb{R}$, $(c + d)\vec{x} = c\vec{x} + d\vec{x}$.
(7) For all vectors $\vec{x} \in V$, and all scalars c and $d \in \mathbb{R}$, $(cd)\vec{x} = c(d\vec{x})$.
(8) For all vectors $\vec{x} \in V$, $1\vec{x} = \vec{x}$.

Definition. Let V be a vector space and let $W \subseteq V$ be a subset. Then W is a (vector) *subspace* of V if W is a vector space itself under the operations of vector sum and scalar multiplication from V.

Theorem (Subspace Theorem)**.** Let V be a vector space. A subset $W \subseteq V$ is a subspace if it satisfies the following properties:

(1) $W \neq \emptyset$
(2) For all $\vec{x}, \vec{y} \in W$ and all $c \in \mathbb{R}$, we have $c\vec{x} + \vec{y} \in W$.